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Asset Prices, Epstein Zin Utility, and Endogenous Economic Disasters

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

The financial and economic crisis of 2007-2009 has emphasized the importance of understanding the interplay between asset markets and goods and factor markets. Macroeconomic models, which are consulted to analyze this interplay and to quantitatively assess policy options, have to be consistent with empirical regularities which characterize these markets.

Since the publication of Mehra and Prescott (1985), however, many papers have further confirmed the diagnosis that, for a reasonable degree of risk aversion, the historically observed U.S. risk premium (excess of the return on a stock market index over the return of a relatively riskless security) of over 6 percent is an order of magnitude greater than what can be explained within the paradigm of modern macroeconomics, the neoclassical stochastic growth model. This fact has been named ”the equity premium puzzle” in the literature. The inconsistency between the model and empirical data seriously questions ”the viability of using this class of models for any quantitative assessment–say, to gauge the welfare implications of alternative stabilization policies”.[1]

The source of the puzzle within a framework where assets are held by an infinitely-lived representative household with rational expectations, who maximizes his expected lifetime utility drawn from consumption streams, can be summarized as follows. An equilibrium condition for asset prices in this framework demands that the loss in utility from reducing consumption in the current period in order to buy an asset and the expected gain in utility from the future payoff of the asset must be equal. Consequently, an asset with an undesirable payoff structure, i.e. an asset which is expected to pay off comparatively much in states where the marginal utility of consumption will be low but will pay off less in states where the marginal utility of consumption will be high, will sell only for a lower price compared to an asset which pays equally well in all future states. In other words, such an asset must offer a higher equity premium in order to be held. The key to the magnitude of the equity premium, therefore, lies in the size of the negative covariation between future marginal utility (or identical the stochastic discount factor) and the payoff of a risky asset. A too small equity premium in this framework is the result from a lack of such covariation given a reasonable degree of risk aversion.

Over the past years, different attempts have been made in order to solve the puzzle.[2] One approach is concerned with modifying the preference structure from Mehra and Prescott (1985) who assume that the representative household's lifetime utility is determined as the expected discounted sum of within period CRRA[3] utilities. The standard additive time separable CRRA preference structure implies that the household's attitude towards uneven consumption paths over time, measured by the elasticity of intertemporal substitution (EIS), and the attitude towards risk from varying consumption levels in different future states, measured by the RRA, are inversely connected via $\text{EIS} = \text{RRA}^{-1}$. The class of generalized recursive preferences introduced

[1] See Mehra (2003), p. 18.

[2] See, among others, the surveys of Abel (1990), Kocherlakota (1996), and Mehra (2003).

[3] I.e. a utility function with a constant Arrow-Pratt relative risk aversion (RRA).

by Epstein and Zin (1989) (EZ) allows for disentanglement of theses quantities. While the equity premium arises since a variety of studies suggest that the coefficient of RRA is a rather small number[4]—Mehra and Prescott (1985) therefore restrict the value to be below 10 in their analysis—there is less evidence against a low EIS close to zero[5]. Hence, even when the RRA is set to a plausible low value, EZ preferences provide an additional, to some degree free parameter in the EIS which can be used to improve the equity premium puzzle. In particular, a low EIS implicates that further deviations from a smooth consumption path associated with procyclical asset payments may cause higher variation in the stochastic discount factor and therefore may generate a higher equity premium.

Another promising approach to explain a sizeable equity premium was introduced by Rietz (1988). He modifies the distribution of consumption streams in such way to allow for the possibility of rare but severe economic disasters where consumption is drastically lower. Even when the volatility of consumption (growth) remains in line with the data, the concavity of the utility function guarantees that the risk in the lottery over the marginal utility from consumption increases. Again, a procyclical asset has to offer a higher risk premium.

The three essays summarized in this thesis can be understood to be commonly concerned with these two approaches for explaining the historically high equity premium.

Chapter 2 contains the current version of the working paper ”Applied Macroeconomic Analysis with Epstein-Zin Utility” (2014a) from joint work with Halvor Ruf. This paper provides a self contained guide to the approximate solution of dynamic stochastic general equilibrium (DSGE) macroeconomies that feature EZ utility. We first summarize the crucial elements of the EZ representation as it is increasingly applied in the literature adding intuitive remarks and illustrative examples. While in the standard framework of temporal decision making under uncertainty choice is made between probability distributions over consumption sequences, EZ preferences add additional timing structure regarding the resolution of uncertainty. The additional timing structure is introduced in form of temporal lotteries describing infinite probability trees of consumption. Such an infinite probability tree is naturally characterized as a tuple of current consumption and a probability distribution over nodes of infinite probability trees emanating next period. This characterization naturally leads to a recursive utility representation. Uncertainty displayed in the distribution over utility drawn from potential probability trees emanating in the next period is evaluated with a certainty equivalent which defines the decision maker's attitude towards risk. Typically, a certainty equivalent exhibiting a constant RRA is assumed. In a second step, current consumption and the certainty equivalent for the next period are summarized with a CES time aggregator. The CES time aggregator determines the decision maker's EIS. A recursive formulation for the utility representation emerges which disentangles the RRA from the EIS. Thereafter, we demonstrate within a basic representative agent economy how EZ utility naturally lends itself to dynamic programming and apply the Schmitt-Grohe and Uribe (2004) approach to find a second order perturbation. We conclude by discussing the immediate implications of employing the EZ representation in applied work by computing an actual numerical example. In particular, we analyze the role of the RRA and the EIS on the comovement of the stochastic discount factor with the return on equity under a second order perturbation solution, providing a first intuition on the implications for the equity premium. The paper is accompanied by a flexible `Maple-Matlab` perturbation toolbox. Both authors contributed equally to the paper.

Chapter 3 is the current version of the working paper ”Epstein-Zin Utility, Asset Prices, and the Business Cycle Revisited” (2014b) together with Halvor Ruf. In this paper we analyze DSGE models with several frictions in the allocation of labor in their ability to resolve the equity premium puzzle while at the same time being consistent with the stylized facts of business cycles

[4]See Mehra and Prescott (1985) for a number of studies reporting estimates on the RRA.
[5]See e.g. Hall (1988).

of the German economy. Different from Heer and Maußner (2013), we assume EZ preferences. We set the coefficient of RRA to a low value but consider the EIS to some degree free in order to match the data. We find that within the EZ utility representation, the frictionless model already yields simulation results in good accordance with the German empirical data if the EIS is set to a low enough value. The additional flexibility provides throughout significant improvements in the models' fit to empirical characteristics compared to the results reported in Heer and Maußner (2013) for standard preferences.

Finally, chapter 4 presents the current version of my working paper "Search Frictions in the Labor Market and Endogenous Economic Disasters" (2017). The paper builds on the work of Kuehn et al. (2012, 2015). The authors claim that a standard search and matching labor market model, as developed by Diamond, Mortensen and Pissaridies[6], has the potential to explain a high equity premium. In such framework a newly created job generates a positive surplus in the economy, which is distributed between employer and employee via the wage. The wage is thereby commonly assumed to be the outcome of a bargaining process over the total surplus. Kuehn et al. (2012, 2015) follow Hagedorn and Manovskii (2008) and assume that the household's bargaining power in this process is small, yet his fall back value in case of unemployment is high and fixed. Consequently, a reservation demand for the wage close to labor productivity in steady state, but also completely inflexible over the business cycle is implied. The model can then endogenously generate severe economic disasters. Similar to the idea of Rietz (1988), it can therefore also provide a framework which explains the historically high equity premium found in the data. I analyze the driving forces behind the mechanism leading to endogenous diasters in the model and identify the crucial assumptions behind the mechanism. I then check whether these assumptions seem plausible and how the results change if they are relaxed to some degree.

[6]See, e.g., Pissarides (1985, 2000) or Mortensen and Pissarides (1994).

Chapter 2

Applied Macroeconomic Analysis with Epstein Zin Utility

— *Christopher Heiberger and Halvor Ruf* —

2.1 Introduction

To us, there are essentially three motivations for applied macroeconomists to study the Epstein and Zin (1989) (EZ) utility representation and its (incomplete) separation of the elasticity of intertemporal substitution (EIS) from the standard risk aversion parameter. First, it provides the researcher with an additional degree of freedom to improve on the empirical performance of his dynamic stochastic general equilibrium (DSGE) models. Second, it is theoretically appealing to loosen those two aspects of preferences because, a priori, there does not seem to be a reason for their reciprocity as implicitly assumed in the paradigmatic framework of additively separable expected utility. Third, probably mostly for these two reasons, the applied literature is increasingly employing EZ utility.[1]

Following the publication in 1989, EZ preferences at first found application primarily in the asset pricing literature. For example, Epstein and Zin (1990) demonstrated that this class of preferences can help to improve the equity premium puzzle reported by Mehra and Prescott (1985). While U.S. data shows an historical average risk premium of 6.18% p.a. over the period 1889-1979, Mehra and Prescott (1985) find that in an exchange economy, where the representative agent's lifetime utility is determined as the expected discounted sum of within period CRRA[2] utilities and where the RRA is restricted to a plausible value below 10, the largest equity premium obtainable is only 0.35%. Introducing EZ preferences into the model, Epstein and Zin (1990) can generate a low risk free rate together with an average equity premium of roughly 2%. Although still less than one third of the historical average risk premium found in the U.S. data, the premium in the model thus rises by an order of magnitude compared to the findings for standard preferences. Further, Kandel and Stambaugh (1991) analyze a representative agent model with EZ preferences in order to separate the effects from either risk aversion or the elasticity of intertemporal substitution on various (first and second) moments of asset returns. However, these earlier studies making use of EZ preferences were mainly restricted to endowment economies where consumption follows an exogenously specified process. Hence, they neglect the effects from the studied variations in the parameter values determining the agent's risk aversion or the elasticity of intertemporal substitution on consumption choice itself as well as feedback effects from adjusted optimal consumption plans on asset returns.

On the other hand, the strand of studies, including e.g. Jermann (1998), Lettau and Uhlig (2000) and Boldrin et al. (2001), that focused on bringing macroeconomic DSGE models into line

[1] Note the introductory remarks in van Binsbergen et al. (2012) and the sources cited therein.

[2] I.e. a utility function with a constant Arrow-Pratt relative risk aversion (RRA).

with both classical real business cycle statistics as well as asset pricing figures found in the data, at first mainly concentrated on consumption habits, as initially proposed by Constantinides (1990), as a possible way to modify the standard preference structure. An early exception is provided by Tallarini (2000). He considers the effects of a varying coefficient of relative risk aversion in a standard stochastic growth model with EZ preferences where the elasticity of intertemporal substitution is fixed to unity. The results are in contrast to earlier findings for production economies with standard additively separable expected utility. Rouwenhorst (1995) found that in the standard setting, increasing risk aversion, coming necessarily hand in hand with a declining elasticity of intertemporal substitution, leads endogenously determined consumption decisions to become smoother. Consequently, as too smooth consumption is already a factor for a too low equity premium in endowment economies, explaining sizeable premia by increasing risk aversion in production economies turns out even more difficult and additionally reduces the model's consumption dynamics significantly. With EZ preferences however, Tallarini (2000) draws the conclusion that the second moment properties of the business cycle are mainly controlled by the elasticity of intertemporal substitution, while they remain almost unaffected by changes in the risk aversion parameter. Therefore, the additional degree of freedom from the disentanglement of the risk aversion parameter from the intertemporal elasticity of substitution helps Tallarini (2000) to improve "the model's performance with regard to asset pricing while not significantly diminishing its ability to account for quantity dynamics."[3]

Consequently, more recent work from, among others, Kaltenbrunner and Lochstoer (2010), Andreasen (2012), Gourio (2012, 2013), Rudebusch and Swanson (2012) and van Binsbergen et al. (2012), has put increasing focus on EZ preferences as a potential mechanism in order to (successfully) replicate both classical real business cycle statistics as well as asset pricing figures.

Now, although there is work on both the rationale behind the EZ representation and its approximation[4], we find the more recent DSGE literature to mainly employ EZ preferences without further discussion of their theoretical background, in particular regarding differences to the standard case of temporal decision making under uncertainty where choice is made between probability distributions over consumption sequences. We find the literature to lack a unified approach which guides the reader from the theoretical framework of preferences over temporal lotteries and their natural representation via a generalized recursive utility function, as introduced by Kreps and Porteus (1978) and Epstein and Zin (1989), to the implementation into standard applied macroeconomic analysis of intertemporal decision problems and their approximate solution. This paper mainly intends to fill this gap and to additionally provide guidance with respect to the expected benefits for applied research.

The remainder is organized as follows. The following two sections summarize the crucial elements of the EZ representation. The material is enhanced with intuitive remarks and illustrative examples. Section 2.4 demonstrates the application of EZ utility to a standard representative agent decision problem as well as the application of the Schmitt-Grohe and Uribe (2004) second order perturbation approach for its approximate solution. A discussion on some immediate implications of the EZ representation for applied work is provided within a numerical exemplification. The employed perturbation routines are collected in a very flexible `Maple–Matlab` toolbox which is briefly documented in section 2.5. Section 2.6 concludes the paper.

2.2 Consumption space

This section introduces the key notion of temporal lotteries and their respective identification as a pair of current consumption and a probability distribution over future temporal lotteries. The

[3]See Tallarini (2000), p. 508.

[4]See e.g. Backus et al. (2008) or Altug and Labadie (2008) for a description of the representation or Caldara et al. (2012) on the approximation techniques.

inherent recursiveness of this identification will give rise to a recursive utility representation of preferences over such temporal lotteries in section 2.3.

2.2.1 Preliminary remarks

Most of the applied DSGE literature relies on a common and thus standard framework of temporal decision making under uncertainty: choice is made between probability distributions over consumption sequences, i.e. over stochastic consumption processes.

The underlying idea of the approach applied in the present work is the introduction of additional structure to the fundamental consumption decision problem. Not only is the probability distribution of consumption sequences considered but also the time at which the uncertainty concerning future consumption is resolved. This is done via the concept of probability trees, so called *temporal lotteries*.[5] This idea and the ensuing construction of the consumption space was introduced by Kreps and Porteus (1978) and extended to an infinite horizon setting by Epstein and Zin (1989). While the former provide an axiomatization for a recursive utility representation over finite temporal lotteries, the latter prove the existence of a recursive utility function over some space of infinite horizon temporal lotteries. This section is concerned with the presentation of these ideas and is thereby intended to summarize particularly crucial results. We thereby often sacrifice the generality of the original treatment in order to keep a focus on applicability. This will provide us with the necessary basis for the application of the EZ approach to neoclassical macroeconomic analysis in the remainder of this paper.

2.2.2 Notation

Let X be a metric space. Denote by

$$\mathscr{B}(X) := \sigma\left(\{O \subset X \mid O \text{ open}\}\right)$$

the induced Borel σ–algebra on X and by

$$\mathscr{M}(X) := \{p : \mathscr{B}(X) \to [0,1] \mid p \text{ probability measure}\}$$

the set of Borel probability measures. Particularly, for $x \in X$ let

$$\begin{aligned} \delta_x : \quad \mathscr{B}(X) \quad &\to \quad [0,1] \\ B \quad &\mapsto \quad \delta_x(B) := \begin{cases} 1, & x \in B \\ 0, & x \notin B \end{cases}, \quad \forall B \in \mathscr{B}(X) \end{aligned}$$

be the Dirac probability measures. Moreover, time is discrete and the planning horizon is infinite. Hence, by $t \in \mathbb{N}$ we denote a point in time or its respective period.

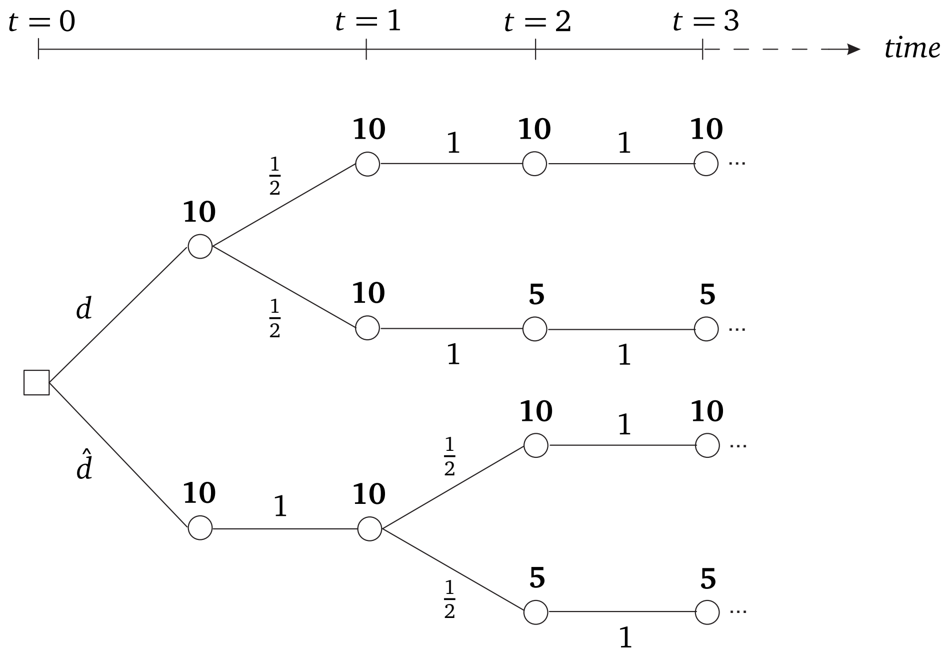
2.2.3 Temporal lotteries

Let D denote the space of temporal lotteries. Elements of D can be pictured as infinite probability trees and can thus naturally be identified with a tuple of current consumption and a probability distribution over nodes of infinite probability trees emanating next period. Accordingly, we will shortly find the space of temporal lotteries D to be homeomorphic to $\mathbb{R}_+ \times \mathscr{M}(D)$. In order to motivate the analysis of temporal lotteries, consider the following example.[6]

[5] Accordingly, their atemporal analog mentioned above will sometimes be called an *atemporal lottery*.

[6] This example is a modified version of the coin flip example originally provided by Kreps and Porteus (1978). We use our formulation instead for the sake of exposition.

Figure 2.1: Temporal decision problem



Example 1. Robinson Crusoe is facing the problem of choosing between two seed technologies. Both possible technologies offer an aggregate period t harvest (output) Y_t as a response to this period's aggregate input K_t according to

$$Y_t = A_t \cdot K_t, \text{ where } A_2 = \begin{cases} 2, & \text{with probability } \frac{1}{2} \\ 1, & \text{with probability } \frac{1}{2} \end{cases} \text{ and } A_t = 2 \; \forall t \neq 2.$$

The difference between these two technologies is the time at which the technologies' period 2–types are revealed.

Suppose for the first technology, its period type is revealed at the beginning of period 1 whereas for the second technology Robinson does not know about its type before period 2. Further suppose that for all periods, Robinson is rigidly bound to a somewhat primitive allocation as he distributes a constant fraction $\frac{1}{2}$ of (produced) resources to both consumption and next period's capital stock.

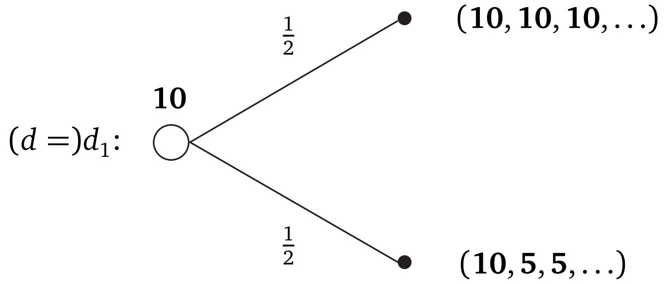
He now decides between these two alternative technologies according to their resulting future consumption prospects, his utility argument. We identify such a decision between actions with the decision between the probability trees induced by these actions. The decision problem is pictured in figure 2.1, where d and $\hat{d}$ denote the consumption probability trees that correspond to an initial endowment of 20.[7] Observe that from a period 0 point of view, both technologies result in the same *atemporal* distribution over future consumption, namely $\frac{1}{2}\delta_{(10,10,...)} + \frac{1}{2}\delta_{(10,10,5,5,...)}$. Hence, if we take the consumption space to be the space of atemporal lotteries over consumption streams—as it is implied by the standard model—there is no way to distinguish between the consequences of these alternative technologies. However evidently, the two pictured probability trees are not identical.

We conclude this introductory example with a brief discussion on its strength as a motivation for the upcoming analysis. For the case of lotteries over *income* sequences there is very little controversy about whether different temporal lotteries inducing identical atemporal lotteries ought to be modelled in a way that allows the decision maker to prefer one over the other.[8]

[7] In such graphs, squares denote action nodes while circles denote uncertainty nodes, cf. Raiffa (1970), p. 11.

[8] Thereby, preference for income sequences is understood as being induced from the primitive preference for consumption sequences as the ultimate source of utility.

Figure 2.2: Temporal lottery with no uncertainty after period 1



For *consumption* sequences, the situation is less evident. To see this, take a second look at our example. If Robinson allocated consumption according to an optimality criterion such as the maximization of his lifetime utility, it would typically be of value for him to know about the future *production* in advance in order to improve on the allocation by situational consumption adaption.[9] However, there is no apparent planning advantage for consumption sequences, as the one just pictured.

We eventually provide two remarks on that. First, even if there is no immediate reason for why it *must* be the case that a decision maker would favor prior or later resolution of consumption uncertainty, it also seems odd to insist on the view that any decision maker would *never* be able to appreciate it. Above all, introspection enhances the latter doubts. As Epstein (1992), p. 23, notes e.g. later knowledge about the future to come might very well be preferred by a person who likes to "defer resolution in order to [be able to sustain] the hope [...] for a favorable outcome for a risky prospect." Moreover, the "rationality" of nonindifference towards the timing of resolution of consumption uncertainty is nicely exemplified in Chew and Epstein (1989). Second, as it turns out, it is exactly this nonindifference that allows to loosen the strict entanglement of risk attitudes and intertemporal consumption substitution as implied by the standard model such that it leaves macroeconomists with an additional degree of freedom in replicating empirical data as noted above. A preference for earlier or later resolution of consumption uncertainty can hence also be interpreted as a cost of the last mentioned advantages, cf. Epstein et al. (2014).

The construction of the consumption space carried out in Epstein and Zin (1989), p. 941-944, is mathematically involved. Since the ideas behind are nevertheless indispensable for our intended discussion of the actual application of EZ utility to a basic DSGE economy, the remainder of this section summarizes their treatment in detail and complements it with some examples and additional intuition. Still, at some places more rigorous remarks supplement our summary.

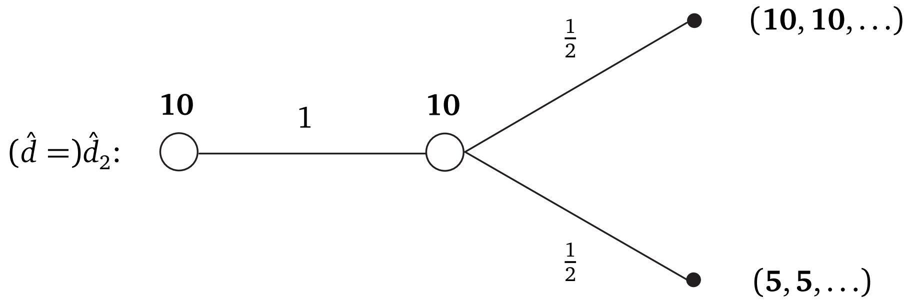
2.2.3.1 Infinite trees of finite length

The example above shows the following. If we identify each branch of a probability tree with a consumption sequence and if—as in the case of the example probability tree d—there is no uncertainty after period 1, then no structure is lost by simply considering the "finite" probability tree d_1 that results through considering only the probability distribution of the consumption sequence starting at period 1 next to initial consumption. We thus say that the infinite tree d_1 has length 1 and generally call an infinite tree of finite length t a finite (t-stage) tree. Since we only consider infinite horizon decisions, this should not cause confusion but shorten the ensuing analysis.

In particular, figure 2.2 reveals that the probability tree d_1 is unambiguously represented by the tuple (c_0, m_1) consisting of current consumption $c_0 = 10$ and the probability measure

[9]This point is nicely illustrated in Spence and Zeckhauser (1972).

Figure 2.3: Temporal lottery with no uncertainty after period 2



$m_1 = \frac{1}{2}\delta_{(10,10,10,\dots)} + \frac{1}{2}\delta_{(10,5,5,\dots)} \in \mathscr{M}(\mathbb{R}_+^\infty)$ over future consumption sequences. Accordingly, we identify the infinite probability tree d with the finite tree $d_1 = (c_0, m_1) \in \mathbb{R}_+ \times \mathscr{M}(\mathbb{R}_+^\infty)$.

If we now turn to the infinite probability tree $\hat{d}$ instead (cf. figure 2.3), we observe that in this case there is no uncertainty after period 2. Analogously, $\hat{d}$ can be pictured as a finite two–stage probability tree $\hat{d}_2$. The one–stage tree emerging at its second node, $\hat{d}_1$, is similarly identified with the tuple $(\hat{c}_1, \hat{m}_1)$ out of period 1's consumption level $\hat{c}_1 = 10$ and the probability distribution $\hat{m}_1 = \frac{1}{2}\delta_{(10,10,\dots)} + \frac{1}{2}\delta_{(5,5,\dots)} \in \mathscr{M}(\mathbb{R}_+^\infty)$ over consumption as of period 2, i.e. $\hat{d}_1 = (\hat{c}_1, \hat{m}_1) \in \mathbb{R}_+ \times \mathscr{M}(\mathbb{R}_+^\infty)$. Consequently, the whole two–stage probability tree $\hat{d}_2$ now comes up to a tuple of current consumption $\hat{c}_0 = 10$ and a degenerate probability distribution over nodes of one–stage trees $\hat{m}_2 = \delta_{\hat{d}_1} \in \mathscr{M}\left(\mathbb{R}_+ \times \mathscr{M}(\mathbb{R}_+^\infty)\right)$. I.e. we have $\hat{d}_2 = (\hat{c}_0, \hat{m}_2) \in \mathbb{R}_+ \times \mathscr{M}\left(\mathbb{R}_+ \times \mathscr{M}(\mathbb{R}_+^\infty)\right)$.

In general, continuing this reasoning inductively we can define finite probability trees of length t. In such trees, the way in which uncertainty resolves over time is only displayed until period $t-1$ and the only information encoded about future periods' consumption as of t is their joint probability distribution. Precisely, a probability tree of length t can be described completely by a pair of today's consumption and a probability measure over nodes of trees of remaining length $t-1$. I.e. we recursively define

$$
\begin{aligned}
&D_0 := R_+^\infty, \\
&D_t := \mathbb{R}_+ \times \mathscr{M}(D_{t-1}) \ \text{ f.a. } t \geq 1.
\end{aligned}
$$

Observe that any infinite probability tree which does not contain any uncertainty from period t on can be represented by such a t–stage probability tree without loss of information.

As mentioned above, we will from time to time add more rigorous remarks to round off our treatment. In this spirit, note that since R_+^∞ is a complete, separable metric space, i.e. Polish, so is $\mathscr{M}(R_+^\infty)$ with the weak topology. Therefore, $D_1 = R_+ \times \mathscr{M}(R_+^\infty)$ is also a Polish space with the product topology on it. Inductively it follows that D_t is a Polish space f.a. $t \geq 1$ if we recursively endow each $\mathscr{M}(D_{t-1})$ with the weak topology and D_t with the induced product topology. For each $t \in \mathbb{N}$ we denote by $\mathscr{B}_t := \mathscr{B}(D_t)$ the respective Borel σ–algebra.

2.2.3.2 Infinite trees of arbitrary length: consistent reduction

The idea behind formally defining an arbitrary infinite probability tree d is to approximate it stepwise by the t–stage probability trees, which arise from d by reducing its temporal structure in such a way that the reduced tree has the identical distribution over consumption sequences as of period t. Since the reduced tree does not contain any information of how the uncertainty regarding these consumption sequences as of t resolves over time, considering the reduced tree is like pretending that all uncertainty about consumption will have been resolved by period t.

This way we get a sequence $(d_1, d_2, \dots)$ of finite probability trees $d_t \in D_t$, each describing the structure of d with increasing accuracy. Since for each $t \in \mathbb{N}$, the probability tree d_t describes the temporal structure of d up to period t, the whole sequence $(d_1, d_2, \dots)$ describes the entire

Figure 2.4: Consistent reduction (example)

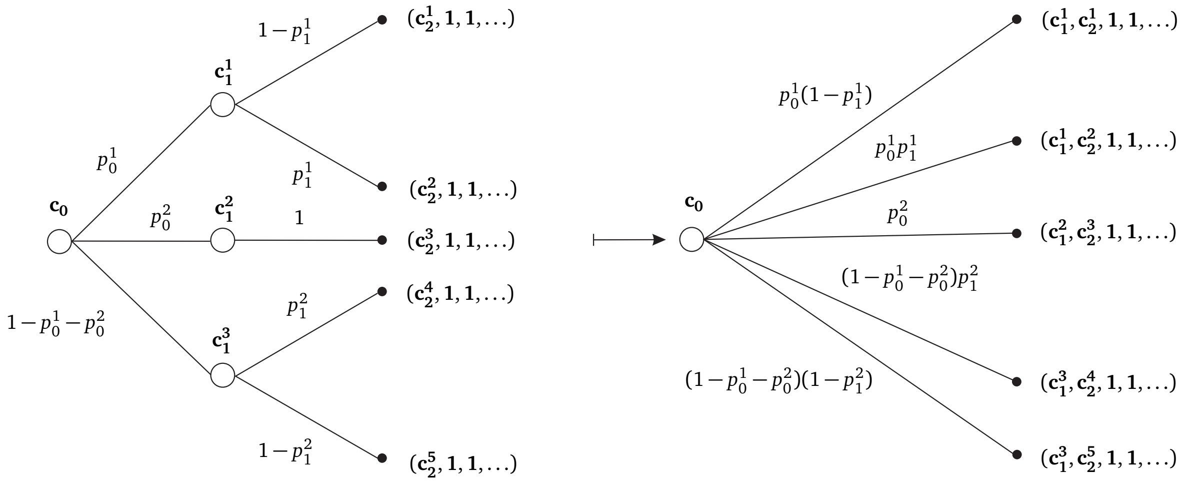
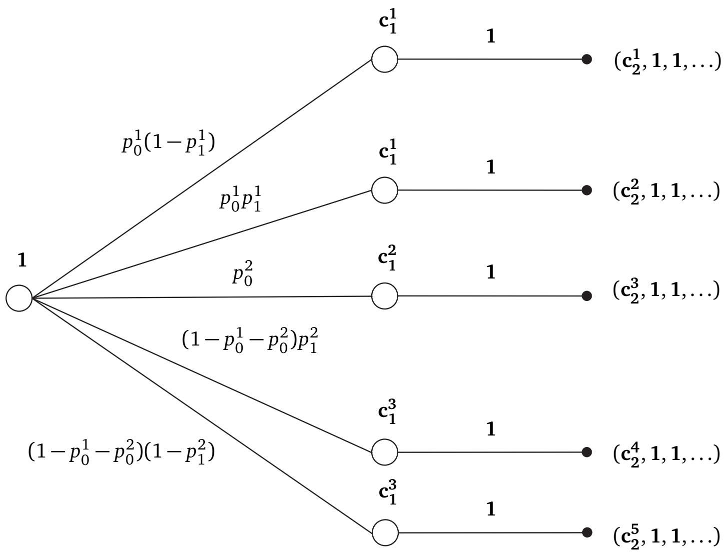


Figure 2.5: Trivial D_2 tree (example)



structure of d. Therefore, we can identify each infinite probability tree with exactly one such sequence. Accordingly, we define the set D of all infinite probability trees as the set of all sequences

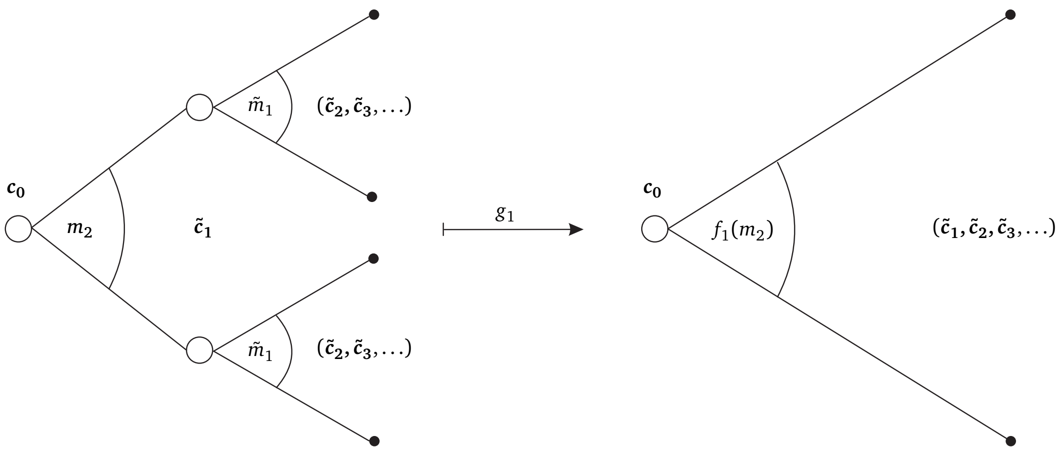
$$d = (d_1, d_2, \ldots),\ d_t \in D_t \text{ f.a. } t \in \mathbb{N},$$

which are consistent in the following sense. For $d_t \in D_t$ and $d_{t+1} \in D_{t+1}$ to be consistent it must hold that up until period t they both obey the same structure and that d_t can be imagined as d_{t+1} folded back one period. To put it another way, d_t and d_{t+1} are consistent if d_t results from d_{t+1} by neglecting its temporal structure already in period t and merely considering the joint distribution of consumption to come. The process of consistently reducing probability trees is first illustrated in the following example and then outlined generally.

Example 2. Consider a two-stage probability tree. As depicted in figure 2.4, consistent reduction of such a tree demands the computation of the induced distribution of consumption to come as of period 1. Note that e.g. the two-stage tree pictured in figure 2.5 is trivially consistently reduced to the same D_1 tree as the two-stage tree of figure 2.4. Hence, consistent reduction cannot be an injection. In particular, for all trees that share this consistent reduction, the induced probability measure must e.g. for $B = \{(c_1^1, c_2^1, 1, 1, \ldots), (c_1^3, c_2^4, 1, 1, \ldots)\} \in \mathscr{B}\left(\mathbb{R}_+^\infty\right)$ yield

$$P(B) = p_0^1(1-p_1^1) + (1-p_0^1-p_0^2)p_1^2.$$

Figure 2.6: Consistent reduction



In general, consistent reduction can formally be stated as follows. Starting with the reduction of a two–stage probability tree to a tree of length 1, we first note that such a tree is given by $d_2 = (c_0, m_2)$, where $m_2 \in \mathcal{M}\left(\mathbb{R}_+ \times \mathcal{M}(\mathbb{R}_+^\infty)\right)$ is the probability measure over the random tuple $(\tilde{c}_1, \tilde{m}_1)$, i.e. over tomorrow's consumption level $\tilde{c}_1$ and the joint probability measure for $(\tilde{c}_2, \tilde{c}_3, \ldots)$. Hence, as exemplified above, for every $B \in \mathcal{B}(\mathbb{R}_+^\infty)$ we get

$$\begin{aligned} P(B) \equiv & P(\{(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \ldots) \in B\}) = \\ & \int\limits_{\mathbb{R}_+ \times \mathcal{M}(R_+^\infty)} \tilde{m}_1\left(\left\{(\tilde{c}_2, \tilde{c}_3, \ldots) \in \mathbb{R}_+^\infty \mid (\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \ldots) \in B\right\}\right) dm_2(\tilde{c}_1, \tilde{m}_1). \end{aligned}$$

Accordingly, we define

$$\begin{array}{llll} f_1: & \mathcal{M}(D_1) & \to & \mathcal{M}(D_0) \\ & m_2 & \mapsto & f_1(m_2): \begin{array}{ll} \mathcal{B}(\mathbb{R}_+^\infty) & \to [0,1] \\ B & \mapsto \int \tilde{m}_1\left(\left\{y \in \mathbb{R}_+^\infty \mid (\tilde{c}_1, y) \in B\right\}\right) dm_2. \end{array} \end{array}$$

The mapping f_1 directly yields a probability measure over consumption sequences $(\tilde{c}_1, \tilde{c}_2, \ldots)$ out of a probability measure m_2 over nodes of one–stage trees. Thus, we define

$$\begin{array}{llll} g_1: & D_2 & \to & D_1 \\ & (c_0, m_2) & \mapsto & (c_0, f_1(m_2)). \end{array}$$

This is the desired mapping for consistently reducing a two–stage tree to its one–stage counterpart (cf. figure 2.6). Note again that, as illustrated in the example above, g_1 is not injective. One can furthermore show that f_1 and g_1 are both continuous and therefore measurable.

We now pursue inductively. Suppose the desired continuous mappings

$$\begin{array}{llll} f_i: & \mathcal{M}(D_i) & \to & \mathcal{M}(D_{i-1}) \\ g_i: & D_{i+1} & \to & D_i \end{array}$$

have already been constructed for $i = 1, \ldots, t-1, t > 1$, such that $g_i(d_{i+1})$ is the resulting tree of length i that consistently corresponds to d_{i+1}. Consider an arbitrary probability tree $d_{t+1} \in D_{t+1}$ of length $t+1$. By definition, d_{t+1} is a tuple (c_0, m_{t+1}) of a non-negative real number c_0 representing today's consumption and a probability measure m_{t+1} over nodes of probability trees of length t,

$$d_{t+1} = (c_0, m_{t+1}) \in \mathbb{R}_+ \times \mathcal{M}(D_t).$$

Let $\tilde{d}_t$ denote the random variable of trees of length t emerging at stage 1 distributed according to m_{t+1}. If we want to consistently shorten the $(t+1)$–stage tree d_{t+1} by one step, we have to shorten the random t–stage tree $\tilde{d}_t$ by one step. Now, we already know that the latter reduction is executed via the mapping g_{t-1}. So by $g_{t-1}(\tilde{d}_t)$ we get the random $(t-1)$–stage tree emanating from the first node of the desired t–stage tree d_t that consistently corresponds to d_{t+1}. The wanted induced probability measure over $(t-1)$–stage trees is thus given by the distribution of $g_{t-1}(\tilde{d}_t)$, i.e. by

$$m_{t+1} \circ g_{t-1}^{-1} : \mathscr{B}_{t-1} \to [0,1].$$

We accordingly set

$$\begin{array}{cccc} f_t: & \mathscr{M}(D_t) & \to & \mathscr{M}(D_{t-1}) \\ & m_{t+1} & \mapsto & m_{t+1} \circ g_{t-1}^{-1} \end{array}$$

and

$$\begin{array}{cccc} g_t: & D_{t+1} & \to & D_t \\ & (c_0, m_{t+1}) & \mapsto & (c_0, f_t(m_{t+1})) \,. \end{array}$$

It follows inductively, that all f_t and g_t are continuous and thus measurable and also that g_t is not injective for all t. Observe that it is this non-injectivity that gives rise to the notion of nonindifference towards the timing of uncertainty resolution.

Now that we formally described what it means for a sequence of trees $(d_1, d_2, \ldots), d_t \in D_t$, to be consistent, we round off this subsection replicating a result in Epstein and Zin (1989) proving what we have already stated intuitively at the beginning of this section. Namely, every infinite probability tree can be identified unambiguously with a tuple of current consumption and a probability measure over nodes of probability trees emerging at period 1.

Definition 1. *We define $(D, \mathscr{B})$ as the inverse limit of the separable measurable spaces $(D_t, \mathscr{B}_t), t = 1, 2, \ldots$, relative to the measurable mappings $g_t : D_{t+1} \to D_t$. I.e.*

(i) $D = \left\{ (d_1, d_2, \ldots) \in \prod_{t=1}^{\infty} D_t \mid d_t = g_t(d_{t+1}) \text{ f.a. } t \geq 1 \right\}$

(ii) $\mathscr{B}$ is the smallest σ–algebra of subspaces of D that renders the canonical projection

$$\pi_t : D \to D_t, (d_1, d_2, \ldots) \mapsto d_t$$

measurable for all $t = 1, 2, \ldots$, i.e. $\mathscr{B} = \sigma\left(\bigcup_{t=1}^{\infty} \{\pi_t^{-1}(B_t) \mid B_t \in \mathscr{B}_t\} \right)$.

Theorem 1. *First, D is a Polish space relative to the subspace topology that is induced by the product topology on $\prod_{t=1}^{\infty} D_t$ and $\mathscr{B}$ equals the Borel σ–algebra that is generated by this topology on D. Second, D is homeomorphic to $\mathbb{R}_+ \times \mathscr{M}(D)$.*

Proof. The first part follows from Parthasarathy (1967), Theorem 2.6. For the second part, let $d = (d_1, d_2, \ldots) \in D, d_t = (c_0, m_t) \in D_t, d_t = g_t(d_{t+1})$ f.a. $t \in \mathbb{N}$ be arbitrary. Thus, by definition we have for all $t \in \mathbb{N}$ a probability measure m_{t+1} on $\mathscr{B}_t$ satisfying $m_{t+1} = f_{t+1}(m_{t+2}) = m_{t+2} \circ g_t^{-1}$. Following Parthasarathy (1967), Theorem 3.2, there exists a unique probability measure $m : \mathscr{B} \to [0,1]$ such that $m\left(\pi_t^{-1}(B_t)\right) = m_{t+1}(B_t)$ f.a. $t \in \mathbb{N}, B_t \in \mathscr{B}_t$. By setting $\Phi(d) := (c_0, m)$ we define the mapping $\Phi : D \to \mathbb{R}_+ \times \mathscr{M}(D)$. One can now show that this mapping is a homeomorphism. □

Before we proceed, let us provide some additional intuition about the constructed homeomorphism and its significance for the upcoming utility analysis. Consider again an arbitrary probability tree $d = (d_1, d_2, \ldots) \in D, d_t = (c_0, m_t) \in D_t, d_t = g_t(d_{t+1})$ f.a. $t \in \mathbb{N}$. Approximated until stage $(t+1)$, this tree is given by $\pi_{t+1}(d) = d_{t+1} = (c_0, m_{t+1})$ and the trees of length t emerging at the first stage of d_{t+1} are distributed according to m_{t+1}. Yet on the other hand, the probability trees originating at the first stage of d_{t+1} also correspond to the t–stage approximations of the infinite probability trees emerging at the first stage of the whole tree d. Since the infinite probability trees emerging at stage 1 of d are distributed according to the probability measure m and their t–stage approximation is given via the mapping π_t, these t–stage approximations are distributed according to the probability distribution $m \circ \pi_t^{-1}$. So it must hold that

$$m \circ \pi_t^{-1} = m_{t+1}.$$

Moreover, observe that the fact that D is homeomorphic to $\mathbb{R}_+ \times \mathscr{M}(D)$ importantly says that for every tree $d \in D$ it holds that every "subtree" that emanates from some of its intermediary nodes necessarily also lies in D. This "stationarity" of the consumption space is necessary for the existence of a recursive utility function on D.[10]

2.2.3.3 Additional restrictions to the lottery space

The space D will provide the fundament for our consumption space, i.e. the space over which decisions will be made. In order to describe such decisions, we will introduce eligible utility functions in the next section. However, these utility functions can generally not be defined on the whole space D but only on particular subspaces. In fact, consumption has to be bounded in some sense. Therefore, we have to further narrow the lottery space appropriately.

For that purpose, Epstein and Zin (1989) define the space of consumption sequences such that the gross growth rate is capped by some $b \geq 1$, i.e. for $l > 0$

$$Y(b, l) := \left\{ (c_0, c_2, \ldots) \in \mathbb{R}_+^\infty \,\middle|\, \frac{c_t}{b^t} \leq l \text{ f.a. } t \in \mathbb{N} \right\} = \prod_{t=0}^{\infty} [0, b^t l].$$

Endow $Y(b, l)$ with the product topology and note that according to Tychonoff's theorem $Y(b, l)$ is compact. The subspaces $D(b)$ of D consisting only of probability trees $d = (d_1, d_2, \ldots), d_t = (c_0, m_t)$, such that the atemporal probability measure m_1 gives rise to consumption sequences $(c_1, c_2, \ldots)$ in $Y(b, l)$ for some $l > 0$ with probability 1, are suitable as domains for recursive utility functions.

Definition 2. *For $b \geq 1$ define*

$$D(b) := \{d = (d_1, d_2, \ldots) \in D \mid d_1 = (c_0, m_1) \text{ s.t. } \exists l > 0 : \operatorname{supp}(m_1) \subset Y(b, l)\}$$

and endow $D(b)$ with the subspace topology.

Note that since $D(b)$ is a subspace of a separable metric space, it is thus a separable metric space itself. Moreover, as a subset of D it is homeomorphic to a subset of $\mathbb{R}_+^\infty \times \mathscr{M}(D)$, i.e. via Φ every probability tree $d \in D(b)$ can uniquely be represented as a tuple of the consumption level c_0 today and a probability measure over nodes of trees emanating at period 1. However, as the next example will illustrate, not every probability measure $m \in \mathscr{M}(D(b))$ is in question for this identification.

Example 3. For $n \geq 1$ consider the probability tree d^n depicted in figure 2.7. The corresponding

[10] Cf. Epstein and Zin (1989), p. 941, and the introductory remarks to section 2.3.

Figure 2.7: Degenerate temporal lottery

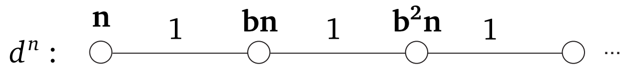
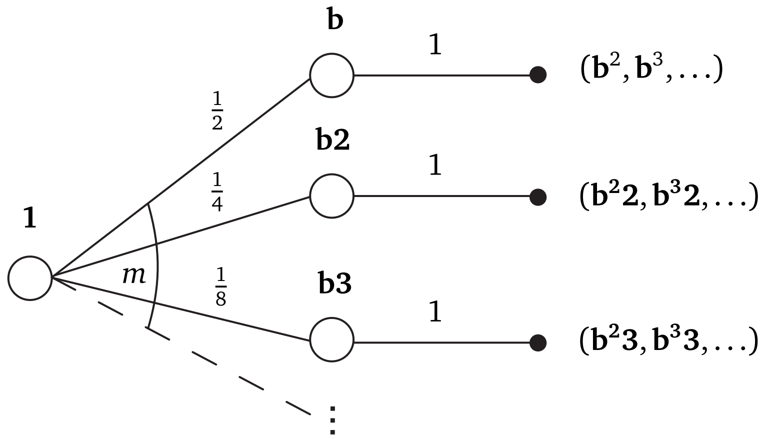


Figure 2.8: Temporal lottery *not* in $D(b)$



1–stage trees are given by $d_1^n = (n, m_1^n) \in D_1$, where $m_1^n = \delta_{(bn,b^2n,b^3n,\ldots)}, n \geq 1$. Since $\operatorname{supp}(m_1^n) \subset Y(b,n), n \geq 1$, it holds that $d^n \in D(b)$ f.a. $n \geq 1$. But now consider the probability defined by

$$m = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_{d^{bn}} \in \mathcal{M}(D(b)).$$

We want to show that the tree $d \in D$ that via Φ corresponds to the tuple $(1, m) \in \mathbb{R}_+ \times \mathcal{M}(D(b))$ does not lie in $D(b)$. The tree is pictured in figure 2.8 and it can be seen that its 1-stage approximation is given by $d_1 = (1, m_1)$, where $m_1 = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_{(bn,b^2n,b^2n,\ldots)}$. Since for every $b \geq 1$ there is no $l > 0$ such that $\operatorname{supp}(m_1) \subset Y(b,l)$, it follows that $d \notin D(b)$.

This example makes clear that $D(b)$ cannot be homeomorphic to $\mathbb{R}_+ \times \mathcal{M}(D(b))$ but that only a subset of $\mathcal{M}(D(b))$ is in question. In order to see which probability measures these are, let us again consider an arbitrary tuple $(c_0, m) \in \mathbb{R}_+ \times \mathcal{M}(D(b))$. The corresponding tree $d = (d_1, d_2, \ldots) \in D, d_t = (c_0, m_t)$, is identified through $m_{t+1} = m \circ \pi_t^{-1}, t \geq 1$ (cf. Theorem 1). In particular, we have that $m_2 = m \circ \pi_1^{-1}$ and hence $m_1 = f_1(m \circ \pi_1^{-1})$. In order for d to lie in $D(b)$ it is necessary and sufficient that $\operatorname{supp}(f_1(m \circ \pi_1^{-1})) \subset Y(b,l)$ for some $l > 0$. If we restrict the homeomorphism $\Phi : D \to \mathbb{R}_+ \times \mathcal{M}(D)$ to $D(b)$ we get a homeomorphism $\Phi\big|_{D(b)}$:

$$D(b) \to \mathbb{R}_+ \times \{m \in \mathcal{M}(D(b)) \mid \operatorname{supp}(f_1(m \circ \pi_1^{-1})) \subset Y(b,l), \text{ for some } l > 0\}.$$

We finish this section by summarizing these considerations with the following Theorem.

Theorem 2. *$D(b)$ is homeomorphic to $\mathbb{R}_+ \times \hat{\mathcal{M}}(D(b))$ with*

$$\hat{\mathcal{M}}(D(b)) := \{m \in \mathcal{M}(D(b)) \mid \operatorname{supp}(f_1(m \circ \pi_1^{-1})) \subset Y(b,l), \textit{ for some } l > 0\},$$

where $\hat{\mathcal{M}}(D(b))$ is endowed with the subspace topology induced from $\mathcal{M}(D(b))$.

2.3 EZ utility

Having reached a formal description of the concept of an infinite probability tree, we next want to describe the decision making over such trees by means of the EZ utility representation.

Thereby, we parallel the approach taken above in that we identify the decision over such infinite trees with the two-stage decision over current consumption and future utility prospects. I.e., precisely how we represented an infinite probability tree for the infinite random consumption sequence $(c_0, c_1, c_2 \ldots)$ by a pair (c_0, m) of period 0 consumption and a probability measure over nodes of probability trees emanating next period, we now identify the utility of such an infinite consumption sequence $(c_0, c_1, c_2 \ldots)$ by a recursive two–period utility stemming from today's consumption level c_0 and future utility contingent on next period's random node.

The mechanics of an EZ utility function $U : D(b) \to \mathbb{R}_+$ can be described as follows. Let $d \in D(b)$ denote an arbitrary infinite probability tree identified with the tuple $(c_0, m) \in \mathbb{R}_+ \times \hat{\mathscr{M}}(D(b))$ via the above constructed homeomorphism (cf. Theorem 2). Hence, the random node of a probability tree $\tilde{d}$ emerging at period 1, is distributed according to the probability measure m. Thus, if we evaluate possible next period nodes by their utility through $U(\tilde{d})$, we get an induced probability distribution over random real utility levels as of tomorrow. Given measurability of U, this random future utility is distributed according to

$$m_U := m \circ U^{-1} : \mathscr{B}(\mathbb{R}_+) \to [0, 1].$$

In other words, the probability measure m_U describes an atemporal lottery over next period's random utility. Given monotonic preferences over such atemporal utility lotteries,[11] we will assume these to be representable through a certainty equivalence functional

$$\mu : \mathscr{M}(\mathbb{R}_+) \to \mathbb{R}_+,$$

meaning for any lottery $P \in \mathscr{M}(\mathbb{R}_+)$ the decision maker is indifferent between the lottery itself and a certain utility level $\mu(P)$. Note that μ thereby aggregates the decision maker's evaluation of uncertain future utility along next period's nodes. This perspective gives rise to the intuitive notion of uncertainty aggregation as coined by Träger (2011). Finally, this certainty equivalent is combined with today's consumption c_0 via

$$W : \mathbb{R}^2_+ \to \mathbb{R}_+,$$

acting as a time aggregator of the utility contributions of the elements of (c_0, m). To sum up, for all $d \in D(b)$

$$U(d) = W\left(\Phi_1(d), \mu\left(\Phi_2(d) \circ U^{-1}\right)\right) = W(c_0, \mu(m_U)). \tag{2.3.1}$$

A utility function U over $D(b)$ that satisfies the above equation is called recursive. Koopmans (1960) introduced the notion of a recursive utility function on infinite deterministic consumption programs by means of a function that aggregates current consumption and future continuation utility. On the other hand, Selden (1978) provided a representation for two-period "certain×uncertain" consumption programs, where utility is defined as an aggregation of current consumption and a certainty equivalent of next period's random consumption level. A time-consistent multi-period extension of the Selden representation is found in Kreps and Porteus (1978).[12] The stochastic generalization to the infinite horizon recursive Koopmans representation is eventually provided by Epstein and Zin (1989) as described above. Note thereby that assuming an EZ representation is sufficient for the underlying preference ordering over temporal consumption lotteries to satisfy stationarity and intertemporal consistency.[13] Further note the relation of such stationarity of preferences and the "stationarity" of the consumption space as proven above.

[11] By assuming monotonic preferences, we plausibly model the decision maker as favoring higher utility levels over lower, but see later.

[12] Cf. the introductory remarks in Weil (1990).

[13] Cf. Epstein and Zin (1989), p. 945.

2.3.1 Uncertainty aggregation

As already sketched above, the idea of a certainty equivalent with regard to random continuation utility is to assign a deterministic level of appreciation (utility level) to every atemporal lottery over continuation utilities, which renders the decision maker indifferent to the latter. In order for this to make sense, we first have to introduce preferences over atemporal continuation utility lotteries. Thereby, we discuss a set of assumptions about such preferences and their respectively implied utility representations. The employed axiomatization is provided by Chew (1989). It is flexible enough to account for prominent behavioral peculiarities that have been reported as violations of the classic von Neumann/Morgenstern (vNM) independence axiom.[14]

Thereafter, we will specify the functional form of certainty equivalents that come with such utility representations. We conclude with two parametric examples of particular interest.

2.3.1.1 Preferences over utility lotteries

Let $I \subset \mathbb{R}_+$ denote a compact interval and again write $\mathscr{B}(I)$ and $\mathscr{M}(I)$. We assume that preferences over lotteries in $\mathscr{M}(I)$ are given by a binary relation $\preceq$. Further let $\prec$ and $\sim$ denote the induced strict preference relation and indifference relation, respectively. Consider the following behavioral axioms:

(O) $\preceq$ is a weak order, i.e. complete and transitive.

(C) $\forall P \in \mathscr{M}(I) : \{Q \in \mathscr{M}(I) \mid P \prec Q\}$ and $\{Q \in \mathscr{M}(I) \mid Q \prec P\}$ are open with respect to the weak topology on $\mathscr{M}(I)$.

(VWS) $\forall P, Q \in \mathscr{M}(I)$:

$$P \sim Q \Rightarrow \forall R \in \mathscr{M}(I), \lambda \in (0,1) : \exists \theta \in (0,1) : \lambda P + (1-\lambda)R \sim \theta Q + (1-\theta)R.$$

(WS) $\forall P, Q \in \mathscr{M}(I)$:

$$P \sim Q \Rightarrow \forall \lambda \in (0,1) : \exists \theta \in (0,1) : \forall R \in \mathscr{M}(I) : \lambda P + (1-\lambda)R \sim \theta Q + (1-\theta)R.$$

(S) $\forall P, Q, R \in \mathscr{M}(I)$:

$$P \sim Q \Rightarrow \forall \lambda \in (0,1) : \lambda P + (1-\lambda)R \sim \lambda Q + (1-\lambda)R.$$

Weak order (O) and continuity (C) are the standard requirements for there to exist a continuous numerical function representing $\preceq$ over $\mathscr{M}(I)$.[15] We will thus always demand preferences to satisfy (O) and (C). Assuming $\preceq$ to also obey one of the remaining three axioms imposes considerably more structure on the decision making. Thereby, (VWS), (WS) and (S) increasingly facilitate the analytical implementation of the resulting utility representation in applied work. Nevertheless, their empirical appeal is decreasing in the same order along their respective degree of reconcilability with behavioral data.[16]

Starting with the most restrictive axiom, substitution (S) demands that whenever the decision maker is indifferent between two lotteries P and Q, he is also indifferent between the mixture of P with a third lottery R and the mixture of Q with that third lottery R, both mixtures by the same ratio.

[14] See e.g. the Allais (1953) paradox as the most prominent of such violations.
[15] Cf. Debreu (1954).
[16] See how Chew (1983) motivates his inquiry into utility representations over lottery spaces that do not necessarily satisfy the classic independence axiom, which is rephrased to (S) in the Chew (1989) framework.

Now, weak substitution (WS) weakens (S) as it additionally permits the mixture ratios at which the decision maker is indifferent between a mixture of P or Q with a third lottery R to differ. Yet, these mixtures cannot depend on the third lottery R. In other words, it is possible that one of the two indifferent lotteries can be mixed more "easily" with the third leaving room for complementarity between P and R or Q and R.

Eventually, with very weak substitution (VWS), the ratio at which indifference is attained is additionally allowed to depend on R. Hence, (S) implies (WS), which in turn implies (VWS).

2.3.1.2 Chew certainty equivalents

General representation Chew (1989), Theorem 3, proves necessity and sufficiency of the "very weak" constellation of the above axioms for there to exist a continuous utility representation of $\preceq$ over $\mathscr{M}(I)$ of the so called implicit-weighted kind. Thereby, the utility level $u(P)$ associated with a lottery $P \in \mathscr{M}(I)$ is the unique root of a mapping

$$y \mapsto \int_I \Psi(x,y)dP(x),$$

where $\Psi : I \times \mathbb{R} \to \mathbb{R}$ has to satisfy certain continuity requirements. Implicitly defining the continuous weight function $w : I \times \mathbb{R} \to \mathbb{R}$ such that

$$\Psi(x,y) = w(x,y)(v(x) - y),$$

where the continuous mapping $v : I \to \mathbb{R}$ is defined via $v(x) := u(\delta_x)$,[17] thus yields

$$0 = \int_I \Psi(x,u(P))dP(x) = \int_I w(x,u(P))v(x)dP(x) - u(P)\int_I w(x,u(P))dP(x),$$

such that

$$u(P) = \int_I \frac{w(x,u(P))}{\int_I w(x,u(P))dP(x)} v(x)dP(x). \tag{2.3.2}$$

In a sense, the utility u associated with a lottery $P \in \mathscr{M}(I)$ is given by a weighted expected value of the utility index v over deterministic outcomes. The fact that the weights in this representation also depend on the utility level $u(\cdot)$ of the considered lottery itself gives rise to the notion of implicit weighting.[18]

Next, since our analysis is concerned with lotteries over utility levels, the preference ordering $\preceq$ is also assumed to always satisfy the following strict monotonicity axiom:

(M) For all $x, y \in I$ it holds that $\delta_x \prec \delta_y \Leftrightarrow x < y$.

Hence, v is strictly monotonically increasing on I and therefore invertible such that for every $P \in \mathscr{M}(I)$ there is a unique real number in I, denoted by $\mu(P)$ and called lottery P's certainty equivalent, that satisfies

$$P \sim \delta_{\mu(P)} \Leftrightarrow u(P) = u(\delta_{\mu(P)}) = v(\mu(P)). \tag{2.3.3}$$

Hence, by (M), preferences over $\mathscr{M}(I)$ can equivalently be stated in terms of certainty equivalents, i.e. for all $P, Q \in \mathscr{M}(I)$ it holds that

$$P \preceq Q \Leftrightarrow \mu(P) \leq \mu(Q).$$

[17] I.e. $v(\cdot)$ is a utility index over deterministic outcomes in I.

[18] It is the particular functional form of w that comprises behavioral implications about the underlying decision making, et vice versa, but see shortly.

With regard to the notion of uncertainty aggregation, note that μ makes explicit its ingredients. Next to the description of uncertainty $P \in \mathscr{M}(I)$, μ incorporates both the appreciation of deterministic continuation utility levels $v(\cdot)$ and the respective implicit weights of the utility representation.

Next, (2.3.3) implies that

$$\int_I \Psi(x, v(\mu(P)))dP(x) = 0 \iff \int_I \psi(x, \mu(P))dP(x) = 0,$$

where $\psi : I \times I \to \mathbb{R}, (x, y) \mapsto \Psi(x, v(y))$. I.e., given preferences $\preceq$ over $\mathscr{M}(I)$ which satisfy (O), (C), (VWS) and (M), the certainty equivalent $\mu(P)$ of a lottery $P \in \mathscr{M}(I)$ is given by the unique root of the mapping

$$y \mapsto \int_I \psi(x, y)dP(x).$$

Note that by construction we have

$$\psi(x, x) \equiv \Psi(x, v(x)) = \int_I \Psi(\tilde{x}, v(x))d\delta_x(\tilde{x}) = \int_I \Psi(\tilde{x}, u(\delta_x))d\delta_x(\tilde{x}) = 0.$$

Thus, to put it explicitly, the considered certainty equivalents satisfy the "consistency with certainty" property listed by Chew (1983) as a crucial requirement for mean value functionals, i.e.

$$\mu(\delta_x) = x.$$

Moreover, Chew (1989), Theorem 5, proves that such certainty equivalents are consistent with first (resp. second) degree stochastic dominance if and only if for all $y \in I$ the mapping $x \mapsto \psi(x, y)$ is strictly monotonically increasing (resp. concave).

Eventually, if we assume preferences to have a Chew-type utility representation over $\mathscr{M}(I)$, with $I = \mathbb{R}_+$, we further restrict the certainty equivalents to satisfy homogeneity:

(H) For all $P \in \mathscr{M}(I)$ and $\lambda > 0$ it holds that $\mu(P_\lambda) = \lambda\mu(P)$,

where P_λ denotes the probability measure defined by $P_\lambda(B) := P(x \in I | \lambda x \in B)$. Intuitively, (H) requires the assignment of a λ-fold certainty equivalent to a lottery that yields a λ-fold utility. This implies

$$0 = \int_I \psi(x, \mu(P_\lambda))dP_\lambda(x) = \int_I \psi(\lambda x, \lambda\mu(P))dP(x),$$

which is satisfied in particular for ψ linear homogenous. Thus, by defining $\zeta(x) := \psi(x, 1)$, we find the sought for certainty equivalent $\mu(P)$ of a continuation utility lottery $P \in \mathscr{M}(I)$ to be the unique root of the mapping

$$y \mapsto \int_I \zeta\left(\frac{x}{y}\right) dP(x).$$

Special cases We finish this subsection on Chew-type implicit-weighted certainty equivalents for some utility lottery $P \in \mathscr{M}(I)$ by presenting two parametric examples explicitly considered by Epstein and Zin (1989).

First, let $\zeta(x) = \frac{x^\alpha - 1}{\alpha} + a(x-1), 0 \neq \alpha < 1, a \geq 0$. This gives rise to a so called Chew/Dekel (CD) certainty equivalent implicitly defined as the solution to

$$\mu_{CD}(P)(1 + a\alpha) - a\alpha \int_I x dP(x)) = (\mu_{CD}(P))^{1-\alpha} \int_I x^\alpha dP(x). \tag{2.3.4}$$

Second, consider the case of $a = 0$ in the above example, which allows us to explicitly solve for the certainty equivalent,

$$\mu_{KP}(P) = \left(\int_I x^\alpha dP(x) \right)^{\frac{1}{\alpha}}. \tag{2.3.5}$$

This is the certainty equivalent of a decision maker with so called Kreps/Porteus (KP) preferences. Observe by equation (2.3.2) that this μ-specification arises in the case of a constant (explicit) weight function w and for $v(x) = x^\alpha$, $0 \neq \alpha < 1$. To put it another way, next to a CRRA-type utility index $v(\cdot)$, such a representation demands the decision maker to obey (O), (C) and (S) by the vNM Theorem.[19]

2.3.2 Time aggregation

So far, we saw how the notion of recursive utility "reduces" the problem of evaluating intertemporal consumption tradeoffs within an infinite horizon framework to a two–period problem of assessing the tradeoff between current consumption and a certainty equivalent of random utility prospects. In order to merge these two ingredients we already introduced the time aggregator

$$W: \quad \mathbb{R}^2_+ \quad \to \quad \mathbb{R}_+.$$

Epstein and Zin (1989) explicitly demand this aggregator to have the form

$$W(c, \mu) = [c^\rho + \beta \mu^\rho]^{\frac{1}{\rho}}, \quad 0 \neq \rho < 1, \beta \in (0, 1). \tag{2.3.6}$$

Note that it is made sure that both today's consumption and future utility enter the modelled decision maker's evaluation positively.

To complement our analysis of parametric examples of Chew certainty equivalents above, finally consider the special case of the KP functional further restricted to $\alpha = \rho$, where ρ is the time aggregation parameter above

$$\mu_{EU}(P) := \left(\int_I x^\rho dP(x) \right)^{\frac{1}{\rho}}. \tag{2.3.7}$$

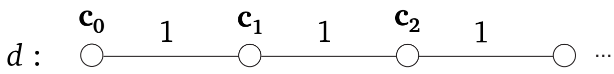
This specification finally gives rise to an expected utility (EU) representation over temporal lotteries and thus yields the standard model as described in the introductory remarks to section 2.2.

Let us next restrict our attention to probability trees that correspond to deterministic consumption sequences $(c_0, c_1, c_2, \ldots)$, such as the tree d depicted in figure 2.9. Since the tree d is characterized by

$$\begin{aligned} d &\cong (c_0, m), && \text{where } m = \delta_{d^1} \\ d^1 &\cong (c_1, m^1), && \text{where } m^1 = \delta_{d^2} \\ d^2 &\cong (c_2, m^2), && \text{where } m^2 = \delta_{d^3} \\ & \text{etc.} \end{aligned}$$

[19] See Chew (1989), Theorem 1, for a formal statement of the vNM Theorem in this context.
Note that Chew (1989), Theorem 2, also proves necessity and sufficiency of the "weak" constellation for a "(explicit) weighted utility" representation of intermediate generality. We skip this part because we will not refer to it in our ensuing analysis. The (WS) axiom is thus listed merely for the sake of exposition.

Figure 2.9: Deterministic consumption sequence



it follows from (2.3.1) that

$$U(d)=[c_0^\rho+\beta\mu(\delta_{U(d^1)})^\rho]^{\frac{1}{\rho}}=[c_0^\rho+\beta U(d^1)^\rho]^{\frac{1}{\rho}}=[c_0^\rho+\beta[c_1^\rho+\beta\mu(\delta_{U(d^2)})^\rho]^{\frac{1}{\rho}\rho}]^{\frac{1}{\rho}}=$$
$$=[c_0^\rho+\beta c_1^\rho+\beta^2U(d^2)^\rho]^{\frac{1}{\rho}}=\ldots=\left[\sum_{t=0}^{\infty}\beta^t c_t^\rho\right]^{\frac{1}{\rho}}. \tag{2.3.8}$$

I.e., the approach taken here results in a utility function of the constant elasticity of substitution (CES) class as an evaluator of *deterministic* consumption sequences.[20] In this context, the expression

$$\frac{1}{1-\rho}$$

measures the EIS.

We conclude with the following Theorem that ensures existence of the considered EZ utility functions.[21]

Theorem 3. *If W has the CES form (2.3.6), then, for the three parametric examples of μ considered above, the functional equation*

$$U(c_0,m)=W(c_o,\mu(m_U))$$

has a solution

1. *for $\rho>0: V: D(b)\to\mathbb{R}_+$, where b satisfies $\beta b^\rho<1$*
2. *for $\rho<0: V: D\to\mathbb{R}_+$.*

2.3.3 Timing and risk preferences

The major advantage of adopting the more general EZ utility for applied work stems from the disentanglement (however incomplete) of the decision maker's attitude towards risk and towards the timing of consumption. This subsection demonstrates their separation.

2.3.3.1 Definitions

We begin by defining the notion of timing and risk preferences, respectively. Both definitions are mutually abstract in the following sense. In defining timing preferences we abstract from uncertainty, while in defining risk preferences we keep the analysis atemporal.

Moreover, we are particularly interested in a comparative assessment of preferences. We therefore consider two recursive decision makers with utility representations U^I and U^{II}. Specifically, for $i=I,II$, $0\neq\rho^i<1$ and $0<\beta^i<1$

$$V^i(c_0,m)=W^i\left(c_0,\mu^i(V_m^i)\right)=\left[c_0^{\rho^i}+\beta^i\left(\mu^i(V_m^i)\right)^{\rho^i}\right]^{\frac{1}{\rho^i}},$$

where $\mu^i:\mathscr{M}(\mathbb{R}_+)\to\mathbb{R}_+$ is one of the three parametric examples of the Chew class studied above.

[20] Observe that, since $\mathrm{id}^{\frac{1}{\rho}}$ is a strictly monotonically increasing transformation, a standard Samuelson (1937) Discounted Utility function $\sum_{t=0}^{\infty}\beta^t c_t^\rho$ is an alternative representation of the underlying preferences if restricted to degenerate temporal lotteries.

[21] Epstein and Zin (1989) prove a more general version in their Theorem 3.1.

Timing preferences The deterministic treatment of consumption timing and the associated intertemporal consumption substitution is at least two-dimensional in (modern) economics.[22] First, there is mere impatience such that a decision maker c.p. favors current consumption. Second, he also assesses the "relative abundance" of consumption available to him over time.[23]

Now, impatience is formally reflected in the discount parameter β. We thus regard the decision maker V^I as less patient than V^{II} if he discounts future consumption more strongly, i.e. if it holds that

$$\beta^I \leq \beta^{II}.^{24} \tag{2.3.9}$$

As far as deterministic consumption substitution is concerned, we note that, by means of the utility U, the EIS controls the decision maker's affinity towards a smooth consumption profile. We thus interpret V^I to be more averse towards deviations from smooth consumption than V^{II} if

$$\text{EIS}^I := \frac{1}{1-\rho^I} \leq \frac{1}{1-\rho^{II}} =: \text{EIS}^{II},$$

or equivalently

$$\rho^I \leq \rho^{II}. \tag{2.3.10}$$

Risk preferences As opposed to timing preferences, a readily operationalized measure of comparative risk aversion is less evident. However, it seems natural to focus on the employed certainty equivalent functional as it serves as the uncertainty aggregation device in the EZ framework. Accordingly, for $W^I = W^{II}$, Epstein and Zin (1989) define V^I to be more risk averse than V^{II} if it holds that

$$\mu^I \leq \mu^{II}.^{25} \tag{2.3.11}$$

2.3.3.2 Disentangling attitudes towards risk and timing

Separation By further examining the above definitions we first find for the CD-class that (2.3.11) is equivalent to

$$\alpha^I \leq \alpha^{II} \text{ and } a^I \leq a^{II}.$$

Second, for the KP-class ($a^I = a^{II} = 0$) this condition reduces to

$$\alpha^I \leq \alpha^{II}.$$

Thus, the EZ utility representation with CD- and KP-certainty equivalents yields a parametric disentanglement of comparative risk aversion and timing preferences.[26]

Eventually, with the additional restriction of $\alpha^i = \rho^i, i = I, II$, it is obvious that such separation is impossible in the case of EU certainty equivalents.

[22] A thorough discussion on the notion of time preference, its historic development and also some serious reservations against discounted utility models is provided by Frederick et al. (2002).

[23] Cf. Fisher (1930), p. 67.

[24] Note that the notion of comparative impatience can be defined more formally in terms of preferences along the lines of Olson and Bailey (1981) by means of the decision makers' marginal rate of substitution between consumption levels in two different periods after excluding "the effect of a difference in marginal utility."

[25] Note that, as remarked by Epstein and Zin (1989), for the case of KP certainty equivalents, by the assumed consistency with second order stochastic dominance it follows that the least risk averse decision maker aggregates uncertainty additionally obeying $\alpha = 1$. I.e. his utility index satisfies $v(\cdot) = \text{id}(\cdot)$ such that his certainty equivalent is a plain expected value.

[26] Note that this separation as well as the utility representation is akin to Selden (1978)'s result in a two period environment.

Nonindifference towards the timing of uncertainty resolution Even though the EZ representation allows for a parametric disentanglement of the above notions of risk and timing preferences, it is important to note that this separation is only partly in nature. Specifically, departing from an EU certainty equivalent necessarily gives rise to nonindifference towards the timing of the resolution of consumption uncertainty as it is illustrated in the introductory example to the present chapter.[27]

Precisely, Epstein and Zin (1989) conclude in the sense of Kreps and Porteus (1978), Theorem 3, that a decision maker V^i with KP preferences over atemporal continuation utility lotteries prefers earlier (later) resolution if and only if it holds that $\alpha^i < (>) \rho^i$. Moreover, V^i is indifferent to the timing of uncertainty resolution if and only if it holds that $\alpha^i = \rho^i$, i.e. if and only if he has EU preferences.

2.3.4 The EZ/KP representation

We conclude this section on the EZ representation by the application of the KP case to our introductory example. The focus on KP preferences is natural in the sense that our ensuing analysis is focused on applied macroeconomics. For such work it is important to have an explicit functional form of U to parameterize. Summing up, the EZ/KP utility representation reads

$$U(d) = \left[(\Phi_1(d))^\rho + \beta \left(\mathbb{E}_{\Phi_2(d)} [U^\alpha] \right)^{\frac{\rho}{\alpha}} \right]^{\frac{1}{\rho}}, \quad 0 \neq \rho < 1, 0 \neq \alpha < 1, \beta \in (0,1).^{28} \tag{2.3.12}$$

To illustrate its application, consider Robinson's two alternatives as presented in example 1 and compare his behavior for the three cases of him preferring earlier or later resolution or being indifferent about the timing of uncertainty resolution, i.e. following the consequentialist hypothesis implied by the standard model.

Example 1 (Continued). Robinson's preferences over temporal lotteries are assumed to be representable by an EZ utility function of the KP form as stated above. He has to choose between the two (finite) temporal lotteries displayed in figure 2.1, i.e. between $d = (10, m)$ and $\hat{d} = (10, \hat{m})$, with $d, \hat{d} \in D$. Since Robinson decides in favor of the temporal lottery that results in a relatively higher utility evaluation, his decision making can be described as comparing the utility levels

$$U(d) = \left[10^\rho + \beta \mu (m \circ U^{-1})^\rho \right]^{\frac{1}{\rho}} = \left[10^\rho + \beta \left(\mathbb{E}_m [U^\alpha] \right)^{\frac{\rho}{\alpha}} \right]^{\frac{1}{\rho}}, \tag{2.3.13}$$

and

$$U(\hat{d}) = \left[10^\rho + \beta \mu (\hat{m} \circ U^{-1})^\rho \right]^{\frac{1}{\rho}} = \left[10^\rho + \beta \left(\mathbb{E}_{\hat{m}} [U^\alpha] \right)^{\frac{\rho}{\alpha}} \right]^{\frac{1}{\rho}}, \tag{2.3.14}$$

where $\beta \in (0,1)$, $0 \neq \rho < 1$.

Now, note again that m is already a lottery over degenerate trees (deterministic consumption sequences) while $\hat{m}$ is a degenerate lottery $\delta_{\hat{d}_1}$, with $\hat{d}_1 = (10, \hat{m}_1)$. Moreover, $\hat{m}_1$ is a lottery

[27] Note, as sketched above, that looked at it this way such nonindifference may appear as a cost of the achieved disentanglement, cf. Epstein et al. (2014). Interestingly, Kreps and Porteus (1978), who provided the fundament of the EZ framework and thus for the studied separation of risk aversion and EIS, were looking for a temporal utility representation that allowed for the explicit modelling of such nonindifference and did not motivate their analysis through the issue of entangled risk aversion and consumption substitutability. Nevertheless, they already indicated on some relation between risk aversion and nonindifference, see Kreps and Porteus (1978), p. 198.

[28] Note that Weil (1990)'s "generalized isoelastic" utility provides an equivalent representation.

over degenerate trees. We thus compute

$$\begin{aligned}\mu(m\circ U^{-1}) &= \left(\frac{1}{2}U^{\alpha}(10,10,\ldots)+\frac{1}{2}U^{\alpha}(10,5,5,\ldots)\right)^{\frac{1}{\alpha}}\\ &\overset{(2.3.8)}{=} \left(\frac{1}{2}\left(\left[\sum_{t=0}^{\infty}\beta^t 10^{\rho}\right]^{\frac{1}{\rho}}\right)^{\alpha}+\frac{1}{2}\left(\left[10^{\rho}+\sum_{t=1}^{\infty}\beta^t 5^{\rho}\right]^{\frac{1}{\rho}}\right)^{\alpha}\right)^{\frac{1}{\alpha}}\\ &= \left(\frac{1}{2}\left(10\left[\frac{1}{1-\beta}\right]^{\frac{1}{\rho}}\right)^{\alpha}+\frac{1}{2}\left(\left[10^{\rho}+5^{\rho}\frac{\beta}{1-\beta}\right]^{\frac{1}{\rho}}\right)^{\alpha}\right)^{\frac{1}{\alpha}}\end{aligned}$$

and

$$\begin{aligned}\mu(\hat{m}\circ U^{-1}) =& U(\hat{d}_1)=\left[10^{\rho}+\beta\mu(\hat{m}_1\circ U^{-1})^{\rho}\right]^{\frac{1}{\rho}}\\ &= \left[10^{\rho}+\beta\left(\frac{1}{2}U^{\alpha}(10,10,\ldots)+\frac{1}{2}U^{\alpha}(5,5,\ldots)\right)^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}\\ &\overset{(2.3.8)}{=} \left[10^{\rho}+\beta\left(\frac{1}{2}\left(10\left[\frac{1}{1-\beta}\right]^{\frac{1}{\rho}}\right)^{\alpha}+\frac{1}{2}\left(5\left[\frac{1}{1-\beta}\right]^{\frac{1}{\rho}}\right)^{\alpha}\right)^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}.\end{aligned}$$

Now, plugging these two results in (2.3.13) and (2.3.14) makes it possible to determine Robinson's decision for any given set of parameter values. As foreshadowed above, we here want to parameterize three situations. Therefore, we set $\beta = 0.9$ and $\rho = -1$ and consider three different degrees of risk aversion implied by $\alpha_1 = -\frac{1}{2}$, $\alpha_2 = -1$ and $\alpha_3 = -2$. This respectively yields

$$U(d;\alpha)=\begin{cases}0.7284, & \text{if } \alpha=\alpha_1\\ 0.7117, & \text{if } \alpha=\alpha_2\\ 0.6819, & \text{if } \alpha=\alpha_3\end{cases} \quad\text{and}\quad U(\hat{d};\alpha)=\begin{cases}0.7298, & \text{if } \alpha=\alpha_1\\ 0.7117, & \text{if } \alpha=\alpha_2\\ 0.6799, & \text{if } \alpha=\alpha_3.\end{cases}$$

Note that, as generally stated above, Robinson prefers earlier over later resolution of uncertainty, i.e. d over $\hat{d}$, in the case of $\rho \geq \alpha$. Analogously, preference for later resolution is calibrated via $\rho \leq \alpha$. Eventually, $\rho = \alpha$ parameterizes indifference towards the timing of uncertainty resolution (giving rise to the standard model).

2.4 Solving a basic DSGE model with EZ/KP utility

This section is intended to illustrate the implementation of the theoretical considerations so far into standard applied macroeconomic analysis, which we understand as the approximate solution of the intertemporal decision problem characterizing some model economy by means of value and policy functions and their usage in simulating artificial data to be contrasted with the stylized facts describing the real economy. Since simulation is a computational exercise independent of the EZ specification, our presentation is only concerned with the approximate solution of EZ economies.

For this purpose, we once more come back to a Robinson Crusoe decision problem. This time, we embed it into a more complete model economy in that we describe the interdependence of current consumption and future consumption opportunities by the means of a savings equation while his output is again subject to an exogenous stochastic influence. Formally, Robinson's situation is described as a stochastic control system. Given an initial capital stock, the way how Robinson chooses his consumption path thereby induces a probability tree. The utility of the latter can thus be found through the representation described in section 2.3 so that the decision problem is stated in terms of utility maximization. Thereafter, we briefly describe how to apply the perturbation methodology of Schmitt-Grohe and Uribe (2004) to approximate this control system's solution. Eventually, we actually compute a second order perturbation for a given parametrization and discuss the most prominent implications of the EZ/KP framework for such applied work.

2.4.1 Representative agent environment

Robinson uses capital to produce a final good. He can either consume the final good or use it as investment in the capital stock. His planning horizon is infinite,[29] his personal utility stems solely from consumption and his preferences over uncertain consumption paths have an EZ/KP representation as in (2.3.12). To keep the notation parsimonious, we avoid time indices wherever possible.

Now, for each $t \in \mathbb{N}$, we consider the state space

$$X := \mathbb{R}_+ \times \mathbb{R}.$$

An element $x = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T \in X$ denotes a tuple of capital stock x_1 and productivity level x_2. Further, for every $t \in \mathbb{N}$ Robinson chooses from the control space

$$Y := \mathbb{R}_+.$$

The control $y \in Y$ is interpreted as his consumption level. We endow X and Y with their respective standard topologies.

Moreover, let the triplet $(\Omega, \mathscr{A}, P)$ denote a probability space with a stochastic process $\{\epsilon_t\}_{t=1}^{\infty}$ satisfying $\epsilon_t \sim \text{iidN}(0,1)$ for all $t \in \mathbb{N}$. The control system is now determined by its dynamic

$$\begin{aligned} f: \quad X \times Y \times \mathbb{R} &\to \mathbb{R}^2 \\ (x, y, \epsilon) &\mapsto \begin{pmatrix} e^{x_2} x_1^{\eta} + (1-\delta)x_1 - y \\ \lambda x_2 + \sigma\epsilon \end{pmatrix}, \end{aligned} \tag{2.4.1}$$

where $\eta, \lambda \in (0,1), \delta \in [0,1]$ and $\sigma \geq 0$. Note that f is continuous and thus measurable. The idea behind this construction is that, given a state x, a choice of the control variable y and a stochastic influence ϵ, next period's state is determined through the mapping f. Thereby,

$$e^{x_2} x_1^{\eta}$$

denotes the produced output and δ determines material wear. Thus, the first component of f says that next period's capital stock amounts to output plus the not worn part of this period's capital stock less chosen consumption. In order to make sure that next period's capital stock is nonnegative, the chosen consumption level has to satisfy

$$y \leq e^{x_2} x_1^{\alpha} + (1-\delta)x_1.$$

We accordingly define for every $x \in X$

$$Y(x) := \left\{ y \in Y \mid y \leq e^{x_2} x_1^{\alpha} + (1-\delta)x_1 \right\}$$

[29] Apparently, we cannot simply refer to a bequest motive in order to motivate this assumption so we rather interpret this as an approximation of Robinson's actual situation, which is just as much characterized by his knowledge about the finiteness of his horizon as of the lack of knowledge about the exact duration (maybe complemented with his hope for (or fear of) a long life).

as the feasible control space.[30] On the other hand, f's second component says that the productivity level x_2 evolves as a stationary AR(1) process, i.e. next period's productivity additively depends on its current level and an exogenous stochastic iid influence. While λ determines the persistence of the productivity process, σ is the standard deviation of its stochastic influence and therefore scales the uncertainty in our economy.

We next define a strategy as a measurable mapping

$$h : X \to Y,$$

where X and Y are each endowed with their respective Borel σ-algebra. In other words, a strategy assigns a control value $h(x) \in Y$ to every state $x \in X$, i.e. the agent chooses $h(x)$ whenever confronted with the state x. Note that our definitions of state space and strategy do not allow for the consideration of the state or consumption history. We further call a strategy admissible if it results in a feasible control choice. Accordingly, define the space of admissible strategies as

$$\Pi := \{h \colon X \to Y \mid h \text{ measurable, } h(x) \in Y(x) \text{ f.a. } x \in X\}.$$

Now, given such a discrete time stochastic control system, an initial state $x_0 \in X$ and an admissible strategy $h \in \Pi$, we recursively define the system's solution under x_0 and h as the state process $\left\{X_t^{x_0,h}\right\}_{t=0}^{\infty}$ and the control process $\left\{Y_t^{x_0,h}\right\}_{t=0}^{\infty}$, where

$$\begin{aligned}
X_0^{x_0,h} &:= x_0,\\
Y_t^{x_0,h} &:= h(X_t^{x_0,h}), \text{ f.a. } t \in \mathbb{N},\\
X_{t+1}^{x_0,h} &:= f(X_t^{x_0,h}, Y_t^{x_0,h}, \epsilon_{t+1}), \text{ f.a. } t \in \mathbb{N}.
\end{aligned}$$

Note first that $X_t^{x_0,h}$ and $Y_t^{x_0,h}$ are well-defined random variables for all $t \in \mathbb{N}$ because of the measurability of f, h and ϵ_t, for all $t \in \mathbb{N}$. Second, the solution's recursive construction further reveals that, for all $t \in \mathbb{N}$, $X_t^{x_0,h}$ and $Y_t^{x_0,h}$ are already measurable in the random variables $\epsilon_1, \ldots, \epsilon_t$. I.e., if we denote by

$$\mathscr{F}_t := \sigma(\epsilon_1, \ldots, \epsilon_t),\ t \in \mathbb{N},$$

the σ-algebra generated by $\epsilon_1, \ldots, \epsilon_t$, with $\mathscr{F}_0 := \{\emptyset, \Omega\}$, and further by $\mathscr{F} := \{\mathscr{F}_t\}_{t=0}^{\infty}$ the filtration generated by $\{\epsilon_t\}_{t=0}^{\infty}$, we find the solution $\left\{X_t^{x_0,h}\right\}_{t=0}^{\infty}$ and $\left\{Y_t^{x_0,h}\right\}_{t=0}^{\infty}$ to be $\mathscr{F}$-adapted. Thus, our modeling reasonably assumes Robinson to base his control choice only on already observed ϵ-values. Thereby, the realization of ϵ_t becomes observable for him at the beginning of period t. Moreover, because of our AR(1)/iid assumption it holds for all $s > t$ that given $X_t^{x_0,h}$, the random state $X_s^{x_0,h}$ is independent of $\mathscr{F}_t$. Hence, the solution $\left\{X_t^{x_0,h}\right\}_{t=0}^{\infty}$ and therefore $\left\{Y_t^{x_0,h}\right\}_{t=0}^{\infty}$ both have the Markov property.

2.4.2 Induced temporal lotteries

Having laid out the basic framework and notation, we next want to describe how to assign a temporal lottery to a tuple of initial state and admissible strategy. Therefore, arbitrarily fix $x_0 \in X$ and $h \in \Pi$. To find the probability tree in D that is induced by this tuple, we consider the solution of the stochastic control system for the control process under x_0 and h. We proceed

[30] The basic terminology mostly follows Kreps and Porteus (1979).

inductively as we treat the solution in the t^{th} step as if, regarding the uncertainty of future realizations as of period t, we were only interested in their joint distribution.

In the first step, we thus restrict attention with regard to the solution of the control process $\left\{Y_t^{x_0,h}\right\}_{t=0}^{\infty}$ only to $Y_0^{x_0,h} = h(x_0)$ in $t = 0$ and the induced joint probability distribution over $\mathbb{R}_+^\infty$ from period 1 on. More precisely, define the stochastic process

$$Y^{x_0,h} \colon \Omega \to \mathbb{R}_+^\infty,\ \omega \mapsto (Y_1^{x_0,h}(\omega), Y_2^{x_0,h}(\omega), \dots).$$

Next, set $m_1^{x_0,h}$ as the induced image measure over $\mathscr{B}\left(\mathbb{R}_+^\infty\right)$, i.e.

$$m_1^{x_0,h} := P \circ (Y^{x_0,h})^{-1},$$

such that for all $B \in \mathscr{B}\left(\mathbb{R}_+^\infty\right)$ it holds that

$$\begin{aligned} m_1^{x_0,h}(B) &= P\left(\{\omega \in \Omega \mid Y^{x_0,h}(\omega) \in B\}\right) \\ &= P\left(\{\omega \in \Omega \mid (Y_1^{x_0,h}(\omega), Y_2^{x_0,h}(\omega), \dots) \in B\}\right). \end{aligned}$$

This way, we find the mapping

$$\begin{array}{rccc} \iota_1 : & X \times \Pi & \to & D_1 \\ & (x_0, h) & \mapsto & \left(Y_0^{x_0,h}, m_1^{x_0,h}\right) = \left(h(x_0,h), m_1^{x_0,h}\right). \end{array}$$

We now pursue inductively. Suppose the desired mappings $\iota_1, \dots, \iota_t$ have already been constructed and we now want to also consider the structure of the induced probability tree until period t. We therefore define for all $B \in \mathscr{B}_t$

$$m_{t+1}^{x_0,h}(B) := P(\{\omega \in \Omega \mid \iota_t(X_1^{x_0,h}(\omega), h) \in B\}) = P \circ \iota_t(X_1^{x_0,h}(\cdot), h)^{-1}(B) = P_{\iota_t(X_1^{x_0,h},h)}(B).$$

The idea behind this definition is the following. Next period's state under x_0 and h is the random variable $X_1^{x_0,h}$. The random t-stage probability tree that is induced by $X_1^{x_0,h}$ and h is given by $\iota_t\left(X_1^{x_0,h}, h\right)$. Thus, the probability distribution over such trees is given by the image measure that is induced by $\iota_t\left(X_1^{x_0,h}(\cdot), h\right)$. Consequently, we define

$$\begin{array}{rccc} \iota_{t+1} : & X \times \Pi & \to & D_{t+1} \\ & (x_0, h) & \mapsto & \left(Y_0^{x_0,h}, m_{t+1}^{x_0,h}\right) = \left(h(x_0,h), m_{t+1}^{x_0,h}\right). \end{array}$$

Finally, we set

$$\begin{array}{rccc} \iota : & X \times \Pi & \to & D \\ & (x_0, h) & \mapsto & (\iota_1(x_0,h), \iota_2(x_0,h), \dots). \end{array}$$

Consistency In order for the mapping $\iota(\cdot,\cdot)$ to be well-defined, the elements of the image sequence have to be consistent in the sense introduced in subsection 2.2.3.2. To put it another way, if the previous construction was carried out correctly, it has to hold for all $t \in \mathbb{N}$ that

$$\iota_t(x_0, h) = g_t\left(\iota_{t+1}(x_0, h)\right).$$

Since on the one hand

$$g_t\left(\iota_{t+1}(x_0,h)\right) = g_t\left(h(x_0), m_{t+1}^{x_0,h}\right) = \left(h(x_0), f_t(m_{t+1}^{x_0,h})\right)$$

and on the other hand

$$\iota_t(x_0,h) = \left(h(x_0), m_t^{x_0,h}\right),$$

it remains to be shown for all $t \in \mathbb{N}$ that

$$f_t(m_{t+1}^{x_0,h}) = m_t^{x_0,h}.$$

We prove this by induction in appendix A. Thus, via ι we can naturally assign the corresponding temporal lottery in D to a tuple of initial state and admissible strategy.

Unique measure over induced trees From Theorem 1 we further know that we can identify every tree in D with a tuple of current consumption and a probability measure over trees emanating next period via a homeomorphism

$$\Phi : D \to \mathbb{R}_+ \times \mathscr{M}(D).$$

I.e., it holds that $\Phi(\iota(x_0,h)) = (h(x_0), m)$, where $m \in \mathscr{M}(D)$ is the unique measure, which satisfies for all $t \in \mathbb{N}$ and $B \in \mathscr{B}_t$

$$m(\pi_t^{-1}(B)) = m_{t+1}^{x_0,h}(B).$$

We eventually want to show that it holds that

$$\Phi(\iota(x_0,h)) = (h(x_0), P \circ \iota(X_1^{x_0,h}(\cdot),h)^{-1}),$$

i.e.

$$P \circ \iota(X_1^{x_0,h}(\cdot),h)^{-1}(\pi_t^{-1}(B)) = m_{t+1}^{x_0,h}(B).$$

This readily follows from the fact that for all $B \in \mathscr{B}_t$

$$\begin{aligned} P \circ \iota(X_1^{x_0,h}(\cdot),h)^{-1}(\pi_t^{-1}(B)) &= P\left(\left\{\omega \in \Omega \mid \iota(X_1^{x_0,h}(\omega),h) \in \pi_t^{-1}(B)\right\}\right) = \\ &= P\left(\left\{\omega \in \Omega \mid \pi_t(\iota(X_1^{x_0,h}(\omega),h)) \in B\right\}\right) = P\left(\left\{\omega \in \Omega \mid \iota_t(X_1^{x_0,h}(\omega),h) \in B\right\}\right) = \\ &= m_{t+1}^{x_0,h}(B). \end{aligned}$$

2.4.3 Consumption choice

We are now ready to describe Robinson's decision making in this economy. Therefore, define

$$\hat{\iota} := \Phi \circ \iota,$$

i.e. the mapping

$$\begin{array}{rccc} \hat{\iota}: & X \times \Pi & \to & Y \times \mathscr{M}(D) \\ & (x_0,h) & \mapsto & (h(x_0), P \circ \iota(X_1^{x_0,h}(\cdot),h)^{-1}). \end{array}$$

This allows us to indirectly assign a corresponding utility level to the pair (x_0,h). For this purpose, let $U : D \to \mathbb{R}_+$ denote a solution to the recursive functional equation

$$U(\Phi^{-1}(c_0,m)) = \left[c_0^\rho + \beta(\mathbb{E}_m U^\alpha)^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}, \quad \beta \in (0,1),\ 0 \neq \alpha < 1,\ 0 \neq \rho < 1,$$

as introduced in section 2.3. We then define the utility mapping

$$\begin{array}{rccc} \hat{U}: & X \times \Pi & \to & \mathbb{R} \\ & (x_0, h) & \mapsto & U(\iota(x_0, h)). \end{array}$$

Using the fact that U is a solution to the recursive equation above, we find

$$\begin{aligned} \hat{U}(x_0,h) &= U(\iota(x_0,h)) = U(\Phi^{-1}(\hat{\iota}(x_0,h))) = U(\Phi^{-1}(h(x_0), P\circ\iota(X_1^{x_0,h}(\cdot),h)^{-1})) \\ &= \left[h(x_0)^\rho + \beta\left(\mathbb{E}_{P\circ\iota(X_1^{x_0,h}(\cdot),h)^{-1}}\left[U^\alpha\right]\right)^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} = \left[h(x_0)^\rho + \beta\left(\mathbb{E}_P\left[U^\alpha(\iota(X_1^{x_0,h},h))\right]\right)^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \\ &= \left[h(x_0)^\rho + \beta\left(\mathbb{E}_P\left[\hat{U}^\alpha(X_1^{x_0,h},h)\right]\right)^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} = \left[h(x_0)^\rho + \beta\left(\mathbb{E}_P\left[\hat{U}^\alpha(f(x_0,h(x_0),\epsilon_1),h)\right]\right)^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}. \end{aligned}$$

Hence, $\hat{U}$ is itself a solution to the recursive functional equation

$$\hat{U}(x,h) = \left[h(x)^\rho + \beta\left(\mathbb{E}_P\left[\hat{U}^\alpha(f(x,h(x),\epsilon),h)\right]\right)^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}, \tag{2.4.2}$$

with $\epsilon \sim N(0,1)$. I.e.,

$$\begin{aligned} \hat{U}(x,h) &= \left[h(x)^\rho + \beta\left(\int_\Omega \hat{U}^\alpha(f(x,h(x),\epsilon(\tilde{\omega})),h)dP(\tilde{\omega})\right)^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \\ &= \left[h(x)^\rho + \beta\left(\int_{\mathbb{R}} \hat{U}^\alpha(f(x,h(x),\tilde{\epsilon}),h)\phi(\tilde{\epsilon})d\tilde{\epsilon}\right)^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}, \end{aligned}$$

where $\phi(\cdot)$ denotes the standard normal density function.

2.4.3.1 Decision problem

Having shown how to assign a utility level to a pair of initial state and admissible strategy, we are now able to formulate Robinson's choice problem. Given an initial state x_0, his objective is to find a strategy that maximizes the utility associated with it. I.e. formally Robinson has to solve

$$\max_{h\in\Pi} \hat{U}(x_0,h), \text{ given } x_0.$$

An admissible strategy $h^* \in \Pi$ is thereby called an optimal policy, if it satisfies

$$\hat{U}(x,h^*) \geq \hat{U}(x,h) \text{ f.a. } x \in X, h \in \Pi.$$

Further, we write

$$V(x) := \sup_{h\in\Pi} \hat{U}(x,h)$$

for this consumption problem's value function. Assuming the existence of both, a maximum value, given some $x \in X$, and an optimal policy, we note that the recursive formulation (2.4.2) of Robinson's problem directly lends itself to the application of dynamic programming. Thus, from Bellman's Principle of Optimality it follows that an optimal consumption policy has to comply with

$$h^*(x) = \underset{y\in Y(x)}{\arg\max}\Big[y^\rho + \beta(E_P[V^\alpha(f(x,y,\epsilon))])^{\frac{\rho}{\alpha}}\Big]^{\frac{1}{\rho}},\ \epsilon \sim N(0,1), \tag{2.4.3}$$

and that the value function must satisfy the intertemporal relation demanded by the (generalized) Bellman equation

$$V(x) = \max_{y\in Y(x)}\Big[y^\rho + \beta(E_P[V^\alpha(f(x,y,\epsilon))])^{\frac{\rho}{\alpha}}\Big]^{\frac{1}{\rho}},\ \epsilon \sim N(0,1). \tag{2.4.4}$$

Necessary optimality conditions In order to actually find an optimal policy, we next use (2.4.3) and (2.4.4) to derive conditions, which h^* and the induced optimal dynamic have to meet necessarily. We thereby assume the value function to be differentiable. First, an optimal consumption policy must satisfy the Euler equation

$$\begin{aligned}0 = \mathbb{E}_P\Bigg[\beta\left(\frac{V^\alpha(f(x,h^*(x),\epsilon))}{\mathbb{E}_P[V^\alpha(f(x,h^*(x),\epsilon))]}\right)^{1-\frac{\rho}{\alpha}}\left(\frac{h^*(f(x,h^*(x),\epsilon))}{h^*(x)}\right)^{\rho-1}\cdot \\ \cdot\,(\eta e^{\lambda x_2+\sigma\epsilon}(e^{x_2}x_1^\eta+(1-\delta)x_1-h^*(x))^{\eta-1}+(1-\delta))-1\Big].\end{aligned} \tag{2.4.5}$$

We derive this result in appendix B.

Additionally, in order to clearly distinguish between what can be controlled by Robinson and what is left to pure chance, we introduce further notation. Therefore, denote the deterministic part of the system's dynamic that is induced by following the optimal policy by

$$f^*(x) := \begin{pmatrix} e^{x_2}x_1^\eta - h^*(x) + (1-\delta)x_1 \\ \lambda x_2 \end{pmatrix}.$$

Accordingly, we write

$$f(x,h^*(x),\epsilon) = f^*(x) + \Sigma\epsilon,$$

with

$$\Sigma := \begin{pmatrix} 0 \\ \sigma \end{pmatrix}$$

to denote the optimal dynamic.

Summing up, we have the following conditions, which have to be satisfied by an optimal policy $h^* \in \Pi$, the value function, and the resulting deterministic part f^* of the dynamic

$$V(x) - \Big[h^*(x)^\rho + \beta\left(\mathbb{E}_P[V(f_1^*(x), f_2^*(x)+\sigma\epsilon)^\alpha]\right)^{\frac{\rho}{\alpha}}\Big]^{\frac{1}{\rho}} = 0,$$

$$\begin{aligned}\mathbb{E}_P\Bigg[\beta\left(\frac{V^\alpha(f_1^*(x), f_2^*(x)+\sigma\epsilon)}{\mathbb{E}_P[V^\alpha(f_1^*(x), f_2^*(x)+\sigma\epsilon)]}\right)^{1-\frac{\rho}{\alpha}}\left(\frac{h^*(f_1^*(x), f_2^*(x)+\sigma\epsilon)}{h^*(x)}\right)^{\rho-1}\cdot \\ \cdot\,(\eta e^{f_2^*(x)+\sigma\epsilon}(f_1^*(x))^{\eta-1}+(1-\delta))-1\Big] = 0,\end{aligned}$$

$$f_1^*(x) - e^{x_2}x_1^\eta + h^*(x) - (1-\delta)x_1 = 0,$$

$$f_2^*(x) - \lambda x_2 = 0.$$

It is generally not possible to solve for the functions h^*, V, and f^* analytically. In the next subsection, we therefore describe a popular method of finding approximations for these functions from this system of equations. Note that neither uniqueness nor existence is ensured by the above set of conditions. This would e.g. demand us to add a transversality condition. The approach taken here, however, is more direct in that it imposes a stability restriction directly on our approximate solution.[31] Further note that the actual feasibility of the approximate solution is typically checked ex post in the simulation results.

2.4.4 Perturbation

This subsection is intended to illustrate the application of the perturbation approach to EZ environments. Now, due to the fact that the non-linear time aggregation in the EZ representation translates itself directly into the generalized Bellman equation, the DSGE economy studied here has to be slightly modified to match the class of models studied by Schmitt-Grohe and Uribe (2004). In particular, by defining an auxiliary variable for Robinson's expected evaluation of continuation utility as of next period, an EZ economy can be fitted into their required structure.

Thus, the following subsection outlines the application of second order perturbation to an EZ economy along the lines of Schmitt-Grohe and Uribe (2004). Thereby, it is detailed enough in order to be self contained and to serve as a complement to the analysis of Caldara et al. (2012), who document on the appropriateness of the perturbation approach to EZ economies.

2.4.4.1 Method

The method of perturbation relies on the approximation of the optimal policy h^*, the resulting deterministic part of the dynamic f^* and the value function V by means of Taylor polynomials. Thereby, the variance parameter σ in the dynamic f is understood as variable. To make the dependence of the optimal policy, the induced deterministic part of the dynamic and the value function from σ explicit, σ is considered as an additional argument of those functions. I.e. we write

$$h^*(x,\sigma),\ f^*(x,\sigma) \text{ and } V(x,\sigma).$$

The Taylor polynomials are thereby expanded around a deterministic fixpoint of the dynamic, i.e. a point $(x,\sigma) = (x_{ss}, 0)$, with $x_{ss} \in X$ satisfying

$$f^*(x_{ss}, 0) = x_{ss}.$$

This point is called (deterministic) steady state. Now, a simulated solution of the state and control variables along the computed Taylor approximations that results from perturbing the control system from its steady state by allowing $\sigma \neq 0$ is appropriately called a perturbation. For this example, we are computing second order approximations and are hence executing a second order perturbation.

For feasibility, we thus assume h^*, f^* and V to be continuously differentiable in x and σ up to second order. In order to be able to apply the Schmitt-Grohe and Uribe methodology, we further define the auxiliary function

$$W(x,\sigma) := \mathbb{E}_P\left[V(f_1^*(x,\sigma), f_2^*(x,\sigma) + \sigma\epsilon, \sigma)^\alpha\right],$$

for the expectation of the value function to the power of α at next period's random state.

[31] See the next subsection on its implementation in numerical work.

Next, we use the necessary optimality conditions derived above to define the functions $F_i : X \times \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, 5$ by

$$
\begin{aligned}
F_1(x,\sigma) &:= V(x,\sigma) - \left[h^*(x,\sigma)^\rho + \beta\,(W(x,\sigma))^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}},\\
F_2(x,\sigma) &:= \mathbb{E}_P\left[\beta\left(\frac{V^\alpha(f_1^*(x,\sigma), f_2^*(x)+\sigma\epsilon,\sigma)}{W(x,\sigma)}\right)^{1-\frac{\rho}{\alpha}}\left(\frac{h^*(f_1^*(x,\sigma), f_2^*(x,\sigma)+\sigma\epsilon,\sigma)}{h^*(x,\sigma)}\right)^{\rho-1}\cdot\right.\\
&\qquad\left.\cdot\,(\eta e^{f_2^*(x)+\sigma\epsilon} f_1^*(x)^{\eta-1} + (1-\delta))\right] - 1,\\
F_3(x,\sigma) &:= W(x,\sigma) - \mathbb{E}_P\left[V(f_1^*(x,\sigma), f_2^*(x,\sigma)+\sigma\epsilon,\sigma)^\alpha\right],\\
F_4(x,\sigma) &:= f_1^*(x,\sigma) - e^{x_2}x_1^\eta - (1-\delta)x_1 + h^*(x,\sigma),\\
F_5(x,\sigma) &:= f_2^*(x,\sigma) - \rho x_2.
\end{aligned}
\tag{2.4.6}
$$

We thus know that

$$F_i(x,\sigma) \equiv 0, \text{ for all } i = 1,\dots,5.$$

Evaluating $F_i(x_{ss}, 0) = 0$ for $i = 1, \dots, 5$ and imposing the above condition for a fixpoint then yields

$$
\begin{aligned}
&V(x_{ss},0) = \left(\frac{1}{1-\beta}\right)^{\frac{1}{\rho}} h(x_{ss},0),\\
&\beta(\eta e^{x_{ss,2}} x_{ss,1}^{\eta-1} + 1 - \delta) = 1,\\
&W(x_{ss},0) - V(x_{ss},0)^\alpha = 0,\\
&x_{ss,1} - e^{x_{ss,2}} x_{ss,1}^\eta + h^*(x_{ss},0) - (1-\delta)x_{ss,1} = 0,\\
&x_{ss,2} - \rho x_{ss,2} = 0.
\end{aligned}
$$

From the last equation it follows that

$$x_{ss,2} = 0.$$

Hence, the second equation delivers Robinson's steady state capital stock

$$x_{ss,1} = \left(\frac{1-\beta(1-\delta)}{\beta\eta}\right)^{\frac{1}{\eta-1}}.$$

The point of expansion for the Taylor polynomials therefore is $(x_{ss}, 0)$ with

$$x_{ss} = \begin{pmatrix} \left(\frac{1-\beta(1-\delta)}{\beta\eta}\right)^{\frac{1}{\eta-1}} \\ 0 \end{pmatrix}.$$

Further, we use the remaining three equations to calculate Robinson's steady state consumption

$$h^*(x_{ss},0) = x_{ss,1}^\eta - \delta x_{ss,1},$$

the steady state value of the value function

$$V(x_{ss},0) = \left(\frac{1}{1-\beta}\right)^{\frac{1}{\rho}} h^*(x_{ss},0),$$

plus the auxiliary function

$$W(x_{ss},0)=V(x_{ss},0)^{\alpha}.$$

In order to find the Taylor approximations of the optimal policy, the induced deterministic part of the dynamic and the value function around the steady state, we have to compute the derivatives of h^*, f^*, V and W in $(x_{ss},0)$ with respect to x_1, x_2 and σ at the steady state.

Now, from $F_i(x,\sigma)\equiv 0, i=1,\dots,5$, it follows that all partial derivatives must be zero, too. I.e. especially at the deterministic steady state it holds that

$$\begin{aligned}
&\frac{F_i}{\partial x_1}(x_{ss},0)=0, i=1,\dots,5,\\
&\frac{F_i}{\partial x_2}(x_{ss},0)=0, i=1,\dots,5,\\
&\frac{F_i}{\partial \sigma}(x_{ss},0)=0, i=1,\dots,5.
\end{aligned} \tag{2.4.7}$$

By plugging in the values for $h^*(x_{ss},0)$, $f_1^*(x_{ss},0)$, $f_2^*(x_{ss},0)$, $V(x_{ss},0)$ and $W(x_{ss},0)$ derived above, (2.4.7) is a system of 15 polynomial equations of at most second order in 15 unknowns, namely the partial derivatives of h^*, f_1^*, f_2^*, V and W with respect to x_1, x_2 and σ at the steady state. Solving this system of equations thus yields the sought for first derivatives. In order to be able to pin down the polynomial coefficients uniquely, we additionally demand the dynamic to be stable, or equivalently demand its Jacobian, with respect to x_1, x_2 and σ evaluated at the steady state, to only have eigenvalues of modulus less than unity.

Next, to find the second order derivatives of h^*, f_1^*, f_2^*, V and W, we accordingly compute the second order derivatives of all F_i, $i=1,\dots,5$, at the steady state with respect to x_1, x_2 and σ and additionally plug in the already calculated values of the first order derivatives. This yields a (now linear) system of equations in the unknown second order derivatives of h^*, f_1^*, f_2^*, V and W with respect to x_1, x_2 and σ at the steady state. Its necessarily unique solution completes the required computations for a second order perturbation.

2.4.4.2 Why at least second order?

In this subsection, we want to briefly summarize why it is sensible to at least perform second order approximations of EZ economies. This is a consequence of the certainty equivalence property of first order perturbations as proved in Schmitt-Grohe and Uribe (2004). It states that the coefficients of a linear approximation are independent of the degree of uncertainty in the economy. This is already problematic in general. Most prominently, the expectation of a linearly perturbed variable turns out to equal its deterministic steady state value, entirely independent of σ. Thus, in terms of artificial data generated by a linear approximation of our model, simulating ergodic return time series from pseudorandom iidN(0,1) shocks will yield vanishing simulated risk premia on average, independent of the assumed degree of risk aversion.

Moreover, as demonstrated by van Binsbergen et al. (2012), first order perturbation coefficients are also independent of the risk aversion parameter α. Thus, generalizing a model towards the EZ/KP class leads to identical results as with standard EU preferences ($\alpha=\rho$) if both variants feature the same EIS.[32]

Eventually, (relative) welfare cost measures of business cycle volatility are typically based on “risky steady state” comparisons of the approximated value function, i.e. on

$$V(x_{ss},\sigma).$$

Such evaluations thus also demand at least second order approximation.

[32] Note that while this renders the calibration of α ineffective, the EZ/KP representation (at least) still allows α to be set independently of the EIS.

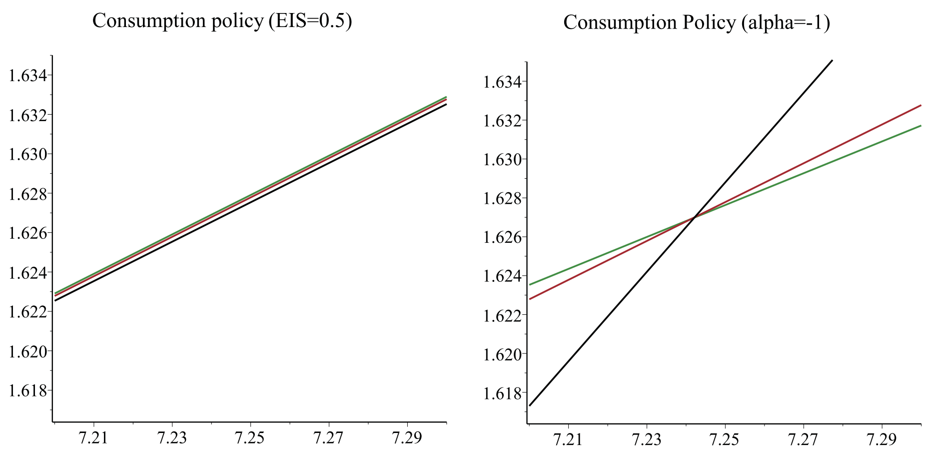
2.4.4.3 Numerical example

In this subsection, we use an actual parametric example of our model's second order perturbation to demonstrate where the assumption of nonindifference towards the timing of uncertainty resolution explicitly impacts applied work. Additionally, we begin with pointing at the different ways through which the two parameters of primary interest, EIS and α, affect such second order perturbations.

All figures in this subsection display three different parameterizations of Robinson's attitude towards the timing of uncertainty resolution. Thereby, green lines denote a scenario in which Robinson prefers later resolution, i.e. $\text{EIS}^{-1} > 1-\alpha$, red lines denote the indifference scenario $\text{EIS}^{-1} = 1-\alpha$, and black lines display an early resolution case $\text{EIS}^{-1} < 1-\alpha$. Besides, we fix a quarterly calibration $\beta = 0.95, \eta = 0.27, \lambda = 0.9, \sigma = 0.0072, \delta = 0.011$.

Key parameters First, figure 2.10 compares the respective effects of α and EIS on the computed approximation of the optimal consumption policy h^*, displayed at the steady state value of productivity as a function of capital only. The left graph results from the calibrations EIS $=$ 0.5 with $\alpha \in \{-4, -1, 0.5\}$ to calibrate early, indifference and late resolution preferences, respectively. Similarly, the right graph displays the resulting approximation for $\alpha = -1$ with EIS $\in \{0.25, 0.5, 5\}$. It shows that while the risk aversion parameter only shifts the consumption

Figure 2.10: Key parameters (1/2)



policy, the EIS also exerts influence on its slope. This is generally true.[33] Moreover, scaling uncertainty via σ also only affects the policy's ordinate intercept.

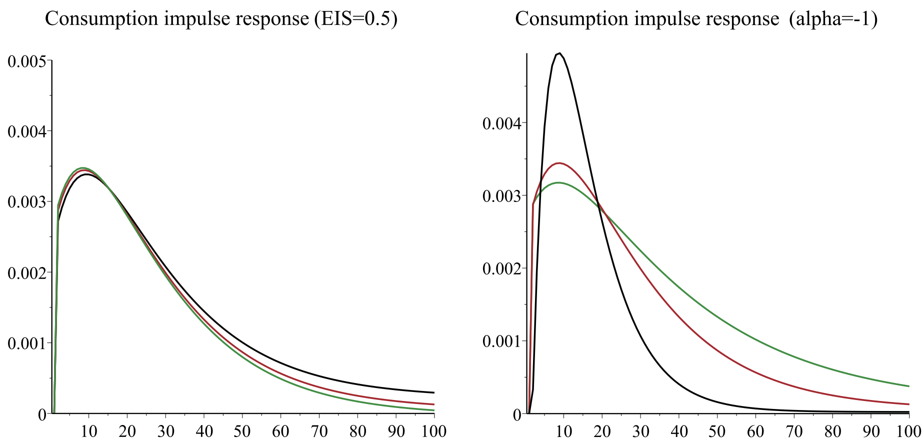
Figure 2.11 offers another perspective on each parameter's impact on the solution displaying the consumption policy's response to a once only positive shock in ϵ of magnitude 1. The studied scenarios are identical to the figure above. It shows how it is less the attitude towards uncertainty resolution but much rather the EIS directly that impacts the response of macroeconomic quantities. Note how the nature of the effect of EIS is evident in the right graph. The smaller his EIS, the more Robinson strives for a smooth consumption path. Accordingly is a smaller EIS (and therefore rather late resolution preferences for some fixed calibration of α) reflected in larger persistence in the consumption's response.[34]

Nonindifference Another crucial variable in empirical macroeconomics is the equity premium. We finish this discussion with some remarks on its replicability. Therefore, note that Robinson's

[33] See Schmitt-Grohe and Uribe (2004) and van Binsbergen et al. (2012) on this limited effect of α on second order perturbations.

[34] Note the differences in the scenarios' convergence levels, i.e. their *risky* steady state.

Figure 2.11: Key parameters (2/2)



risky next period return on equity (RoE) is given by

$$\text{RoE} := \eta e^{\lambda x_2+\sigma\epsilon}(e^{x_2}x_1^{\eta}+(1-\delta)x_1-h^*(x))^{\eta-1}+(1-\delta).$$

Consequently, Robinson's Euler condition (2.4.5) has the interpretation of a Lucas (1978) equation such that his stochastic discount factor (SDF) is

$$\text{SDF} := \beta\left(\frac{V^{\alpha}(f_1^*(x,\sigma),f_2^*(x)+\sigma\epsilon)}{\mathbb{E}_P[V^{\alpha}(f_1^*(x,\sigma),f_2^*(x)+\sigma\epsilon)]}\right)^{1-\frac{\rho}{\alpha}}\left(\frac{h^*(f_1^*(x,\sigma),f_2^*(x,\sigma)+\sigma\epsilon)}{h^*(x)}\right)^{\rho-1}.$$

Accordingly, if there was a risk free asset available to Robinson, its next period return (r^f) would have to satisfy

$$1=\mathbb{E}_P\,[\text{SDF}]\,r^f.$$

Therefore, his expected premium for bearing equity risk

$$EP := \mathbb{E}_P\left[\text{RoE}-r^f\right]$$

would be of magnitude

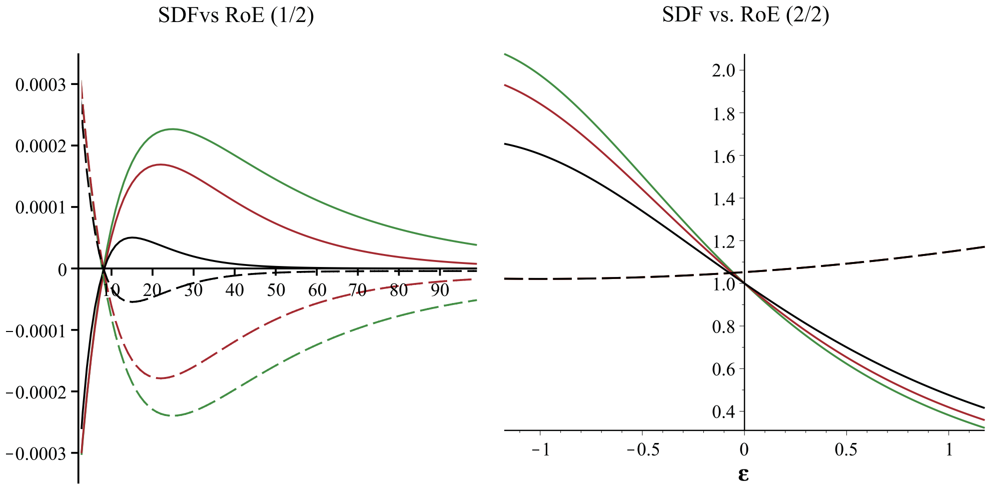
$$EP = -r^f\,\text{Cov}\,[\text{SDF},\text{RoE}]$$

such that the covariance between SDF and RoE is found to be its key driver.

It is evident from the primitive perturbation system (2.4.6) that an assumed nonindifference with respect to the timing of uncertainty resolution enters Robinson's solution through the SDF. Figure 2.12 therefore illustrates how different preference scenarios affect the SDF and the covariance between SDF and RoE. Both its graphs are generated from a second order perturbation at $\alpha=-1$ and display the SDF (solid lines) and the RoE (dashed lines) in the three parametric scenarios from above, i.e. EIS $=\{0.25,0.5,5\}$.

The left graph displays impulse responses to an ϵ shock as above. It first again shows how the convergence to respective *risky* steady state levels is slower for later resolution calibrations. It secondly also shows a more pronounced countercycality of SDF and RoE indicating a larger negative covariance. This is further confirmed by the right hand side graph. It explicitly plots both SDF and RoE as functions in (possible realizations of) ϵ with capital and technology at their respective deterministic steady state values. While the RoE is largely unaffected, the SDF shows a stronger negative comovement for smaller EIS (later resolution). This shows how the EZ representation offers applied macroeconomists a channel for the replication of the empirical equity premium. One may impose a strong enough aversion to nonsmooth consumption without having to set the risk aversion parameter unreasonably high.

Figure 2.12: Nonindifference



2.5 `Maple-Matlab` toolbox

As outlined above, for perturbation it is necessary to find the derivatives of the functions that constitute our model's equilibrium, all evaluated at the steady state, up to the desired order of the Taylor polynomials. As suggested by Judd (1996) in his introductory remarks, we delegate such computation to a computer algebra system. This section provides some noteworthy details about the associated files, which can be downloaded from the authors homepages.[35]

2.5.1 Overview

We provide a `Maple`-based toolbox for perturbation of DSGE models. The core `mw`-file has an intuitive structure that fits a wide variety of model economies. The systems of equations that determine the perturbation coefficients are derived analytically. Hence, in comparison to purely numerical perturbation packages, our solution is more precise and importantly allows for an intuitive and easy implementation of approximations up to any desired order.[36] In fact, the provided procedures already allow for third order perturbation but the code is straightforwardly extended to higher orders.

The quadratic system of equations for the first order coefficients can be solved either through a general (analytical or a numerical) `Maple`-internal nonlinear solver or using the generalized Schur decomposition implemented in `Matlab`.[37]

Alongside, our toolbox features a number of test devices to check for the quality of the solution and provides all necessary information about the solution in the form of `txt`-output. In order for these files to be correctly stored, the user must create a folder named `output` in the same directory where the core `mw`-file is located.

2.5.2 Brief documentation

Functionality The core `mw`-file requires the user to enter the set of equilibrium conditions that defines the DSGE model under consideration. This is done conveniently using the worksheet's `Math` mode. Some variable x is entered with the suffix 1 if associated with the current period

[35] `www.wiwi.uni-augsburg.de/vwl/maussner/lehrstuhl/heiberger_en.html` or `http://www.wiwi.uni-augsburg.de/vwl/maussner/lehrstuhl/ruf_en.html`.

[36] Of course, the practicability of higher order perturbation is nevertheless restricted by computation time.

[37] This solution method makes use of the `Matlab`-link provided by `Maple`. Generally, the programs were run on `Maple 17` and `Matlab 2013a`.

($x1$) and with suffix 2 for next period ($x2$). The toolbox thereafter demands the user to list the equation numbers (`fktn`), the endogenous states (`xname`), exogenous states (`zname`), the control variables (`uname`), and the shocks (`shocks`). Static equations within the set of equilibrium conditions that are easily solved for (e.g. definitional equations) can be listed as auxiliary functions (`hilfsf`) alongside the associated auxiliary variables (`hilfsvar`) to facilitate the nonlinear solving procedures. Next, the user is asked to enter the model's calibration (`parameter`) and finally its deterministic steady state solution (`ss`) in terms of parameter names (not values). The remaining steps are automatized and briefly outlined in the following paragraphs.

The toolbox's core procedure is `getlsg_fneu`. Using the information entered as described above, it solves for the Taylor polynomials coefficients up to third order (`thrd`) using the chosen solution method for the first order system (`mode`), where `mode` $= 1$ selects the analytical solver, `mode` $= 4$ the numerical solver, and `mode` $= \{2,3\}$ respectively executes a generalized Schur decomposition using the `Maple-Matlab` link. `getlsg_fneu` itself calls six subprocedures to be sketched below.

First, `transf` transforms the conveniently entered equilibrium conditions into the perturbation logic, i.e. control variables and next period's states as functions of the current states and the perturbation parameter, subject to the model's shocks. Next, `getss` computes the deterministic steady state and writes it into the `steadystate.txt` file.

The subsequent step depends on the chosen solution method. For `mode` $= \{1,4\}$, the solve routine calls `getgls`. This subprocedure in turn calls `glsys` which generates the basic systems of equations by differentiating the equilibrium conditions and calling `ew` to compute the expected value of all equations. The latter is done analytically making use of the assumed (mutual) independence and standard normality of the shocks. In particular, it iteratively factors the equations as polynomials in the respective shocks and then multiplies the resulting coefficients by these shocks' moments which are determined by the double factorial formula. Thereafter, the equations are returned at the calibrated parameter values. For `mode` $= \{2,3\}$, `getmatrix` is called. This subprocedure again uses the system of equations generated by `glsys` but now makes further use of its formal structure. In particular, it generates the matrix pencil A, B as in Schmitt-Grohe and Uribe (2009) and writes it into the two respectively named `txt` files.

In the remaining steps, `getlsg_fneu` computes the sought for derivatives of the optimal policy and dynamic by the chosen method and collects them in the arrays $J = \begin{bmatrix} J_1 & J_2 & J_3 \end{bmatrix}$ and $H = \begin{bmatrix} H_1 & H_2 & H_3 \end{bmatrix}$. The three elements of J are the Jacobians of the endogenous dynamic (J_1), the exogenous dynamic (J_2), and the optimal policy (J_3), while the three elements of H are the associated cubes consisting of the respective Hessian matrices in the same order. These Jacobians and Hessian cubes are finally written into accordingly named `txt` files. Note that in the Schur-based method (`mode` $= \{2,3\}$), the uniqueness of the quadratic first order system's solution is attained by construction. For `mode` $= \{1,4\}$, uniqueness is forced through additionally imposing the dynamic's Jacobian to only have eigenvalues within the unit circle.

Quality of solution In order to check and document the quality of the computed solution, the toolbox features the following instruments. First, the deterministic steady state values are inserted into the primitive equilibrium conditions and the equations numerical deviation from 0 is printed. Second, for all solution methods, `getlsg_fneu` checks the uniqueness of the first order solution and aborts the computation with an error message in case of multiple equilibria. Third, we also provide the maximum norms of the differences $A - Q^T A Z^T$ and $B - Q^T B Z^T$, respectively, to check whether the computed matrices Q and Z actually constitute a Schur decomposition. Fourth, the QZ decomposition can be executed in both orders, A, B (`mode` $= 2$) and B, A (`mode` $= 3$), such that a cross check may support confidence in the solution. This, of course, holds just as much for crosschecking over the other available solution methods.

2.6 Conclusion

In this paper, we summarized the crucial elements of the EZ representation, demonstrated the application of the Schmitt-Grohe and Uribe (2004) approach, and provide a flexible computer algebra toolbox for its application. As an immediate implication for applied work, we find the EIS to play a more prominent role than the risk aversion parameter α and (consequently) suspect the *late* resolution case to be likely to give rise to higher equity premia.

In light of the latter results, we would like to close with two remarks. First, the incomplete disentanglement of comparative risk aversion from EIS, which gives rise to nonindifference with respect to resolution timing, might itself be undesirable.[38] A more general representation in which risk aversion and EIS are completely separated would not only provide more flexibility for applied work but also help to clarify the roles played by each of these key aspects of preferences.[39] Second, the parameter of atemporal consumption risk aversion might simply be an inadequate measure of risk aversion in intertemporal decision problems.[40]

[38] Note that it is not only difficult to rationalize why preferences for later resolution generate larger premia, it is even less evident how to correctly calibrate nonindifference.

[39] See Weil (1990), p. 33, for a conjecture on how to achieve further disentanglement. Note however, that there might also be an "inherent inseparability" as suspected by Epstein and Zin (1989), p. 953.

[40] See Träger (2011) and Swanson (2012) for related theoretical work.

Appendix

A Consistency of induced trees

We want to prove by induction that the induced trees are indeed consistent. Therefore, we additionally introduce further notation. Define for any $\tau \in \mathbb{N}$ the solution of the state process $\{X_t^{x_0,h,\tau}\}_{t=0}^{\infty}$ and the control process $\{Y_t^{x_0,h,\tau}\}_{t=0}^{\infty}$ under the initial state x_0 and the admissible strategy h that arises if we shift the stochastic process $\{\epsilon_t\}_{t=1}^{\infty}$ by τ periods, i.e.

$$
\begin{aligned}
X_0^{x_0,h,\tau} &:= x_0 \\
Y_t^{x_0,h,\tau} &:= h(X_t^{x_0,h,\tau}), \text{ f.a. } t \in \mathbb{N}, \\
X_{t+1}^{x_0,h,\tau} &:= f(X_t^{x_0,h,\tau}, Y_t^{x_0,h,\tau}, \epsilon_{\tau+t+1}), \text{ f.a. } t \in \mathbb{N}.
\end{aligned}
$$

Note that because of the iid assumption of the stochastic process $\{\epsilon_t\}_{t=0}^{\infty}$, the probability distribution for this process is the same as for the non-shifted solution. Yet, the specific realizations for some arbitrary $\omega \in \Omega$ may differ.

Now, starting with $t = 1$, we–according to subsection 2.2.3.2–find for all $B \in \mathscr{B}\left(\mathbb{R}_+^{\infty}\right)$

$$
\begin{aligned}
f_1(m_2^{x_0,h})(B) &= \int_{\mathbb{R}_+ \times \mathscr{M}(\mathbb{R}_+^{\infty})} \tilde{m}_1\left(\left\{y \in \mathbb{R}_+^{\infty} \mid (\tilde{c}_1, y) \in B\right\}\right) dm_2^{x_0,h}(\tilde{c}_1, \tilde{m}_1) \\
&= \int_{\mathbb{R}_+ \times \mathscr{M}(\mathbb{R}_+^{\infty})} \tilde{m}_1\left(\left\{y \in \mathbb{R}_+^{\infty} \mid (\tilde{c}_1, y) \in B\right\}\right) dP \circ \left(\iota_1(X_1^{x_0,h}(\cdot), h)\right)^{-1} (\tilde{c}_1, \tilde{m}_1) \\
&= \int_{\Omega} m_1^{X_1^{x_0,h}(\tilde{\omega}),h}\left(\left\{y \in \mathbb{R}_+^{\infty} \mid (Y_0^{X_1^{x_0,h}(\tilde{\omega}),h}, y) \in B\right\}\right) dP(\tilde{\omega}).
\end{aligned}
$$

Now, note that for an arbitrary but fixed $\tilde{\omega} \in \Omega$ it holds for all $B \in \mathscr{B}\left(\mathbb{R}_+^{\infty}\right)$ that

$$
\begin{aligned}
& m_1^{X_1^{x_0,h}(\tilde{\omega}),h}\left(\left\{y \in \mathbb{R}_+^{\infty} \mid (Y_0^{X_1^{x_0,h}(\tilde{\omega}),h}, y) \in B\right\}\right) = \\
&= P\left(\left\{\omega \in \Omega \mid (Y_1^{X_1^{x_0,h}(\tilde{\omega}),h}(\omega), Y_2^{X_1^{x_0,h}(\tilde{\omega}),h}(\omega), \ldots) \in \left\{y \in \mathbb{R}_+^{\infty} \mid (Y_0^{X_1^{x_0,h}(\tilde{\omega}),h}, y) \in B\right\}\right\}\right) \\
&= P\left(\left\{\omega \in \Omega \mid (Y_0^{X_1^{x_0,h}(\tilde{\omega}),h}(\omega), Y_1^{X_1^{x_0,h}(\tilde{\omega}),h}(\omega), Y_2^{X_1^{x_0,h}(\tilde{\omega}),h}(\omega), \ldots) \in B\right\}\right) \\
&= P\left(\left\{\omega \in \Omega \mid (Y_0^{X_1^{x_0,h}(\tilde{\omega}),h,1}(\omega), Y_1^{X_1^{x_0,h}(\tilde{\omega}),h,1}(\omega), \ldots) \in B\right\}\right) \\
&= P\left(\left\{\omega \in \Omega \mid (Y_0^{X_1^{x_0,h}(\tilde{\omega}),h,1}(\omega), Y_1^{X_1^{x_0,h}(\tilde{\omega}),h,1}(\omega), \ldots) \in B\right\} \middle| \left\{\omega \in \Omega \mid X_1^{x_0,h}(\omega) = X_1^{x_0,h}(\tilde{\omega})\right\}\right) \\
&= P\left(\left\{\omega \in \Omega \mid (Y_1^{x_0,h}(\omega), Y_2^{x_0,h}(\omega), \ldots) \in B\right\} \middle| \left\{\omega \in \Omega \mid X_1^{x_0,h}(\omega) = X_1^{x_0,h}(\tilde{\omega})\right\}\right).
\end{aligned}
$$

Thereby, the third last equality makes use of the above definition of a solution that is shifted by $\tau = 1$ steps. The second last equality then follows from the fact that in this shifted solution there is no ϵ_1 such that it is stochastically independent of the condition. Thereby, the expressions in the last two lines denote the conditional probability for the respective random sequence given the event $X_1^{x_0,h} = X_1^{x_0,h}(\hat{\omega})$. Because of the shifting, the random sequences in the last two lines are identical. The base case now follows from the properties of conditional expectations. To see this, we define

$$C := \left\{\omega \in \Omega \mid (Y_1^{x_0,h}(\omega), Y_2^{x_0,h}(\omega), \ldots) \in B\right\}$$

and find

$$\begin{aligned}
&f_1(m_2^{x_0,h})(B) = \\
&= \int_\Omega P\left(\left\{\omega \in \Omega \mid (Y_1^{x_0,h}(\omega), Y_2^{x_0,h}(\omega), \ldots) \in B\right\} \,\middle|\, \left\{\omega \in \Omega \mid X_1^{x_0,h}(\omega) = X_1^{x_0,h}(\tilde{\omega})\right\}\right) dP(\tilde{\omega}) \\
&= \int_\Omega P\left(C \,\middle|\, \left\{\omega \in \Omega \mid X_1^{x_0,h}(\omega) = X_1^{x_0,h}(\tilde{\omega})\right\}\right) dP(\tilde{\omega}) \\
&= \int_\Omega \mathbb{E}\left[\mathbb{1}_C \,\middle|\, X_1^{x_0,h} = X_1^{x_0,h}(\tilde{\omega})\right] dP(\tilde{\omega}) = \int_\Omega \mathbb{E}\left[\mathbb{1}_C \,\middle|\, X_1^{x_0,h}\right](\tilde{\omega}) dP(\tilde{\omega}) \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_C \,\middle|\, X_1^{x_0,h}\right]\right] = \mathbb{E}[\mathbb{E}[\mathbb{1}_C \mid \mathscr{F}_1]] = \mathbb{E}[\mathbb{1}_\Omega \mathbb{E}[\mathbb{1}_C \mid \mathscr{F}_1]] \overset{\Omega \in \mathscr{F}_1}{=} \mathbb{E}[\mathbb{1}_\Omega \mathbb{1}_C] = P(C) \\
&= P\left(\left\{\omega \in \Omega \mid (Y_1^{x_0,h}(\omega), Y_2^{x_0,h}(\omega), \ldots) \in B\right\}\right) \\
&= m_1^{x_0,h}(B).
\end{aligned}$$

Thus, the induction hypothesis (IH) reads

$$f_t(m_{t+1}^{x_0,h}) = m_t^{x_0,h},$$

for some $t \in \mathbb{N}$. Recalling the constructed f_t and g_t from subsection 2.2.3.2, the inductive step follows for all $t \geq 1$ and $B \in \mathscr{B}_t$, i.e.

$$\begin{aligned}
f_{t+1}(m_{t+2}^{x_0,h})(B) = m_{t+2}^{x_0,h}(g_t^{-1}(B)) &= P\left(\left\{\omega \in \Omega \mid \iota_{t+1}(X_1^{x_0,h}(\omega), h) \in g_t^{-1}(B)\right\}\right) \\
&= P\left(\left\{\omega \in \Omega \mid g_t\left(\iota_{t+1}(X_1^{x_0,h}(\omega), h)\right) \in B\right\}\right) \\
&= P\left(\left\{\omega \in \Omega \mid g_t\left(Y_1^{x_0,h}(\omega), m_{t+1}^{X_1^{x_0,h}(\omega)}\right) \in B\right\}\right) \\
&= P\left(\left\{\omega \in \Omega \mid (Y_1^{x_0,h}(\omega), f_t(m_{t+1}^{X_1^{x_0,h}(\omega)})) \in B\right\}\right) \\
&\overset{\text{(IH)}}{=} P\left(\left\{\omega \in \Omega \mid (Y_1^{x_0,h}(\omega), m_t^{X_1^{x_0,h}(\omega)}) \in B\right\}\right) \\
&= m_{t+1}^{x_0,h}(B).
\end{aligned}$$

This shows that the sequence $(\iota_1(x_0, h), \iota_2(x_0, h), \ldots)$ is consistent and therefore lies in D.

B Euler equation

Differentiating the bracketed right hand side expression of (2.4.3) with respect to y, we find the necessary condition for a maximum

$$\begin{aligned}
0 =& \frac{1}{\rho}[\dots]^{\frac{1}{\rho}-1}\Big(\rho y^{\rho-1}+ \\
&+\beta\frac{\rho}{\alpha}(\mathbb{E}_P[V^\alpha(f(x,y,\epsilon))])^{\frac{\rho}{\alpha}-1}\mathbb{E}_P[\alpha V^{\alpha-1}(f(x,y,\epsilon_t))\frac{\partial V}{\partial x_1}(f(x,y,\epsilon_t))\frac{\partial f_1}{\partial y}(x,y,\epsilon)]\Big) \\
=& V^{1-\rho}(x)\Big(y^{\rho-1}-\beta(\mathbb{E}_P[V^\alpha(f(x,y,\epsilon))])^{\frac{\rho}{\alpha}-1}\mathbb{E}_P[V^{\alpha-1}(f(x,y,\epsilon))\frac{\partial V}{\partial x_1}(f(x,y,\epsilon_t))]\Big) \\
\Leftrightarrow& y^{\rho-1}-\beta(\mathbb{E}_P[V^\alpha(f(x,y,\epsilon))])^{\frac{\rho}{\alpha}-1}\mathbb{E}_P[V^{\alpha-1}(f(x,y,\epsilon))\frac{\partial V}{\partial x_1}(f(x,y,\epsilon))] = 0.
\end{aligned}$$

Introducing the abbreviation

$$X^{(1)} := f(x,h^*(x),\epsilon),$$

h^* must hence satisfy

$$h^*(x)^{\rho-1}-\beta\left(\mathbb{E}_P[V^\alpha(X^{(1)})]\right)^{\frac{\rho}{\alpha}-1}\mathbb{E}_P\left[V^{\alpha-1}(X^{(1)})\frac{\partial V}{\partial x_1}(X^{(1)})\right]=0. \tag{B.1}$$

Further, by the envelope theorem, we find the derivative of the value function with respect to x_1 from (2.4.4)

$$\begin{aligned}
\frac{\partial V}{\partial x_1}(x) =& \frac{1}{\rho}[\dots]^{\frac{1}{\rho}-1}\beta\frac{\rho}{\alpha}\mathbb{E}_P[V^\alpha(f(x,h^*(x),\epsilon)])^{\frac{\rho}{\alpha}-1}\cdot \\
&\cdot\mathbb{E}_P\left[\alpha V^{\alpha-1}(f(x,h^*(x),\epsilon)\frac{\partial V}{\partial x_1}(f(x,h^*(x),\epsilon)\frac{\partial f_1}{\partial x_1}(x,h^*(x),\epsilon)\right] \\
=& V^{1-\rho}(x)\beta\mathbb{E}_P[V^\alpha(f(x,h^*(x),\epsilon))])^{\frac{\rho}{\alpha}-1}\cdot \\
&\cdot\mathbb{E}_P\left[V^{\alpha-1}(f(x,h^*(x),\epsilon))\frac{\partial V}{\partial x_1}(f(x,h^*(x),\epsilon))(\eta e^{x_2}x_1^{\eta-1}+(1-\delta))\right],
\end{aligned}$$

or again abbreviated

$$\frac{\partial V}{\partial x_1}(x)=V^{1-\rho}(x)(\eta e^{x_2}x_1^{\eta-1}+(1-\delta))\beta\mathbb{E}_P[V^\alpha(X^{(1)})])^{\frac{\rho}{\alpha}-1}\mathbb{E}_P\left[V^{\alpha-1}(X^{(1)})\frac{\partial V}{\partial x_1}(X^{(1)})\right]. \tag{B.2}$$

From (B.1) we see

$$\frac{\partial V}{\partial x_1}(x)=V^{1-\rho}(x)(\eta e^{x_2}x_1^{\eta-1}+(1-\delta))\,(h^*(x))^{\rho-1}. \tag{B.3}$$

Now, iterating this equation forward by one period and plugging it into (B.1) yields

$$\begin{aligned}
(h^*(x))^{\rho-1} =& \beta\mathbb{E}_P[V^\alpha(X^{(1)})])^{\frac{\rho}{\alpha}-1}\cdot \\
&\cdot\mathbb{E}_P[V^{\alpha-1}(X^{(1)})V^{1-\rho}(X')(\eta e^{X_2^{(1)}}\left(X_1^{(1)}\right)^{\eta-1}+(1-\delta))\left(h^*(X^{(1)})\right)^{\rho-1}],
\end{aligned}$$

or equivalently

$$\begin{aligned}
0 =& \mathbb{E}_P\left[\beta\mathbb{E}_P[V^\alpha(X^{(1)})])^{\frac{\rho}{\alpha}-1}V^{\alpha-\rho}(X^{(1)})(\eta e^{X_2^{(1)}}\left(X_1^{(1)}\right)^{\eta-1}+(1-\delta))\left(\frac{h^*(X^{(1)})}{h^*(x)}\right)^{\rho-1}-1\right] \\
=& \mathbb{E}_P\left[\beta\left(\frac{V^\alpha(X^{(1)})}{\mathbb{E}_P[V^\alpha(X^{(1)})]}\right)^{1-\frac{\rho}{\alpha}}\left(\frac{h^*(X^{(1)})}{h^*(x)}\right)^{\rho-1}(\eta e^{X_2^{(1)}}\left(X_1^{(1)}\right)^{\eta-1}+(1-\delta))-1\right].
\end{aligned}$$

Chapter 3

Epstein-Zin Utility, Asset Prices, and the Business Cycle Revisited

— *Christopher Heiberger and Halvor Ruf* —

3.1 Introduction

The financial and economic crisis of 2007-2009 has drawn attention to the interplay between asset markets and goods and factor markets. In macroeconomics, this interplay is considered within the framework of dynamic stochastic general equilibrium (DSGE) models, which have grown out of the neoclassical (stochastic) growth model. To be useful for this purpose DSGE models have to be consistent with empirical findings that characterize these markets.

Since the work of Mehra and Prescott (1985) and the bulk of literature spurred by this paper[1] it is well known that the stochastic growth model is quantitatively not able to replicate the equity premia that have been observed in 20th century data. For instance, Mehra and Prescott (2003), updating their calculations from Mehra and Prescott (1985) for the U.S., report an average equity premium (excess of the return on a stock market index over the return of a relatively riskless security) of 6.92% over the period 1989-2000. The standard stochastic growth model, however, predicts an equity premium of only 0.02% according to Jermann (1998).

Jermann (1998) amends the standard stochastic growth model in two directions. Drawing on work by Constantinides (1990) and Campbell and Cochrane (1999) he introduces habits, which increase the household's desire to smooth consumption. Assets with procyclical returns, therefore, must pay a higher return to be willingly held by investors. In a production economy, however, this feature does not suffice to increase the equity premium. In addition, it must be costly for the household to smooth consumption. Jermann (1998) accomplishes this with adjustment costs of capital. His model, with these two components combined, is able to produce an equity premium of about 6.18%. However, this feature of the model breaks down if labor supply is endogenous because the household can smooth his consumption by reducing working hours in response to positive technology shocks. The stochastic growth model with habits, adjustment costs, and endogenous labor supply, thus, has two unattractive features: it is unable to replicate an empirically plausible equity premium and it predicts negative correlations between hours and output and between hours and real wages, both contrary to the correlations found in the data.

Later work therefore has considered frictions in the allocation of labor. Boldrin et al. (2001) consider a two sector model in which the representative household is committed to his choice of labor supply to the two sectors after the productivity shock has been revealed. This model is able to reproduce the equity premium and the positive correlation between hours and output

[1] See, among others, the surveys of Abel (1990), Kocherlakota (1996), Campbell (2003), Mehra and Prescott (2003), and Cochrane (2008).

but still fails to predict a positive correlation between hours and the real wage. Uhlig (2007) combines habits in consumption and leisure with a sticky real wage and successfully reproduces the empirically observed equity premium as well as the correlations between output and hours and hours and the real wage.

The work considered so far has relied on standard preferences over consumption and leisure where the non-time additivity arises from the specification of habits. The parameters which determine the importance of the habit and its evolution over time are then chosen to match serval empirical targets, as e.g., the equity premium or the risk free rate. The class of generalized recursive preferences introduced by Epstein and Zin (1989) (EZ) and Weil (1989) has received attention mainly in studies of the equity premium puzzle within the context of endowment economies. EZ preferences allow to separate the parameter that captures the agent's risk aversion from the parameter that determines his willingness to substitute consumption between periods. While many researcher consider a rather small number close to unity to be a plausible parameterization for the former, there is less restricting evidence for the value of the latter. Therefore, this class of preferences provides an additional degree of freedom to trace empirical regularities. As shown by Weil (1989), however, in order to get both an empirically plausible equity premium of 5.72% and a risk-free rate of 0.85% one must assume a coefficient of relative risk aversion of 45 and an intertemporal elasticity of substitution of 0.10. Reducing the former to the value of unity, the equity premium shrinks to 0.45% while the risk free rate jumps above 20%. Weil (1989) receives this results from a model of an endowment economy in which the process of dividend and consumption growth follows a two-state Markov chain calibrated to the U.S. experience. Thus, his model neglects the interplay between savings, capital accumulation, and the return to capital.

Kaltenbrunner and Lochstoer (2010) consider EZ preferences in a production economy with fixed labor supply and adjustment costs of capital. Their model, thus, deviates from the Jermann (1998) model just in one respect. Roughly speaking, they substitute one free parameter (the habit parameter) for one other (the intertemporal substitution parameter). Their model is able to trace four targets: the volatility of consumption growth, the relative volatilities of consumption and output growth, the equity Sharpe ratio (i.e. the ratio of the equity premium to the standard deviation of the equity return), and the level of the risk-free rate. We follow their approach with regard to real business cycle models with endogenous labor, for the reason mentioned above: useful models should be able to replicate facts from three kinds of markets: asset markets, output, and factor markets, among which the labor market is the most important one. In this respect our study is in line with Heer and Maußner (2013) who consider the ability of the Jermann (1998), the Boldrin et al. (2001), and the Uhlig (2007) model (among others) to replicate plausible asset market and business cycle facts.

Our study focuses on seven empirical targets: the equity premium, the risk free rate, the standard deviations of the cyclical components of investment, hours, and the real wage relative to the standard deviation of output as well as the cross-correlations of output with hours and of hours and the real wage. The models are parameterized to have up to six free parameters: the discount factor $\beta \in (0,1)$, the intertemporal elasticity of substitution $\psi > 0$, the elasticity of Tobin's q with respect to the investment-capital ratio $\kappa \geq 0$ (a measure of the importance of adjustment costs of capital), a habit parameter $\chi \in [0,1)$ and a parameter $\lambda \in [0,1]$ that reflects the speed of adjustment of the habit to actual consumption, and a parameter $\mu \in [0,1]$ that measures the degree of real wage stickiness.

Tables 3.1 and 3.2 summarize our results. The model labeled M0 is the Jermann (1998) model without the consumption habit, which has just three free parameters: β, ψ, and κ which were calibrated to exactly match the equity premium EP, the risk-free rate r^f, and the relative standard deviation of investment s_i/s_y. The model M1 is the Jermann (1998) model with endogenous labor, without a habit (a), with an exogenous habit equal to previous consumption (b), and with

Table 3.1: Summary of Results

	EP	r^f	s_y	s_i/s_y	s_n/s_y	s_w/s_y	r_{yn}	r_{wn}	$Score$
Data									
	5.18	2.49	1.14	2.28	0.69	1.03	0.40	0.27	
M0: Baseline									
	5.18	2.49	0.89	2.28	0				
M1: No labor market frictions									
a	5.19	2.55	1.01	2.27	0.16	0.84	1.00	1.00	1.23
b	5.27	2.50	0.92	2.16	0.12	0.96	0.37	0.26	0.40
c	5.27	2.50	0.92	2.16	0.12	0.96	0.37	0.26	0.40
M2: Sticky real wages									
a	5.13	2.50	1.23	2.18	0.47	0.64	0.87	0.63	0.58
b	5.27	2.50	0.92	2.16	0.12	0.96	0.37	0.26	0.40
c	5.17	2.50	1.03	2.35	0.43	0.84	0.55	0.15	0.16
M3: Predetermined labor supply									
a	5.15	2.49	0.96	2.30	0.14	0.90	0.71	0.62	0.57
b	5.19	2.52	0.92	2.35	0.09	0.97	0.41	0.34	0.43
c	5.19	2.52	0.92	2.35	0.09	0.97	0.41	0.34	0.43
M4: Sectoral frictions in the allocation of labor									
a	5.24	2.49	1.03	2.33	0.26	1.73	0.75	0.13	0.84
b	5.24	2.49	1.03	2.33	0.26	1.73	0.75	0.13	0.84
c	5.24	2.49	1.03	2.33	0.26	1.73	0.75	0.13	0.84

a habit slowly adapting to previous consumption (c). The score of the model is defined as the unweighed sum of squares of the differences of the model implied moments and our empirical targets. The further models considered are a model with sticky wages as in Uhlig (2007), M2, a model with predetermined labor supply as in Boldrin et al. (2001), and a two sector model with adjustment costs as in Heer and Maußner (2013).

The model that is closest to the targets is the sticky wage model with a slowly adjusting consumption habit (M2.c) with a score of 0.16. Since this is the model with the most free parameters, this should not be too surprising. Except for this model, labor market frictions do not improve the fit to the data. The model M1.b with a small consumption habit and a very small intertemporal elasticity of substitution (and thus with four free parameters) performs at least as good as the other models. Note also that the sluggish adjustment of the habit does not improve the performance of the respective models. When we compare the simulated statistics in Table 3.1 to those reported in Table 1 of Heer and Maußner (2013) the success of EZ preferences over standard ones becomes even more apparent. Their benchmark model with endogenous labor supply has a score of 26.43 (excluding the deviation of the risk free rate). It fails in several dimensions: it implies an equity premium of 0.52% and strong negative correlations between output and hours (-0.68) and between hours and the real wage (-0.96).

The remainder of our paper consists of five sections. Section 3.2 lays out those parts of our framework that are common to all models. In particular, section 3.2.1 studies the production side and the decision problem of the firm, while section 3.2.2 develops the model of the household with EZ preferences and derives the stochastic discount factor applied to evaluate returns. The remaining subsections consider the equilibrium dynamics, the deterministic steady state and the various forms of habit formation. We discuss the employed solution procedure as well as the models' common calibration and introduce our empirical targets in section 3.3. Section 3.4 considers the various model variants and their respective results, while section 3.5 discusses

Table 3.2: Free Parameters

	β	ψ	κ	χ	λ	μ
M0: Benchmark						
	0.9874	0.04265	7.05	-	-	-
M1: No labor market frictions						
a	0.9873	0.00635	6.3		-	-
b	0.988	0.00715	6.25	0.3	-	-
c	0.988	0.00715	6.25	0.3	0	-
M2: Sticky wages						
a	0.9875	0.00588	5.25	-	-	0.5
b	0.988	0.00715	6.25	0.3	-	0
c	0.988	0.007775	5.4	0.49	0.8	0.65
M3: Predetermined labor supply						
a	0.987485	0.00643	6.5	-	-	-
b	0.98872	0.0073	5.3	0.53	-	-
c	0.98872	0.0073	5.3	0.53	0	-
M4: Sectoral frictions in the allocation of labor						
a	0.9884	0.0086	3.0625	-	-	-
b	0.9884	0.0086	3.0625	0	-	-
c	0.9884	0.0086	3.0625	0	0	-

these findings in summary. Section 3.6 concludes the paper. The accompanying appendix collects the more tedious derivations and also a complete list of equilibrium conditions for every considered model (class).

3.2 Analytical framework

This section is primarily concerned with the analytical fundament of the ensuing analysis, which is the description of the behavior of our model economies' respective agents.

3.2.1 Household

Throughout this paper, we will assume the existence of an infinitely-lived representative household with preferences represented by a recursive utility function of EZ's Kreps/Porteus class.[2] We employ this representation because it allows us to address the household's attitude towards intertemporal consumption substitution and towards the uncertainty associated with future lifetime utility somewhat separately. This considerably helps us in replicating our return targets.[3] Yet, the additional degree of freedom also comes at a cost. It namely most prominently implies a preference for either earlier or later resolution of the uncertainty regarding consumption (or, more generally, the composite good) that may be tricky to justify.[4] We will get back to this issue at the end of the paper. The household's infinite planning horizon can be motivated by intergenerational altruism.

[2] Cf. Epstein and Zin (1989), p. 947 et. seq.

[3] In fact, the development of more flexible utility representations was, to a certain degree, driven by the standard framework's bad empirical performance mentioned above, cf. Epstein and Zin (1989), p. 938.

[4] Cf. Epstein et al. (2014).

Essentially, the representative household maximizes his lifetime utility as of period τ, denoted by U_τ, stemming from consumption, $c_t, t \geq \tau$, and leisure, $1-n_t, t \geq \tau$, yet to come, where n denotes labor normalized to a maximum level of 1. Thereby, U_τ is stated as a recursive two–period utility that aggregates today's within period utility from c_τ and $1-n_\tau$, denoted by $u(c_\tau, n_\tau)$, with a certainty equivalent of random future lifetime utility depending on tomorrow's state.[5]

Thereby, we assume the certainty equivalent, μ, of a risk averse expected utility maximizer with a constant rate of relative risk aversion to serve as our household's uncertainty aggregation rule for lotteries over random future lifetime utility, i.e.

$$\mu_\tau := \left(\mathbb{E}_\tau[U_{\tau+1}^{1-\gamma}]\right)^{\frac{1}{1-\gamma}}, \quad \gamma \in \mathbb{R}_{>0} \setminus \{1\}.$$

The time aggregation of these two components, resulting in the lifetime utility as of period τ, is of the CES form

$$W(u,\mu) = [(1-\beta)u^{1-\frac{1}{\psi}} + \beta\mu^{1-\frac{1}{\psi}}]^{\frac{1}{1-\frac{1}{\psi}}}, \quad \psi \in \mathbb{R}_{>0} \setminus \{1\}, \beta \in (0,1).$$

Summing up, we are led to the following recursive formulation of the representative household's preferences over intertemporal consumption lotteries

$$U_\tau = W(u(c_\tau, n_\tau), \mu_\tau) = [(1-\beta)u(c_\tau, n_\tau)^{1-\frac{1}{\psi}} + \beta(\mathbb{E}_\tau[U_{\tau+1}^{1-\gamma}])^{\frac{1-\frac{1}{\psi}}{1-\gamma}}]^{\frac{1}{1-\frac{1}{\psi}}}.$$

Note that the change in notation from the original treatment in Epstein and Zin (1989) e.g. allows to directly parameterize the EIS and CRRA by ψ and γ, respectively. Following Caldara et al. (2012), we will additionally introduce

$$\theta := \frac{1-\gamma}{1-\frac{1}{\psi}}$$

as a parameter measuring the relative deviation from the "classic" case, where the coefficient of relative risk aversion coincides with the reciprocal of the intertemporal elasticity of substitution. We thus write

$$W(u,\mu) = [(1-\beta)u^{\frac{1-\gamma}{\theta}} + \beta\mu^{\frac{1-\gamma}{\theta}}]^{\frac{\theta}{1-\gamma}}$$

and

$$U_\tau = [(1-\beta)u(c_\tau, n_\tau)^{\frac{1-\gamma}{\theta}} + \beta(\mathbb{E}_\tau[U_{\tau+1}^{1-\gamma}])^{\frac{1}{\theta}}]^{\frac{\theta}{1-\gamma}}. \tag{3.2.1}$$

In the ensuing analysis, we consider different specifications of the composite good u. The core difference between the corresponding models will be their implied stochastic discount factor. For the sake of exposition, we thus begin with the household's necessary optimality conditions for a general composite good. In the models to come, the respective decision problem basically[6] is

$$\begin{aligned} \max \quad & U_\tau = W(u(c_\tau, n_\tau), (\mathbb{E}_\tau U_{\tau+1}^{1-\gamma})^{\frac{1}{1-\gamma}}) \\ \text{s.t.} \quad & c_t \leq w_t n_t + d_t s_t - v_t(s_{t+1} - s_t), \\ & c_t \geq 0,\ 0 \leq n_t \leq 1, \ \text{for all } t \geq \tau, \\ \text{given} \quad & s_\tau, \end{aligned} \tag{3.2.2}$$

[5] More precisely, any period's composite good is an aggregation of its consumption and leisure and the within period utility mapping of this composite good is the identity mapping. We may thus, for the sake of intuition, switch between the two notions of within period utility and the composite good.

[6] Although the decision problems of the more complicated models differ from this basic framework, their treatment closely parallels the one to be outlined. See the respective sections on the necessary changes.

where c_t, w_t, n_t, d_t, s_t, and v_t denote period t's consumption, wage, working hours, dividend payments, the number of shares held by the household, and the share price, respectively.

We can summarize the necessary conditions for an interior optimum to the representative household's problem as follows.[7] For all $t \geq \tau$ it has to hold that

$$V_t = [(1-\beta)u(c_t, n_t)^{\frac{1-\gamma}{\theta}} + \beta(\mathbb{E}_t[V_{t+1}^{1-\gamma}])^{\frac{1}{\theta}}]^{\frac{\theta}{1-\gamma}}, \tag{3.2.3}$$

$$\mathbb{E}_t\left[m_{t+1,t}\frac{d_{t+1}+v_{t+1}}{v_t} - 1\right] = 0, \tag{3.2.4}$$

$$\frac{\partial u}{\partial c}(c_t, n_t)w_t = -\frac{\partial u}{\partial n}(c_t, n_t), \tag{3.2.5}$$

$$c_t = w_t n_t + d_t s_t - v_t(s_{t+1} - s_t), \tag{3.2.6}$$

where V_t denotes the problem's period t value function and

$$m_{t+1,t} := \beta\left(\frac{V_{t+1}^{1-\gamma}}{\mathbb{E}_t V_{t+1}^{1-\gamma}}\right)^{1-\frac{1}{\theta}}\left(\frac{u(c_{t+1}, n_{t+1})}{u(c_t, n_t)}\right)^{\frac{1-\gamma}{\theta}-1}\frac{\frac{\partial u}{\partial c}(c_{t+1}, n_{t+1})}{\frac{\partial u}{\partial c}(c_t, n_t)} \tag{3.2.7}$$

is the household's stochastic discount factor. It reflects his marginal lifetime utility evaluation of the implications of setting aside v_t units of the consumption good in period t in order to receive the uncertain reward of $d_{t+1} + v_{t+1}$ consumption units next period. Note how

$$\left(\frac{V_{t+1}^{1-\gamma}}{\mathbb{E}_t V_{t+1}^{1-\gamma}}\right)^{1-\frac{1}{\theta}} = \left(\frac{V_{t+1}}{\left(\mathbb{E}_t V_{t+1}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}}\right)^{\frac{1}{\psi}-\gamma} = \left(\frac{V_{t+1}}{\mu_t}\right)^{\frac{1}{\psi}-\gamma}$$

makes explicit the effect of timing preferences on the household's asset valuation, especially how the standard discount factor emerges from the classic consequentialist indifference assumption $\frac{1}{\psi} = \gamma$.[8]

3.2.2 Firm

Next, we accordingly assume the existence of a representative firm. In period t it produces the amount y_t of the final good employing the households' labor force and capital k_t via a constant returns to scale Cobb-Douglas technology

$$y_t = e^{z_t} n_t^{1-\alpha} k_t^{\alpha}, \quad \alpha \in (0,1),$$

with α determining the factors' output elasticities. The firm's period t total factor productivity evolves randomly with z_t. The latter is modeled as a stationary first order autoregressive process, i.e.

$$z_{t+1} = \rho z_t + \sigma\epsilon_{t+1}, \ \ \epsilon_t \sim \text{iidN}(0,1), \quad |\rho| < 1. \tag{3.2.8}$$

Hence, ϵ_t can be interpreted as a technology shock.

Capital is owned and produced by the firm whose capital stock evolves as

$$k_{t+1} - (1-\delta)k_t = \Phi\left(\frac{i_t}{k_t}\right)k_t, \tag{3.2.9}$$

[7] A detailed derivation of these conditions can be found in appendix A.

[8] Compare the discussion on the discounting implications of nonindifference towards the timing of uncertainty resolution in Heiberger and Ruf (2014a).

where δ measures depreciation. Φ is a concave function introducing capital adjustment costs in the form employed by Jermann (1998).[9] We define

$$\Phi(x) := \frac{b_1}{1-\kappa}x^{1-\kappa} + b_2, \quad \kappa > 1, \; b_1 > 0. \tag{3.2.10}$$

In order for equation (3.2.9) to be well-defined, we therefore additionally demand $i_t > 0$ and $k_t > 0$ for all $t \geq \tau$. While κ controls the speed of investment, the parameters b_1 and b_2 are chosen in order to render the steady state unaffected by adjustment costs.[10] The concavity of $\Phi(\cdot)$ implies capital adjustment costs in that it both limits the growth rate of capital and makes abrupt changes in the capital stock more investment intensive. More specifically, by rearranging (3.2.9),

$$k_{t+1} = \left(1 - \delta + \Phi\left(\frac{i_t}{k_t}\right)\right)k_t,$$

we find the function $\Phi(\cdot)$ to model the part of the capital stock's growth rate controlled through investment effort. With larger values of κ, positive deviations from i_t to δk_t have a decreasing, less than proportional effect on the capital stock while the effect of negative deviations is increasing and more than proportional. Hence, the firm's management has an incentive to avoid large deviations from i_t to δk_t. To put it another way, investment variability decreases in κ. Eventually note that $\kappa > 1$ implies that $\Phi(\frac{i_t}{k_t})$ is bound above by b_2 but falls without any bound if investment approaches 0 and that the case of no adjustment costs, i.e. $\kappa = 0$, makes the "standard" specification of $\Phi = \mathrm{id}$ emerge.

Next, since capital is owned by the firm, period t's profit amounts to revenue less labor costs, $y_t - w_t n_t$.[11] The firm's owners—i.e. households—participate in these profits via dividend payments, d_t per share. Investment in the capital stock is financed through profits beyond dividend payments plus the issuance of new shares

$$i_t = y_t - w_t n_t - d_t s_t + v_t(s_{t+1} - s_t). \tag{3.2.11}$$

Eventually, period t's cash flow cf_t is defined as profits less investment expenditures, both in t,

$$cf_t := y_t - w_t n_t - i_t.$$

Using this definition, the financing equation (3.2.11) can be equivalently stated as

$$d_t s_t - v_t(s_{t+1} - s_t) = cf_t. \tag{3.2.12}$$

Now, next to its dividend and stock policy, the firm's management decides over capital investment, its next period capital stock and its demand for labor. It has to balance the tradeoff between current profits and future capital resources knowing that its investment funding depends on the share price process. The management is thus not statically maximizing profits or cashflows period by period but rather maximizing its firm value as of t, denoted by fv_t. The latter is classically defined as the firm's current period cash flow plus its ex dividend market capitalization, i.e.

$$fv_t := cf_t + v_t s_{t+1}.$$

Next, the shareholders' infinite scope requires us to impose an additional constraint. It demands that, from period τ on, their appreciation of any market capitalization in infinite future vanishes.

[9] Following Hayashi (1982), we interpret Φ as an "installation function" for it describes adjustment costs by means of capital accumulation rather than by means of a negative summand in the firm's definition of profit.

[10] Note that Φ could also be defined for $\kappa \in \mathbb{R}_{\geq 0} \setminus \{1\}$. See later on empirical evidence against $\kappa \leq 1$ and on more details about b_1, b_2.

[11] Revenue equals output because we assume that firms never store any of their output and take y_t as numeraire.

I.e. the growth rate of the firm's market capitalization has to be capped by the household's discounting behavior,

$$\lim_{t\to\infty} \mathbb{E}_\tau \left[m_{t,\tau} v_t s_{t+1}\right] = 0, \tag{3.2.13}$$

where $m_{t,\tau} := m_{\tau+1,\tau} \cdot \ \ldots \ \cdot m_{t,t-1}$ for $t \geq \tau + 1$ with $m_{\tau,\tau} \equiv 1$ is the stochastic discount factor from period t to τ. Hence, following Altug and Labadie (2008), p. 265, repeatedly using (3.2.4), (3.2.12) and (3.2.13) we find

$$\begin{aligned} v_\tau s_{\tau+1} &= \mathbb{E}_\tau \left[m_{\tau+1,\tau}(d_{\tau+1} + v_{\tau+1})s_{\tau+1}\right] = \mathbb{E}_\tau \left[m_{\tau+1,\tau}\left(d_{\tau+1}s_{\tau+1} - v_{\tau+1}(s_{\tau+2} - s_{\tau+1}) + v_{\tau+1}s_{\tau+2}\right)\right] \\ &= \mathbb{E}_\tau \left[m_{\tau+1,\tau} cf_{\tau+1} + m_{\tau+1,\tau} v_{\tau+1} s_{\tau+2}\right] = \ldots = \mathbb{E}_\tau \left[\sum_{t=\tau+1}^{\infty} m_{t,\tau} cf_t\right], \end{aligned} \tag{3.2.14}$$

so that the firm value as of period τ is the expected present value of its cash flows to come,

$$f v_\tau = \mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} m_{t,\tau} cf_t\right].$$

In period τ, the firm's management has to choose the amount of working hours employed, the investment expenditures and next period's capital stock, while k_τ is given. In other words, the maximization problem of the representative firm is

$$\begin{aligned} \max \quad & \mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} m_{t,\tau}(e^{z_t} n_t^{1-\alpha} k_t^\alpha - w_t n_t - i_t)\right] \\ \text{s.t.} \quad & k_{t+1} = (1-\delta)k_t + \Phi\left(\frac{i_t}{k_t}\right)k_t, \\ & k_{t+1} > 0,\ i_t > 0,\ 0 \leq n_t \leq 1, \ \text{ for all } t \geq \tau, \\ \text{given} \quad & k_\tau > 0. \end{aligned} \tag{3.2.15}$$

Note that we do not explicitly consider the firm's financing in the statement of its maximization problem. To see why, note that after having determined the optimal level of investment, next period's capital stock and labor demand, for the firm it is always possible to find a respective financing that satisfies (3.2.11). To see this, note that given optimal i_t, k_{t+1} and n_t the firm's cash flow $cf_t = e^{z_t} n_t^{1-\alpha} k_t^\alpha - w_t n_t - i_t$ is determined. The constraint (3.2.11) then is satisfied for any d_t and s_{t+1}, which meet

$$d_t s_t - v_t(s_{t+1} - s_t) = cf_t,\ t \geq \tau,$$

where by iterating equation (3.2.4)

$$\begin{aligned} v_t &= \mathbb{E}_t[m_{t+1,t}(d_{t+1} + v_{t+1})] = \mathbb{E}_t \left[m_{t+1,t} d_{t+1} + m_{t+1,t} v_{t+1})\right] = \\ &= \mathbb{E}_t \left[m_{t+1,t} d_{t+1} + m_{t+1,t} \mathbb{E}_{t+1}[m_{t+2,t+1}(d_{t+2} + v_{t+2})]\right] = \ldots = \mathbb{E}_t \left[\sum_{s=1}^{\infty} m_{t+s,t} d_{t+s}\right],\ t \geq \tau, \end{aligned}$$

if we additionally assume $\lim_{s\to\infty} \mathbb{E}_t[m_{t+s,t} v_{t+s}] = 0$. In general, however, the resulting s_{t+1}, d_t and v_t are not uniquely determined without imposing a particular dividend policy for the firm.

Differentiating with respect to n_t, i_t and k_{t+1} and assuming an interior solution yields the first order conditions for the firm's maximization problem. First,

$$w_t = (1-\alpha)e^{z_t} n_t^{-\alpha} k_t^\alpha \ \text{ for all } t \geq \tau,$$

i.e. wages have to equal the marginal product of labor.[12] Second,

$$q_t = \frac{1}{\Phi'\left(\frac{i_t}{k_t}\right)} \quad \text{for all } t \geq \tau,$$

where q_t is period t's Lagrange multiplier for the capital accumulation constraint (3.2.9) divided by $m_{t,\tau}$. Third, the Euler equation

$$q_t = \mathbb{E}_t\left[m_{t+1,t}\left(\alpha e^{z_{t+1}} n_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} - \frac{i_{t+1}}{k_{t+1}} + q_{t+1}\left(1-\delta+\Phi\left(\frac{i_{t+1}}{k_{t+1}}\right)\right)\right)\right]$$

must hold for all $t \geq \tau$.

Summing up, the list of optimality conditions for an interior solution to the firm's problem are for all $t \geq \tau$,

$$w_t = (1-\alpha)e^{z_t} n_t^{-\alpha} k_t^{\alpha}, \tag{3.2.16}$$

$$q_t = \frac{1}{\Phi'\left(\frac{i_t}{k_t}\right)}, \tag{3.2.17}$$

$$q_t = \mathbb{E}_t\left[m_{t+1,t}\left(\alpha e^{z_{t+1}} n_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} - \frac{i_{t+1}}{k_{t+1}} + q_{t+1}\left(1-\delta+\Phi\left(\frac{i_{t+1}}{k_{t+1}}\right)\right)\right)\right], \tag{3.2.18}$$

$$k_{t+1} = (1-\delta)k_t + \Phi\left(\frac{i_t}{k_t}\right)k_t, \tag{3.2.19}$$

$$i_t = y_t - w_t n_t - d_t s_t + v_t(s_{t+1} - s_t), \tag{3.2.20}$$

$$y_t = e^{z_t} k_t^{\alpha} n_t^{1-\alpha}. \tag{3.2.21}$$

We add a first (informal) remark on q. Since the price of the investment good is 1 and additional investment in period τ increases $k_{\tau+1}$ by $\Phi'\left(\frac{i_\tau}{k_\tau}\right)$, we find additional $k_{\tau+1}$ to have a price of $\left(\Phi'\left(\frac{i_\tau}{k_\tau}\right)\right)^{-1}$. Next, using the envelope theorem, in $\tau+1$, the increase of the then given state $k_{\tau+1}$ increases the maximum firm value as of period $\tau+1$ by

$$\alpha e^{z_{\tau+1}} n_{\tau+1}^{1-\alpha} k_{\tau+1}^{\alpha-1} - \frac{i_{\tau+1}}{k_{\tau+1}} + q_{\tau+1}\left(1-\delta+\Phi\left(\frac{i_{\tau+1}}{k_{\tau+1}}\right)\right),$$

all variables evaluated at the optimum. By (3.2.18) we thus find q_τ to measure the expected discounted change in period τ's maximum firm value induced by an exogenous increase in the capital stock at the end of τ. Hence, equation (3.2.17) states that, in an optimum, the value of a unit of capital to the firm has to be equal to its price.[13]

3.2.3 General equilibrium

In a general equilibrium, all markets in the model economy are cleared simultaneously and the representative household as well as the representative firm mutually act optimally. In

[12] Note that this will imply profits to be positive in equilibrium due to the assumed linear homogeneity of production by Euler's Theorem, because both capital and its marginal product are positive for all t given z_t. Precisely,

$$k_t \frac{\partial y_t}{\partial k_t} = \alpha y_t > 0, \text{ for all } t.$$

[13] Hayashi (1982) rigorously develops this crucial role of the capital accumulation constraint's shadow price in economies with installation costs.

anticipation of the ensuing general equilibrium analysis, we have already denoted demand and supply variables identically. Imposing these identities on the agents' respective optimality conditions therefore already ensures a cleared labor market. Additionally, the goods market has to be cleared, i.e.

$$y_t = c_t + i_t.$$

This equation together with (3.2.3)-(3.2.6) and (3.2.16)-(3.2.21) define a general equilibrium in period t.

As mentioned above, the values for s_{t+1}, d_t and v_t are not uniquely determined in equilibrium without assuming the firm to follow a particular dividend policy. Note however, that it follows from equations (3.2.6) and (3.2.20), that the goods market clearing condition is already necessary and sufficient for the stock market to clear, too. Hence, we can ignore (3.2.4), (3.2.6) and (3.2.20) without any loss, if we are not interested in s_{t+1}, d_t and v_t.

To sum up, we list our fundamental equilibrium conditions: For all $t \geq \tau$ it must hold that

$$V_t - [(1-\beta)u(c_t, n_t)^{\frac{1-\gamma}{\theta}} + \beta(\mathbb{E}_t[V_{t+1}^{1-\gamma}])^{\frac{1}{\theta}}]^{\frac{\theta}{1-\gamma}} = 0, \tag{3.2.22}$$

$$\frac{\partial u}{\partial c}(c_t, n_t)w_t = -\frac{\partial u}{\partial n}(c_t, n_t), \tag{3.2.23}$$

$$w_t - (1-\alpha)e^{z_t} n_t^{-\alpha} k_t^{\alpha} = 0, \tag{3.2.24}$$

$$q_t - \frac{1}{\Phi'\left(\frac{i_t}{k_t}\right)} = 0, \tag{3.2.25}$$

$$q_t - \mathbb{E}_t\left[m_{t+1,t}\left(\alpha e^{z_{t+1}} n_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} - \frac{i_{t+1}}{k_{t+1}} + q_{t+1}\left(1-\delta+\Phi\left(\frac{i_{t+1}}{k_{t+1}}\right)\right)\right)\right] = 0, \tag{3.2.26}$$

$$k_{t+1} - (1-\delta)k_t - \Phi\left(\frac{i_t}{k_t}\right)k_t = 0, \tag{3.2.27}$$

$$y_t - e^{z_t} k_t^{\alpha} n_t^{1-\alpha} = 0, \tag{3.2.28}$$

$$y_t - c_t - i_t = 0, \tag{3.2.29}$$

where the sequence $\{z_t\}$ follows (3.2.8) and the stochastic discount factor is determined by (3.2.7).

3.2.4 Steady state

We next characterize the model's deterministic steady state, i.e. the solution with

$$\sigma = 0 \;\text{ and }\; x_t = x_{t+1} =: x_{ss} \quad \text{for all } t,$$

where $x_t := \begin{pmatrix} k_{t+1} & z_t & V_t & c_t & n_t & w_t & y_t & i_t & q_t \end{pmatrix}^T$. First, we see that this implies $z_{ss} = 0$. Second, using (3.2.28) and (3.2.24) we find

$$y_{ss} = k_{ss}^{\alpha} n_{ss}^{1-\alpha} \quad \text{and} \quad w_{ss} = (1-\alpha)\frac{y_{ss}}{n_{ss}}.$$

Equations (3.2.27) and (3.2.25) further yield

$$\delta k_{ss} = \Phi\left(\frac{i_{ss}}{k_{ss}}\right)k_{ss} \quad \text{and} \quad q_{ss} = \frac{1}{\Phi'\left(\frac{i_{ss}}{k_{ss}}\right)}.$$

Next, as mentioned above, we do not want adjustment costs to play a role in the steady state. Thus, we have to make sure that our parametric specification of $\Phi(\cdot)$ makes

$$i_{ss} = \delta k_{ss} \quad \text{and} \quad q_{ss} = 1$$

emerge. This is achieved by demanding

$$\Phi(\delta) = \delta \;\text{ and }\; \Phi'(\delta) = 1 \quad \overset{(3.2.10)}{\Longleftrightarrow} \quad \frac{b_1}{1-\kappa}\delta^{1-\kappa} + b_2 = \delta \;\text{ and }\; b_1\delta^{-\kappa} = 1,$$

which is satisfied for

$$b_1 = \delta^{\kappa}$$

and thus

$$b_2 = \delta\left(1 - \frac{1}{1-\kappa}\right) = -\delta\frac{\kappa}{1-\kappa}.$$

Additionally, one might want the adjustment cost function $\Phi(\cdot)$ to be positive.[14] This is equivalent to demanding a period's capital stock never to fall short its last period's value less depreciation. For that to be the case, the investment-to-capital ratio has to always satisfy the condition

$$\frac{i_t}{k_t} > \kappa^{\frac{1}{1-\kappa}}\delta,$$

i.e. it must always exceed $\kappa^{\frac{1}{1-\kappa}}$ times its steady state value.[15]

For the value function (3.2.22) we find

$$V_{ss} = u(c_{ss}, n_{ss})$$

and, via the goods market equilibrium condition,

$$c_{ss} = y_{ss} - i_{ss}.$$

We can now express all variables' steady state values in terms of k_{ss} and n_{ss}. While the latter is generally set to some specific level, k_{ss} is eventually determined via the model's Euler equation (3.2.26)

$$k_{ss} = \left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{\frac{1}{\alpha-1}} n_{ss}.$$

[14] Cf. Jermann (1998), p. 260, who actually lists positivity as a defining property of Φ.

[15] Hence, if positivity is demanded, it is necessary to either impose this condition directly or to check for it in the simulation results. Note that for the factor $\kappa^{\frac{1}{1-\kappa}} = e^{\frac{\ln\kappa}{1-\kappa}}$, where $\kappa > 1$, we find

$$\frac{\ln\kappa}{1-\kappa} = -\frac{\ln\kappa - \ln 1}{\kappa - 1} \underset{\kappa\downarrow 1}{\rightarrow} -\ln'(1) = -1 \;\text{ and }\; \frac{\ln\kappa}{1-\kappa} \underset{\kappa\to\infty}{\rightarrow} 0$$

and hence the limits

$$\lim_{\kappa\downarrow 1}\kappa^{\frac{1}{1-\kappa}} = e^{-1} \;\text{ and }\; \lim_{\kappa\to\infty}\kappa^{\frac{1}{1-\kappa}} = 1,$$

demonstrating the implications of different parameterizations with respect to the hazard of generating simulation results that violate this restriction.

3.2.5 Composite good aggregation

This subsection introduces the nontrivial variations of the composite good (bundle) considered in this paper in increasing order of generality. Economic intuition and a brief discussion of their respective implications for the interpretation of our key parameters, ψ and γ, are provided.

Just as the real business cycle phenomenon and also fundamental asset prices can be viewed as a consequence of very few core mechanisms of economic activity, the models considered here generate corresponding fluctuations and return series mainly through technology shocks and the induced intertemporal substitution behavior of households. The shock variability and the EIS are thus recognized as the pivotal parameters in that respect.

While our framework leaves no room to work on σ, the chosen EZ specification of lifetime utility, however, allows for a rather liberal parameterization of ψ as it loosens the strict entanglement of EIS and RRA. Hence, we are e.g. able to make the household as averse to a non-smooth composite good path as is needed to reach our targeted equity premium. Yet, we would rather want to work on the household's *consumption* behavior more directly for we also want to separately target a particular labor variability found in the data.

Primarily for this purpose, we will additionally consider habit formation solely in consumption. Precisely, after analyzing the model with a classical composite good aggregation, we will additionally allow for external habits in the sense of Campbell and Cochrane (1999). The external habit process thereby is either standard or also allows for slowly adjusting consumption habits as in Uhlig (2007).

3.2.5.1 MX.a: No habits

First, we will consider models using the linearly homogenous Cobb–Douglas aggregator

$$u(c_t, n_t) := c_t^{\nu}(1-n_t)^{1-\nu}, \quad \nu \in (0,1),$$

where ν controls the relative weight of consumption and leisure in the composite good, i.e. in the within period utility. In particular, we may interpret the case $\nu \geq (\leq)\frac{1}{2}$ as consumption having a larger (smaller) impact than leisure on the composite good.

It thus becomes necessary to distinguish the household's attitude towards intertemporal substitution of consumption from his attitude towards intertemporally substituting the composite good. While the latter is determined by ψ, the former must also take c_t's "importance parameter" ν into account. In the present case e.g., the computation reads $\psi_c := \frac{1}{1-\nu(1-\frac{1}{\psi})}$. Hence, when we speak of the EIS, we relate to the notion of substituting the composite good. By means of ν, this, however, can be directly translated into a statement about consumption substitution. Precisely, as ψ increases c.p., so does ψ_c. Further, the chosen composite good aggregation analogously yields $\psi_{(1-n)} := \frac{1}{\frac{1}{\psi}+\nu(1-\frac{1}{\psi})}$.

Accordingly, we have to be aware of the fact that the composite good's importance parameter ν must be considered just as much when interpreting the degree of risk aversion associated with each ingredient. E.g. in the special case of $\theta = 1$, while the composite good CRRA is γ, the consumption CRRA and leisure CRRA are $\nu\gamma$ and $(1-\nu)\gamma$, respectively.[16]

It is more cumbersome to derive the implications of the respective composite good's specification within the habit variants to be outlined in the following. Nevertheless, the gist of the above reasoning carries over. We thus skip an explicit discussion of this issue in the following.

From equation (3.2.7) we can calculate the representative household's stochastic discount factor as

$$m_{t+1,t} := \beta \left(\frac{V_{t+1}^{1-\gamma}}{\mathbb{E}_t V_{t+1}^{1-\gamma}} \right)^{1-\frac{1}{\theta}} \left(\frac{c_{t+1}^{\nu}(1-n_{t+1})^{1-\nu}}{c_t^{\nu}(1-n_t)^{1-\nu}} \right)^{\frac{1-\gamma}{\theta}} \frac{c_t}{c_{t+1}}. \tag{3.2.30}$$

[16] Cf. Swanson (2012).

3.2.5.2 MX.b: Standard consumption habits

In this variation, we allow for consumption habits in the aggregator u. The employed version of habits is in some sense naive, as the household now regards the current period's excess over last period's consumption level as his utility argument but does not consider this behavior in advance. From a representative agent perspective one might interpret this as the household comparing his current consumption level to the economy's previous period's overall average consumption, giving rise to the popular notion of catching up with the Joneses.[17] Following Campbell and Cochrane (1999), we will thus refer to the studied form of habit formation as external. Interpreted either way, the fact that current consumption increases future consumption "obligations"–however recognized by the household–is meant to further smoothen the consumption behavior in equilibrium.

Formally, we now consider the aggregator

$$u(c_t, n_t) := (c_t - c_t^h)^\nu (1-n_t)^{1-\nu}, \quad \nu \in (0,1),$$

where c_t^h is a habit process that is strictly exogenous to the household. Note that for this composite good to be always well-defined we must impose the more restrictive constraint

$$c_t \geq c_t^h,\ t \geq \tau, \tag{3.2.31}$$

instead of $c_t \geq 0$. The exogenous habit process here follows

$$c_t^h := \chi c_{t-1}, \quad \chi \in [0,1]. \tag{3.2.32}$$

Given this functional form of u the stochastic discount factor results in

$$m_{t+1,t} = \beta \left(\frac{V_{t+1}^{1-\gamma}}{\mathbb{E}_t V_{t+1}^{1-\gamma}} \right)^{1-\frac{1}{\theta}} \left(\frac{(c_{t+1} - c_{t+1}^h)^\nu (1-n_{t+1})^{1-\nu}}{(c_t - c_t^h)^\nu (1-n_t)^{1-\nu}} \right)^{\frac{1-\gamma}{\theta}} \frac{c_t - c_t^h}{c_{t+1} - c_{t+1}^h}. \tag{3.2.33}$$

Steady state The steady state value for the habit variable is determined through (3.2.32),

$$c_{ss}^h = \chi c_{ss}.$$

3.2.5.3 MX.c: Slowly adapting consumption habits

In this modification we generalize the process specifying the exogenous consumption habit as in Uhlig (2007). More precisely, the consumption habit is no longer some given fraction of the previous period's consumption level, but also depends on its own anteceding value and hence adjusts more slowly to variations in consumption. This behavioral generalization allows us to more finely calibrate the household's smoothing behavior.

While the composite good aggregation again takes the form as in MX.b, i.e.

$$u(c_t, n_t) := (c_t - c_t^h)^\nu (1-n_t)^{1-\nu}, \quad \nu \in (0,1),$$

the exogenous habit process c_t^h now follows

$$c_t^h := \lambda c_{t-1}^h + (1-\lambda)\chi c_{t-1}, \quad \chi \in [0,1],\ \lambda \in [0,1).$$

Thus, the stochastic discount factor remains the same as in the previous variant, i.e.

$$m_{t+1,t} = \beta \left(\frac{V_{t+1}^{1-\gamma}}{\mathbb{E}_t V_{t+1}^{1-\gamma}} \right)^{1-\frac{1}{\theta}} \left(\frac{(c_{t+1} - c_{t+1}^h)^\nu (1-n_{t+1})^{1-\nu}}{(c_t - c_t^h)^\nu (1-n_t)^{1-\nu}} \right)^{\frac{1-\gamma}{\theta}} \frac{c_t - c_t^h}{c_{t+1} - c_{t+1}^h}.$$

[17] Cf. Abel (1990).

Steady state Note that it follows from $\lambda \neq 1$ that still

$$c_{ss}^h = \chi c_{ss}.$$

3.3 Numerical framework

In this section, we report in detail on the employed simulation approach, the chosen empirical targets, and on how our results were computed.[18]

3.3.1 Solution

Our goal is to simulate a time path of the model's equilibrium outcomes emerging from a series $\{\epsilon_t\}$ of iidN(0, 1) distributed pseudorandom shocks to the productivity level z_t. In order to do this, for each period t we have to find the solution to the stochastic dynamic system implicitly defined by the equilibrium conditions (3.2.22)-(3.2.29), given this period's states k_t and z_t. Following Schmitt-Grohe and Uribe (2004), we denote by $h(k_t, z_t, \sigma)$ the solution for k_{t+1} and by $g_i(k_t, z_t, \sigma), i = 1, \ldots, 7$, the solution for the remaining variables of this system of equations except z_{t+1}. I.e. we make explicit that the functions characterizing the solution depend on the states k_t and z_t and also on the standard deviation σ of the AR(1) noise term but are time-independent.[19] The g_is are called policy functions, while h is called the capital stock's dynamic. Note that the technology's dynamic is already explicitly given by (3.2.8).

Rather than solving for their exact solutions, we use the perturbation method and thus search for a local Taylor approximation of the functions h and g_i.[20] The Taylor polynomials' point of expansion is the deterministic steady state $(k_{ss}, z_{ss}, 0)$. More precisely, in our equilibrium conditions, we replace k_{t+1} by the dynamic h and the remaining variables by their respective policy functions g_i. Hence, differentiating these conditions with respect to k, z and σ at the steady state yields a system of equations in which the derivatives of the dynamic and the policy functions at the steady state are the unknowns. Solving for the stable solution to this system of equations, we find the coefficients of the first order Taylor polynomials.[21] Finally, computing the equilibrium conditions' second derivatives at the deterministic steady state and inserting the already determined first derivatives of h and g_i yields a linear system of equations in the dynamic's and policies' second order derivatives. Its solution completes our necessary computations for a second order perturbation.

The appropriateness of perturbation in a DSGE model with EZ utility is documented by Caldara et al. (2012), who compare on different solution methods with regard to accuracy and computing time for several calibrations.

3.3.2 Computation of the return series

The upcoming analysis places particular interest on the equity premium and the risk free rate. This section demonstrates their respective computation.

[18] In the appendix, we additionally provide a brief documentation on the employed computation routines.

[19] Of course, the solution also depends on the other parameters. σ, however, plays a special role with regard to the solution as it scales the uncertainty in our model. Explicitly considering σ as an argument of h and g_i makes it possible to examine the effect of uncertainty on our solution. Note that time-independence is necessary for optimality.

[20] We thereby assume sufficient smoothness of the functions h and g_i.

[21] A solution is regarded "stable" if all eigenvalues of the Jacobian of the system's dynamic are less than unity in absolute value so that the states' processes are bounded.

Return on Equity Our models' return on equity is[22]

$$r^e_{t+1} := \frac{d_{t+1}+v_{t+1}}{v_t}.$$

As already stated, we do not compute v_t and d_t in our solution. Therefore, in order to compute the return on equity in our simulations nonetheless, we follow Heer and Maußner (2013) and make use of the equality

$$\frac{d_{t+1}+v_{t+1}}{v_t} = \frac{y_{t+1}-w_{t+1}n_{t+1}-i_{t+1}+q_{t+1}k_{t+2}}{q_t k_{t+1}}.$$

This equation holds along an equilibrium path and can be deduced as follows. First, remember that the financing constraint of the firm as well as the household's budget constraint in equilibrium determine d_t and s_{t+1} up to

$$d_t s_t - v_t(s_{t+1}-s_t) = cf_t.$$

Second, by equation (3.2.24)

$$w_t n_t = (1-\alpha)e^{z_t} n_t^{1-\alpha} k_t^\alpha = (1-\alpha)y_t$$

and hence

$$cf_t = y_t - w_t n_t - i_t = \alpha y_t - i_t.$$

Therefore, by the fact that k_{t+1} is known at the beginning of period t and by equations (3.2.26) and (3.2.27), we may write

$$\begin{aligned} q_t k_{t+1} &= \mathbb{E}_t\left[m_{t+1,t}\left(\alpha e^{z_{t+1}} n_{t+1}^{1-\alpha} k_{t+1}^\alpha - i_{t+1} + q_{t+1}\left(1-\delta+\Phi\left(\frac{i_{t+1}}{k_{t+1}}\right)\right)k_{t+1}\right)\right] \\ &= \mathbb{E}_t\left[m_{t+1,t}\left(\alpha y_{t+1} - i_{t+1} + q_{t+1}k_{t+2}\right)\right] = \mathbb{E}_t\left[m_{t+1,t}\left(cf_{t+1} + q_{t+1}k_{t+2}\right)\right] \end{aligned}$$

and thus by continuing inductively

$$q_t k_{t+1} = \mathbb{E}_t\left[\sum_{s=1}^{\infty} m_{t+s,t}\, cf_{t+s}\right],$$

if we additionally assume $\lim_{s\to\infty} \mathbb{E}_t\left[m_{t+s,t} q_{t+s} k_{t+1+s}\right] = 0$.[23] Thus, according to (3.2.14), the term on the right hand side equals $v_t s_{t+1}$, so that

$$q_t k_{t+1} = v_t s_{t+1}. \tag{3.3.1}$$

The claim now follows from

$$\begin{aligned} \frac{d_{t+1}+v_{t+1}}{v_t} &= \frac{d_{t+1}s_{t+1}+v_{t+1}s_{t+1}}{v_t s_{t+1}} = \frac{d_{t+1}s_{t+1}-v_{t+1}(s_{t+2}-s_{t+1})+v_{t+1}s_{t+2}}{v_t s_{t+1}} \\ &= \frac{d_{t+1}s_{t+1}-v_{t+1}(s_{t+2}-s_{t+1})+q_{t+1}k_{t+2}}{q_t k_{t+1}} = \frac{cf_{t+1}+q_{t+1}k_{t+2}}{q_t k_{t+1}} \\ &= \frac{y_{t+1}-w_{t+1}n_{t+1}-i_{t+1}+q_{t+1}k_{t+2}}{q_t k_{t+1}}. \end{aligned}$$

This allows us to compute the return on equity without having to determine d_t or v_t. To put it another way, under the assumed transversality conditions it plausibly holds that the return on the firm's capital investment equals the return on the households' (i.e. firm owners') share investment.[24] Besides, in the steady state the return on equity is $\frac{1}{\beta}$.

[22] Note that the two sector economies studied in subsection 3.4.5 demand a slight modification with respect to the computation of the risky return, but see later.

[23] This is another transversality condition, akin to (3.2.13). It imposes a growth cap on the value of the firm's capital stock.

[24] Cf. Kaltenbrunner and Lochstoer (2010), who directly define the risky return via the capital investment Euler equation.

A second remark on q. By (3.3.1), the price of one share divided by the book value of the firm's capital stock per share, turns out as

$$q_t = \frac{v_t}{\frac{k_{t+1}\cdot 1}{s_{t+1}}}.$$

Hence, following the first remark on q above, in our model q in fact (also) measures the figure Tobin (1969) already found to be central in any agent's investment decision. Also does the chosen capital adjustment friction meet his suggestion, p. 21, that "the speed at which investors wish to increase the capital stock should be related [...] to q," for it turns out that

$$\begin{aligned}
\Phi'(x) = b_1 x^{-\kappa} \quad &\Leftrightarrow \quad x = \left(b_1 \frac{1}{\Phi'(x)}\right)^{\frac{1}{\kappa}} = (b_1 q)^{\frac{1}{\kappa}} \\
&\Rightarrow \quad \frac{\partial x}{\partial q} = \frac{1}{\kappa} b_1 (b_1 q)^{\frac{1}{\kappa}-1} \;\text{ and }\; \frac{q}{x} = \frac{1}{b_1} x^{\kappa-1} = \frac{1}{b_1}(b_1 q)^{\frac{\kappa-1}{\kappa}} \\
&\Rightarrow \quad \frac{\partial x}{\partial q}\frac{q}{x} = \kappa^{-1}.
\end{aligned}$$

I.e. the parameter controlling the severity of the adjustment friction, and thus the "speed of investment", κ, is reciprocal to the elasticity of the investment-to-capital ratio with respect to q, "Tobin's q".

Risk Free Return In order to be able to also approximate the risk free rate, we add it to our list of variables and find its respective necessary equilibrium condition. The latter is achieved by applying the Euler-Lucas equation (3.2.4) to evaluate a claim on one unit of the final good with certainty at the end of *next* period. Following the reasoning before, such an asset's price v^f would have to satisfy

$$v_t^f = \mathbb{E}_t\left[m_{t+1,t}\cdot 1\right] \text{ for all } t \quad \Leftrightarrow \quad \mathbb{E}_t\left[m_{t+1,t}\right]\frac{1}{v_t^f} - 1 = 0 \text{ for all } t.$$

Defining $r_t^f := \frac{1}{v_t^f}$ yields the sought for conditions

$$\mathbb{E}_t\left[m_{t+1,t}\right] r_t^f = 1 \text{ for all } t. \tag{3.3.2}$$

Note that this also yields a steady state value of $r_{ss}^f = \frac{1}{\beta}$.

Equity Premium The equity premium is finally computed as the expected excess return on equity beyond the risk free rate,

$$ep_{t+1} := \mathbb{E}_t\left[r_{t+1}^e - r_t^f\right], \tag{3.3.3}$$

which implies a zero steady state premium.

3.3.3 Empirical targets

We examine all models along their ability to replicate factual German quarterly business cycle statistics. Additionally, we try to match two asset pricing figures, the annual equity premium (EP) and the annual risk free rate (r^f).

Since related versions of the models considered in the present paper are also examined in Heer and Maußner (2013), we decided to stick with their empirical targets to be able to compare our results. Specifically, the chosen RBC statistics are output volatility s_y, relative volatility of investment to output s_i/s_y, working hours to output s_n/s_y, wages to output s_w/s_y, and the contemporary correlation of output to working hours r_{yn} and wages to working hours r_{wn}.[25]

[25]Note that all macro variables are understood as the respective real aggregates' cyclical components, i.e. HP-filtered.

The respective numerical target values are thus taken from Heer and Maußner (2009), while the empirical equity premium is from Kyriacou et al. (2004). Consequently, we are left with the task of finding a reasonable target for the real risk free rate of the German economy.

Yet, the way in which Kyriacou et al. compute their figure of 5.18 for the German equity premium is hard to trace. As a consequence, we determine the German risk free rate target indirectly as follows. First, we take the German prime standard share index, DAX, as our approximation of the German market portfolio and calculate its mean real return over an extended historical performance, including dividend payments.[26] In particular, we find a real annual return on equity of 7.67. Now, this figure must, by definition, exceed the sought for risk free rate by 5.18 in order to be consistent with the chosen equity premium target.

We want to remark that our results, i.e. the "goodness of fit" found possible for the considered models, are not particularly sensitive to the chosen risk free target rate. Our empirical targets are summarized in table 3.3.

Table 3.3: Empirical targets

EP	r^f	s_y	s_i/s_y	s_n/s_y	s_w/s_y	r_{yn}	r_{wn}
5.18	2.49	1.14	2.28	0.69	1.03	0.4	0.27

3.3.4 Calibration and simulation

Calibration In our models, a unit of time equals three months. The numerical results are heavily driven by the values chosen for the parameters $\alpha, \delta, \rho, \sigma$, the RRA parameter γ and β, ψ, κ. There is direct empirical evidence for the first set of parameters so these are usually chosen very similarly by researchers.[27] Therefore, we consider them as fixed throughout the whole paper at the values displayed in table 3.4. For γ, the possible bandwidth seems less restraint. An authoritative range, $[0, 10]$, is e.g. provided by Mehra and Prescott (1985). However, in order to emphasize on the impact of the household's attitude towards intertemporal consumption substitution, we nevertheless regard the RRA parameter also as fixed. In particular, we choose $\gamma = 2$.[28]

Table 3.4: Fixed Parameters

α	δ	ρ	σ	γ
0.27	0.011	0.9	0.0072	2

[26] Although the DAX only covers 30 firm shares, it already "represent[s] around 80 percent of the market capitalization listed in Germany." Cf. the official information from Deutsche Börse AG, as in December 2013, `http://dax-indices.com/EN/MediaLibrary/Document/120611_DeutscheBoerse_E_WEB.pdf`. The DAX history was officially prolonged backwards until December 1959. However, issues with the chaining of the preceding indices have been discussed, cf. Strehle et al. (1996). Hence, we do not want to pull the available data too far into the past. On the other hand, more good data would improve on the mean as our return on equity estimator. We exogenously balance this tradeoff by taking the 1973/74 oil embargo as our cutting date and thus only consider data as of 1974:Q2.

[27] Cf. Heer and Maußner (2013).

[28] E.g. Caldara et al. (2012) also consider this RRA calibration amongst others.

The remaining parameters, as well as the parameters to be introduced alongside the upcoming extensions to this framework, are understood as free within particular intervals. These intervals are fixed descriptively in the first place. The chosen calibration's (quantitative) implications will be discussed only after the parameters' final determination.

With regard to the calibration of β, the DSGE literature displays disagreement, at least within the boundaries of 0.95 (e.g. Schmitt-Grohe and Uribe (2004)) and 0.99999 (Boldrin et al. (2001)). These values thus span our considered interval.

In the case of ψ, a comparably tight interval of possible values is much less evident in the literature. Moreover, as ψ is our key parameter, we decided to leave it less restrained a priori. Thus, ψ is centered around the standard case of reciprocal EIS and RRA with 0 and 1 as excluded boundaries. However, note already that the results in Heiberger and Ruf (2014a) suggest smaller values of ψ to be more likely to give rise to a good empirical performance.

Further, the range of values for κ^{-1} estimated by Abel (1980) in a somewhat different setting provides us with initial empirical evidence regarding the magnitude of κ, suggested to be around 3.[29] In accordance to that, we choose our interval for κ as the union of the respective intervals in Jermann (1998) and Heer and Maußner (2013), also considering (3.2.10), i.e. from above 1 to 9.

Eventually, the weighting parameter ν is chosen such that steady state working hours meet

$$n_{ss} = 0.13,$$ [30]

while the habit parameters χ and λ are considered free within their respective domains.

Simulation Our approach is to set the free parameters' values within the intervals above in order to match the respective models' simulation results to the German real economy at the best. We discretely optimize this match over a grid $\mathscr{A} \subset \mathbb{R}^k$, where k is the number of free parameters in the considered model, calculating the second order approximations of the policy functions and the dynamic of the model as described above for each parametrization $a \in \mathscr{A}$.

With these approximated solutions, we follow Heer and Maußner (2013) and simulate 300 time series, each of length 80, of the models' variables and compute the average outcomes of their moments $\sigma_y, \frac{\sigma_i}{\sigma_y}, \frac{\sigma_n}{\sigma_y}, \frac{\sigma_w}{\sigma_y}, \rho_{yn}$ and ρ_{wn} as the models' counterparts to our empirical targets in obvious notation.[31]

The model's risk free rate and the equity premium are computed as the annualized time series averages of a simulation of 500,000 periods along the formulae derived in subsection 3.3.2. Hence, we actually have to compute *ex post* risk premia, since the computation of any period's return on equity requires knowledge over later periods' quantities.[32] Note that this is just in line with the typical computation of empirical return targets.

Altogether, this yields a vector $S_m(a) \in \mathbb{R}^8$ of values implied by the simulation of the model that corresponds to our chosen targets. We accordingly evaluate the models' fit to the empirical

[29] Cf. Abel (1980), p. 75. Note that in order to find the estimates that correspond best to our framework, we must choose the time preference parameter that lies within our allowed interval, 0.95, and an elasticity of substitution between capital and labor of 1, due to our Cobb-Douglas technology. Abel's corresponding interval for κ^{-1} is $[0.272, 0.516]$, i.e $[1.938, 3.677]$ for κ.

[30] Cf. Heer and Maußner (2013).

[31] Note that also the model's moments are calculated from HP-filtered (artificial) time series.

[32] For the computation of the risk free asset's return, no such complication arises. This is the case because its ex post return coincides with the ex ante return due to its risk free nature. To see this, note that the return computation can be decomposed into two steps. First, we derive the price of the risk free asset via its Euler-Lucas equation. Next period, we calculate the return by relating the payoff to this price. Yet the payoff is risk free and already known to be 1 with certainty by the time of the purchase such that the ex post return is identical to the reciprocal of the price.

data as displayed in table 3.3, denoted by $S_d \in \mathbb{R}^8$, via some distance measure of the form

$$\operatorname{dist}_A(S_m(a), S_d) := \langle A(S_m(a) - S_d), S_m(a) - S_d \rangle,$$

where A is a positive definite matrix. Within the grid, we search for the parameter values $\hat{a} \in \mathscr{A}$ minimizing this distance. The resulting minimum value is called the model's *score* and is reported alongside the respective models' artificial moments and return figures. In many cases, cf. e.g. in Boldrin et al. (2001), the weighting matrix is chosen as diagonal with the reciprocal of the estimates' respective error variances on the diagonal. We, however, follow Heer and Maußner (2013) and Uhlig (2007), in that we weight all statistics equally but quote the asset pricing quantities in percentage notation. Hence, our matching criterion is a slightly modified sum of squared differences between the model's simulated results and the respective empirical targets, where the modification is executed via

$$A = \begin{bmatrix} 100^2 \cdot I_2 & 0_{2\times 6} \\ 0_{6\times 2} & I_6 \end{bmatrix}.$$

3.4 Model analysis

This section provides the description of the model economies considered. For the sake of exposition, tedious derivations and the final list of respective equilibrium conditions are collected in the accompanying appendix.

3.4.1 M0: Baseline

We start our model analysis with an EZ variation of Jermann (1998). We also use this less complicated baseline model to demonstrate how we will constantly refer to the results obtained in the previous sections in order to keep the presentation of all models to come brief and well-arranged.

The representative household faces the decision problem (3.2.2). For the baseline case we choose the functional form of the within-period utility function as

$$u(c_t, n_t) := c_t,$$

i.e. the household does not value leisure.

From equation (3.2.7) we can thus calculate this model's stochastic discount factor as

$$m_{t+1,t} = \beta \left(\frac{V_{t+1}^{1-\gamma}}{\mathbb{E}_t V_{t+1}^{1-\gamma}} \right)^{1-\frac{1}{\theta}} \left(\frac{c_{t+1}}{c_t} \right)^{\frac{1-\gamma}{\theta}-1}. \tag{3.4.1}$$

Since the household does not care for leisure, it is obvious that the optimal solution here has to satisfy $n_\tau = 1$ instead of equation (3.2.5) for an interior solution. The remaining optimality conditions are unchanged. Hence, in the equilibrium conditions, (3.2.23) is replaced by

$$n_t = 1, \; t \geq \tau.$$

The representative firm faces the basic decision problem described in (3.2.15).

The equilibrium conditions for the baseline model, (3.2.23) and (3.2.24) in subsection 3.2.3 have to be replaced by

$$n_t - 1 = 0, \quad \text{and} \quad w_t \leq (1-\alpha)e^{z_t} n_t^{-\alpha} k_t^{\alpha}, \tag{3.4.2}$$

respectively. The complete list can be found in appendix B. Note that this model's general equilibrium cannot yield a unique wage because every wage that satisfies (3.4.2) solves the problem.

The steady state can be computed by the equations given in subsection 3.2.4, now complemented with

$$n_{ss} = 1.$$

Results Our baseline model's free parameters are β, ψ and κ. First, all considered parameterizations within our grid led to nearly the same output volatility. Second, correlation is not defined for n and third, because of (3.4.2), the equilibrium wage is not determined. We are thus left with three targets, namely EP, r^f, and s_i/s_y and choose the free parameters in order to exactly match the data with respect to these. The simulation results and the corresponding parameter values are displayed in tables 3.5 and 3.6.

Table 3.5: Results M0

EP	r^f	s_y	s_i/s_y	s_n/s_y	s_w/s_y	r_{yn}	r_{wn}	$Score$
				Data				
5.18	2.49	1.14	2.28	0.69	1.03	0.4	0.27	
				M0				
5.18	2.49	0.89	2.28	0				

Table 3.6: Free Parameters M0

β	ψ	κ
0.9874	0.04265	7.05

Additionally, we exemplify our previous discussion on the positivity of $\Phi(\cdot)$ for M0. The found parametrization would demand the investment never to go below eight thousandth of the current capital stock, i.e. the investment-to-capital ratio never to go below 72% its steady state value.[33]

As a consequence of the reduced list of targets and their matching mentioned above, we do not compute a score value for M0 and neither consider consumption habits in order to improve on the model's empirical performance.

[33] $\kappa^{\frac{1}{1-\kappa}}\delta = 0.7241 \cdot 0.011 = 0.008$.

3.4.2 M1: No labor market frictions

In this section's class of models, we focus on the effect of making the household appreciate leisure. This will result in a fluctuating labor supply below 1 and a unique equilibrium wage allowing us to also target our selected labor market statistics. As announced before, we will analyze this and the upcoming model classes within the three variations presented in subsection 3.2.5. The corresponding variants, M1.a - M1.c, will thereafter serve as our benchmark models.

We start with a brief summary of this class' structure. The following paragraphs' general statements hold up to potential habit formation.

The representative household faces exactly the decision problem (3.2.2) within all of the upcoming three settings and the specification of u will guarantee an interior solution.

The representative firm decides upon the decision problem (3.2.15) as already presented in the framework section.

The equilibrium conditions for all three variants therefore are fully characterized by the system of equations in subsection 3.2.3.

The steady state is computed as described in subsection 3.2.4.

3.4.2.1 M1.a: No habits

The stochastic discount factor $m_{t+1,t}$ is determined by (3.2.30). Further, by (3.2.23), the equilibrium wages necessarily satisfy

$$w_t = \frac{1-\nu}{\nu}\frac{c_t}{1-n_t}.$$

Hence, we compute ν as

$$\nu = \frac{c_{ss}}{w_{ss}(1-n_{ss})+c_{ss}}.$$

Results The best fit to the data that was achievable for the respective variants within M1 is collectively summarized in table 3.7 while the score minimizing parameter values can be read from table 3.8. Both tables can be found at the end of this subsection.

With regard to M1.a, we emphasize on the fact that–in line with the intuition sketched in the introductory remarks–making the household appreciate leisure requires a notable change in the EIS parametrization in order to still be able to generate a sizeable equity premium. Specifically, in comparison to M0 the resulting ψ drops by 85 percent. This way, we are able to reproduce the empirical returns. With regard to the RBC targets, apart from the relative volatility of working hours that is too low and the almost perfectly positive labor market correlations, this model's results are already roughly in line with the empirical data.

3.4.2.2 M1.b: Standard consumption habits

Here the stochastic discount factor $m_{t+1,t}$ is given by (3.2.33) and condition (3.2.23) reads

$$w_t = \frac{1-\nu}{\nu}\frac{c_t - c_t^h}{1-n_t}.$$

Hence,

$$\nu = \frac{(1-\chi)c_{ss}}{w_{ss}(1-n_{ss})+(1-\chi)c_{ss}}.$$

Results Introducing standard consumption habits, the additional free parameter χ primarily allows us to improve the fit of the labor market correlations. The score drops by nearly two thirds. With consumption habits in the model, we do not have to choose the EIS as low as in M1.a in order to replicate the return figures.

3.4.2.3 M1.c: Slowly adapting consumption habits

Neither the stochastic discount factor nor the equilibrium conditions are changed in comparison to M1.b.

Results The additional free parameter λ, does not help in further lowering the score. The best fit is found for a standard habit process with $\lambda = 0$ and the remaining optimal parameter values found for M1.b, so that the simulation results are identical to those of M1.b.

Table 3.7: Summary of Results M1

	EP	r^f	s_y	s_i/s_y	s_n/s_y	s_w/s_y	r_{yn}	r_{wn}	*Score*
					Data				
	5.18	2.49	1.14	2.28	0.69	1.03	0.40	0.27	
					M1: No labor market frictions				
a	5.19	2.55	1.01	2.27	0.16	0.84	1.00	1.00	1.23
b	5.27	2.50	0.92	2.16	0.12	0.96	0.37	0.26	0.40
c	5.27	2.50	0.92	2.16	0.12	0.96	0.37	0.26	0.40

Table 3.8: Free Parameters M1

	β	ψ	κ	χ	λ
		M1: No labor market frictions			
a	0.9873	0.00635	6.3		-
b	0.988	0.00715	6.25	0.3	-
c	0.988	0.00715	6.25	0.3	0

3.4.3 M2: Sticky real wages

In this section we add a friction to the labor market by introducing a type of stickiness to the real wages as in Uhlig (2007). Primarily, this stickiness is expected to decrease the volatility of wages and thus increase the volatility of working hours, as the wages' "buffering" of changes in the productivity of labor is limited. We first analyze the changes in the general framework, again up to potential habit formation, before reporting on the individual variants' results.

The representative household Due to some nonmodeled friction, the household's optimality condition (3.2.5) is not necessarily fulfilled in equilibrium anymore. Instead, we implicitly define

$$\frac{\partial u}{\partial c}(c_t, n_t) w_t^f = -\frac{\partial u}{\partial n}(c_t, n_t), \tag{3.4.3}$$

introducing a separate symbol for the marginal rate of substitution, w_t^f, denoting the equilibrium wage in an economy that is free of such frictions.

The representative firm again faces the basic decision problem (3.2.15).

The equilibrium conditions As mentioned above, the wage w_t in the economy's equilibrium is no longer necessarily equal to the marginal rate of substitution w_t^f of the household. Instead it evolves as a geometric mean of the previous period's wage w_{t-1} and the marginal rate of substitution, i.e.

$$w_t = w_{t-1}^{\mu} (w_t^f)^{1-\mu}, \quad \mu \in [0, 1). \tag{3.4.4}$$

This way, high changes of the wage between two subsequent periods become less likely.

Summing up, with regard to the equilibrium conditions in subsection 3.2.3, condition (3.2.23) is jointly replaced by both wage equations above. The full list of equations characterizing the equilibrium is laid out in appendix D.

The Steady state Since $\mu \neq 1$, it follows from (3.4.4) that in the steady state

$$w_{ss}^f = w_{ss}.$$

We then parameterize ν again using (3.4.3) in order to ensure a steady state value of $n_{ss} = 0.13$. Hence, for all considered variants of our sticky wages economy, all the remaining steady state values are identical to their frictionless counterparts of the previous section.

Calibration The additional parameter controlling the degree of wage stickiness is considered free within its domain, i.e. $\mu \in [0, 1)$.

3.4.3.1 M2.a: No habits

The first variant's stochastic discount factor is given by (3.2.30) and the equilibrium condition (3.4.3) reads

$$w_t^f = \frac{1-\nu}{\nu} \frac{c_t}{1-n_t}. \tag{3.4.5}$$

Results The optimal parameter values and the corresponding fit for M2 are again collectively summarized in tables 3.9 and 3.10. As in model M1.a, a low value for the EIS is necessary in order to replicate the empirical equity premium. The additional stickiness parameter μ helps in dissolving the strict correlation structure between hours, output and wages found in M1.a. Plus, as anticipated, the relative volatility of hours can also be increased, while on the other hand the relative volatility of wages falls—in fact to about two thirds of its empirical value.

Altogether, the additional degree of freedom allows us to reduce the score of M1.a by more than 50 percent.

3.4.3.2 M2.b: Standard consumption habits

For the second setting, the stochastic discount factor is given by (3.2.33). Further, equation (3.4.3) now takes the form

$$w_t^f = \frac{1-\nu}{\nu}\frac{c_t - c_t^h}{1-n_t}. \tag{3.4.6}$$

Results Despite the fact that the possibility of real wage stickiness results in a considerable score reduction from M1.a to M2.a, allowing for standard consumption habits leads to a model with $\mu = 0$. The optimal values of the remaining parameters are thus identical to those found in M1.b, just as the simulation results can be read from table 3.8.

3.4.3.3 M2.c: Slowly adapting consumption habits

As in the previous subsection, only alternating the external habit process compared to M2.b does neither change the stochastic discount factor nor the exact form of (3.4.3).

Results The consideration of real wage stickiness together with slowly adjusting consumption habits now again leads to $\mu \neq 0$. In comparison to M1.c and M2.b, simultaneously allowing

Table 3.9: Summary of Results M2

	EP	r^f	s_y	s_i/s_y	s_n/s_y	s_w/s_y	r_{yn}	r_{wn}	$Score$
					Data				
	5.18	2.49	1.14	2.28	0.69	1.03	0.40	0.27	
					M2: Sticky real wages				
a	5.13	2.50	1.23	2.18	0.47	0.64	0.87	0.63	0.58
b	5.27	2.50	0.92	2.16	0.12	0.96	0.37	0.26	0.40
c	5.17	2.50	1.03	2.35	0.43	0.84	0.55	0.15	0.16

Table 3.10: Free Parameters M2

	β	ψ	κ	χ	λ	μ
			M2: Sticky wages			
a	0.9875	0.00588	5.25	-	-	0.5
b	0.988	0.00715	6.25	0.3	-	0
c	0.988	0.007775	5.4	0.49	0.8	0.65

for $\mu \neq 0$ and $\lambda \neq 0$ increases the relative volatility of hours by such an amount that the model score can be reduced by nearly 60 percent even though the remaining labor market statistics are hit less exactly. As displayed in table 3.10, the corresponding parametrization features a rather high persistence in the habit formation and a larger degree of wage stickiness as in M2.a.

3.4.4 M3: Predetermined labor supply

We now introduce a different friction with respect to labor supply flexibility as we follow Boldrin et al. (2001) and require the representative household to commit himself to a certain labor supply one period in advance. Hence, he cannot respond to changes in productivity directly but with a time lag of one period. Again, we first discuss the implications of this change with respect to the general framework up to potential habit formation, before stating the results for the three individual variations considered.

The representative household has to fix his labor supply before the technology shock is revealed, i.e. we consider period τ's working hours n_τ as a given state variable and the household's decision on $n_{\tau+1}$ may not depend on $\epsilon_{\tau+1}$. Summing up, the representative household's problem reads

$$\begin{array}{ll} \max & U_\tau = W(u(c_\tau, n_\tau), (\mathbb{E}_\tau U_{\tau+1}^{1-\gamma})^{\frac{1}{1-\gamma}}) \\ \text{s.t.} & c_t \leq w_t n_t + d_t s_t - v_t(s_{t+1} - s_t), \\ & c_t \geq 0,\ 0 \leq n_t \leq 1, \quad \text{for all } t \geq \tau, \\ \text{given} & s_\tau, n_\tau. \end{array} \tag{3.4.7}$$

While the necessary optimality conditions (3.2.3), (3.2.4), (3.2.6) remain unchanged, the condition for next period's labor supply is now given by[34]

$$\mathbb{E}_t\left[m_{t+1,t}\left(w_{t+1} + \frac{\frac{\partial u}{\partial n}(c_{t+1}, n_{t+1})}{\frac{\partial u}{\partial c}(c_{t+1}, n_{t+1})}\right)\right] = 0,\ t \geq \tau. \tag{3.4.8}$$

The representative firm again faces the basic decision problem (3.2.15).

The equilibrium conditions in this model are obtained by replacing equation (3.2.23) of subsection 3.2.3 with (3.4.8). The full list is again presented in the appendix.

The steady state With respect to the steady state values, there are no changes to the general framework described in subsection 3.2.4.

3.4.4.1 M3.a: No habits

The stochastic discount factor is given by (3.2.30). Further, equation (3.4.8) becomes

$$\mathbb{E}_t\left[m_{t+1,t}\left(w_{t+1} - \frac{1-\nu}{\nu}\frac{c_{t+1}}{1-n_{t+1}}\right)\right] = 0. \tag{3.4.9}$$

Results Again, this class' fit and the respective parameterizations are collectively summarized in table 3.11 and in table 3.12.

In comparison to the frictionless counterpart M1.a, the fit of both correlation targets can be improved in M3.a. Yet, in contrast to the sticky wages variant M2.a, we cannot achieve a better fit for the relative volatility of hours while the relative volatility of wages is matched a little more exactly. Altogether, this model's score is virtually the same as in M2.a, even with one free parameter less.

[34] The detailed derivation can be found in appendix.

3.4.4.2 M3.b: Standard consumption habits

We now combine the assumptions of predetermined labor supply and consumption habits to see whether we can further improve our model score particularly with respect to the labor market targets. Now, the stochastic discount factor is given by (3.2.33). Moreover with $u(c_t, n_t) = (c_t - c_t^h)^\nu (1-n_t)^{1-\nu}$ equation (3.4.8) turns out as

$$\mathbb{E}_t \left[m_{t+1,t} \left(w_{t+1} - \frac{1-\nu}{\nu} \frac{(c_{t+1} - c_{t+1}^h)}{1-n_{t+1}} \right) \right] = 0. \tag{3.4.10}$$

Results In comparison to model M1.b the relative volatility of hours decreases even more. Also are the correlation between wages and working hours not matched as exactly. Concluding, in our grid, the score for the variant with standard consumption habits cannot be made smaller than in its frictionless counterpart.

3.4.4.3 M3.c: Slowly adapting consumption habits

In this variant, again, the stochastic discount factor and the exact form of equation (3.4.8) are the same as for M3.b.

Results Adding the possibility of slowly adapting consumption habits to the model with predetermined hours does not further lower the achievable score. The best fit is again found by setting the additional parameter $\lambda = 0$ and the simulation results are therefore identical to those of M3.b.

Table 3.11: Summary of Results M3

	EP	r^f	s_y	s_i/s_y	s_n/s_y	s_w/s_y	r_{yn}	r_{wn}	$Score$
					Data				
	5.18	2.49	1.14	2.28	0.69	1.03	0.40	0.27	
					M3: Predetermined labor supply				
a	5.15	2.49	0.96	2.30	0.14	0.90	0.71	0.62	0.57
b	5.19	2.52	0.92	2.35	0.09	0.97	0.41	0.34	0.43
c	5.19	2.52	0.92	2.35	0.09	0.97	0.41	0.34	0.43

Table 3.12: Free Parameters M3

	β	ψ	κ	χ	λ
		M3: Predetermined labor supply			
a	0.987485	0.00643	6.5	-	-
b	0.98872	0.0073	5.3	0.53	-
c	0.98872	0.0073	5.3	0.53	0

3.4.5 M4: Sectoral frictions in the allocation of labor

In this section, we follow the approach of Boldrin et al. (2001) and further extend our framework in that we decompose the economy's productive part into two sectors, both of which are assumed to be representable by one stand-in firm. The consumption good is produced in one sector, the investment good in the other.[35]

As in subsection 3.4.4, there is a representative household who is assumed to be unable to adapt his labor supply to technology shocks in the respective periods but is committed to the hours of labor contracted prior to that period. We now, additionally, require labor to be contracted sector-specific, i.e. the household can neither switch intersectorally within a given period. To sum up, while we already analyzed the effect of predetermination of labor supply in M3, the sector mobility constraint introduces an additional friction into the framework discussed so far.

This model class' structure is considerably different to our basic framework and will thus be introduced in detail in the next paragraphs, again up to potential habit formation.

The representative household For the representative household there are two changes. Since there are two representative firms, one for the consumption sector and one for the investment good sector, he may now allocate working hours, hold shares and receive dividends from either of these. Hence, with the obvious notation, his budget constraint becomes

$$c_t \leq w_t^I n_t^I + w_t^C n_t^C + d_t^I s_t^I + d_t^C s_t^C - v_t^I (s_{t+1}^I - s_t^I) - v_t^C (s_{t+1}^C - s_t^C),\ t \geq \tau.$$

Further, just like in M3, the household also has to decide on his labor supply one period ahead so that his decision problem reads

$$\begin{array}{ll} \max & U_\tau = W(u(c_\tau, n_\tau), (\mathbb{E}_\tau U_{\tau+1}^{1-\gamma})^{\frac{1}{1-\gamma}}) \\ \text{s.t.} & c_t \leq w_t^I n_t^I + w_t^C n_t^C + d_t^I s_t^I + d_t^C s_t^C - v_t^I (s_{t+1}^I - s_t^I) - v_t^C (s_{t+1}^C - s_t^C), \\ & n_t = n_t^I + n_t^C, \\ & c_t \geq 0,\ n_t^I \geq 0,\ n_t^C \geq 0,\ 0 \leq n_t \leq 1, \ \text{ for all } t \geq \tau, \\ \text{given} & s_\tau^I, s_\tau^C, n_\tau^I, n_\tau^C. \end{array} \tag{3.4.11}$$

With these changes, the optimality conditions (3.2.3)-(3.2.6) for a solution where last row's constraints do not bind are

$$V_t = [(1-\beta) u(c_t, n_t)^{\frac{1-\gamma}{\theta}} + \beta (\mathbb{E}_t [V_{t+1}^{1-\gamma}])^{\frac{1}{\theta}}]^{\frac{\theta}{1-\gamma}},$$

$$\mathbb{E}_t \left[m_{t+1,t} \frac{d_{t+1}^I + v_{t+1}^I}{v_t^I} - 1 \right] = 0,$$

$$\mathbb{E}_t \left[m_{t+1,t} \frac{d_{t+1}^C + v_{t+1}^C}{v_t^C} - 1 \right] = 0,$$

$$\mathbb{E}_t \left[m_{t+1,t} \left(w_{t+1}^I + \frac{\frac{\partial u}{\partial n}(c_{t+1}, n_{t+1})}{\frac{\partial u}{\partial c}(c_{t+1}, n_{t+1})} \right) \right] = 0, \tag{3.4.12}$$

$$\mathbb{E}_t \left[m_{t+1,t} \left(w_{t+1}^C + \frac{\frac{\partial u}{\partial n}(c_{t+1}, n_{t+1})}{\frac{\partial u}{\partial c}(c_{t+1}, n_{t+1})} \right) \right] = 0, \tag{3.4.13}$$

[35] Note that M4 is not an EZ variation of the original Boldrin et al. (2001) model since we also stick with our adjustment cost assumption.

$$n_t = n_t^I + n_t^C,$$
$$c_t = w_t^I n_t^I + w_t^C n_t^C + d_t^I s_t^I + d_t^C s_t^C - v_t^I (s_{t+1}^I - s_t^I) - v_t^C (s_{t+1}^C - s_t^C),$$

with the stochastic discount factor given by (3.2.7).

Note that we are only interested in interior solutions with respect to the last row of constraints because sticking to a Cobb-Douglas production technology implies labor demand to always be strictly positive in both sectors, so that in general equilibrium wages have to be set in such a way that also labor supply is strictly positive in both sectors. Particulary, as stated above, the household has to be indifferent in expectation between the wages and returns in both sectors. I.e. it must hold for all $t \geq \tau$ that

$$\mathbb{E}_t \left[m_{t+1,t} (w_{t+1}^I - w_{t+1}^C) \right] = 0$$

and further, for the problem to not be unbounded,

$$\mathbb{E}_t \left[m_{t+1,t} \left(\frac{d_{t+1}^I + v_{t+1}^I}{v_t^I} - \frac{d_{t+1}^C + v_{t+1}^C}{v_t^C} \right) \right] = 0.$$

The latter is a no arbitrage condition on the sector-specific stocks.

The representative firm in the consumption good sector produces the consumption good via the technology

$$c_t = e^{z_t} (n_t^C)^{1-\alpha} (k_t^C)^\alpha, \quad \alpha \in (0,1),$$

where the sequence $\{z_t\}$ follows (3.2.8), and accumulates capital according to

$$k_{t+1}^C - (1-\delta) k_t^C = \Phi \left(\frac{i_t^C}{k_t^C} \right) k_t^C,$$

with $\Phi(\cdot)$ as defined in (3.2.10). Investment goods now have to be purchased from the representative firm in the investment sector. Let p_t denote the price of investment relative to consumption. These investment expenditures are again assumed to be financed through profits beyond dividend payments plus the issuance of new shares. Hence, the equivalent to (3.2.11) here is

$$p_t i_t^C = c_t - w_t^C n_t^C - d_t^C s_t^C + v_t^C (s_{t+1}^C - s_t^C)$$

and period t's cash flow is given by

$$cf_t^C := c_t - w_t^C n_t^C - p_t i_t^C.$$

The firm's management again maximizes its firm value, which is defined as above and can, under the respective transversality condition

$$\lim_{t \to \infty} \mathbb{E}_\tau \left[m_{t,\tau} v_t^C s_{t+1}^C \right] = 0,$$

thus be written as

$$f v_\tau^C := cf_\tau^C + v_\tau^C s_{\tau+1}^C = \mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} m_{t,\tau} cf_t^C \right].$$

In other words, the maximization problem of this sector's representative firm is

$$\begin{aligned}
\max \quad & \mathbb{E}_\tau\left[\sum_{t=\tau}^{\infty} m_{t,\tau}\left(e^{z_t}(n_t^C)^{1-\alpha}(k_t^C)^{\alpha} - w_t^C n_t^C - p_t i_t^C\right)\right] \\
\text{s.t.} \quad & k_{t+1}^C = (1-\delta)k_t^C + \Phi\left(\frac{i_t^C}{k_t^C}\right)k_t^C, \quad \text{for all } t \geq \tau, \\
\text{given} \quad & k_\tau^C.
\end{aligned} \tag{3.4.14}$$

The equivalent optimality conditions to (3.2.16)-(3.2.21) hence are

$$w_t^C = (1-\alpha)e^{z_t}(n_t^C)^{-\alpha}(k_t^C)^{\alpha}, \tag{3.4.15}$$

$$q_t^C = \frac{p_t}{\Phi'\left(\frac{i_t^C}{k_t^C}\right)},$$

$$q_t^C = \mathbb{E}_t\left[m_{t+1,t}\left(\alpha e^{z_{t+1}}(n_{t+1}^C)^{1-\alpha}(k_{t+1}^C)^{\alpha-1} - p_{t+1}\frac{i_{t+1}^C}{k_{t+1}^C} + q_{t+1}^C\left(1-\delta+\Phi\left(\frac{i_{t+1}^C}{k_{t+1}^C}\right)\right)\right)\right], \tag{3.4.16}$$

$$k_{t+1}^C = (1-\delta)k_t^C + \Phi\left(\frac{i_t^C}{k_t^C}\right)k_t^C,$$

$$p_t i_t^C = c_t - w_t^C n_t^C - d_t^C s_t^C + v_t^C(s_{t+1}^C - s_t^C),$$

$$c_t = e^{z_t}(k_t^C)^{\alpha}(n_t^C)^{1-\alpha}, \tag{3.4.17}$$

with $\{z_t\}$ following (3.2.8).

The representative firm in the investment good sector produces the investment good via the production function

$$i_t = e^{z_t}(n_t^I)^{1-\alpha}(k_t^I)^{\alpha} \qquad \alpha \in (0,1),$$

where the sequence $\{z_t\}$ follows (3.2.8), and also accumulates capital according to

$$k_{t+1}^I - (1-\delta)k_t^I = \Phi\left(\frac{i_t^I}{k_t^I}\right)k_t^I.$$

This firm sells an amount of i_t^C of the investment good to the firm in the consumption good sector. The remaining i_t^I is used for own investments. Its respective equivalent to (3.2.11) hence is

$$p_t i_t^I = p_t i_t - w_t^I n_t^I - d_t^I s_t^I + v_t^I(s_{t+1}^I - s_t^I),$$

or equivalently

$$p_t i_t^C - w_t^I n_t^I - d_t^I s_t^I + v_t^I(s_{t+1}^I - s_t^I) = 0$$

and period t's cash flow is

$$cf_t^I := p_t i_t - w_t^I n_t^I - p_t i_t^I = p_t i_t^C - w_t^I n_t^I.$$

This firm's management maximizes its firm value, again defined as above. Under the respective transversality condition

$$\lim_{t\to\infty} \mathbb{E}_\tau\left[m_{t,\tau} v_t^I s_{t+1}^I\right] = 0,$$

this can be written as

$$fv_\tau^I := cf_\tau^I + v_\tau^I s_{\tau+1}^I = \mathbb{E}_\tau\left[\sum_{t=\tau}^{\infty} m_{t,\tau} cf_t^I\right].$$

Consequently, the maximization problem of the representative firm in the investment good sector is

$$\begin{aligned}
\max \quad & \mathbb{E}_\tau\left[\sum_{t=\tau}^{\infty} m_{t,\tau}\left(p_t e^{z_t}(n_t^I)^{1-\alpha}(k_t^I)^\alpha - w_t^I n_t^I - p_t i_t^I\right)\right] \\
\text{s.t.} \quad & k_{t+1}^I = (1-\delta)k_t^I + \Phi\left(\frac{i_t^I}{k_t^I}\right)k_t^I, \;\; \text{for all } t \geq \tau, \\
\text{given} \quad & k_\tau^I.
\end{aligned} \tag{3.4.18}$$

Hence, this sector's equivalent optimality conditions to (3.2.16)-(3.2.21) are

$$w_t^I = (1-\alpha)p_t e^{z_t}(n_t^C)^{-\alpha}(k_t^C)^\alpha, \tag{3.4.19}$$

$$q_t^I = \frac{p_t}{\Phi'\left(\frac{i_t^I}{k_t^I}\right)},$$

$$q_t^I = \mathbb{E}_t\left[m_{t+1,t}\left(\alpha p_{t+1}e^{z_{t+1}}(n_{t+1}^I)^{1-\alpha}(k_{t+1}^I)^{\alpha-1} - p_{t+1}\frac{i_{t+1}^I}{k_{t+1}^I} + q_{t+1}^I\left(1-\delta+\Phi\left(\frac{i_{t+1}^I}{k_{t+1}^I}\right)\right)\right)\right], \tag{3.4.20}$$

$$k_{t+1}^I = (1-\delta)k_t^I + \Phi\left(\frac{i_t^I}{k_t^I}\right)k_t^I,$$

$$p_t i_t^I = p_t i_t - w_t^I n_t^I - d_t^I s_t^I + v_t^I(s_{t+1}^I - s_t^I),$$

$$i_t = e^{z_t}(k_t^I)^\alpha(n_t^I)^{1-\alpha}, \tag{3.4.21}$$

with $\{z_t\}$ following (3.2.8).

The equilibrium conditions The general equilibrium for this two sector model is characterized by the optimality conditions listed in the paragraphs above plus the condition

$$i_t = i_t^I + i_t^C.$$

Again, this condition already guarantees a cleared stock market, even if we do not solve for a solution for s_{t+1}^i, d_t^i and v_t^i, $i \in \{C, I\}$. The full list of equilibrium conditions can be found in appendix F.

The steady state According to our basic framework, for the steady state we demand that

$$i_{ss}^C = \delta k_{ss}^C \quad \text{and} \quad i_{ss}^I = \delta k_{ss}^I$$

as well as

$$q_{ss}^C = 1 \quad \text{and} \quad q_{ss}^I = 1 \quad \text{and} \quad p_{ss} = 1$$

and thus parameterize Φ as before. Next, equations (3.4.16) and (3.4.20) first yield

$$k_{ss}^C = \left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{\frac{1}{\alpha-1}} n_{ss}^C \quad \text{and} \quad k_{ss}^I = \left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{\frac{1}{\alpha-1}} n_{ss}^I. \tag{3.4.22}$$

Taking the sum,

$$k_{ss} := k_{ss}^C + k_{ss}^I = \left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{\frac{1}{\alpha-1}} n_{ss}$$

and hence

$$i_{ss} = i_{ss}^C + i_{ss}^I = \delta k_{ss} = \delta \left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{\frac{1}{\alpha-1}} n_{ss}.$$

Using (3.4.21) and (3.4.22) we calculate

$$\begin{aligned} i_{ss} &= (k_{ss}^I)^\alpha (n_{ss}^I)^{1-\alpha} = \left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{\frac{\alpha}{\alpha-1}} (n_{ss}^I)^\alpha (n_{ss}^I)^{1-\alpha} \\ \Leftrightarrow n_{ss}^I &= \left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{-\frac{\alpha}{\alpha-1}} i_{ss} = \delta \left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{-1} n_{ss} \end{aligned} \tag{3.4.23}$$

and

$$n_{ss}^C = n_{ss} - n_{ss}^I.$$

With (3.4.22) and (3.4.23), we find

$$k_{ss}^I = \left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{\frac{1}{\alpha-1}} \delta \left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{-1} n_{ss} = \delta \left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{\frac{2-\alpha}{\alpha-1}} n_{ss}$$

and

$$k_{ss}^C = k_{ss} - k_{ss}^I.$$

From (3.4.17) we can determine

$$c_{ss} = (k_{ss}^C)^\alpha (1 - n_{ss}^C)^{1-\alpha}.$$

(3.4.15) and (3.4.19) further yield

$$\begin{aligned} w_{ss}^C &= (1-\alpha)\left(\frac{k_{ss}^C}{n_{ss}^C}\right)^\alpha = (1-\alpha)\left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{\frac{\alpha}{\alpha-1}}, \\ w_{ss}^I &= (1-\alpha)\left(\frac{k_{ss}^I}{n_{ss}^I}\right)^\alpha = (1-\alpha)\left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{\frac{\alpha}{\alpha-1}} = w_{ss}^C. \end{aligned}$$

Now, all steady state variables are expressed in terms of n_{ss}. We use (3.4.12) or (3.4.13) to again set ν for all three variants of u in such way that a steady state value of $n_{ss} = 0.13$ arises.

Computation of the return series The risk free return is, of course, unaffected by the extension of our basic framework to two productive sectors. What does change, though, is the computation of the return on equity in this economy.

As in the one sector case, in order to be able to derive the formulae needed to compute both sectors' return on equity, we have to impose additional conditions on our two sector economy. Precisely, for both sectors, i.e. for $i \in \{I, C\}$, we assume

$$\lim_{s\to\infty} \mathbb{E}_t \left[m_{t+s,t} q_{t+s}^i k_{t+1+s}^i\right] = 0.$$

By the same reasoning as above, we thus find

$$v_t^i s_{t+1}^i = q_t^i k_{t+1}^i \tag{3.4.24}$$

so that the two sectors' period $t+1$ return on equity both satisfy

$$\begin{aligned}
r_{t+1}^{e,C} &:= \frac{d_{t+1}^C + v_{t+1}^C}{v_t^C} = \frac{c_{t+1} - w_{t+1}^C n_{t+1}^C - p_{t+1} i_{t+1}^C + q_{t+1}^C k_{t+2}^C}{q_t^C k_{t+1}^C},\\
r_{t+1}^{e,I} &:= \frac{d_{t+1}^I + v_{t+1}^I}{v_t^I} = \frac{p_{t+1} i_{t+1}^C - w_{t+1}^I n_{t+1}^I + q_{t+1}^I k_{t+2}^I}{q_t^I k_{t+1}^I}.
\end{aligned}$$

Thus, as the overall gross return on firm shares over both sectors is naturally computed as

$$r_{t+1}^{e,C} v_t^C s_{t+1}^C + r_{t+1}^{e,I} v_t^I s_{t+1}^I,$$

we finally reach

$$r_{t+1}^e := \frac{r_{t+1}^{e,C} v_t^C s_{t+1}^C + r_{t+1}^{e,I} v_t^I s_{t+1}^I}{v_t^C s_{t+1}^C + v_t^I s_{t+1}^I} \overset{(3.4.24)}{=} r_{t+1}^{e,C} \frac{q_t^C k_{t+1}^C}{q_t^C k_{t+1}^C + q_t^I k_{t+1}^I} + r_{t+1}^{e,I} \frac{q_t^I k_{t+1}^I}{q_t^C k_{t+1}^C + q_t^I k_{t+1}^I}$$

as this economy's return on equity.[36]

3.4.5.1 M4.a: No habits

The stochastic discount factor remains the same as in (3.2.30). Further, equations (3.4.12) and (3.4.13) can be written as

$$\mathbb{E}_t\left[m_{t+1,t}\left(w_{t+1}^I - \frac{1-\nu}{\nu}\frac{c_{t+1}}{1-n_{t+1}}\right)\right] = 0 \quad \text{and} \quad \mathbb{E}_t\left[m_{t+1,t}\left(w_{t+1}^C - \frac{1-\nu}{\nu}\frac{c_{t+1}}{1-n_{t+1}}\right)\right] = 0.$$

From these equations it (again) follows that

$$\nu = \frac{c_{ss}}{w_{ss}^C(1-n_{ss}) + c_{ss}} = \frac{c_{ss}}{w_{ss}^I(1-n_{ss}) + c_{ss}}.$$

Results Again, this class' variants' fit and the corresponding parametrization are collectively summarized in table 3.13 and in table 3.14.

With respect to M1.a, the lower score is again primarily ascribable to the improvement on the labor market correlation, despite the notable "overshooting" in the relative volatility of wages.

3.4.5.2 M4.b: Standard consumption habits

The stochastic discount factor is given by (3.2.33). Equations (3.4.12) and (3.4.13) become

$$\mathbb{E}_t\left[m_{t+1,t}\left(w_{t+1}^I - \frac{1-\nu}{\nu}\frac{c_{t+1} - c_{t+1}^h}{1-n_{t+1}}\right)\right] = 0 \text{ and } \mathbb{E}_t\left[m_{t+1,t}\left(w_{t+1}^C - \frac{1-\nu}{\nu}\frac{c_{t+1} - c_{t+1}^h}{1-n_{t+1}}\right)\right] = 0.$$

Therefore, we set

$$\nu = \frac{(1-\chi)c_{ss}}{w_{ss}^C(1-n_{ss}) + (1-\chi)c_{ss}} = \frac{(1-\chi)c_{ss}}{w_{ss}^I(1-n_{ss}) + (1-\chi)c_{ss}}.$$

[36] Note that this departs from Heer and Maußner (2013), who weight each sector's return on equity by the respective sectors' capital shares only.

Results Standard consumption habits cannot improve on the achieved fit. Thus, the best fit is found at $\chi = 0$, with the remaining parameter values chosen identically to M4.a. We observe that the more complicated two sector framework is not able to empirically perform as well as the comparable frictionless economy modeled in M1.b.

3.4.5.3 M4.c: Slowly adapting consumption habits

Generalizing the habit defining process with respect to M4.b, does neither change the discount factor nor the form of equations (3.4.12) and (3.4.13).

Results The consideration of slowly adjusting consumption habits neither helps to improve the data fit. Hence, the optimal fit is achieved at $\lambda = 0$, with the other parameters as in M4.b.

Table 3.13: Summary of Results M4

	EP	r^f	s_y	s_i/s_y	s_n/s_y	s_w/s_y	r_{yn}	r_{wn}	$Score$
				Data					
	5.18	2.49	1.14	2.28	0.69	1.03	0.40	0.27	
				M4: Sectoral frictions in the allocation of labor					
a	5.24	2.49	1.03	2.33	0.26	1.73	0.75	0.13	0.84
b	5.24	2.49	1.03	2.33	0.26	1.73	0.75	0.13	0.84
c	5.24	2.49	1.03	2.33	0.26	1.73	0.75	0.13	0.84

Table 3.14: Free Parameters M4

	β	ψ	κ	χ	λ
		M4: Sectoral frictions in the allocation of labor			
a	0.9884	0.0086	3.0625	-	-
b	0.9884	0.0086	3.0625	0	-
c	0.9884	0.0086	3.0625	0	0

3.5 Results and discussion

This section is devoted to the collective presentation of the respective models' results and to their comparative discussion. First, the best fits achievable and the corresponding parameterizations were already summarized in tables 3.1 and 3.2.

Since all models in the present paper are EZ variations with Cobb-Douglas composite good aggregation of the corresponding models analyzed in Heer and Maußner (2013), we first want to point out that, due to the more general utility representation, we were able to considerably improve on their reported data fit. Note that next to the extra degree of freedom associated with the EZ representation with regard to ψ, we thereby also considered β as free in order to additionally target the German real risk-free rate.

Model evaluation As foreshadowed in the analysis of M0 above, we must flexibilize the household's labor decision in order to arrive at models that allow for the targeting of labor market statistics. Now, M1.a shows that it is still possible to match our return targets but the corresponding calibration yields simulation results with nearly perfectly positive correlations between output and working hours and between wages and working hours as well as a rather low relative volatility of working hours. Allowing for standard consumption habits clears away these perfect correlations. As a matter of fact, the corresponding simulation results already might very well be regarded in line with the empirical evidence. Generalizing the assumed habit formation towards slowly adjustment, however, does not yield any improvement for the data fit. We again stress the fact that the range we allowed for ψ was broad enough in order to prevent the equity premium from dropping, as it dramatically happens in Heer and Maußner (2013).

In M2, M3 and M4, we study to what extent different real labor market frictions are—within this framework—able to help in improving on the data fit already achieved by M1.

We start with the analysis of real wage stickiness. The comparison of M1.a and M2.a shows that allowing for wage stickiness also dissolves the rigid correlation structure between output, hours and wages—albeit not as much as the introduction of habits within M1—and improves on the volatility of working hours. Now, while the consideration of real wage stickiness does not improve on the data fit under standard consumption habits, also allowing for these habits' slow adjustment most notably further improves on the considered correlations, leading to our overall minimum score.

An alternative friction, predetermined labor supply (M3), also initially moves ρ_{yn} and ρ_{wn} towards the data. As within M1, the assumption of standard habits again further improves on these correlations while allowing for $\lambda \neq 0$ does not help in lowering the model score. It is worth pointing at the fact that M3.b and M3.c do not empirically outperform their frictionless counterparts.

Eventually, in the two sector framework, M4.a also yields better simulated labor market correlations than M1.a. Yet, the relative volatility of wages rises to nearly 170 percent of its empirical value rendering the overall fit inferior to the pure predetermined hours model. Moreover, the considered forms of habit formation cannot improve on the score. We want to stress on the fact that M4.b and M4.c are not able to yield simulation results that are as well in accordance with the data as M1.b and M1.c.

The pairwise comparison of the three considered labor market frictions clearly attributes the largest score improvement to the modeling device of sticky wages, which introduces an additional free parameter μ. Checked against the pure predetermined labor class, M4 cannot justify its more complicated structure through empirical performance.

Implications So far, our analysis was kept descriptive in that we objectively fixed intervals for the free parameters via their respective domains and partly via observable consensus in the literature. In particular, we hitherto did not bother about the found parameterizations' behavioral implications on our representative household. We thus want to complement our analysis with a few—partly summarizing, partly normative—remarks on the resulting values of our free parameters.

First, the range of score minimizing values for β, $[0.9873, 0.98872]$, is much smaller than initially anticipated. To put it another way, the targeting of r^f does not require remarkably different levels of impatience along the models considered.

For reasons laid out in subsection 3.2.5, the parametrization of ψ is crucial for our models' data fit as the EIS controls the household's sensitivity to deviations from a smooth composite good path. The smaller ψ, the higher his sensitivity. Now, the only way the household can transfer consumption intertemporally is provided by our models' asset market, namely via the purchase of stocks or the riskfree security. Thus, decreasing ψ makes the household demand a higher compensation for him taking the risk of a stock investment, which leads to a larger return on equity. The range of values for ψ we actually found to optimize the data fit, $[0.00588, 0.04265]$, was already broad enough to match the empirical equity premium in all our models by an accuracy of less than a decimal. Caution must nevertheless be paid to this resulting magnitude of ψ, which is rather close to the lower boundary of the interval initially allowed. Although Hall (1988) reports on confirmatory estimates leading him to the conclusion that "the elasticity is unlikely to be much above 0.1 [...]",[37] we have to recall that the disentanglement of EIS and RRA within the EZ framework can only be partly in nature since any deviation from $\psi = 0.5$, gives rise to nonindifference towards the temporal resolution of uncertainty regarding the composite good. More precisely, since in all our models the score minimizing values of ψ clearly satisfy $\psi^{-1} > \gamma$, we are actually simulating economies where the stand-in agent is assumed to have a preference for *later* resolution of uncertainty. Importantly, the above interval of optimizing values for ψ noticeably indicates a deviation from the typically assumed expected utility framework.

As pointed out above, the reciprocal of κ is the elasticity of the investment-to-capital ratio with respect to Tobin's q. Thus, M0 and the classes M1-M3 roughly span its interval as $[0.14, 0.19]$, close to the value found by Jermann (1998). The two sector class M4 yields a notably higher elasticity of about 0.33.[38]

Regarding standard consumption habits, by (3.2.31) and (3.2.32), the chosen value for χ seems to critically affect the plausibility of the assumed habit formation. The resulting range of values, $[0.3, 0.53]$, indicate a moderate and thus maybe a more easily agreeable degree of habit formation as e.g. found by Jermann (1998), Uhlig (2007) or Heer and Maußner (2013).

Among the four models M1.c, M2.c, M3.c and M4.c, only the sticky wages framework actually indicates slow adjustment of consumption habits. The score minimizing value of $\lambda = 0.8$ is of notable magnitude and close to Uhlig (2007)'s calibration with 0.9. The resulting values for μ, 0.5 and 0.65, display a medium degree of real wage stickiness, again well below the score minimizers found in Uhlig (2007) or Heer and Maußner (2013).

3.6 Conclusion

Within the EZ utility representation, frictionless models already yield simulation results in good accordance with the German empirical data. Amongst the considered labor market frictions, allowing for real wage stickiness leads to the most remarkable improvement in fit, while, under habit formation, predetermined labor supply, with or without additionally decomposing the production sector into two parts, could not further improve on the frictionless models' empirical performance.

In a sense, a researcher considering policy evaluation on the basis of an EZ framework might look at the information collected in tables 3.1 and 3.2 as initial guidance with respect to the specification of his DSGE economy.

With respect to the standard additive power utility model, the additional flexibility of the EZ framework seems to help in avoiding such extreme parameterizations as found necessary in Heer and Maußner (2013). The degree of additional flexibility, however, primarily hinges on the allowed magnitude of deviations from the standard case of $\theta = 1$. Yet, since there is no obvious reason for such a nonindifference towards the timing of uncertainty resolution, any large deviation from $\theta = 1$ calls for justification. It would therefore be interesting to quantitatively asses the plausibility of the implied preference for later resolution that results in our analysis. This could e.g. be done along the lines of Epstein et al. (2014) and Kaltenbrunner and Lochstoer (2010). While the former authors present such a quantitative measure within a long run risk (LRR) framework, the latter study how endogenous long run consumption risk arises in M0.

[37] Cf. Hall (1988), p. 340.

[38] Note that the two sector framework forces Heer and Maußner (2013), p. 19, to assume "negligible adjustment costs", i.e. an enormous elasticity of 200.

Appendix

A Framework

We derive the optimality conditions from (3.2.2) for the representative household's maximization problem. Since an optimal solution has to fulfill the first constraint with equality, we can plug it into the objective function. Also, in almost all of the considered cases it will be obvious that the solution has to be interior with respect to the remaining two constraints, i.e. it satisfies $c_t > 0$ and $n_t \in (0,1)$. Hence, we state the corresponding necessary optimality conditions, i.e. we set the derivatives of the objective function U_τ, with the first constraint plugged in, equal to zero.

With respect to $s_{\tau+1}$, we find the first condition for an interior optimum

$$\begin{aligned}
0 &= \frac{\partial W}{\partial u}\frac{\partial u}{\partial c}(-v_\tau) + \frac{\partial W}{\partial \mu}\frac{1}{1-\gamma}(\mathbb{E}_\tau U_{\tau+1}^{1-\gamma})^{\frac{1}{1-\gamma}-1}(1-\gamma)\mathbb{E}_\tau\left[U_{\tau+1}^{-\gamma}\frac{\partial U_{\tau+1}}{\partial s_{\tau+1}}\right] \\
&= -\frac{\partial W}{\partial u}\frac{\partial u}{\partial c}v_\tau + \frac{\partial W}{\partial \mu}(\mathbb{E}_\tau U_{\tau+1}^{1-\gamma})^{\frac{1}{1-\gamma}-1}\mathbb{E}_\tau\left[U_{\tau+1}^{-\gamma}\frac{\partial U_{\tau+1}}{\partial s_{\tau+1}}\right],
\end{aligned}$$

where $\frac{\partial W}{\partial u}$ is short for $\frac{\partial W}{\partial u}(u(c_\tau,n_\tau),(\mathbb{E}_\tau U_{\tau+1}^{1-\gamma})^{\frac{1}{1-\gamma}})$, analogously for $\frac{\partial W}{\partial \mu}$. $\frac{\partial u}{\partial c}$ abbreviates $\frac{\partial u}{\partial c}(c_\tau,n_\tau)$. We thus calculate

$$\begin{aligned}
\frac{\partial W}{\partial u} &= \frac{\theta}{1-\gamma}[\ldots]^{\frac{\theta}{1-\gamma}-1}(1-\beta)\frac{1-\gamma}{\theta}u(c_\tau,n_\tau)^{\frac{1-\gamma}{\theta}-1} = (1-\beta)[\ldots]^{\frac{\theta}{1-\gamma}(1-\frac{1-\gamma}{\theta})}u(c_\tau,n_\tau)^{\frac{1-\gamma}{\theta}-1} \\
&= (1-\beta)U_\tau^{1-\frac{1-\gamma}{\theta}}u(c_\tau,n_\tau)^{\frac{1-\gamma}{\theta}-1},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial W}{\partial \mu} &= \frac{\theta}{1-\gamma}[\ldots]^{\frac{\theta}{1-\gamma}-1}\beta\frac{1-\gamma}{\theta}(\mathbb{E}_\tau U_{\tau+1}^{1-\gamma})^{\frac{1}{1-\gamma}(\frac{1-\gamma}{\theta}-1)} = \beta[\ldots]^{\frac{\theta}{1-\gamma}(1-\frac{1-\gamma}{\theta})}(\mathbb{E}_\tau U_{\tau+1}^{1-\gamma})^{\frac{1}{\theta}-\frac{1}{1-\gamma}} \\
&= \beta U_\tau^{1-\frac{1-\gamma}{\theta}}(\mathbb{E}_\tau U_{\tau+1}^{1-\gamma})^{\frac{1}{\theta}-\frac{1}{1-\gamma}}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial U_{\tau+1}}{\partial s_{\tau+1}} &= \frac{\partial W}{\partial u}(u(c_{\tau+1},n_{\tau+1}),(\mathbb{E}_{\tau+1}U_{\tau+2}^{1-\gamma})^{\frac{1}{1-\gamma}})\frac{\partial u}{\partial c}(c_{\tau+1},n_{\tau+1})(d_{\tau+1}+v_{\tau+1}) \\
&= (1-\beta)U_{\tau+1}^{1-\frac{1-\gamma}{\theta}}u(c_{\tau+1},n_{\tau+1})^{\frac{1-\gamma}{\theta}-1}\frac{\partial u}{\partial c}(c_{\tau+1},n_{\tau+1})(d_{\tau+1}+v_{\tau+1}).
\end{aligned}$$

Combining these equations, we finally reach at

$$\begin{aligned}
0 = &-(1-\beta)U_\tau^{1-\frac{1-\gamma}{\theta}}u(c_\tau,n_\tau)^{\frac{1-\gamma}{\theta}-1}\frac{\partial u}{\partial c}(c_\tau,n_\tau)v_\tau + \\
&+\beta U_\tau^{1-\frac{1-\gamma}{\theta}}(\mathbb{E}_\tau U_{\tau+1}^{1-\gamma})^{\frac{1}{\theta}-\frac{1}{1-\gamma}}(\mathbb{E}_\tau U_{\tau+1}^{1-\gamma})^{\frac{1}{1-\gamma}-1}\cdot \\
&\quad\mathbb{E}_\tau\left[U_{\tau+1}^{-\gamma}(1-\beta)U_{\tau+1}^{1-\frac{1-\gamma}{\theta}}u(c_{\tau+1},n_{\tau+1})^{\frac{1-\gamma}{\theta}-1}\frac{\partial u}{\partial c}(c_{\tau+1},n_{\tau+1})(d_{\tau+1}+v_{\tau+1})\right]
\end{aligned}$$

$$
\begin{aligned}
&= (1-\beta)U_\tau^{1-\frac{1-\gamma}{\theta}} u(c_\tau, n_\tau)^{\frac{1-\gamma}{\theta}-1} \frac{\partial u}{\partial c}(c_\tau, n_\tau) v_\tau \cdot \\
&\quad \left(\beta (\mathbb{E}_\tau U_{\tau+1}^{1-\gamma})^{\frac{1}{\theta}-1} \mathbb{E}_\tau \left[U_{\tau+1}^{(1-\gamma)(1-\frac{1}{\theta})} \left(\frac{u(c_{\tau+1}, n_{\tau+1})}{u(c_\tau, n_\tau)} \right)^{\frac{1-\gamma}{\theta}-1} \frac{\frac{\partial u}{\partial c}(c_{\tau+1}, n_{\tau+1})}{\frac{\partial u}{\partial c}(c_\tau, n_\tau)} \frac{(d_{\tau+1} + v_{\tau+1})}{v_\tau} \right] - 1 \right) \\
&= (1-\beta)U_\tau^{1-\frac{1-\gamma}{\theta}} u(c_\tau, n_\tau)^{\frac{1-\gamma}{\theta}-1} \frac{\partial u}{\partial c}(c_\tau, n_\tau) v_\tau \cdot \\
&\quad \mathbb{E}_\tau \left[\beta \left(\frac{U_{\tau+1}^{1-\gamma}}{\mathbb{E}_\tau U_{\tau+1}^{1-\gamma}} \right)^{1-\frac{1}{\theta}} \left(\frac{u(c_{\tau+1}, n_{\tau+1})}{u(c_\tau, n_\tau)} \right)^{\frac{1-\gamma}{\theta}-1} \frac{\frac{\partial u}{\partial c}(c_{\tau+1}, n_{\tau+1})}{\frac{\partial u}{\partial c}(c_\tau, n_\tau)} \frac{(d_{\tau+1} + v_{\tau+1})}{v_\tau} - 1 \right].
\end{aligned}
$$

Writing V_t for the value function as of period t to the dynamic optimization problem above and using $V_\tau = U_\tau$, if we evaluate U_τ at the optimal processes it follows that[39]

$$
\mathbb{E}_\tau \left[m_{\tau+1,\tau} \frac{d_{\tau+1} + v_{\tau+1}}{v_\tau} - 1 \right] = 0,
$$

where

$$
m_{\tau+1,\tau} := \beta \left(\frac{V_{\tau+1}^{1-\gamma}}{\mathbb{E}_\tau V_{\tau+1}^{1-\gamma}} \right)^{1-\frac{1}{\theta}} \left(\frac{u(c_{\tau+1}, n_{\tau+1})}{u(c_\tau, n_\tau)} \right)^{\frac{1-\gamma}{\theta}-1} \frac{\frac{\partial u}{\partial c}(c_{\tau+1}, n_{\tau+1})}{\frac{\partial u}{\partial c}(c_\tau, n_\tau)} \tag{A.1}
$$

is the household's stochastic discount factor.

Second, differentiating with respect to n_τ reveals the second optimality condition

$$
\begin{aligned}
& 0 = \frac{\partial W}{\partial u}(u(c_\tau, n_\tau), (\mathbb{E}_\tau U_{\tau+1}^{1-\gamma})^{\frac{1}{1-\gamma}}) \left(\frac{\partial u}{\partial c}(c_\tau, n_\tau) w_\tau + \frac{\partial u}{\partial n}(c_\tau, n_\tau) \right) \\
\Leftrightarrow \quad & \frac{\partial u}{\partial c}(c_\tau, n_\tau) w_\tau = -\frac{\partial u}{\partial n}(c_\tau, n_\tau).
\end{aligned} \tag{A.2}
$$

B Model M0: Baseline

For a general equilibrium in M0 it has to hold that for all $t \geq \tau$

$$
\begin{aligned}
& V_t - [(1-\beta)c_t^{\frac{1-\gamma}{\theta}} + \beta(\mathbb{E}_t[V_{t+1}^{1-\gamma}])^{\frac{1}{\theta}}]^{\frac{\theta}{1-\gamma}} = 0, \\
& n_t - 1 = 0, \\
& w_t \leq (1-\alpha)e^{z_t} n_t^{-\alpha} k_t^\alpha, \\
& q_t - \frac{1}{\Phi'\left(\frac{i_t}{k_t}\right)} = 0, \\
& q_t - \mathbb{E}_t \left[m_{t+1,t} \left(\alpha e^{z_{t+1}} n_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} - \frac{i_{t+1}}{k_{t+1}} + q_{t+1} \left(1 - \delta + \Phi\left(\frac{i_{t+1}}{k_{t+1}} \right) \right) \right) \right] = 0, \\
& k_{t+1} - (1-\delta)k_t - \Phi\left(\frac{i_t}{k_t} \right) k_t = 0, \\
& y_t - e^{z_t} k_t^\alpha n_t^{1-\alpha} = 0, \\
& y_t - c_t - i_t = 0,
\end{aligned}
$$

where the sequence $\{z_t\}$ follows (3.2.8) and $m_{t+1,t}$ is given by (3.4.1).

[39] Note also the positivity of lifetime utility, within period utility, marginal utility and the stock price.

C Model M1: No labor market frictions

For a general equilibrium in M1 it has to hold that for all $t \geq \tau$

$$V_t - [(1-\beta)u(c_t, n_t)^{\frac{1-\gamma}{\theta}} + \beta(\mathbb{E}_t[V_{t+1}^{1-\gamma}])^{\frac{1}{\theta}}]^{\frac{\theta}{1-\gamma}} = 0,$$

$$\frac{\partial u}{\partial c}(c_t, n_t)w_t = -\frac{\partial u}{\partial n}(c_t, n_t),$$

$$w_t - (1-\alpha)e^{z_t}n_t^{-\alpha}k_t^{\alpha} = 0,$$

$$q_t - \frac{1}{\Phi'\left(\frac{i_t}{k_t}\right)} = 0,$$

$$q_t - \mathbb{E}_t\left[m_{t+1,t}\left(\alpha e^{z_{t+1}}n_{t+1}^{1-\alpha}k_{t+1}^{\alpha-1} - \frac{i_{t+1}}{k_{t+1}} + q_{t+1}\left(1-\delta+\Phi\left(\frac{i_{t+1}}{k_{t+1}}\right)\right)\right)\right] = 0,$$

$$k_{t+1} - (1-\delta)k_t - \Phi\left(\frac{i_t}{k_t}\right)k_t = 0,$$

$$y_t - e^{z_t}k_t^{\alpha}n_t^{1-\alpha} = 0,$$

$$y_t - c_t - i_t = 0,$$

where the sequence $\{z_t\}$ follows (3.2.8) and the stochastic discount factor is determined by (3.2.7).

D Model M2: Sticky wages

For a general equilibrium in M2 it has to hold that for all $t \geq \tau$

$$V_t - [(1-\beta)u(c_t, n_t)^{\frac{1-\gamma}{\theta}} + \beta(\mathbb{E}_t[V_{t+1}^{1-\gamma}])^{\frac{1}{\theta}}]^{\frac{\theta}{1-\gamma}} = 0,$$

$$\frac{\partial u}{\partial c}(c_t, n_t)w_t^f + \frac{\partial u}{\partial n}(c_t, n_t) = 0,$$

$$w_t - w_{t-1}^{\mu}(w_t^f)^{1-\mu} = 0,$$

$$w_t - (1-\alpha)e^{z_t}n_t^{-\alpha}k_t^{\alpha} = 0,$$

$$q_t - \frac{1}{\Phi'\left(\frac{i_t}{k_t}\right)} = 0,$$

$$q_t - \mathbb{E}_t\left[m_{t+1,t}\left(\alpha e^{z_{t+1}}n_{t+1}^{1-\alpha}k_{t+1}^{\alpha-1} - \frac{i_{t+1}}{k_{t+1}} + q_{t+1}\left(1-\delta+\Phi\left(\frac{i_{t+1}}{k_{t+1}}\right)\right)\right)\right] = 0,$$

$$k_{t+1} - (1-\delta)k_t - \Phi\left(\frac{i_t}{k_t}\right)k_t = 0,$$

$$y_t - e^{z_t}k_t^{\alpha}n_t^{1-\alpha} = 0,$$

$$y_t - c_t - i_t = 0,$$

where the sequence $\{z_t\}$ follows (3.2.8) and $m_{t+1,t}$ is given by (3.2.7).

E Model M3: Predetermined labor supply

E.1 Model M3: Household

Equation (3.4.8) can be derived as follows. First, note that the first restriction again has to be fulfilled with equality in an optimum. Hence substituting for $c_t, t \geq \tau$, in the objective function and differentiating with respect to $n_{\tau+1}$ yields

$$
\begin{aligned}
0 &= \frac{\partial W}{\partial \mu}\frac{1}{1-\gamma}(\mathbb{E}_\tau V_{\tau+1}^{1-\gamma})^{\frac{1}{1-\gamma}-1}\mathbb{E}_\tau\left[(1-\gamma)V_{\tau+1}^{-\gamma}\frac{\partial V_{\tau+1}}{\partial n_{\tau+1}}\right] && \text{(E.3)}\\
&= \beta V_\tau^{1-\frac{1-\gamma}{\theta}}(\mathbb{E}_\tau V_{\tau+1}^{1-\gamma})^{\frac{1}{\theta}-\frac{1}{1-\gamma}}(\mathbb{E}_\tau V_{\tau+1}^{1-\gamma})^{\frac{1}{1-\gamma}-1}\mathbb{E}_\tau\left[V_{\tau+1}^{-\gamma}\frac{\partial V_{\tau+1}}{\partial n_{\tau+1}}\right] && \text{(E.4)}\\
&= V_\tau^{1-\frac{1-\gamma}{\theta}}\mathbb{E}_\tau\left[\beta(\mathbb{E}_\tau V_{\tau+1}^{1-\gamma})^{\frac{1}{\theta}-1}V_{\tau+1}^{-\gamma}\frac{\partial W}{\partial u}\left(\frac{\partial u}{\partial c}(c_{\tau+1},n_{\tau+1})w_{\tau+1}+\frac{\partial u}{\partial n}(c_{\tau+1},c_{\tau+1})\right)\right] && \text{(E.5)}\\
&= V_\tau^{1-\frac{1-\gamma}{\theta}}\mathbb{E}_\tau\big[\beta(\mathbb{E}_\tau V_{\tau+1}^{1-\gamma})^{\frac{1}{\theta}-1}V_{\tau+1}^{-\gamma}(1-\beta)V_{\tau+1}^{1-\frac{1-\gamma}{\theta}}u(c_{\tau+1},n_{\tau+1})^{\frac{1-\gamma}{\theta}-1}\cdot && \text{(E.6)}\\
&\qquad\cdot\left(\frac{\partial u}{\partial c}(c_{\tau+1},n_{\tau+1})w_{\tau+1}+\frac{\partial u}{\partial n}(c_{\tau+1},c_{\tau+1})\right)\Big] && \text{(E.7)}\\
&= (1-\beta)V_\tau^{1-\frac{1-\gamma}{\theta}}\mathbb{E}_\tau[\beta(\mathbb{E}_\tau V_{\tau+1}^{1-\gamma})^{\frac{1}{\theta}-1}V_{\tau+1}^{(1-\gamma)(1-\frac{1}{\theta})}u(c_{\tau+1},n_{\tau+1})^{\frac{1-\gamma}{\theta}-1}\cdot && \text{(E.8)}\\
&\qquad\cdot\left(\frac{\partial u}{\partial c}(c_{\tau+1},n_{\tau+1})w_{\tau+1}+\frac{\partial u}{\partial n}(c_{\tau+1},c_{\tau+1})\right)\Big] && \text{(E.9)}\\
&= (1-\beta)V_\tau^{1-\frac{1-\gamma}{\theta}}\mathbb{E}_\tau[\beta\left(\frac{V_{\tau+1}^{(1-\gamma)}}{\mathbb{E}_\tau V_{\tau+1}^{1-\gamma}}\right)^{1-\frac{1}{\theta}}u(c_{\tau+1},n_{\tau+1})^{\frac{1-\gamma}{\theta}-1}\cdot && \text{(E.10)}\\
&\qquad\cdot\left(\frac{\partial u}{\partial c}(c_{\tau+1},n_{\tau+1})w_{\tau+1}+\frac{\partial u}{\partial n}(c_{\tau+1},c_{\tau+1})\right)\Big] && \text{(E.11)}\\
&= (1-\beta)V_\tau^{1-\frac{1-\gamma}{\theta}}\mathbb{E}_\tau\Bigg[m_{\tau+1,\tau}u(c_\tau,n_\tau)^{\frac{1-\gamma}{\theta}-1}\frac{\frac{\partial u}{\partial c}(c_\tau,n_\tau)}{\frac{\partial u}{\partial c}(c_{\tau+1},n_{\tau+1})}\cdot && \text{(E.12)}\\
&\qquad\cdot\left(\frac{\partial u}{\partial c}(c_{\tau+1},n_{\tau+1})w_{\tau+1}+\frac{\partial u}{\partial n}(c_{\tau+1},c_{\tau+1})\right)\Bigg] && \text{(E.13)}\\
&= (1-\beta)V_\tau^{1-\frac{1-\gamma}{\theta}}u(c_\tau,n_\tau)^{\frac{1-\gamma}{\theta}-1}\frac{\partial u}{\partial c}(c_\tau,n_\tau)\mathbb{E}_\tau\left[m_{\tau+1,\tau}\left(w_{\tau+1}+\frac{\frac{\partial u}{\partial n}(c_{\tau+1},c_{\tau+1})}{\frac{\partial u}{\partial c}(c_{\tau+1},n_{\tau+1})}\right)\right], && \text{(E.14)}
\end{aligned}
$$

where in the next to last step we used equation (3.2.7) for the stochastic discount factor.

E.2 Model M3: Equilibrium conditions

For a general equilibrium in M3 it has to hold that for all $t \geq \tau$

$$V_t - [(1-\beta)u(c_t,n_t)^{\frac{1-\gamma}{\theta}} + \beta(\mathbb{E}_t[V_{t+1}^{1-\gamma}])^{\frac{1}{\theta}}]^{\frac{\theta}{1-\gamma}} = 0,$$

$$\mathbb{E}_t\left[m_{t+1,t}\left(w_{t+1}+\frac{\frac{\partial u}{\partial n}(c_{t+1},n_{t+1})}{\frac{\partial u}{\partial c}(c_{t+1},n_{t+1})}\right)\right] = 0,$$

$$w_t - (1-\alpha)e^{z_t}n_t^{-\alpha}k_t^{\alpha} = 0,$$

$$q_t - \frac{1}{\Phi'\left(\frac{i_t}{k_t}\right)} = 0,$$

$$
\begin{aligned}
&q_t - \mathbb{E}_t\left[m_{t+1,t}\left(\alpha e^{z_{t+1}} n_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} - \frac{i_{t+1}}{k_{t+1}} + q_{t+1}\left(1-\delta+\Phi\left(\frac{i_{t+1}}{k_{t+1}}\right)\right)\right)\right] = 0,\\
&k_{t+1} - (1-\delta)k_t - \Phi\left(\frac{i_t}{k_t}\right)k_t = 0,\\
&y_t - e^{z_t} k_t^{\alpha} n_t^{1-\alpha} = 0,\\
&y_t - c_t - i_t = 0,
\end{aligned}
$$

where the sequence $\{z_t\}$ follows (3.2.8) and the stochastic discount factor is given by (3.2.7).

F Model M4: Sectoral frictions in the allocation of labor

For a general equilibrium in M4 it has to hold that for all $t \geq \tau$

$$
\begin{aligned}
&V_t = [(1-\beta)((c_t - c_t^h)^{\nu}(1-n_t)^{1-\nu})^{\frac{1-\gamma}{\theta}} + \beta(\mathbb{E}_t[V_{t+1}^{1-\gamma}])^{\frac{1}{\theta}}]^{\frac{\theta}{1-\gamma}},\\
&\mathbb{E}_t\left[m_{t+1,t}\left(w_{t+1}^I + \frac{\frac{\partial u}{\partial n}(c_{t+1}, n_{t+1})}{\frac{\partial u}{\partial c}(c_{t+1}, n_{t+1})}\right)\right] = 0,\\
&\mathbb{E}_t\left[m_{t+1,t}\left(w_{t+1}^C + \frac{\frac{\partial u}{\partial n}(c_{t+1}, n_{t+1})}{\frac{\partial u}{\partial c}(c_{t+1}, n_{t+1})}\right)\right] = 0,\\
&n_t = n_t^I + n_t^C,\\
&w_t^C = (1-\alpha)e^{z_t}(n_t^C)^{-\alpha}(k_t^C)^{\alpha},\\
&q_t^C = \frac{p_t}{\Phi'\left(\frac{i_t^C}{k_t^C}\right)},\\
&q_t^C = \mathbb{E}_t\left[m_{t+1,t}\left(\alpha e^{z_{t+1}}(n_{t+1}^C)^{1-\alpha}(k_{t+1}^C)^{\alpha-1} - p_{t+1}\frac{i_{t+1}^C}{k_{t+1}^C} + q_{t+1}^C\left(1-\delta+\Phi\left(\frac{i_{t+1}^C}{k_{t+1}^C}\right)\right)\right)\right],\\
&k_{t+1}^C = (1-\delta)k_t^C + \Phi\left(\frac{i_t^C}{k_t^C}\right)k_t^C,\\
&c_t = e^{z_t}(k_t^C)^{\alpha}(n_t^C)^{1-\alpha},\\
&w_t^I = (1-\alpha)p_t e^{z_t}(n_t^I)^{-\alpha}(k_t^I)^{\alpha},\\
&q_t^I = \frac{p_t}{\Phi'\left(\frac{i_t^I}{k_t^I}\right)},\\
&q_t^I = \mathbb{E}_t\left[m_{t+1,t}\left(\alpha p_{t+1} e^{z_{t+1}}(n_{t+1}^I)^{1-\alpha}(k_{t+1}^I)^{\alpha-1} - p_{t+1}\frac{i_{t+1}^I}{k_{t+1}^I} + q_{t+1}^I\left(1-\delta+\Phi\left(\frac{i_{t+1}^I}{k_{t+1}^I}\right)\right)\right)\right],\\
&k_{t+1}^I = (1-\delta)k_t^I + \Phi\left(\frac{i_t^I}{k_t^I}\right)k_t^I,\\
&i_t = e^{z_t}(k_t^I)^{\alpha}(n_t^I)^{1-\alpha},\\
&i_t = i_t^C + i_t^I,
\end{aligned}
$$

where the sequence $\{z_t\}$ follows (3.2.8) and the stochastic discount factor is given by (3.2.7).

G Documentation of computation routines

In order to find the models' respective perturbations, we employed the `Maple`-`Matlab` toolbox introduced in Heiberger and Ruf (2014a). For the simulation and evaluation, we essentially added two procedures. On the one hand, `mom2` computes the second moments of our models' variables. In particular, after either loading or generating 300 pseudorandom iid N(0,1) shock series of length 80, it simulates the induced time paths of the state and control variables from their respective (second order) approximations. Second, depending on the user's choice, the procedure computes the second moments from the plain time paths ($\texttt{mode} = 0$) or particular manipulations thereof such as e.g. their natural log ($\texttt{mode} = 1$), their growth rates ($\texttt{mode} = 3$), or log differences ($\texttt{mode} = 5$). Thereby, if $\texttt{hp} = 1$, the HP-filter is applied by calling the respective `Matlab` routine.

On the other hand, `prem_mxx_lang` computes model Mx.x's simulated ex post return figures. Therefore, it first loads a pseudorandom iid N(0,1) shock series of length 500,000 and then simulates the induced time paths of all variables along their (second order) approximations.[40] Second, it uses the models' return formulae and accordingly computes ex post averages of the risk free rate, the return on equity and the equity premium.[41]

[40] Note that in both `mom2` and `prem_mxx_lang`, the path to the shock series has to be specified correctly.

[41] The employed version of the `Maple`/`Matlab` toolbox can be downloaded from `http://www.wiwi.uni-augsburg.de/vwl/maussner/lehrstuhl/heiberger_en.html` and `http://www.wiwi.uni-augsburg.de/vwl/maussner/lehrstuhl/ruf_en.html`. The programs were run on `Maple 17` and `Matlab 2013a`.

Chapter 4

Search Frictions in the Labor Market and Endogenous Economic Disasters

— *Christopher Heiberger*[*] —

4.1 Introduction

The benchmark business cycle model of (amongst others) Kydland and Prescott (1982), Long and Plosser (1983), Hansen (1985), Prescott (1986) and King et al. (1988) features a Walrasian labor market. Working hours are determined by the intersection of a downward sloping labor demand curve and an upward sloping labor supply curve. Hence, involuntary unemployment in the sense of Keynes (1936), p.15, where at a given real wage labor supply exceeds labor demand, cannot exist. Small departures from this framework are models with either sticky real wages as, e.g., in Blanchard and Galí (2007) and Uhlig (2007), or with nominal price and wage rigidities as, e.g., in the New Keynesian models of Christiano et al. (2005) and Smets and Wouters (2003). While in these models workers might want to work more hours at the given real wage, they still are unable to explain a well-documented feature of labor markets: the coexistence of unemployed workers and open vacancies.

The search and matching literature, developed by Diamond, Mortensen and Pissaridies and reviewed in Rogerson et al. (2005) and Rogerson and Shimer (2011), can account for this fact. It considers labor turnover as a resource consuming process. Neither the search of unemployed workers for new jobs nor the recruiting activities of firms are costless. Search and recruiting effort are viewed as inputs into a technology that matches unemployed workers to newly created jobs. Since this process is not frictionless, as is the working of the Walrasian auctioneer, there will be both unemployed workers and unfilled vacancies. In this framework the real wage distributes the value of a newly created job between employer and employee. The outcome depends on the assumptions about the bargaining process.

The papers by Merz (1995), Andolfatto (1996), and den Haan et al. (2000) embed this framework into real business cycle models. Walsh (2005), Trigari (2009), and Christiano et al. (2010), to name but a few, introduced search and matching into monetary models of the New Keynesian variety.

The ability of the benchmark search and matching model to account for the stylized facts of the labor market has been seriously challenged by Shimer (2005, 2010). In the data unemployment and vacancies are 10 times more volatile than in simulations of the model. Shimer attributes this failure to the flexibility of real wages. They absorb a large part of the shocks to labor productivity so that there is little room for fluctuations in unemployment and vacancies. Since then, many attempts have been made to solve the puzzle. Cardullo (2010) distinguishes three strands of the respective literature. The first group modifies the process of wage determination, the second

[*] I continue to use the pronoun ”we” in order to refer to author and reader in this paper.

group enriches the matching process, while the third group argues that the puzzle vanishes if the model is properly calibrated.

This paper is concerned with the third group. Hagedorn and Manovskii (2008) suggest a parametrization of the benchmark search and matching model, where the period value of unemployment is close to labor productivity and where the household's bargaining power is small. As a consequence, the worker's reservation wage is close to labor productivity and bargained wages are insensitive to productivity shocks. Accordingly, these shocks trigger large swings in the firm's effort to recruit new workers, and thus, in posted vacancies and unemployment. Kuehn et al. (2012) argue that this model is even able to explain periods of extremely high unemployment. This feature of the model has remained unobserved because researchers have employed inadequate solution methods.

The most frequently employed technique to solve dynamic general equilibrium (DSGE) models are perturbation methods. They provide linear or higher order Taylor polynomial approximations to the model's policy functions. Near the point of approximation these solutions are usually sufficiently accurate (see Heer and Maußner (2008), Aruoba et al. (2006)). Yet, if a model drifts away from this point, global methods are called for. Kuehn et al. (2012) therefore use a projection method (see Heer and Maußner (2009), Chapter 6) to study the Hagedorn and Manovskii (2008) version of the benchmark search and matching model.

The model's potential to endogenously explain rare but severe economic downturns raises hope that it may also contribute to resolve the equity premium puzzle. In their seminal work Mehra and Prescott (1985) argue that the neoclassical stochastic growth model is unable to explain the historically observed risk premium on stocks of about 6 percent. A promising line of research from Rietz (1988) over Barro (2006) to Gourio (2012, 2013) introduces disaster risk exogenously in otherwise standard DSGE models and is able to predict sizeable risk premia. The question, thus, is whether Hagedorn and Manovskii (2008)'s model with endogenous disasters is also able to replicate the empirically observed risk premia. Kuehn et al. (2012) answer this question in the affirmative.

It is therefore of particular interest to understand what drives the results of Kuehn et al. (2012, 2015) and whether their results change if those crucial assumptions are modified or extended. Accordingly, the present paper is structured as follows.

We start with the same model as in Kuehn et al. (2012) as our baseline model. First, in order to allow for an accurate analysis of the model, we describe in great detail different approximation methods for the model's policy functions within the framework of mean weighted residuals. We find that, in order to accurately display the model's dynamics in artificial simulations, a high amount of degrees of freedom for the underlying approximation function is necessary. For example, a comparison of the simulation results based on an approximation with a total of 35 Chebyshev polynomials to the results obtained with an approximation with over 11000 piecewise (bi)cubic polynomials yields deviations in the computed time paths of the employment rate by over 40 percent points. As found by Kuehn et al. (2012), the model gives rise to periods of extremely high unemployment rates. A second order perturbation solution fails to reproduce these results.

The model's dynamics yield a significantly more volatile labor market than reported by Shimer (2005). The standard deviation of the unemployment rate in the model is similar to the value found in the data while the standard deviation of vacancies is only moderately lower than empirically observed. Moreover, the periods of extremely high unemployment imply rare but huge declines in consumption. The model generates a sizeable equity premium if generalized recursive preferences of the class introduced by Epstein and Zin (1989) are assumed. However, for standard additive time separable expected utility preferences this is not the case. Different from models where disaster risk is introduced by an exogenous shock leading to a sudden and drastic decline in consumption, risk about next period's consumption is only limited in the labor market model. Economic downturns may evolve only gradually over a longer time span of increasing unemployment. Consequently, with only comparable small variability in next period's marginal utility of consumption, the model fails to predict a sizeable equity premium under additive time separable preferences. Yet, the risk of entering an economic downturn in the upcoming periods may differ substantially amongst different states of the economy in the next period. As a result, the lottery over the household's lifetime utility in the next period exhibits much more risk. Since this lottery enters the stochastic discount factor under Epstein-Zin preferences, the model can generate a significantly higher equity premium with this preference structure.

In the next step, we consider a corresponding social planner problem, where the employment rate does not fluctuate much around the deterministic steady state. By adding the distortions from social optimum, which are incorporated in the baseline model, separately to the model, we find the interaction of three mechanisms to play a particular important role for generating the huge unemployment rates in the model. First, a high and fixed period value of unemployment activities implies that the workers' reservation wage in the model is close to labor productivity and does not adjust to the state of the economy. Consequently, reflecting the results from Hagedorn and Manovskii (2008), negative shocks to labor productivity implicate large relative declines in the excess of labor productivity over wage costs and therefore in the present value of a worker to the firm. Second, high elasticities of the number of open vacancies posted and of realized job matches with respect to the present value of a worker to the firm further amplify the relative declines of these quantities, while separations result exogenously at a fixed rate. Finally, with unemployment quickly rising in the following periods, more unemployed members enter the matching process and the probability to fill a vacancy from the firm's perspective ceteris paribus increases. However, with increasing marginal utility of consumption, the firm also discounts expected future profits gained from a worker in the long run at a higher rate so that the present value of a worker from the firm's perspective will decline further—even if labor productivity remains the same. Since realized job matches become even more sensitive to changes in the present value of a worker, the last effect turns out to prevail and even less new jobs are created. Consequently, unemployment will rise further and the economy won't stabilize but is destined to enter a downward spiral until productivity sufficiently recovers.

A high and fixed period value of unemployment activities in the model turns out crucial for setting off the mechanism that leads the economy to plunge into disasters. Hagedorn and Manovskii (2008) motivate this high value for the standard search and matching model in that they argue that it should not only reflect unemployment insurance in this framework[1], but also other factors for which a worker demands to be compensated for by the wage and which are not included otherwise in the model, as e.g. the foregone value from leisure over work effort, the value of home production, etc. This point can also be made for our baseline model following Kuehn et al. (2012, 2015), since only consumption but not leisure enters the household's utility function and since home production is not included endogenously. The baseline model introduces the period value of unemployment activities in form of fixed compensation payments to unemployed members in the representative household's budget constraint, therefore establishing his reservation wage. However, this modeling has three consequences. First, the fact that the complete period value of unemployment, which determines the household's reservation wage, is summarized in a fixed parameter, also includes the implicit assumption that this value does not adjust over the business cycle indifferent to the fact how far unemployment eventually rises. Second, it is assumed that the compensation payments unemployed household members receive are redistributed from taxes in equilibrium. While this implies that unemployment cannot yield

[1] Different from our setting with a risk averse household, the standard search and matching model in Hagedorn and Manovskii (2008) considers the case of a risk neutral household maximizing his expected, discounted income stream.

an aggregate income effect which increases total consumption, the model on the other hand ignores the positive effects on the household's utility from unemployment—the same effects which are (implicitly) essential to keep his reservation wage high. Yet, return rates crucially depend on such (cross) effects on the household's marginal utility of consumption, or even on the direct effects on his lifetime utility in case of Epstein-Zin preferences. Third, as direct consequence of the fact that taxes equal total transfer payments to unemployed members, taxes rise to a multitude of the household's labor income during periods of very high unemployment in the model.

We analyze how the model's dynamics change once we modify these assumptions. For example, if we add leisure to the household's utility, the value of unemployment due to leisure arises endogenously in the model. If the marginal rate of substitution (MRS) between leisure and the consumption good is not constant, the value will then adjust over the business cycle. We find the huge declines in the employment rate to disappear from the model in consequence.[2] Even if we set the total period value of unemployment activities to the same high level in the deterministic steady state, the marginal rate of substitution between leisure and consumption will be declining during a recession with increasing unemployment and less consumption. Different from the baseline model, the workers' reservation wage reduces moderately and the mechanism leading the model's economy into a downward spiral does not take effect anymore. This result maintains, if we allow the employed members and the firm to decide about the working hours and assume unemployed members to actively decide about their job searching effort. Finally, a similar argument is also true for the value of unemployment from home production. Since no huge drops in the employment rate appear in the model's extensions, we find that a second order perturbation solution can already provide a much better approximation. Differences to a global solution method in simulations of the models turn out to be only small.

Without the occurrence of periods with extremely high unemployment in the model's extensions, the volatility of the labor market reduces significantly. Moreover, huge but severe declines in consumption, as observed in the baseline model, also disappear. If working hours are variable, employed household members will spend more time working once unemployment increases and consumption decreases. With home production in the model, the output of the home produced good will additionally increase during periods of higher unemployment. Consequently, declines in consumption are reduced even further. As a result, we find that the model extensions can no longer generate a sizeable equity premium even if Epstein-Zin preferences are assumed.

The remainder of the paper is organized as follows. Section 4.2 presents the baseline model. In section 4.3 we give a detailed description of the global solution methods, while the numerical results are provided in section 4.4. Section 4.5 gives an overview of the economic disasters, the equity premium and the second moments in the baseline model. We analyze the effects in the model's economy, which lead to economic disasters in section 4.6, before studying the extensions of the model in section 4.7. Finally, section 4.8 concludes.

4.2 Baseline Model

In the following section, we review the search and matching labor market model used by Kuehn et al. (2012), in which rare disasters in form of periods with extremely high unemployment arise endogenously. The economy consists of a representative household and a representative

[2] Kuehn et al. (2015) also consider an extension of the model with leisure in the utility in their appendix, but arrive at a different conclusion. Yet, their specification of the utility function implies a constant MRS between leisure and consumption. Note however that, as long as the MRS is not too low, the effects of huge declines in the employment rate on the household's lifetime utility should be significantly dampened compared to the baseline model in that case. It is therefore questionable if the model can still generate a sizeable equity premium even under this specification.

firm. Time is discrete and the planning horizon is infinite so that the periods are indexed by $t \in \mathbb{N}$. The household itself consists of a unit mass of members.

4.2.1 Search and Matching

In any time period t, each member of the household is either employed by the representative firm, taking part in the economy's production process, or unemployed, searching for a job. The mass of employed household members is denoted by N_t, while $U_t = 1 - N_t$ denotes the mass of unemployed members. Further, in each period t, the representative firm chooses an amount $V_t \geq 0$ of open vacancies to post. It is assumed that the outcome $M_t = M(U_t, V_t)$ of newly created jobs during the unmodeled process of all unemployed members searching for jobs and the representative firm recruiting is described by a matching function $M\colon [0,1] \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ with

$$M(U_t, V_t) = \begin{cases} \dfrac{U_t V_t}{\left(U_t^\tau + V_t^\tau\right)^{\frac{1}{\tau}}}, & \text{if } (U_t, V_t) \neq (0,0); \\ 0, & \text{if } (U_t, V_t) = (0,0); \end{cases} \tag{4.2.1}$$

where $\tau > 0$. [3] Further, employed workers depart from their jobs at the exogenously given rate $\omega > 0$. Hence, employment in the model evolves according to

$$N_{t+1} = (1-\omega) N_t + M_t. \tag{4.2.2}$$

The functional form of M guarantees that the amount of newly formed jobs can neither exceed the mass of unemployed household members nor the amount of open vacancies posted, i.e.

$$M_t \leq \min\{U_t, V_t\},$$

with equality if and only if one of the two input factors is zero. Together with the strictly positive separation rate ω in (4.2.2) this already implies $U_t > 0$ for all $t \in \mathbb{N}$ (except possibly for the first period, if the economy was set to start with all the household's members employed). We will therefore assume without any loss of generality $U_t > 0$ in the following. From the representative firm's perspective the average probability for an open vacancy to be filled is

$$\kappa_{f,t} := \frac{M_t}{V_t} = \frac{1}{\left(1 + \left(\frac{V_t}{U_t}\right)^\tau\right)^{\frac{1}{\tau}}}, \qquad \text{if } V_t > 0.$$

In case of $V_t = 0$ the probability is formally not defined, but we will set it to the limit $\kappa_{f,t} = 1$ for notational convenience. Analogously, the average probability for an unemployed member of the household to find a job is

$$\kappa_{w,t} := \frac{M_t}{U_t} = \begin{cases} \dfrac{1}{\left(1 + \left(\frac{V_t}{U_t}\right)^{-\tau}\right)^{\frac{1}{\tau}}}, & \text{if } V_t > 0; \\ 0, & \text{if } V_t = 0; \end{cases}$$

We define $\theta_t := \frac{V_t}{U_t}$ as the vacancy-unemployment ratio. In terms of θ_t, we can then equivalently write

$$\kappa_{f,t} = \frac{1}{\left(1 + \theta_t^\tau\right)^{\frac{1}{\tau}}},$$

[3] Note, that the function M is continuous in $(0,0)$.

and

$$\kappa_{w,t} = \begin{cases} \dfrac{1}{\left(1+\theta_t^{-\tau}\right)^{\frac{1}{\tau}}}, & \text{if } \theta_t > 0; \\ 0, & \text{if } \theta_t = 0. \end{cases}$$

Note that $\kappa_{f,t}$ is strictly monotonously decreasing, while $\kappa_{w,t}$ is strictly monotonously increasing in θ_t. We can interpret θ_t as a measure of the labor market tightness. The higher the value of θ_t, the more open vacancies per unemployed member and the lower becomes the probability of an open vacancy being filled from the representative firm's perspective, yet the higher becomes the probability of an unemployed member finding a job.

4.2.2 Representative Household

To avoid heterogeneity amongst the employed and unemployed members of the household, we assume that the members of the representative household pool their income before deciding about per capita consumption. We further assume the household's lifetime utility, derived from a probability distribution over a consumption stream $\{C_{t+s}\}_{s=0}^{\infty}$ as of period t, to be the expected sum of discounted CRRA within period utility, i.e.

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s\frac{C_{t+s}^{1-\eta}-1}{1-\eta}\right], \quad \beta\in(0,1), \eta>0, \eta\neq 1. \tag{4.2.3}$$

At any period t, the household's budget constraint is given by

$$C_t + v_t(S_{t+1}-S_t) \leq w_tN_t + b(1-N_t) + d_tS_t - T_t, \tag{4.2.4}$$

where w_t, b, d_t, S_t, v_t and T_t denote period t's wage, the (fixed) period value of unemployment activities (measured in consumption units)[4], dividend payment per share, the number of shares hold by the household, the share price and lump sum taxes, respectively. Employed members of the household depart from their jobs at the rate ω, while unemployed members find a job with probability $\kappa_{w,t}$. I.e. from the representative household's perspective the mass of employed members evolves according to

$$N_{t+1} = (1-\omega)N_t + \kappa_{w,t}U_t = (1-\omega)N_t + \kappa_{w,t}(1-N_t). \tag{4.2.5}$$

Households in the economy do not coordinate in any way in their job searching activities and have no influence on the average job finding rate on their own. Consequently, the representative household treats the probability $\kappa_{w,t}$ as strictly exogenous in all of his decisions. The representative household chooses consumption and stock holdings to maximize (4.2.3) under the series of budget constraints (4.2.4) and the dynamic (4.2.5) given initial values N_t and S_t. If $J^h(N_t,S_t)$ denotes the household's value function, then

$$\begin{aligned} J^h(N_t,S_t) = \quad & \max_{C_t,S_{t+1}} \quad \frac{C_t^{1-\eta}-1}{1-\eta} + \beta\mathbb{E}_t\left[J^h\left((1-\omega)N_t+\kappa_{w,t}(1-N_t),S_{t+1}\right)\right] \\ & \text{s.t.} \quad C_t \leq w_tN_t + d_tS_t - v_t(S_{t+1}-S_t) + b(1-N_t) - T_t, \\ & \text{given} \quad N_t, S_t. \end{aligned}$$

[4]As will become apparent immediately, the parameter b defines the workers' reservation wage in the model. It will therefore be understood to include all the value from unemployment a worker demands to be compensated for at minimum, i.e. unemployment benefits but also the foregone value from leisure, home production etc. Hence, we will call b the period value of unemployment activities instead of unemployment benefits.

Observing that the budget constraint has to be fulfilled with equality in an optimum, the first order condition of the maximization problem on the right hand side with respect to C_t is

$$\lambda_t = C_t^{-\eta}, \tag{4.2.6}$$

where λ_t denotes the Lagrange multiplier of the budget constraint. For a bounded solution with respect to S_{t+1} to exist it must hold that

$$\lambda_t v_t = \beta \mathbb{E}_t \left[\frac{\partial J^h}{\partial S}(N_{t+1}, S_{t+1}) \right].$$

Further, by the envelope theorem, we find the derivative of the value function with respect to S_t to be

$$\frac{\partial J^h}{\partial S}(N_t, S_t) = \lambda_t (d_t + v_t).$$

Plugging this expression for $(t+1)$ into the first order condition for S_{t+1} results in the following Euler condition for the share price

$$v_t = \mathbb{E}_t \left[\beta \frac{\lambda_{t+1}}{\lambda_t} (d_{t+1} + v_{t+1}) \right]. \tag{4.2.7}$$

Further, the value of employment over unemployment to the household can also be derived via the envelope theorem by

$$\frac{\partial J^h}{\partial N}(N_t, S_t) = \lambda_t (w_t - b) + \beta \mathbb{E}_t \left[\frac{\partial J^h}{\partial N}(N_{t+1}, S_{t+1})(1 - \omega - \kappa_{w,t}) \right].$$

Measuring this value in consumption units by defining $\xi_t^h := \frac{1}{\lambda_t} \frac{\partial J^h}{\partial N}(N_t, S_t)$, we arrive at the recursive formulation

$$\xi_t^h = w_t - b + (1 - \omega - \kappa_{w,t}) \mathbb{E}_t \left[\beta \frac{\lambda_{t+1}}{\lambda_t} \xi_{t+1}^h \right]. \tag{4.2.8}$$

Equation (4.2.8) is interpreted as follows. The household's instantaneous within period value from employment over unemployment is given by the excess of the wage over the period value of unemployment activities, i.e. $w_t - b$. Moreover, a worker keeps his job with probability $1-\omega$ in the next period, whereas he would have found a job with probability $\kappa_{w,t}$ if he was unemployed. Thus the total value of employment (over unemployment) from the household's perspective is the sum of $w_t - b$ and the continuation value of next period's expected discounted value of employment weighted with $(1 - \omega - \kappa_{w,t})$.

4.2.3 Representative Firm

At any period t, the representative firm produces the amount Y_t of the final good with the employed workers via a linear technology

$$Y_t = \exp(Z_t) N_t. \tag{4.2.9}$$

Labor productivity (LP) $\exp(Z_t)$ evolves randomly with the stationary and causal fist order autoregressive process Z_t, i.e.

$$Z_{t+1} = \rho Z_t + \epsilon_t, \quad \epsilon_t \sim \text{iidN}(0, \sigma_\epsilon^2), \quad |\rho| < 1.$$

In order to adjust the labor input for the next period, the firm can post vacancies $V_t \geq 0$, which are filled with probability $\kappa_{f,t}$. Posting an open vacancy is associated with fixed costs $c > 0$, which are then lost from total output. Job arrangements are only quit at the exogenous rate ω. I.e. from the firm's perspective employment evolves according to

$$N_{t+1} = (1-\omega)N_t + \kappa_{f,t}V_t. \tag{4.2.10}$$

The representative firm takes the probability $\kappa_{f,t}$ as exogenously given in its decisions reflecting the assumptions that firms in the economy are assumed to not coordinate their recruiting efforts and to be too small to have influence on the average probability on their own. The representative firm's profit π_t in period t amounts to revenue less labor and recruiting costs, i.e

$$\pi_t := Y_t - w_t N_t - cV_t. \tag{4.2.11}$$

Profits beyond dividend payments are used to buy back shares, while any eventual loss after dividend payments is covered by the issuance of new shares

$$-v_t(S_{t+1} - S_t) = \pi_t - d_t S_t. \tag{4.2.12}$$

The ex-dividend firm value at the end of the current period t, denoted by FV_t, is defined as the number of outstanding shares S_{t+1} times the current stock price v_t. Repeatedly using the Euler equation (4.2.7) for the share price together with

$$S_t d_t + S_t v_t = \pi_t + S_{t+1} v_t$$

from (4.2.12) implies

$$\begin{aligned} FV_t := S_{t+1} v_t &\overset{(4.2.7)}{=} \mathbb{E}_t\left[\beta \frac{\lambda_{t+1}}{\lambda_t}(S_{t+1}d_{t+1} + S_{t+1}v_{t+1})\right] = \mathbb{E}_t\left[\beta \frac{\lambda_{t+1}}{\lambda_t}(\pi_{t+1} + S_{t+2}v_{t+1})\right] = \ldots = \\ &= \mathbb{E}_t\left[\sum_{s=1}^{\infty} \beta^s \frac{\lambda_{t+s}}{\lambda_t}\pi_{t+s}\right], \end{aligned} \tag{4.2.13}$$

if we additionally assume that

$$\lim_{s\to\infty} \mathbb{E}_t\left[\beta^s \frac{\lambda_{t+s}}{\lambda_t} v_{t+s} S_{t+s+1}\right] = 0.$$

Thus, the firm value at the end of period t is the expected present value of its profits to come. Subject to the production technology (4.2.9) and the dynamics of employment (4.2.10), the representative firm decides about the number of open vacancies in order to maximize the beginning-of-period firm value FV_t^{bop}, which is defined as the firm's current period profits plus the ex-dividend firm value, i.e.

$$FV_t^{bop} := \pi_t + FV_t = \mathbb{E}_t\left[\sum_{s=0}^{\infty} \beta^s \frac{\lambda_{t+s}}{\lambda_t}\pi_{t+s}\right]. \tag{4.2.14}$$

Let $J^f(N_t)$ denote the value function of the firm's maximization problem. Then

$$\begin{aligned} J^f(N_t) = \quad & \max_{V_t} \quad \exp(Z_t)N_t - w_t N_t - cV_t + \mathbb{E}_t\left[\beta \frac{\lambda_{t+1}}{\lambda_t} J^f\left((1-\omega)N_t + \kappa_{f,t}V_t\right)\right] \\ & \text{s.t.} \quad V_t \geq 0, \\ & \text{given} \quad N_t. \end{aligned}$$

Note that we do not impose any particular dividend policy or strategy regarding the number of outstanding shares. After determining the optimal amount of open vacancies, any d_t, S_{t+1} and v_t which satisfy (4.2.12) and (4.2.7) are admissible. The Karush-Kuhn-Tucker (KKT) conditions for a bounded solution of the maximization problem on the right hand side are

$$c = \kappa_{f,t}\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{\partial J^f}{\partial N}(N_{t+1})\right] + \mu_t, \tag{4.2.15}$$

$$V_t \geq 0, \tag{4.2.16}$$

$$\mu_t \geq 0, \tag{4.2.17}$$

$$\mu_t V_t = 0, \tag{4.2.18}$$

with μ_t denoting the KKT multiplier of the non-negativity constraint. Next, we are introducing additional notation to keep the upcoming derivations short

$$\xi_t^f := \frac{\partial J^f}{\partial N}(N_t),$$

$$\hat{\xi}_t^f := \mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^f\right] = \mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{\partial J^f}{\partial N}(N_{t+1})\right].$$

Hence, ξ_t^f denotes this period's marginal value of a worker to the firm, while $\hat{\xi}_t^f$ denotes the present value of a worker in the next period, or equivalently of a filled position in the next period, to the firm. Equation (4.2.15) can then also be written as

$$c = \kappa_{f,t}\hat{\xi}_t^f + \mu_t. \tag{4.2.19}$$

If the non-negativity constraint on open vacancies is non-binding, i.e. $V_t > 0$ and $\mu_t = 0$, the present value of a filled position in the next period times the probability of a vacancy getting filled must equal the costs of posting an open vacancy in order for a bounded solution to exist. The representative firm is then indifferent between posting any amount of open vacancies. In case of the non-negativity constraint binding, $V_t = 0$ (and $\kappa_{f,t} = 1$ in equilibrium), the Lagrange multiplier μ_t measures the amount by which the costs of posting an open vacancy exceed the expected return from it. Applying again the envelope theorem, we can derive

$$\begin{aligned}\xi_t^f &= \frac{\partial J^f}{\partial N}(N_t) = \exp(Z_t) - w_t + \mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{\partial J^f}{\partial N}(N_{t+1})(1-\omega)\right] = \\ &= \exp(Z_t) - w_t + (1-\omega)\hat{\xi}_t^f.\end{aligned} \tag{4.2.20}$$

Hence, the value of an employed worker to the representative firm can be expressed recursively as the sum of the within period profit obtained from a worker, i.e. labor productivity less wage costs, $\exp(Z_t) - w_t$, and the present value of a worker in the next period weighted with the probability $1-\omega$ of the worker keeping his job. Moreover,

$$\hat{\xi}_t^f = \mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\left(\exp(Z_{t+1}) - w_{t+1} + (1-\omega)\hat{\xi}_{t+1}^f\right)\right]. \tag{4.2.21}$$

4.2.4 Wage Bargaining

The wage rate is the outcome of a bargaining process between the household and the firm over the total surplus from the employment of a household member. More specifically, we assume the wage to be the result of maximizing the Nash product of the value of employment to the household ξ_t^h and to the firm ξ_t^f, i.e.

$$\max_{w_t} \quad (\xi_t^h)^{\varphi}(\xi_t^f)^{1-\varphi}, \quad \varphi \in (0,1), \tag{4.2.22}$$

where $\varphi \in (0,1)$ can be interpreted as a measure of the household's bargaining power. First, using (4.2.8) and (4.2.20) the total surplus $\xi_t := \xi_t^h + \xi_t^f$ generated by the employment of a worker in the economy is

$$\xi_t = \exp(Z_t) - b - \kappa_{w,t}\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^h\right] + (1-\omega)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}\right]. \tag{4.2.23}$$

Since the total surplus is independent of this period's wage, maximizing (4.2.22) is equivalent to determining the shares ξ_t^h and ξ_t^f maximizing the geometric mean with weights φ and $1-\varphi$ given a fixed total surplus and then solving for the wage leading to this sharing rule. The sharing rule maximizing the Nash product given the total surplus is the outcome of

$$\begin{aligned} \max_{\xi_t^h,\xi_t^f} \quad & (\xi_t^h)^{\varphi}(\xi_t^f)^{1-\varphi} \\ \text{s.t.} \quad & \xi_t^h + \xi_t^f = \xi_t \\ \text{given} \quad & \xi_t. \end{aligned}$$

Plugging in the constraint $\xi_t^f = \xi_t - \xi_t^h$ and differentiating with respect to ξ_t^h yields the first order condition

$$0 = \varphi(\xi_t^h)^{\varphi-1}(\xi_t - \xi_t^h)^{1-\varphi} - (1-\varphi)(\xi_t^h)^{\varphi}(\xi_t - \xi_t^h)^{-\varphi}$$

or equivalently

$$\xi_t^h = \varphi\xi_t,$$

and therefore

$$\xi_t^f = (1-\varphi)\xi_t.$$

The assumed bargaining process thus implies the representative household and the representative firm to receive the shares φ and $1-\varphi$, respectively, of the total surplus from the employment of the worker. We can now use equations (4.2.8) and (4.2.20) to determine the wage resulting in this sharing rule. We have for all periods t

$$\begin{aligned} & \xi_t^h = \varphi\xi_t = \varphi(\xi_t^h + \xi_t^f) \\ \Leftrightarrow & (1-\varphi)\xi_t^h = \varphi\xi_t^f \\ \Leftrightarrow & (1-\varphi)\left(w_t - b + (1-\omega-\kappa_{w,t})\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{\varphi}{1-\varphi}\xi_{t+1}^f\right]\right) = \varphi\left(\exp(Z_t) - w_t + (1-\omega)\hat{\xi}_t^f\right) \\ \Leftrightarrow & (1-\varphi)(w_t - b) + \varphi(1-\omega-\kappa_{w,t})\hat{\xi}_t^f = \varphi\left(\exp(Z_t) - w_t + (1-\omega)\hat{\xi}_t^f\right) \\ \Leftrightarrow & w_t = \varphi\exp(Z_t) + (1-\varphi)b + \varphi\kappa_{w,t}\hat{\xi}_t^f. \end{aligned} \tag{4.2.24}$$

Now, since $\kappa_{f,t} > 0$, equation (4.2.19) yields $\hat{\xi}_t^f = \frac{c}{\kappa_{f,t}} - \frac{\mu_t}{\kappa_{f,t}}$, so that

$$\kappa_{w,t}\hat{\xi}_t^f = \frac{\kappa_{w,t}}{\kappa_{f,t}}c - \frac{\kappa_{w,t}\mu_t}{\kappa_{f,t}}.$$

But, if $V_t = 0$ then $\kappa_{w,t} = 0$ and if $V_t > 0$ then $\mu_t = 0$ so that $\kappa_{w,t}\mu_t = 0$ always holds and therefore

$$\kappa_{w,t}\hat{\xi}_t^f = \frac{\kappa_{w,t}}{\kappa_{f,t}}c = \frac{V_t}{U_t}c. \tag{4.2.25}$$

The wage equation can thus be written as

$$w_t = \varphi \exp(Z_t) + (1-\varphi)b + \varphi \frac{V_t}{1-N_t} c. \tag{4.2.26}$$

I.e. ignoring the last term, the wage is a weighted mean of the revenue gained by the representative firm through the employment arrangement—the marginal product of labor—weighted with the household's bargaining power and the household's fall back value if unemployed—the period value of unemployment activities—weighted with the firm's bargaining power. Moreover, if a position is filled, the representative firm saves the costs c for posting a vacancy. The last term constitutes a compensation to the worker for these saved costs where he is rewarded more the less likely an open vacancy can currently be filled.

Note that we did not additionally impose the restriction that labor productivity always exceeds the period value of unemployment b. This is the case for the following reason. Although it follows from (4.2.26) that in the event of $\exp(Z_t) < b$, the wage w_t will exceed labor productivity, it remains optimal for the firm to continue an already *existing* employment arrangement as long as ξ_t^f is non-negative and to even hire some *new* workers if $\hat{\xi}_t^f > c$. Expected discounted future profits realized from the employment of the worker exceed the current loss faced from it in that case. For analogous reasons it remains optimal for a worker to continue an employment arrangement and for an unemployed member to look for jobs, respectively, as long as ξ_t^h is non-negative, indifferent of the fact whether w_t might fall below b in some periods.[5] Hence, only ξ_t^f and ξ_t^h becoming negative would pose a problem in the model since it would then be optimal for the firm to shut down, all the workers to quit their jobs and unemployed members to stop searching for jobs. However, while labor productivity (very rarely) does fall below the period value of unemployment activities in simulations, the value ξ_t^f of a worker to the firm as well as the value ξ_t^h of employment over unemployment to the household remain significantly positive throughout.

4.2.5 Government sector

Unemployment activities of unemployed household members in the model do not increase the aggregate amount of the consumption good C_t available. Accordingly, it is assumed that the complete value bU_t the household receives from his unemployed members in the budget constraint (4.2.4) does not yield an aggregate income effect but is financed completely by transfer payments instead. Further, the government runs a balanced budget so that taxes equal the total period value from unemployment activities, i.e.

$$T_t = b(1-N_t). \tag{4.2.27}$$

Consequently, the model includes no aggregate value from unemployment.

4.2.6 General Equilibrium

In a general equilibrium the goods market has to be cleared, i.e. $Y_t - cV_t = C_t$, the share market clears, the representative household as well as the representative firm mutually act optimally while obeying their respective constraints, the wage equation holds, the state's budget is balanced, employment evolves according to the dynamic implied by the matching process, the probability of filling a vacancy from the firm's perspective equals the average matches per vacancy and the probability of an unemployed member finding a job from the household's

[5] Despite this fact we will refer to the household's fall back value in case of unemployment, given by the value of unemployment activities, as the households's reservation wage throughout this article.

perspective is the average matches per unemployed member. As already mentioned, the values for d_t, S_{t+1} and v_t are not uniquely determined in equilibrium without assuming the firm to follow a particular dividend policy. Note however, that it follows from the household's budget constraint (4.2.4) together with (4.2.12) and (4.2.27) that the goods market clearing condition is already necessary and sufficient for the stock market to clear too. Moreover, the return on equity can be computed from the other variables so that we are not interested in the share price, dividend payment per share and the number of outstanding shares per se. Hence, we can ignore (4.2.4), (4.2.7) and (4.2.12) without any loss.

To sum up, the equilibrium is determined by the following system of equations

$$U_t = 1 - N_t, \tag{4.2.28}$$

$$M_t = \frac{U_t V_t}{\left(U_t^\tau + V_t^\tau\right)^{\frac{1}{\tau}}}, \tag{4.2.29}$$

$$\kappa_{f,t} = \begin{cases} \dfrac{M_t}{V_t} & \text{if } V_t > 0, \\ 1 & \text{if } V_t = 0, \end{cases} \tag{4.2.30}$$

$$\kappa_{w,t} = \frac{M_t}{U_t}, \tag{4.2.31}$$

$$N_{t+1} = (1-\omega)N_t + M_t, \tag{4.2.32}$$

$$Y_t = \exp(Z_t)N_t, \tag{4.2.33}$$

$$Y_t = C_t + cV_t, \tag{4.2.34}$$

$$\lambda_t = C_t^{-\eta}, \tag{4.2.35}$$

$$w_t = \varphi \exp(Z_t) + (1-\varphi)b + \varphi \frac{V_t}{1-N_t} c, \tag{4.2.36}$$

$$c = \kappa_{f,t} \hat{\xi}_t^f + \mu_t, \tag{4.2.37}$$

$$\mu_t \geq 0, \tag{4.2.38}$$

$$V_t \geq 0, \tag{4.2.39}$$

$$\mu_t V_t = 0, \tag{4.2.40}$$

$$\hat{\xi}_t^f = \mathbb{E}_t\left[\beta \frac{\lambda_{t+1}}{\lambda_t}\left(\exp(Z_{t+1}) - w_{t+1} + (1-\omega)\hat{\xi}_{t+1}^f\right)\right]. \tag{4.2.41}$$

Next, we show how all period t variables (and N_{t+1}) can be derived analytically from the state variables (N_t, Z_t) and the present value $\hat{\xi}_t^f$ of a worker in the next period to the firm in such way that all equations except for the last one (4.2.41) are satisfied.

Let us first only consider the KKT conditions (4.2.37)-(4.2.40), where we plug in the definition of $\kappa_{f,t}$ from (4.2.30), of U_t from (4.2.28) and of M_t from (4.2.29) and already bear in mind, that $\mu_t = 0$ in the case of the non-negativity constraint on V_t not binding by (4.2.40). We then get

$$c = \begin{cases} \dfrac{1-N_t}{\left((1-N_t)^\tau + V_t^\tau\right)^{\frac{1}{\tau}}} \hat{\xi}_t^f & \text{if } V_t > 0, \\ \hat{\xi}_t^f + \mu_t & \text{if } V_t = 0, \end{cases} \tag{4.2.42}$$

$$\mu_t \geq 0, \tag{4.2.43}$$
$$V_t \geq 0, \tag{4.2.44}$$
$$\mu_t V_t = 0. \tag{4.2.45}$$

If we solve this reduced system for V_t and μ_t dependent on the state variables N_t and Z_t and the value of $\hat{\xi}^f_t$, all remaining variables can be derived directly from (4.2.28)-(4.2.36). For the reduced system, we can rewrite the first case in (4.2.42) equivalently as

$$\begin{aligned} c = \frac{1-N_t}{\left((1-N_t)^\tau + V_t^\tau\right)^{\frac{1}{\tau}}}\hat{\xi}^f_t \quad &\Leftrightarrow \quad c\left((1-N_t)^\tau + V_t^\tau\right)^{\frac{1}{\tau}} = (1-N_t)\hat{\xi}^f_t \\ &\Leftrightarrow \quad V_t^\tau = (1-N_t)^\tau\left(\left(\frac{\hat{\xi}^f_t}{c}\right)^\tau - 1\right) \\ &\Leftrightarrow \quad V_t = (1-N_t)\left(\left(\frac{\hat{\xi}^f_t}{c}\right)^\tau - 1\right)^{\frac{1}{\tau}}. \end{aligned}$$

But this can be satisfied with $V_t > 0$ iff $\hat{\xi}^f_t > c$. On the other hand, in the second case of $V_t = 0$, (4.2.42), yields

$$\mu_t = c - \hat{\xi}^f_t,$$

so that (4.2.43) is satisfied iff $\hat{\xi}^f_t \leq c$. Summing up, setting

$$V_t = \begin{cases} 0, & \text{if } \hat{\xi}^f_t \leq c, \\ (1-N_t)\left(\left(\frac{\hat{\xi}^f_t}{c}\right)^\tau - 1\right)^{\frac{1}{\tau}}, & \text{if } \hat{\xi}^f_t > c, \end{cases} \quad \text{and} \quad \mu_t = \begin{cases} c - \hat{\xi}^f_t, & \text{if } \hat{\xi}^f_t \leq c, \\ 0, & \text{if } \hat{\xi}^f_t > c, \end{cases}. \tag{4.2.46}$$

the KKT conditions are met. As noted above, $U_t, M_t, \kappa_{ft}, \kappa_{wt}, N_{t+1} Y_t, C_t, \lambda_t$ and w_t can then be computed successively from (4.2.28)-(4.2.36). By plugging all the derived expressions into (4.2.41), the system of equations defining the equilibrium thus reduces to a single equation in the single remaining variable $\hat{\xi}^f_t$ (next to the predetermined state variables). We will exploit this fact in the solution methods described in the next section.

4.3 Solution methods

We proceed to describe the framework for the methods used in order to find the solution to the stochastic dynamic system implicitly defined by the equilibrium conditions (4.2.28)-(4.2.41).

We already showed in the previous section how the system of equations defining the economy's equilibrium can be reduced analytically to the single Euler equation (4.2.41) in the unknown variable $\hat{\xi}^f_t$. The model's equilibrium is therefore fully characterized once we have solved the Euler equation for $\hat{\xi}^f_t$ and set the remaining variables as described. Since the policy function of $\hat{\xi}^f_t$ identifies the variable dependent on the state variables, the Euler equation constitutes an equation the policy function of $\hat{\xi}^f_t$ has to solve. The equation involves an integral since the expectation on the right hand side is taken with respect to a normal distributed random variable. The task of finding the policy function for $\hat{\xi}^f_t$ is thus equivalent to solving a functional integral equation. More specifically, the policy function of $\hat{\xi}^f_t$ is characterized as follows.

Let

$$g : [0,1] \times \mathbb{R} \to \mathbb{R}$$

denote the (time invariant) policy function for next period's expected discounted marginal value of a worker to the firm, i.e.

$$\hat{\xi}_t^f = g(N_t, Z_t).$$

Then, according to the Euler equation (4.2.41) and the characterization of the other variables described in the preceding subsection, g must solve the equation

$$R(g,x,z) := lhs(g,x,z) - rhs(g,x,z) = 0 \text{ for all } x \in [0,1], z \in \mathbb{R}, \tag{4.3.1}$$

with

$$lhs(g,x,z) := g(x,z) \tag{4.3.2}$$

and

$$rhs(g,x,z) := \mathbb{E}\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\eta}(\exp(Z_{t+1}) - w_{t+1} + (1-\omega)g(N_{t+1}, Z_{t+1}))\right], \tag{4.3.3}$$

where $C_t, N_{t+1}, Z_{t+1}, C_{t+1}$ and w_{t+1} in rhs are short for the expressions of the respective variables dependent on $x = N_t, z = Z_t, g(x,z) = \hat{\xi}_t^f$ and the innovation $\epsilon \sim N(0, \sigma_\epsilon^2)$, i.e.

$$V_t \quad := V(g,x,z) := \begin{cases} 0, & \text{if } g(x,z) \leq c \\ (1-x)\left(\left(\frac{g(x,z)}{c}\right)^\tau - 1\right)^{\frac{1}{\tau}}, & \text{if } g(x,z) > c \end{cases} \tag{4.3.4}$$

$$M_t \quad := M(g,x,z) = \frac{(1-x)V(g,x,z)}{((1-x)^\tau + V(g,x,z)^\tau)^{\frac{1}{\tau}}}, \tag{4.3.5}$$

$$C_t \quad := C(g,x,z) := \exp(z)x - cV(g,x,z), \tag{4.3.6}$$

$$w_t \quad := w(g,x,z) := \varphi \exp(z) + (1-\varphi)b + \varphi c \frac{V(g,x,z)}{1-x}, \tag{4.3.7}$$

$$N_{t+1} := N(g,x,z) := (1-\omega)x + M(g,x,z), \tag{4.3.8}$$

$$Z_{t+1} := \rho z + \epsilon, \quad \epsilon \sim N(0, \sigma_\epsilon^2), \tag{4.3.9}$$

$$C_{t+1} := C(g, N(g,x,z), \rho z + \epsilon), \tag{4.3.10}$$

$$w_{t+1} := w(g, N(g,x,z), \rho z + \epsilon). \tag{4.3.11}$$

4.3.1 Method of mean weighted residuals

We identify the policy function g of $\hat{\xi}_t^f$ as the solution to the above stated functional integral equation. Since there is no analytical solution available, we will rely on approximation methods applicable to solutions of functional equations as the one at hand. The approximation methods we used all fall within the framework of mean weighted residuals.[6] We will therefore first describe the common basic idea of the framework, before laying out the details for the specific methods used.

[6]See, for instance Judd (1998), Chapter 11, Heer and Maußner (2009), Chapter 6, Judd (1992) or McGrattan (1999).

Let $X \subset \mathbb{R}^n$ denote a compact subset of $\mathbb{R}^n$ and $\mathscr{F}(X,\mathbb{R})$ the vector space of real valued functions on X. Further, let

$$R\colon \mathscr{F}(X,\mathbb{R}) \times X \to \mathbb{R}$$

and consider the problem of finding a function $g \in \mathscr{F}(X,\mathbb{R})$ with

$$R(g,x) = 0, \text{ for all } x \in X. \tag{4.3.12}$$

In a first step, the complexity of the problem to find a function which adequately approximates the unknown solution to (4.3.12) is reduced by restricting the approximation to a specific parametric family $\{\hat{g}_a : X \to \mathbb{R} \mid a \in \mathbb{R}^K\}$. Instead of searching for an appropriate function, only K suitable parameter values must be determined. More specifically, we may choose a set of basis functions $\Phi_k, k = 1,\ldots,K$ and restrict ourselves to approximate the solution g within the subspace spanned by these functions, i.e. by a linear combination

$$\hat{g}_a(x) = \sum_{k=1}^{K} a_k \Phi_k(x), \quad x \in X, \quad a = (a_1,\ldots,a_K)' \in \mathbb{R}^K.$$

More generally, the approximation may also be non-linear in the parameters, i.e. the approximation $\hat{g}_a$ may be found from a class

$$\hat{g}_a(x) = \Phi(a,x), \quad x \in X, \quad a = (a_1,\ldots,a_K)' \in \mathbb{R}^K.$$

In the second step, a criteria which allows to compute the free parameters a_k in a way rendering the approximation fitting to the problem must be provided. The criteria for the choice of the parameter values is such that weighted averages of $R(\hat{g}_a,.)$ on X vanish. Note that $R(\hat{g}_a,.) = 0$ almost everywhere on X would be equivalent to

$$\langle R(\hat{g}_a,.),\Psi\rangle := \int_X R(\hat{g}_a,x)\Psi(x)dx = 0 \quad \text{for all } \Psi \in L^2(X), \tag{4.3.13}$$

where $L^2(X)$ is the space of square-integrable functions on X and $\langle .,.\rangle$ is the canonical inner product on $L^2(X)$. Since the solution in general does not lie within the class we restricted ourselves to, it may not be possible to choose the parameter vector $a \in \mathbb{R}^K$ in such way that (4.3.13) holds for every admissible function Ψ. But given another choice of K test functions $\Psi_i, i = 1,\ldots,K$, we can try to find parameters $a_k, k = 1,\ldots,K$, so that (4.3.13) holds for these test functions, i.e. so that the K equations

$$\langle R(\hat{g}_a,.),\Psi_i\rangle := \int_X R(\hat{g}_a,x)\Psi_i(x)dx = 0 \quad \text{for } i = 1,\ldots,K, \tag{4.3.14}$$

are met. In other words, we choose the parameters such way that $R(\hat{g}_a,.)$ is orthogonal to all of our test functions and hence, although $R(\hat{g}_a,.)$ may not equal zero, the orthogonal projection of $R(\hat{g}_a,.)$ on the subspace spanned by the test functions is zero. Equivalently, the conditions (4.3.14) state that the approximation error $R(\hat{g}_a,.)$ vanishes on average over X when weighted with the test functions Ψ_i. Since evaluating the conditions in (4.3.14) involves the computation of another integral, the inner product on L^2 is often endowed with an additional weight function $w\colon X \to \mathbb{R}_{>0}$ so that the parameters can be determined from a suitable quadrature formula to

$$\langle R(\hat{g}_a,.),\Psi_i\rangle_w := \int_X R(\hat{g}_a,x)\Psi_i(x)w(x)dx = 0, \quad i = 1,\ldots,K. \tag{4.3.15}$$

In the following, we will present four methods based on this framework that allow us to find an approximation to the policy function g of $\hat{\xi}_t^f$ characterized as the solution to the functional equation stated in (4.3.1). Essentially, these four methods differ regarding two aspects, namely the choice of basis functions the approximation is restricted by and the numerical computation of the expectation operator appearing in the functional equation through (4.3.3). Two of the methods will rely on linear combinations of Chebyshev polynomials, while the other two are based on piecewise (bi)cubic polynomials. In the first case, each of the parameters a_k affects the shape of the approximation function $\hat{g}_a$ on the whole domain and $\hat{g}_a$ is called a spectral function, whereas in the second case, a change in one of the parameters affects the shape of $\hat{g}_a$ only on a limited range and $\hat{g}_a$ is called a finite element function. For both choices of basis functions the expectation appearing in the functional equation will be numerically computed in two ways. The first approach uses a Gauss-Hermite quadrature. In the second approach the AR(1) process governing log labor productivity will be replaced by a finite state space Markov chain so that the expectation reduces to a sum.

4.3.2 Spectral Methods

We present the details for the approximation by a spectral method. More specifically, as already mentioned, we choose Chebyshev polynomials as basis functions throughout this approach.

4.3.2.1 A Chebyshev-Galerkin Method

We begin the description for the case where the expectation in the functional equation is computed numerically by a Gauss-Hermite quadrature. First, in order to evaluate the functional $rhs(g,x,z)$ defined by (4.3.3), one needs to compute an integral since

$$rhs(g,x,z) = \int_{\mathbb{R}} f(x,z,\epsilon)\frac{1}{\sqrt{2\pi\sigma_\epsilon^2}}e^{-\frac{\epsilon^2}{2\sigma_\epsilon^2}}d\epsilon,$$

where

$$\begin{aligned} f(x,z,\epsilon) :=& \beta\left(\frac{C(g,N(g,x,z),\rho z+\epsilon)}{C(g,x,z)}\right)^{-\eta}\cdot \\ &\cdot(\exp(\rho z+\epsilon)-w(g,N(g,x,z),\rho z+\epsilon)+(1-\omega)g(N(g,x,z),\rho z+\epsilon)).\end{aligned}$$

By substitution with $\tilde{\epsilon} := \frac{\epsilon}{\sqrt{2\sigma_\epsilon^2}}$, we get

$$\int_{\mathbb{R}} f(x,z,\epsilon)\frac{1}{\sqrt{2\pi\sigma_\epsilon^2}}e^{-\frac{\epsilon^2}{2\sigma_\epsilon^2}}d\epsilon = \int_{\mathbb{R}} f\left(x,z,\sqrt{2\sigma_\epsilon^2}\tilde{\epsilon}\right)\frac{1}{\sqrt{\pi}}e^{-\tilde{\epsilon}^2}d\tilde{\epsilon},$$

so that we can approximate the integral by Gauss-Hermite quadrature with n nodes through

$$\int_{\mathbb{R}} f\left(x,z,\sqrt{2\sigma_\epsilon^2}\tilde{\epsilon}\right)\frac{1}{\sqrt{\pi}}e^{-\tilde{\epsilon}^2}d\tilde{\epsilon} \approx \frac{1}{\sqrt{\pi}}\sum_{i=1}^{n} w_i f\left(x,z,\sqrt{2\sigma_\epsilon^2}\epsilon_i\right),$$

where the weights w_i and nodes ϵ_i can be computed as described in Golub and Welsch (1969). Hence,

$$\begin{aligned} rhs(g,x,z) \approx rhs_{GH}(g,x,z) :=& \frac{1}{\sqrt{\pi}}\sum_{i=1}^{n} w_i\beta\left(\frac{C(g,N(g,x,z),\rho z+\sqrt{2}\sigma_\epsilon\epsilon_i)}{C(g,x,z)}\right)^{-\eta}\cdot \\ \cdot\left(\exp(\rho z+\sqrt{2}\sigma_\epsilon\epsilon_i)-w(g,N(g,x,z),\rho z+\sqrt{2}\sigma_\epsilon\epsilon_i)+(1-\omega)g(N(g,x,z),\rho z+\sqrt{2}\sigma_\epsilon\epsilon_i)\right).& \end{aligned} \tag{4.3.16}$$

We will now apply the framework of mean weighted residuals to find an approximation to the solution of

$$R_{GH}(g,x,z) := lhs(g,x,z) - rhs_{GH}(g,x,z) = 0, \quad \text{for all } x \in [0,1], z \in \mathbb{R}. \tag{4.3.17}$$

instead of the original functional equation (4.3.1).

Let $[\underline{x}, \bar{x}] \subset [0,1]$, $[\underline{z}, \bar{z}] \subset \mathbb{R}$ and $X := [\underline{x}, \bar{x}] \times [\underline{z}, \bar{z}]$. Further, let $T_i : [-1,1] \to \mathbb{R}$ denote the i-th Chebyshev polynomial (of the first kind). We choose degrees $d_x, d_z \in \mathbb{N}$ and set the $K = d_x d_z$ basis functions to

$$\Phi_{k,l}(x,z) := T_{k-1}(\psi_x(x)) T_{l-1}(\psi_z(z)), \quad (x,z) \in [\underline{x}, \bar{x}] \times [\underline{z}, \bar{z}]$$

for $k = 1, \dots, d_x, l = 1, \dots, d_z$ [7], where ψ_x and ψ_z are bijections between the domains of x and z, respectively, and the domain $[-1,1]$ of Chebyshev polynomials

$$\psi_x : [\underline{x}, \bar{x}] \to [-1,1], \quad x \mapsto 2\frac{x - \underline{x}}{\bar{x} - \underline{x}} - 1;$$

$$\psi_z : [\underline{z}, \bar{z}] \to [-1,1], \quad z \mapsto 2\frac{z - \underline{z}}{\bar{z} - \underline{z}} - 1.$$

After having fixed the basis functions, we restrict the approximation $\hat{g}_a$ over the domain X to the class of linear combinations

$$\hat{g}_a(x,z) = \sum_{k=1}^{d_x} \sum_{l=1}^{d_z} a_{k,l} T_{k-1}(\psi_x(x)) T_{l-1}(\psi_z(z)), \quad (x,z) \in X = [\underline{x}, \bar{x}] \times [\underline{z}, \bar{z}], \tag{4.3.18}$$

with $a = \left(a_{k,l}\right)_{\substack{k=1,\dots,d_x,\\ l=1,\dots,d_z}} \in \mathbb{R}^{d_x \times d_z}$ denoting the free parameters in matrix form.

In order to determine parameter values $a_{k,l}$ that render the approximation $\hat{g}_a$ fitting, we will next choose test functions $\Psi_{k,l}$ as well as a weight function w and derive the parameters from the conditions in (4.3.15) where R is replaced by R_{GH}. Yet, solving the equations in (4.3.15) for the free parameters will in general require to evaluate $R_{GH}(\hat{g}_a, x, z)$ and therefore $rhs_{GH}(\hat{g}_a, x, z)$ over the whole domain $X = [\underline{x}, \bar{x}] \times [\underline{z}, \bar{z}]$. When we take a look at (4.3.16), one further problem regarding this becomes evident. Namely, it might be the case that for some values of this period's state variables $(x,z) \in X$ and realizations ϵ_i of the shock, next period's state $(N(\hat{g}_a, x, z), \rho z + \sqrt{2}\sigma_\epsilon \epsilon_i)$ appearing in $rhs_{GH}(\hat{g}_a, x, z)$ might not be included in the domain X anymore, i.e. it may occur that $(N(\hat{g}_a, x, z), \rho z + \sqrt{2}\sigma_\epsilon \epsilon_i) \notin X$. Yet, if this is the case, $\hat{g}_a\left(N(\hat{g}_a, x, z), \rho z + \sqrt{2}\sigma_\epsilon \epsilon_i\right)$ from (4.3.18) and consequently $rhs_{GH}(\hat{g}_a, x, z)$ from (4.3.16) is not well defined up to this point. We therefore continue the approximation $\hat{g}_a$ on $[0,1] \times \mathbb{R}$ as follows by extrapolation. We choose a discrete two-dimensional grid $\Gamma \subset [\underline{x}, \bar{x}] \times [\underline{z}, \bar{z}]$ and calculate the corresponding function values of $\hat{g}_a$ at the grid points. If $(x,z) \notin [\underline{x}, \bar{x}] \times [\underline{z}, \bar{z}]$, we use a 2-dimensional extrapolation method, e.g. linear or by bicubic splines, to compute $\hat{g}_a(x,z)$ from the values at the grid points. This guarantees that $rhs_{GH}(\hat{g}_a, x, z)$ is well-defined for all $(x,z) \in X = [\underline{x}, \bar{x}] \times [\underline{z}, \bar{z}]$ and parameters $a \in \mathbb{R}^{d_x \times d_z}$. [8]

[7] We choose double indexation for the basis functions $\Phi_{k,l}$ to allow easier notation.

[8] Another approach to deal with the eventual problem of next period's state variables not necessarily lying in the chosen domain for the approximation, i.e. $\left(N(\hat{g}_a, x, z), \rho z + \sqrt{2}\sigma_\epsilon \epsilon_i\right) \notin X = [\underline{x}, \bar{x}] \times [\underline{z}, \bar{z}]$, would be to formally define the approximation on a broader domain than initially of interest. That is, we may additionally choose $[\underline{x}_\psi, \bar{x}_\psi] \times [\underline{z}_\psi, \bar{z}_\psi] \supset [\underline{x}, \bar{x}] \times [\underline{z}, \bar{z}]$ and set the bijections ψ_x and ψ_z to

$$\psi_x : [\underline{x}_\psi, \bar{x}_\psi] \to [-1,1],\ x \mapsto 2\frac{x - \underline{x}_\psi}{\bar{x}_\psi - \underline{x}_\psi} - 1 \quad \text{and} \quad \psi_z : [\underline{z}_\psi, \bar{z}_\psi] \to [-1,1],\ z \mapsto 2\frac{z - \underline{z}_\psi}{\bar{z}_\psi - \underline{z}_\psi} - 1.$$

The approximation $\hat{g}_a$ in (4.3.18) is then defined on the larger domain, which should be chosen broad enough to guarantee $\left(N(\hat{g}_a, x, z), \rho z + \sqrt{2}\sigma_\epsilon \epsilon_i\right) \in [\underline{x}_\psi, \bar{x}_\psi] \times [\underline{z}_\psi, \bar{z}_\psi]$ for all $(x,z) \in X = [\underline{x}, \bar{x}] \times [\underline{z}, \bar{z}]$ and $i = 1, \dots, n$. We will add some remarks for the reasons we preferred to use extrapolation over this method at the appropriate places in the following.

We employ the Galerkin method and set the test functions in (4.3.15) equal to the basis functions, i.e.

$$\Psi_{i,j}(x,z) := T_{i-1}(\psi_x(x))T_{j-1}(\psi_z(z)), \quad \text{for all } i=1,\dots,d_x,\ j=1,\dots,d_z.$$

Moreover, the weight function is chosen as

$$w\colon [\underline{x},\bar{x}]\times[\underline{z},\bar{z}]\to\mathbb{R},(x,z)\mapsto \frac{1}{\sqrt{1-\psi_x(x)^2}}\frac{1}{\sqrt{1-\psi_z(z)^2}}.$$

With these choices (4.3.15) becomes

$$\int_{\underline{x}}^{\bar{x}}\int_{\underline{z}}^{\bar{z}} R_{GH}(\hat{g}_a,x,z)T_{i-1}(\psi_x(x))T_{j-1}(\psi_z(z))\frac{1}{\sqrt{1-\psi_x(x)^2}}\frac{1}{\sqrt{1-\psi_z(z)^2}}dzdx=0,$$

for all $i=1,\dots,d_x, j=1,\dots,d_z$. By substituting $r=\psi_x(x)$ and $q=\psi_z(z)$ we get

$$\frac{(\bar{x}-\underline{x})(\bar{z}-\underline{z})}{4}\int_{-1}^{1}\int_{-1}^{1} R_{GH}\left(\hat{g}_a,\psi_x^{-1}(r),\psi_z^{-1}(q)\right)T_{i-1}(r)T_{j-1}(q)$$
$$\frac{1}{\sqrt{1-r^2}}\frac{1}{\sqrt{1-s^2}}dqdr=0.$$

We approximate both integrals by Chebyshev-Gauss quadrature with $m_x\geq d_x$ and $m_z\geq d_z$ nodes, respectively. Therefore, let $r_1,\dots,r_{m_x}$ and $q_1,\dots,q_{m_z}$ denote the roots of the m_x-th and m_z-th Chebyshev polynomial. Further, let $x_t:=\psi_x^{-1}(r_t), t=1,\dots,m_x$, and $z_s:=\psi_z^{-1}(q_s), s=1,\dots,m_z$, denote the corresponding values in $[\underline{x},\bar{x}]$ and $[\underline{z},\bar{z}]$. The Chebyshev-Gauss quadrature then yields

$$\int_{-1}^{1}\int_{-1}^{1} R_{GH}\left(\hat{g}_a,\psi_x^{-1}(r),\psi_z^{-1}(q)\right)T_{i-1}(r)T_{j-1}(q)\frac{1}{\sqrt{1-r^2}}\frac{1}{\sqrt{1-s^2}}dqdr\approx$$
$$\approx\frac{\pi}{m_x}\frac{\pi}{m_z}\sum_{t=1}^{m_x}\sum_{s=1}^{m_z}R_{GH}(\hat{g}_a,x_t,z_s)T_{i-1}(r_t)T_{j-1}(q_s).$$ [9]

Hence, the unknown parameters $a_{k,l}$ are determined as the solution to the system of equations

$$\sum_{t=1}^{m_x}\sum_{s=1}^{m_z}R_{GH}(\hat{g}_a,x_t,z_s)T_{i-1}(r_t)T_{j-1}(q_s)=0, \tag{4.3.19}$$

or equivalently

$$\sum_{t=1}^{m_x}\sum_{s=1}^{m_z}lhs(\hat{g}_a,x_t,z_s)T_{i-1}(r_t)T_{j-1}(q_s)=\sum_{t=1}^{m_x}\sum_{s=1}^{m_z}rhs_{GH}(\hat{g}_a,x_t,z_s)T_{i-1}(r_t)T_{j-1}(q_s) \tag{4.3.20}$$

for $i=1,\dots,d_x, j=1,\dots,d_z$.

[9] If we follow the approach mentioned in footnote 8 and choose a sufficiently broader domain for the bijections ψ_x and ψ_z in order to guarantee that $rhs_{GH}(\hat{g}_a,x_t,z_s)$ is always well-defined, the formula for the Chebyshev-Gauss quadrature would read the same. Yet, with the different definition of ψ_x and ψ_z, the definitions of x_t and z_s must also be adjusted accordingly. The integration for determining the parameter values is still carried out only on the smaller region $[\underline{x},\bar{x}]\times[\underline{z},\bar{z}]$ so that the substitution of variables would yield $x_t=\tilde{\psi}_x^{-1}(r_t)$ and $z_s=\tilde{\psi}_z^{-1}(q_s)$, where $\tilde{\psi}_x$ and $\tilde{\psi}_z$ denote the bijections on the original, smaller domains

$$\tilde{\psi}_x\colon[\underline{x},\bar{x}]\to[-1,1],\ x\mapsto 2\frac{x-\underline{x}}{\bar{x}-\underline{x}}-1 \quad\text{and}\quad \tilde{\psi}_z\colon[\underline{z},\bar{z}]\to[-1,1],\ z\mapsto 2\frac{z-\underline{z}}{\bar{z}-\underline{z}}-1.$$

To write the system of equations more compactly in matrix form, we define for any $d, m \in \mathbb{N}, d \le m$, the matrices

$$T_{m,d} := \begin{pmatrix} T_0(v_1) & \dots & T_{d-1}(v_1) \\ T_0(v_2) & \dots & T_{d-1}(v_2) \\ \vdots & \vdots & \vdots \\ T_0(v_m) & \dots & T_{d-1}(v_m) \end{pmatrix}, \tag{4.3.21}$$

where $v_1, \dots, v_m$ are the roots of the m-th Chebyshev polynomial. Further let

$$lhs(\hat{g}_a) := \begin{pmatrix} lhs(\hat{g}_a, x_1, z_1) & \dots & lhs(\hat{g}_a, x_1, z_{m_z}) \\ lhs(\hat{g}_a, x_2, z_1) & \dots & lhs(\hat{g}, x_2, z_{m_z}) \\ \vdots & \ddots & \vdots \\ lhs(\hat{g}_a, x_{m_x}, z_1) & \dots & lhs(\hat{g}_a, x_{m_x}, z_{m_z}) \end{pmatrix}$$

and analogously

$$rhs_{GH}(\hat{g}_a) := \begin{pmatrix} rhs_{GH}(\hat{g}_a, x_1, z_1) & \dots & rhs_{GH}(\hat{g}_a, x_1, z_{m_z}) \\ rhs_{GH}(\hat{g}_a, x_2, z_1) & \dots & rhs_{GH}(\hat{g}, x_2, z_{m_z}) \\ \vdots & \ddots & \vdots \\ rhs_{GH}(\hat{g}_a, x_{m_x}, z_1) & \dots & rhs_{GH}(\hat{g}_a, x_{m_x}, z_{m_z}) \end{pmatrix}$$

Then the (i, j)-th elements of $T'_{m_x,d_x} lhs(\hat{g}_a) T_{m_z,d_z}$ and $T'_{m_x,d_x} rhs_{GH}(\hat{g}_a) T_{m_z,d_z}$ equal the left hand and right hand side of (4.3.20) so that the system of equations determining the parameter values can be written equivalently in matrix form as

$$T'_{m_x,d_x} lhs(\hat{g}_a) T_{m_z,d_z} = T'_{m_x,d_z} rhs_{GH}(\hat{g}_a) T_{m_z,d_z}. \tag{4.3.22}$$

Moreover,

$$\begin{aligned} lhs(\hat{g}_a) &= \begin{pmatrix} \hat{g}_a(x_1, z_1) & \dots & \hat{g}_a(x_1, z_{m_z}) \\ \hat{g}_a(x_2, z_1) & \dots & \hat{g}_a(x_2, z_{m_z}) \\ \vdots & \ddots & \vdots \\ \hat{g}_a(x_{m_x}, z_1) & \dots & \hat{g}_a(x_{m_x}, z_{m_z}) \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=1}^{d_x} \sum_{l=1}^{d_z} a_{k,l} T_{k-1}(r_1) T_{l-1}(q_1) & \dots & \sum_{k=1}^{d_x} \sum_{l=1}^{d_z} a_{k,l} T_{k-1}(r_1) T_{l-1}(q_{m_z}) \\ \sum_{k=1}^{d_x} \sum_{l=1}^{d_z} a_{k,l} T_{k-1}(r_2) T_{l-1}(q_1) & \dots & \sum_{k=1}^{d_x} \sum_{l=1}^{d_z} a_{k,l} T_{k-1}(r_2) T_{l-1}(q_{m_z}) \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{d_x} \sum_{l=1}^{d_z} a_{k,l} T_{k-1}(r_{m_x}) T_{l-1}(q_{m_z}) & \dots & \sum_{k=1}^{d_x} \sum_{l=1}^{d_z} a_{k,l} T_{k-1}(r_{m_x}) T_{l-1}(q_{m_z}) \end{pmatrix} \\ &= T_{m_x,d_x} a T'_{m_z,d_z}, \end{aligned}$$

[10]

so that equation (4.3.22) can also be written as

$$T'_{m_x,d_x} T_{m_x,d_x} a T'_{m_z,d_z} T_{m_z,d_z} = T'_{m_x,d_x} rhs_{GH}(\hat{g}_a) T_{m_z,d_z}. \tag{4.3.23}$$

[10] It would not be possible to write $lhs(\hat{g}_a)$ this way, if we use the approach of a broader domain for the bijections ψ_x and ψ_z. The different definitions of x_t and z_s pointed out in footnote 9 in this case, would imply the (t, s)-th component of $lhs(\hat{g}_a)$ to read

$$\hat{g}_a(x_t, z_s) = \sum_{k=1}^{d_x} \sum_{l=1}^{d_z} a_{k,l} T_{k-1}(\psi_x(\tilde{\psi}_x^{-1}(r_t))) T_{l-1}(\psi_z(\tilde{\psi}_z^{-1}(q_s))) \neq \sum_{k=1}^{d_x} \sum_{l=1}^{d_z} a_{k,l} T_{k-1}(r_t) T_{l-1}(q_s).$$

The following derivations depending on this particular form of $lhs(\hat{g}_a)$ would then no longer be valid. This is the reason we chose extrapolation to compute $\hat{g}_a$ for states not included in the original domain instead the approach of defining a wider domain for the bijections.

Now note that for any $m, d \in \mathbb{N}, m \geq d$, we have

$$T'_{m,d}T_{m,d} = \begin{pmatrix} \sum_{k=1}^m T_0(v_k)T_0(v_k) & \dots & \sum_{k=1}^m T_0(v_k)T_{d-1}(v_k)) \\ \sum_{k=1}^m T_1(v_k)T_0(v_k) & \dots & \sum_{k=1}^m T_1(v_k)T_{d-1}(v_k) \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^m T_{d-1}(v_k)T_0(v_k) & \dots & \sum_{k=1}^m T_{d-1}(v_k)T_{d-1}(v_k) \end{pmatrix}$$

$$= \begin{pmatrix} m & 0 & \dots & 0 \\ 0 & \frac{m}{2} & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{m}{2} \end{pmatrix} =: D_m.$$

Finally, (4.3.23)is equivalent to

$$a = D_{m_x}^{-1} T'_{m_x,d_x} rhs_{GH}(\hat{g}_a) T_{m_z,d_z} D_{m_z}^{-1} \tag{4.3.24}$$

or

$$a_{1,1} = \frac{1}{m_x m_z} \sum_{t=1}^{m_x} \sum_{s=1}^{m_z} rhs_{GH}(\hat{g}_a, x_t, z_s) T_0(r_t) T_0(q_s);$$

$$a_{1,l} = \frac{2}{m_x m_z} \sum_{t=1}^{m_x} \sum_{s=1}^{m_z} rhs_{GH}(\hat{g}_a, x_t, z_s) T_0(r_t) T_{l-1}(q_s), \qquad 2 \leq l \leq d_z;$$

$$a_{k,1} = \frac{2}{m_x m_z} \sum_{t=1}^{m_x} \sum_{s=1}^{m_z} rhs_{GH}(\hat{g}_a, x_t, z_s) T_{k-1}(r_t) T_0(q_s), \qquad 2 \leq k \leq d_x;$$

$$a_{k,l} = \frac{4}{m_x m_z} \sum_{t=1}^{m_x} \sum_{s=1}^{m_z} rhs_{GH}(\hat{g}_a, x_t, z_s) T_{k-1}(r_t) T_{l-1}(q_s), \quad 2 \leq k \leq d_x, 2 \leq l \leq d_z;$$

where rhs_{GH} is defined by equation (4.3.16). [11]

4.3.2.2 A Chebyshev-Galerkin Method with discretized Labor Productivity (LP)

In the last subsection we replaced the expectation operator in (4.3.3) by a quadrature rule and approximated the solution to the emerging functional equation (4.3.17) on some rectangle $X = [\underline{x}, \bar{x}] \times [\underline{z}, \bar{z}]$. In this subsection, we will describe another approach, where we replace the AR(1) process Z_t by a finite state space Markov chain. The problem of finding an approximation to g on a (connected) two-dimensional domain then reduces to finding an approximation on a subspace of $[0,1]$ for each of the finite states of the Markov chain. Also, if Z_t follows a finite state space Markov chain, the expectation in (4.3.3) reduces to a sum via the transition probabilities.

First, we choose a number of states n and employ the algorithm proposed by Rouwenhorst (1995) to find a Markov chain with finite state space $\mathscr{Z} := \{z_1, \dots, z_n\}$ and a transition matrix $P = (p_{j,s})_{\substack{j=1,\dots,n, \\ s=1,\dots,n}}$, which approximates the AR(1) process governing log LP. Instead of (4.3.1) we then solve for $g : [0,1] \times \mathscr{Z} \to \mathbb{R}$ satisfying

$$R_{RO}(g, x, z_j) := lhs(g, x, z_j) - rhs_{RO}(g, x, z_j) = 0 \quad \text{for all } x \in [0,1],\ j = 1, \dots, n, \tag{4.3.25}$$

where rhs_{RO} is defined analogously to (4.3.3) only with the expression for Z_{t+1} in (4.3.9) replaced by the transition process

$$Pr\left[Z_{t+1} = z_s | z = z_j\right] = p_{js} \quad \text{for } j, s = 1, \dots, n.$$

[11] Note that this is no closed form solution for the coefficients $a_{k,l}$, since they also appear on the right hand side of the equation.

Hence, we can write

$$rhs_{RO}(g,x,z_j) := \sum_{s=1}^{n} p_{js}\beta\left(\frac{C(g,N(g,x,z_j),z_s)}{C(g,x,z_j)}\right)^{-\eta} \cdot \\ \cdot\left(\exp(z_s) - w(g,N(g,x,z_j),z_s) + (1-\omega)g(N(g,x,z_j),z_s)\right). \tag{4.3.26}$$

We choose a subset $[\underline{x},\bar{x}] \subset [0,1]$ and try to approximate $g(.,z_j)$ on that domain for each $j \in \{1,\ldots,n\}$ with Chebyshev polynomials as basis functions. In order to eventually improve the accuracy of the approximation for states z_j, where the approximation turns out to be more difficult, we will allow the number of basis functions used to differ along the states z_j. Formally, we choose a vector of degrees $d := (d_1,\ldots,d_n)'$ and set the $K = d_1 + \ldots + d_n$ basis functions to

$$\Phi_{k,l}(x,z) := T_{k-1}(\psi_x(x))\mathbb{1}_{z_l}(z), \quad (x,z) \in [\underline{x},\bar{x}] \times \mathscr{Z}$$

for $l = 1,\ldots,n$ and $k = 1,\ldots,d_l$. Thereby, ψ_x is the bijection defined in the preceding subsection and $\mathbb{1}_{z_l}$ denotes the indicator function

$$\mathbb{1}_{z_l} : \mathscr{Z} \to \{0,1\}, \quad \mathbb{1}_{z_l}(z) := \begin{cases} 1, & \text{if } z = z_l \\ 0, & \text{if } z \neq z_l. \end{cases}$$

The approximation is then chosen from the set of linear combinations

$$\hat{g}_a(x,z_j) = \sum_{l=1}^{n}\sum_{k=1}^{d_l} a_{k,l}T_{k-1}(\psi_x(x))\mathbb{1}_{z_l}(z_j) = \sum_{k=1}^{d_j} a_{k,j}T_{k-1}(\psi_x(x)), \quad x \in [\underline{x},\bar{x}],\ j = 1,\ldots,n,$$

We use the notation $a_{\cdot j} := (a_{1j},\ldots,a_{d_j j})' \in \mathbb{R}^{d_j}$, $j = 1,\ldots,n$, and $a := (a'_{\cdot 1},\ldots,a'_{\cdot n})' \in \mathbb{R}^{d_1+\ldots+d_n}$ for the vector of all parameters.

Fitting values for the free parameters in the approximation will again be determined from the conditions in (4.3.15) where R is now replaced by R_{RO}. Since we use a finite state space Markov chain, it is now guaranteed by definition that next period's state z_s for log LP appearing in (4.3.26) lies in $\mathscr{Z}$. But it is still possible that some parameter values a, some $x \in [\underline{x},\bar{x}]$ and $j \in \{1,\ldots,n\}$ yield $N(\hat{g}_a,x,z_j) \in [0,1] \setminus [\underline{x},\bar{x}]$ so that $rhs_{RO}(\hat{g}_a,x,z_j)$ as in (4.3.26) is not yet well-defined. We therefore proceed analogously as in the previous subsection. We choose a one-dimensional grid $\Gamma \subset [\underline{x},\bar{x}]$ and determine the corresponding function values of $\hat{g}_a(.,z_j)$ at the grid points for each $j \in \{1,\ldots,n\}$. If $x \notin [\underline{x},\bar{x}]$, we then use a one-dimensional extrapolation method, e.g. linear or by cubic splines, to compute $\hat{g}_a(x,z_j)$ from the values at the grid points.

We employ the Galerkin method with test functions equal to the basis functions, i.e. $\Psi_{i,j}(x,z) := T_{i-1}(\psi_x(x))\mathbb{1}_{z_j}(z)$, and the weight function is set to $w(x,z) := \frac{1}{\sqrt{1-\psi_x(x)^2}}$. Condition (4.3.15) determining the parameter values hence becomes

$$\int_{\underline{x}}^{\bar{x}}\int_{\mathscr{Z}} R_{RO}(\hat{g}_a,x,z)T_{i-1}(\psi_x(x))\mathbb{1}_{z_j}(z)\frac{1}{\sqrt{1-\psi_x(x)^2}}dzdx = 0$$

or equivalently

$$\int_{\underline{x}}^{\bar{x}} R_{RO}(\hat{g}_a,x,z_j)T_{i-1}(\psi_x(x))\frac{1}{\sqrt{1-\psi_x(x)^2}}dx = 0$$

for all $j = 1, \dots, n$ and $i = 1, \dots, d_j$. By substituting $r = \psi_x(x)$, we get

$$\frac{\bar{x} - \underline{x}}{2} \int_{-1}^{1} R_{RO}\left(\hat{g}_a, \psi_x^{-1}(r), z_j\right) T_{i-1}(r) \frac{1}{\sqrt{1-r^2}} dr = 0.$$

We evaluate the integral numerically by Chebyshev-Gauss quadrature with $m_j \geq d_j$ nodes for each $j \in \{1, \dots, n\}$. Hence, let $r_{1j}, \dots, r_{m_j j}$ denote the roots of the m_j-th Chebyshev polynomial and let $x_{tj} := \psi_x^{-1}(r_{tj}), t = 1, \dots, m_j$, denote the corresponding values in $[\underline{x}, \bar{x}]$. Then the Chebyshev-Gauss quadrature yields

$$\int_{-1}^{1} R_{RO}\left(\hat{g}_a, \psi_x^{-1}(r), z_j\right) T_{i-1}(r) \frac{1}{\sqrt{1-r^2}} dr \approx \frac{\pi}{m_j} \sum_{t=1}^{m_j} R_{RO}\left(\hat{g}_a, x_{tj}, z_j\right) T_{i-1}(r_{tj}). \tag{4.3.27}$$

Therefore, the parameter values in the approximation are determined as the solution to the system of equations

$$\sum_{t=1}^{m_j} R_{RO}(\hat{g}_a, x_{tj}, z_j) T_{i-1}(r_{tj}) = 0,$$

or equivalently

$$\sum_{t=1}^{m_j} lhs(\hat{g}_a, x_{tj}, z_j) T_{i-1}(r_{tj}) = \sum_{t=1}^{m_j} rhs_{RO}(\hat{g}_a, x_{tj}, z_j) T_{i-1}(r_{tj})$$

for $j = 1, \dots, n, i = 1, \dots, d_j$.

Again, we can write the system of equations more compactly in matrix notation. For any $j \in \{1, \dots, n\}$, let $T_{m_j d_j}$ be as defined in (4.3.21), let

$$lhs(\hat{g}_a, z_j) := \begin{pmatrix} lhs(\hat{g}_a, x_{1j}, z_j) \\ lhs(\hat{g}_a, x_{2j}, z_j) \\ \vdots \\ lhs(\hat{g}_a, x_{m_j j}, z_j) \end{pmatrix}$$

and let $rhs_{RO}(\hat{g}_a, z_j)$ be defined analogously. Then the system of equations for a can be written as

$$T'_{m_j d_j} lhs(\hat{g}_a, z_j) = T'_{m_j d_j} rhs_{RO}(\hat{g}_a, z_j), \quad \text{for all } j = 1, \dots, n.$$

Note that now $lhs(\hat{g}_a, z_j) = T_{m_j d_j} a_{\cdot j}$ so that finally

$$a_{\cdot j} = D_{m_j}^{-1} T'_{m_j d_j} rhs_{RO}(\hat{g}_a, z_j), \quad \text{for all } j = 1, \dots, n, \tag{4.3.28}$$

or element by element for $j = 1, \dots, n$

$$a_{1,j} = \frac{1}{m_j} \sum_{t=1}^{m_j} rhs_{RO}(\hat{g}_a, x_t, z_j) T_0(r_t),$$
$$a_{k,j} = \frac{2}{m_j} \sum_{t=1}^{m_j} rhs_{RO}(\hat{g}_a, x_t, z_j) T_{k-1}(r_t), \quad 2 \leq k \leq d_j.$$

4.3.3 Finite Element Methods

Next, we present approximation methods with piecewise (bi)cubic polynomials as basis functions. Different from the Chebyshev polynomials, they are non-zero only over a limited range so that a change in one parameter consequently changes the shape of the approximation function only locally. We distinguish the same two cases for handling the expectations operator in (4.3.3) as in the previous subsection.

4.3.3.1 A Cubic Spline Collocation Method

The first approach again makes use of the Gauss-Hermite quadrature (4.3.16) and solves the resulting functional equation (4.3.17).

Let $[\underline{x},\bar{x}] \subset [0,1]$, $[\underline{z},\bar{z}] \subset \mathbb{R}$ and $X := [\underline{x},\bar{x}]\times[\underline{z},\bar{z}]$. Further, we choose some two dimensional rectilinear grid

$$\Delta := \{(x_i, z_j) \mid i = 1,\ldots,d_x, j = 1,\ldots d_z\},$$

where $\underline{x} =: x_1 < x_2 < \ldots < x_{d_x} := \bar{x}$ and $\underline{z} =: z_1 < z_2 < \ldots < z_{d_z} := \bar{z}$. Our goal is to find piecewise bicubic polynomials approximating g on the grid cells. For any set of parameters in matrix notation $a = \left(a_{k,l}\right)_{\substack{k=1,\ldots,d_x,\\ l=1,\ldots,d_z}} \in \mathbb{R}^{d_x\times d_z}$ let

$$S_\Delta(a,.)\colon [\underline{x},\bar{x}] \times [\underline{z},\bar{z}] \to \mathbb{R}$$

denote the bicubic C^2-spline through the points $(x_i, z_j, a_{ij}), i = 1,\ldots,d_x, j = 1,\ldots,d_z$.[12] We then set

$$\hat{g}_a(x,z) := S_\Delta(a,x,z), \quad \text{for all } (x,z) \in [\underline{x},\bar{x}] \times [\underline{z},\bar{z}].^{13}$$

Again, in order for rhs_{GH} as in (4.3.16) to be well-defined on $[\underline{x},\bar{x}]\times[\underline{z},\bar{z}]$, we use extrapolation methods to determine $\hat{g}_a(N(\hat{g}_a,x,z),\rho z+\epsilon_i)$ whenever $(N(\hat{g}_a,x,z),\rho+z\epsilon_i) \in ([0,1]\times\mathbb{R}) \setminus ([\underline{x},\bar{x}] \times [\underline{z},\bar{z}])$.

We employ a collocation method in this approach. For every $(i,j) \in \{1,\ldots,d_x\} \times \{1,\ldots,d_z\}$ the test functions are chosen equal to the dirac delta functions at the respective grid point (x_i,z_j) in Δ, i.e.

$$\Psi_{ij}(x,z) := \delta_{x_i}(x)\delta_{z_j}(z), \quad (x,z) \in [\underline{x},\bar{x}] \times [\underline{z},\bar{z}],$$

The equations in (4.3.15) determining the fitting parameter values thus reduce to

$$R_{GH}(\hat{g}_a, x_i, z_j) = 0 \quad \text{for all } i = 1,\ldots,d_x, j = 1,\ldots,d_z,$$

or equivalently stated

$$lhs(\hat{g}_a, x_i, z_j) = rhs_{GH}(\hat{g}_a, x_i, z_j) \quad \text{for all } i = 1,\ldots,d_x, j = 1,\ldots,d_z.$$

Since $lhs(\hat{g}_a, x_i, z_j) = \hat{g}_a(x_i,z_j) = S_\Delta(a,x_i,z_j) = a_{ij}$, we can also write

$$a_{ij} = rhs_{GH}(\hat{g}_a, x_i, z_j) \quad \text{for all } i = 1,\ldots,d_x, j = 1,\ldots,d_z. \tag{4.3.29}$$

[12] We choose `Matlab`'s not-a-knot end condition.

[13] Note that the approximation $\hat{g}_a$ can be written as the sum of piecewise bicubic polynomials, non-zero only on one gird cell, but $\hat{g}_a$ is non-linear in the parameters in this case.

4.3.3.2 A Cubic Spline Collocation Method with discretized Labor Productivity (LP)

The second approach using finite element functions relies on the discretization of the process governing LP by the method of Rouwenhorst (1995) and subsequently solving the functional equation (4.3.25).

Let $\mathscr{Z} = \{z_1, \dots, z_n\}$ denote the state space and P the transition matrix and let $[\underline{x}, \bar{x}] \subset [0,1]$. For each of the states z_j, $j = 1, \dots, n$, we choose a grid

$$\Delta_j := \{\underline{x} =: x_{1j} < x_{2j} < \ldots < x_{d_j j} := \bar{x}\},$$

of size $d_j \in \mathbb{N}$. Our goal is then to find piecewise cubic polynomials approximating $g(., z_j)$ for each $j = 1, \dots, n$. We allow the amount of grid points to differ along the different states of log LP in order to improve the approximation by adding nodes at states z_j where the approximation turns out to be more difficult. For any $j \in \{1, \dots, n\}$ and $a_{\cdot j} := (a_{1j}, \dots, a_{d_j j})' \in \mathbb{R}^{d_j}$, we denote the cubic C^2-spline through the points $(x_{ij}, a_{ij}), i = 1, \dots, d_j$, by $S_{\Delta_j}(a_{\cdot j}, .)$.[14] We then set

$$\hat{g}_a(x, z_j) := S_{\Delta_j}(a_{\cdot j}, x), \quad \text{for all } x \in [\underline{x}, \bar{x}], j = 1 \dots, n,$$

where $a := (a'_{\cdot 1}, \dots, a'_{\cdot n})' \in \mathbb{R}^{d_1 + \ldots + d_n}$ is the vector of all parameters. Again, in order for rhs_{RO} to be well-defined on $[\underline{x}, \bar{x}] \times \mathscr{Z}$, we use extrapolation methods to compute $\hat{g}_a(N(\hat{g}_a, x, z_j), z_l)$ whenever $N(\hat{g}_a, x, z_j) \in [0,1] \setminus [\underline{x}, \bar{x}]$. Employing a collocation method with the dirac delta functions at the respective grid points as test functions, i.e.

$$\Psi_{ij}(x, z) := \delta_{x_{ij}}(x) \mathbb{1}_{z_j}(z), \quad (x, z) \in [\underline{x}, \bar{x}] \times \mathscr{Z},$$

the system of equations (4.3.15) pinning down the fitting parameter values becomes

$$R_{RO}(\hat{g}_a, x_{ij}, z_j) = 0 \quad \text{for all } j = 1, \dots, n, i = 1, \dots, d_j,$$

or equivalently

$$lhs(\hat{g}_a, x_{ij}, z_j) = rhs_{RO}(\hat{g}_a, x_{ij}, z_j) \quad \text{for all } j = 1, \dots, n, i = 1, \dots, d_j.$$

Since $lhs(\hat{g}_a, x_{ij}, z_j) = \hat{g}_a(x_{ij}, z_j) = S_{\Delta_j}(a_{\cdot j}, x_{ij}) = a_{ij}$, we can also write

$$a_{ij} = rhs_{RO}(\hat{g}_a, x_{ij}, z_j) \quad \text{for all } j = 1, \dots, n, i = 1, \dots, d_j. \tag{4.3.30}$$

We discussed four different approaches for computing an approximation to the policy function g of the present value $\hat{\xi}^f_t$ of a worker (or equivalently of a filled position) in the next period to the representative firm. All of the approaches ultimately result in solving a system of non-linear equations in order to determine fitting values for the free parameters in the approximation function. We will now proceed to present the numerical results.

4.4 Numerical analysis

In this section, we will describe the calibration of the model's parameters, characterize the deterministic steady state and lay out the details for the numerical computation of the approximations introduced in the preceding section. We will further discuss eventual differences between the approximations as well as goodness of fit dependent on the used methods. Moreover, we will study how eventual differences in the approximations manifest in simulations of the model's equilibrium outcomes. Last, we will also compute a second order perturbation solution and compare it to the solutions based on the mean weighted residuals framework.

[14] We again choose `Matlab`'s not-a-knot end condition.

4.4.1 Calibration

The model is calibrated at a monthly frequency in the same way as in Kuehn et al. (2012). They set the discount factor to $\beta = 0.99^{1/3}$, the autocorrelation of log LP to $\rho = 0.95^{1/3}$ and the standard deviation of the innovations to log LP to $\sigma_\epsilon = 0.0077$ in line with Gertler and Trigari (2009). Following den Haan et al. (2000) the probability for an unemployed member to find a job within the period is set to $\kappa_w = 0.45$, while the probability that an open vacancy can be filled is set to $\kappa_f = 0.71$ in steady state. Further, the authors choose a steady state value of $U = 0.1$ for unemployment within the range of 7% in Gertler and Trigari (2009) and 12% in Krause and Lubik (2007).

The authors partly follow the calibration proposed in Hagedorn and Manovskii (2008) for the period value of unemployment activities and the workers' bargaining weight. They set $b = 0.85$, somewhat lower than the value of 0.955 in Hagedorn and Manovskii (2008), while the workers' bargaining weight $\varphi = 0.052$ is the same. Hagedorn and Manovskii (2008) measure the costs of posting vacancies and the cyclicality of wages in the data and find that in order to match these quantities in the model a relative low bargaining weight of the household but a high period value of unemployment activities is necessary. In comparison, Shimer (2005), who identifies the period value from unemployment with unemployment benefits, sets $b = 0.4$ significantly lower while the household's bargaining weight is noticeably higher at 0.72. Hagedorn and Manovskii (2008) motivate the in comparison high period value of unemployment activities as follows. In the standard search and matching model[15] the period value of unemployment activities defines a worker's fall back value if unemployed and consequently his reservation wage. It should therefore not only reflect unemployment insurance but also other factors for which a worker demands compensation by the wage and which are not included otherwise in the model, as e.g. foregone value from leisure over work effort or from home production etc. In consequence b should be close to labor productivity. The same argument can be made for the present model, where only consumption but not leisure enters the household's utility and where home production is not endogenously determined. However, following this logic and summarizing the complete period value of unemployment activities into the fixed parameter b also implies that the value remains fixed over the business cycle. I.e. we implicitly assume that the period value the household associates to unemployment due to leisure or home production may not change indifferent to the fact how far unemployment, leisure and home production may rise and consumption may fall. Comparing to the case where these factors are endogenously determined by adding them to the utility function, a constant marginal rate of substitution (MRS) between leisure and the consumption good as well as between home production and the consumption good is assumed. Moreover, since the whole value bU_t the household receives from unemployed members is redistributed from taxes (see (4.2.27)) the model at the same time ignores the positive effects on the household's utility from increasing unemployment. We analyze how the results in the model change once we renounce on these assumptions in the extensions of the model in section 4.7.[16]

Kuehn et al. (2012) use a recursive utility function of the class introduced by Epstein and Zin

[15] Hagedorn and Manovskii (2008) consider the (standard) case of a risk neutral household maximizing his expected, discounted income stream in this framework.

[16] We want to note that Kuehn et al. (2015) also mention that the high value of b symbolizes some unmodeled wage rigidities from their point of view. However, once leisure is added to the utility function in a way that renders the MRS non-constant, introducing a wage rigidity in this form and size is not possible anymore. First, the fixed period value of unemployment activities in b can no longer be set as high since the household's reservation wage—determined by the sum of b and the MRS—would then, at least for common utility specifications regarding leisure, already exceed labor productivity in steady state. Second, the wage has to become more flexible since the MRS in the household's reservation wage adjusts. Such wage rigidities hence have to be introduced differently into the model and call for alternative motivation in a framework with a non-constant MRS between leisure and consumption.

(1989) and follow Bansal and Yaron (2004) in setting the parameter of relative risk aversion to 10 and the elasticity of intertemporal substitution to 1.5. We distinguish two cases. For the baseline case presented in section 4.2, where the household's lifetime utility is defined as the expected sum of discounted CRRA within period utility, we set $\eta = 2$ but also check results regarding the equity premium for $\eta = 10$. Further, as in Kuehn et al. (2012), we also considered the case of Epstein-Zin preferences with the parameter values from Bansal and Yaron (2004). However, we restrict the discussion of the model in the following to the case of standard preferences with $\eta = 2$ and only add the results for $\eta = 10$ and for Epstein-Zin preferences where necessary.

The numerical values for the elasticity τ in the matching function, the separation rate ω and the costs of posting an open vacancy c are derived in such a way that the chosen steady state values of U, κ_f and κ_w arise. An overview for all the model's parameters is presented in table 4.1.

Table 4.1: Calibration I

Variable	Value	Description
U	0.1	unemployment rate
κ_f	0.71	probability of filling an open vacancy
κ_w	0.45	probability of finding a job
Parameter	**Value**	**Description**
β	$0.99^{1/3}$	discount factor
η	2	relative risk aversion
φ	0.052	workers' bargaining weight
b	0.85	value of unemployment activities
ρ	$0.95^{1/3}$	Autocorrelation of log LP
σ_ϵ	0.0077	Standard deviation of shocks to log LP

4.4.2 Steady State

We will next characterize the model's deterministic steady state as well as the values of the remaining, not calibrated parameters. Note that in the deterministic steady state, where N_t is constant, the non-negativity condition on open vacancies must be non-binding (since the separation rate $\omega > 0$).

In deterministic steady state the natural logarithm of LP equals

$$Z = 0$$

and from the definition of unemployment, we get

$$N = 1 - U,$$

so that the production technology immediately yields

$$Y = N.$$

The vacancy-unemployment-ratio is determined by the probabilities κ_w and κ_f through

$$\theta = \frac{V}{U} = \frac{\frac{M}{U}}{\frac{M}{V}} = \frac{\kappa_w}{\kappa_f}.$$

Job matches in steady state must be

$$M = \kappa_w U,$$

while open vacancies in steady state are then pinned down by

$$V = \frac{M}{\kappa_f}.$$

On the other hand, considering the definition of the matching function (4.2.29), it must also hold that

$$M = \frac{UV}{(U^\tau + V^\tau)^{\frac{1}{\tau}}}.$$

We can numerically solve this equation for the parameter τ from the already determined values of U, V and M. Next, by the dynamics (4.2.32) of employment, the separation rate must be

$$\omega = \frac{M}{N}.$$

At this point, next period's discounted marginal value of a worker to the firm $\hat{\xi}^f$, the costs of posting an open vacancy c, the wage w and consumption C remain to be determined. First, since the non-negativity condition is non-binding in steady state, i.e. $\mu = 0$, we see from (4.2.37) that

$$c = \kappa_f \hat{\xi}^f. \tag{4.4.1}$$

Replacing c in (4.2.36) with the expression on the right hand side in the above equation, we get

$$w = \varphi \exp(Z) + (1-\varphi)b + \varphi \frac{V}{1-N} \kappa_f \hat{\xi}^f.$$

Plugging the term for w into (4.2.41) then yields

$$\hat{\xi}^f = \beta \left(\exp(Z) - \varphi \exp(Z) - (1-\varphi)b - \varphi \frac{V}{1-N} \kappa_f \hat{\xi}^f + (1-\omega)\hat{\xi}^f \right).$$

Solving the equation for $\hat{\xi}^f$ finally results in

$$\hat{\xi}^f = \frac{\beta(1-\varphi)(\exp(Z)-b)}{1-\beta(1-\omega-\varphi\frac{V}{1-N}\kappa_f)} = \frac{\beta(1-\varphi)(\exp(Z)-b)}{1-\beta(1-\omega-\varphi\kappa_w)}, \tag{4.4.2}$$

so that the vacancy costs c as well as the wage w can be computed from the above provided equations.

Last, consumption in steady state is determined by

$$C = Y - cV$$

and marginal utility from consumption by

$$\lambda = C^{-\eta}.$$

Table 4.2: Calibration II

Parameter	Value	Description
τ	1.2897	elasticity in matching function
ω	0.05	separation rate
c	1.3154	cost of posting an open vacancy

Remarks on the Calibration The resulting values for the free parameters are displayed in table 4.2. Note in particular that imposing the steady state and parameter values from table 4.1 leads to $c = 1.3154$, so that costs of approximately 1.3 times the monthly output of a worker are incurred by posting an open vacancy. In comparison, Hagedorn and Manovskii (2008), who measure these costs from the data, arrive at a much smaller value for c equal to 0.584. Yet, the comparably high value for the costs of posting open vacancies is an immediate consequence of the fact that we set the value of unemployment activities to $b = 0.85$, notably lower than the value of $b = 0.955$ used in Hagedorn and Manovskii (2008). The smaller value of b in our calibration implies a higher present value $\hat{\xi}^f$ of a filled position in the next period to the representative firm in (4.4.2) and consequently higher costs c of posting an open vacancy from (4.4.1). Silva and Toledo (2005) report hiring costs (expenses on job advertising, search firm fees, and compensation of applicants) of approximately 3% of the annual labor costs per worker, while training costs of new employees are noticeably higher at approximately 14% of the annual labor costs per worker. Due to the lack of any additional costs in the model faced by the firm only after a new hire is realized, the total costs per hire in the model account to $\frac{c}{\kappa_f}$. If we understand all the aforementioned costs to be included in this term, hiring and training costs in the model would account for $(c/\kappa_f)/(12 \cdot w)$ of the annual labor costs of a worker.[17] Since $w \approx 0.9$ in steady state and therefore $(c/\kappa_f)/(12 \cdot w) \approx 0.17$, hiring and training costs of approximately 17% of the annual labor costs in the model are in accordance to the findings reported by Silva and Toledo (2005).

4.4.3 Numerical Computation

In section 4.3 we derived, for different approaches of finding an approximation to the policy function of $\hat{\xi}^f_t$, in each case a system of non-linear equations pinning down the free parameters in the respective approximation function. In order to solve the system of equations, we proceeded the following way. First, we computed a second order perturbation solution, which we used to derive an initial guess for the parameters in the approximation when restricted only to a small interval $[\underline{x}, \bar{x}]$ around the steady state value of employment. After determining the solution on the small domain around the steady state, we then decreased the lower bound and increased the upper bound step by step until reaching the desired domain for the approximation. In each step the initial guess for the parameters in the approximation defined on the wider domain was derived by extrapolation from the solution computed in the preceding step. Throughout all of the computations, we used the trust-region dogleg algorithm employed in `Matlab` for finding the zeros.

We present the number of basis functions, the number of underlying grid points etc. employed for computing the four approximations.

[17] Note however that, different from the present setting, Silva and Toledo (2005) understand training costs as a percentage loss in productivity of newly hired workers.

Chebyshev-Galerkin and Gauss-Hermite We set the domain for $\hat{g}_a$ to

$$[\underline{x}, \bar{x}] \times [\underline{z}, \bar{z}] := [0.05, 0.97] \times [-0.16, 0.16]$$

and computed the approximation with $d_x = 7$ and $d_z = 5$ Chebyshev polynomials along the employment and the log LP axis, respectively, so that a total of $K = 35$ basis functions are used. We chose $m_x = m_z = 19$ nodes for the Chebyshev-Gauss quadrature in (4.3.19) and $n = 13$ nodes for the Gauss-Hermite quadrature in (4.3.16).[18]

Chebyshev-Galerkin and discretized LP We replaced the AR(1) process for log LP by a Markov chain with $n = 15$ states by the method proposed by Rouwenhorst (1995), resulting in the lowest state $z_1 = -0.1571$ and a highest state $z_{15} = 0.1571$. For each of the 15 states, $j \in \{1, \dots, 15\}$, we allowed $d_j = 7$ Chebyshev polynomials as basis functions on the domain $[\underline{x}, \bar{x}] = [0.05, 0.97]$ and computed the Chebyshev-Gauss quadrature in (4.3.27) with $m_j = 19$ grid points.

Cubic-Spline-Collocation and Gauss-Hermite Since we ultimately relied on this solution in the upcoming analysis of the model, the domain was set somewhat broader to

$$[\underline{x}, \bar{x}] \times [\underline{z}, \bar{z}] := [0.05, 0.97] \times [-0.21, 0.21]$$

guaranteeing that all states encountered during simulations were included. We chose a two-dimensional rectilinear grid with $d_x = 130$ and $d_z = 85$ nodes along the axes, i.e. a total of 11050 grid points. We evaluated the Gauss-Hermite quadrature in (4.3.16) with $n = 13$ nodes.[19]

Cubic-Spline-Collocation and discretized LP As in the Chebyshev-Galerkin approach with discretized labor productivity, the AR(1) process is replaced by a finite state space Markov chain a la Rouwenhorst (1995) with $n = 15$ states. We set $[\underline{x}, \bar{x}] = [0.05, 0.97]$. The number of grid points in $[\underline{x}, \bar{x}]$ are chosen differently and non-equidistant for the different states of log LP in such way to allow for a finer fragmentation in areas where the residuals remained comparatively high. Nonetheless, the number of grid points are relatively high for all of LP's states ranging from 48 for z_1 to 125 for z_{15}.

Both approaches making use of the discretized process for LP generate an approximation $\hat{g}_a$ initially defined only on $[\underline{x}, \bar{x}] \times \mathscr{Z}$. But since we want to use the original AR(1) process assumption for log LP in simulations, we need the solution to be defined also for values of log LP not in $\mathscr{Z}$. This is again achieved through one-dimensional cubic spline interpolation along the z-axis, where the nodes are the states in $\mathscr{Z}$.

As already mentioned, the domain for the cubic spline, where expectations are computed by Gauss-Hermite quadrature, was chosen wide enough to never be exceeded in simulations. For the remaining approximations this is only true with regard to employment. In the rare cases where the bounds for log LP were exceeded in simulations, we therefore used extrapolation methods to compute $\hat{g}_a$ from these approximations.

[18] Note that the fact that we use 13 nodes in the Gauss-Hermite quadrature implies that extrapolation relatively far outside the domain of the approximation is necessary in order to compute the integrals for states with log LP close to the bounds. However, the (monotonic) shape of the policy function in log LP and comparison to results where the domain was widened suggests that extrapolation errors (especially combined with the very low weights for these nodes) should not become too large.

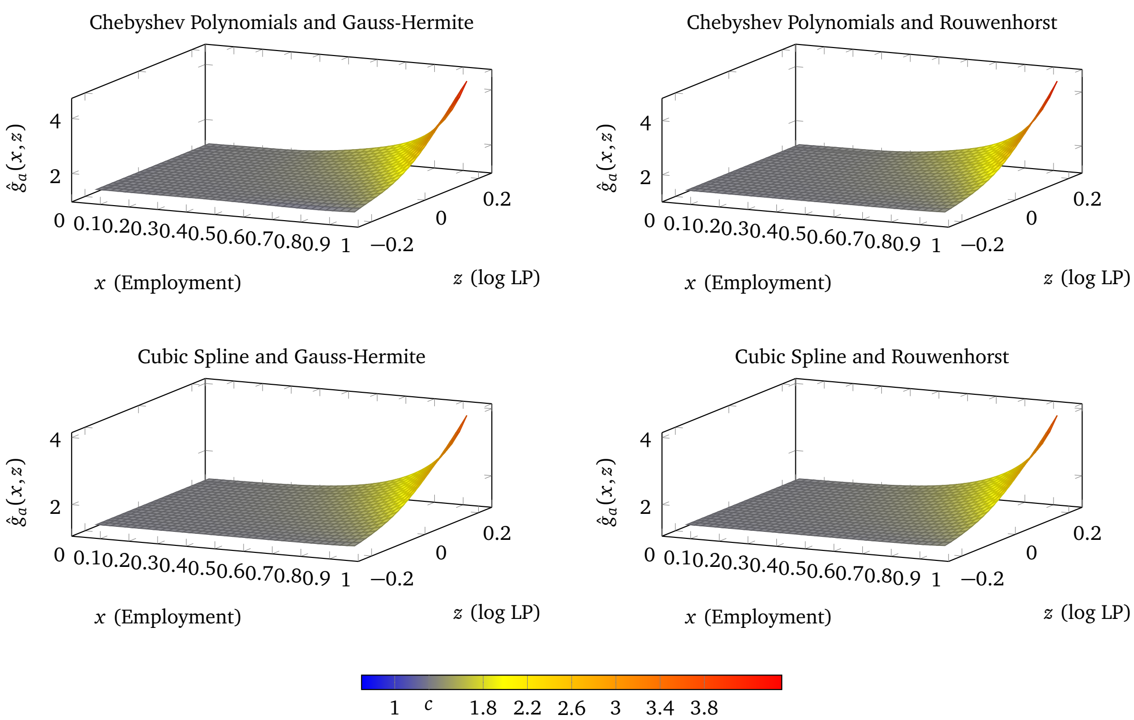
[19] The same remark for the Gauss-Hermite quadrature as in the Chebyshev-Galerkin approach holds here too.

4.4.4 Comparison of the Numerical Results

In this subsection, we will first present the obtained approximations to the policy functions and provide some economic intuition for their shape. We will then analyze eventual discrepancies between the different solution methods and check their accuracy. Lastly, we will also examine if differences in simulations of the model's equilibrium outcomes arise.

Approximations to the Policy Functions Figure 4.1 contains the plots for all four approximations to the policy function of the present value $\hat{\xi}^f_t$ of a worker in the next period to the firm. Before comparing the results from the different methods employed in more detail, some

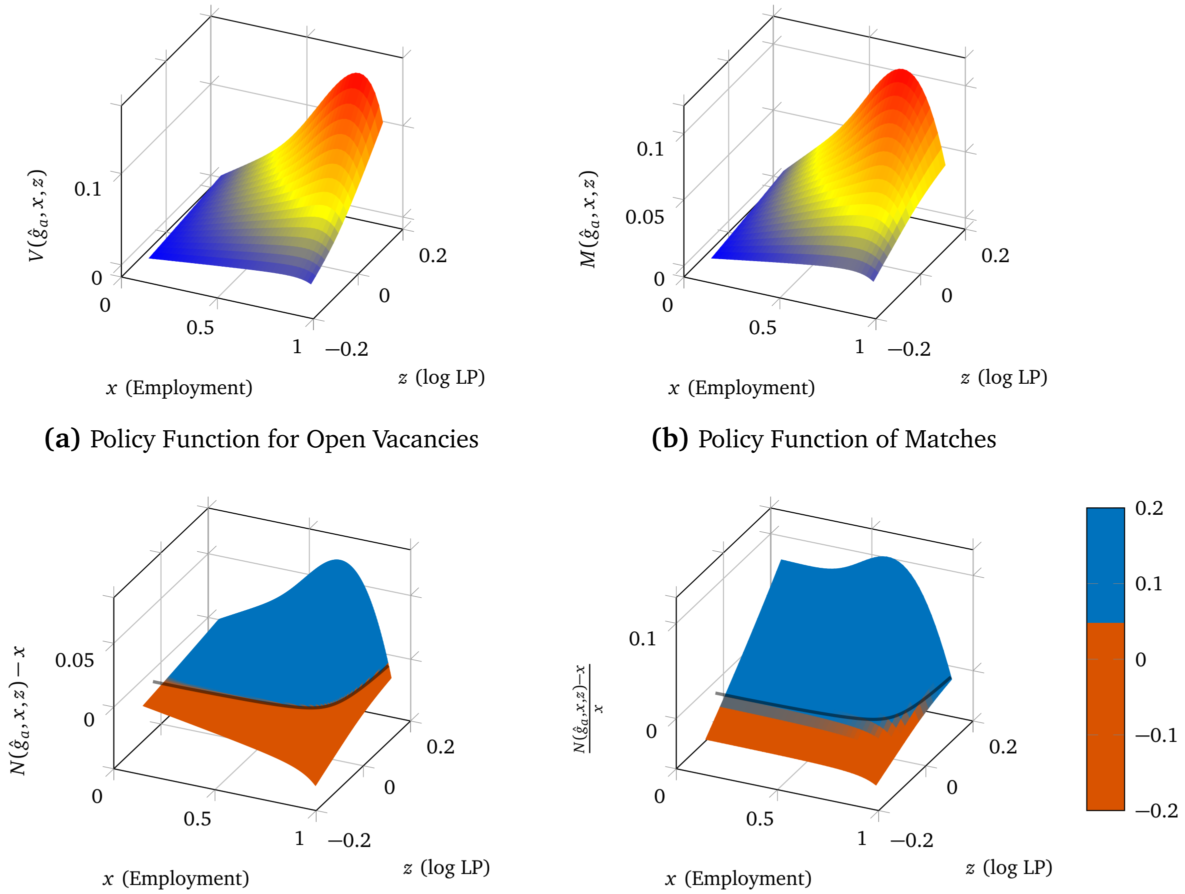
Figure 4.1: Approximations to the Policy Function of $\hat{\xi}^f$



remarks regarding the general shape of the policy function are in order since it will prove helpful in later discussions of the model. First, the present value of a worker in the next period to the representative firm turns out to be strictly increasing in both employment and log LP on the considered domain. We will explain the rationale for this immediately. For the moment, figure 4.2 displays in its four panels a) - d) the resulting policy functions for the amount of open vacancies posted, the realized job matches and the dynamics of employment in the model, respectively (all for the case of a cubic spline and Gauss-Hermite quadrature).

The amount of open vacancies posted crucially depends on the unemployment rate—affecting the average rate $\kappa_{f,t}$ at which posted vacancies will be filled—and the ratio of the expected value of a filled vacancy to the costs of posting (see (4.2.46)). Since $\hat{\xi}^f_t$ is strictly increasing in LP, so must be the amount of open vacancies. Along the employment axis, the slope of the policy function of open vacancies depends on the level of the employment rate. There are two opposing effects which are reflected in the first and second factor in the non-binding case of (4.2.46). With increasing employment it is ceteris paribus harder to fill open vacancies, but $\hat{\xi}^f_t$ is rising on the other hand. More specifically (4.2.46) yields (if the non-negativity constraint on

Figure 4.2: Policy Functions and Dynamics of Employment by Cubic Spline and Gauss-Hermite



(a) Policy Function for Open Vacancies

(b) Policy Function of Matches

(c) Change in Employment

(d) Relative Change in Employment

V_t is non-binding) that

$$\frac{\partial V_t}{\partial N_t} = -\left(\left(\frac{\hat{\xi}^f_t}{c}\right)^{\tau} - 1\right)^{\frac{1}{\tau}} + \frac{1-N_t}{c}\left(\left(\frac{\hat{\xi}^f_t}{c}\right)^{\tau} - 1\right)^{\frac{1}{\tau}-1}\left(\frac{\hat{\xi}^f_t}{c}\right)^{\tau-1}\frac{\partial \hat{\xi}^f_t}{\partial N_t} =$$

$$= -\left(\left(\frac{\hat{\xi}^f_t}{c}\right)^{\tau} - 1\right)^{\frac{1}{\tau}} + \frac{1-N_t}{c}\left(\frac{\left(\frac{\hat{\xi}^f_t}{c}\right)^{\tau}}{\left(\frac{\hat{\xi}^f_t}{c}\right)^{\tau} - 1}\right)^{\frac{\tau-1}{\tau}}\frac{\partial \hat{\xi}^f_t}{\partial N_t}.$$

The negative term captures the effect of a declining probability to fill a vacancy, while the second term reflects the positive effect of the increasing present value $\hat{\xi}^f_t$ of a filled position in the next period. The negative effect becomes smaller the closer $\hat{\xi}_f$ falls to c, while the opposite is true for the positive effect. Hence, taking a look at the shape of the policy function for $\hat{\xi}^f_t$ in figure 4.1 reveals that the positive effect in the second term should dominate in the region for lower and moderate employment rates where $\hat{\xi}^f_t$ is small and close to (but above) c, while the negative effect in the first term should eventually prevail at high employment rates. This is reflected in the slope of the policy function of open vacancies along the employment axis.

The amount of realized job matches is determined by two input factors: the amount of open vacancies and the number of unemployed members entering the matching process. Regarding LP, holding employment constant, matches must react the same way as vacancies, i.e. matches are increasing with LP. Along the employment axis, the amount of open vacancies is first inclining, then declining, while the second input factor of unemployed members is falling throughout.

For lower employment rates, the two variables determining the matches in the economy move in opposite directions, but with many unemployed members in the economy and only few open vacancies V_t exerts the crucial effect on M_t so that realized matches increase with the employment rate. At higher employment rates the input of unemployed workers in the matching process becomes more and more scarce so that changes in U_t affect the number of realized job matches more. Moreover, vacancies also begin to decline. Consequently, realized job matches eventually begin to decrease.

Lastly, we show the resulting change in the economy's employment rate from one period into the next one, i.e. $N_{t+1} - N_t$, in figure 4.2c as well as the corresponding growth rate of employment, i.e. $\frac{N_{t+1}-N_t}{N_t}$, in figure 4.2d. We can see that employment in the model can climb only moderately above its steady state value before declining again. Independent of the behavior in the economy it is even inherent to the model that employment must decline once it exceeds

$$\frac{1}{1+\omega} \approx 0.9524.$$

This applies, since in the case of $N_t \geq \frac{1}{1+\omega}$, input of unemployed members in the matching process is so low that newly realized job matches are always lower than exogenous separations

$$M_t \leq U_t = 1 - N_t \underset{N_t \geq \frac{1}{1+\omega}}{\leq} 1 - \frac{1}{1+\omega} = \frac{\omega}{1+\omega} \underset{N_t \geq \frac{1}{1+\omega}}{\leq} \omega N_t$$

and therefore

$$N_{t+1} = N_t - \omega N_t + M_t \leq N_t.$$

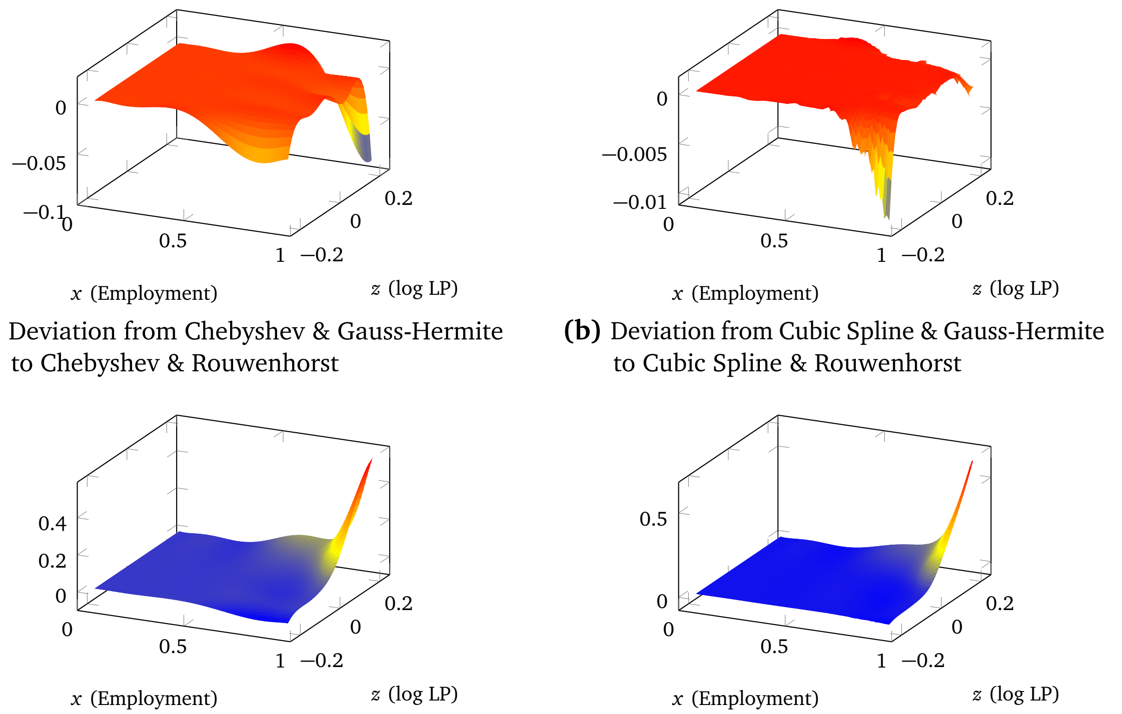
Moreover, for levels of log LP under a threshold of approximately -0.08 the present value $\hat{\xi}_t^f$ of a worker in the next period to the firm falls so low that open vacancies become too scarce to allow for job matches covering the amount of exogenous separations. Note that once the economy arrives at a productivity level under this threshold, the economy's employment will decline until LP recovers even without substantial additional negative shocks lowering productivity any further. This is due to the fact that once employment begins to fall, $\hat{\xi}_t^f$ will be decreasing along the employment axis and so will open vacancies and realized job matches as already described. Hence, the increasing unemployment rate itself yields the economy to enter a downward spiral until LP recovers without the necessity of LP declining any further. This will be an important factor in the explanation of disasters in the model in the later sections.

We have now seen how the dynamics of employment in the model can be explained to a certain level dependent on the shape of the policy function of $\hat{\xi}_t^f$. We have also stressed the importance of the fact that $\hat{\xi}^f$ is increasing with the employment rate, respectively falling with the unemployment rate, for the possibility of the economy drifting to periods with high unemployment. We will now try to provide some intuition for the shape of the policy function of $\hat{\xi}_t^f$. First, observe that the recursive formulation of the Euler equation (4.2.41) can also be written as

$$\hat{\xi}_t^f = \mathbb{E}_t \left[\sum_{s=1}^{\infty} (1-\omega)^{s-1} \beta^s \frac{\lambda_{t+s}}{\lambda_t} (\exp(Z_{t+s}) - w_{t+s}) \right],$$

under the condition that the series on the right hand side converges. I.e. the present value of a worker in the next period from the firm's perspective is the sum of expected discounted future productivity less wage costs weighted with the probability of the worker still being employed in the respective period. Now, since Z_t follows an exogenous, positively autocorrelated AR(1) process and wages do not absorb an increase in productivity completely, an increase in this

Figure 4.3: Deviations amongst Approximations



(a) Deviation from Chebyshev & Gauss-Hermite to Chebyshev & Rouwenhorst

(b) Deviation from Cubic Spline & Gauss-Hermite to Cubic Spline & Rouwenhorst

(c) Deviation from Chebyshev & Gauss-Hermite to Cubic Spline & Gauss-Hermite

(d) Deviation from Chebyshev & Rouwenhorst to Cubic Spline & Rouwenhorst

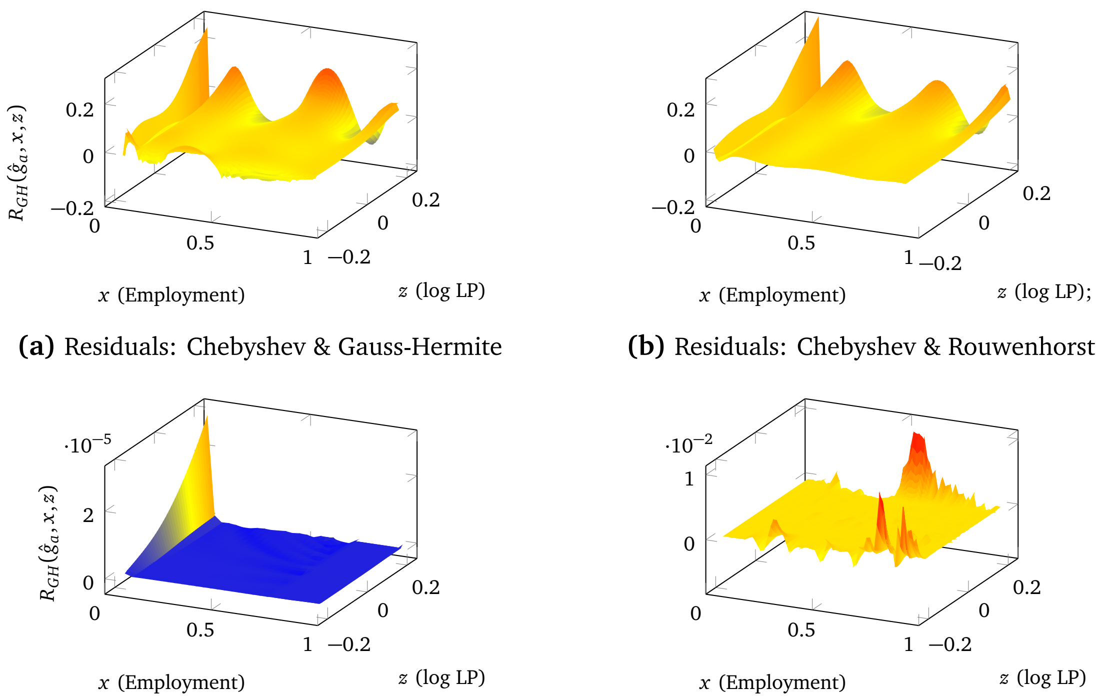
period's productivity must lead to an increase in expected future within period profits obtained by the firm from a worker, i.e. in $\exp(Z_{t+s}) - w_{t+s}$. Moreover, current consumption increases relative to expected future consumption so that future excess returns gain in weight. Consequently, $\hat{\xi}_t^f$ must increase in log LP. The fact that $\hat{\xi}_t^f$ is increasing in employment holding LP constant may seem counterintuitive at first glance. Equation (4.2.46) shows that the labor market tightness depends solely and positively on $\hat{\xi}_t^f$ and must therefore also increase with the employment rate if $\hat{\xi}_t^f$ does. The wage must then do the same (see equation (4.2.36)) and hence expected within period profits from a worker in future periods, productivity level net wage costs $\exp(Z_{t+s}) - w_{t+s}$, must be declining along the employment axis. Although this seems to be contradictory at first to the fact that $\hat{\xi}_t^f$ is increasing along the employment-axis, it is not, since at higher employment levels consumption also increases. Consequently, while expectations in the long run about $\exp(Z_{t+s}) - w_{t+s}$ differ only marginally for large s between different levels of today's employment, they are discounted less the higher employment and ergo consumption is today. The effect of less discounting of excess generated by a worker in the long run due to higher consumption at higher employment rates must dominate the decrease in within period profits to the firm in the short run due to higher wages. This yields $\hat{\xi}_t^f$ to turn out to be increasing in the employment rate.

Deviations in the Approximations We want to compare the approximations in figure 4.1 regarding the different approaches employed. The upper part of figure 4.3 displays the deviations in the solutions for the two methods of numerically computing the expectations operator, while the lower part shows the deviations between the corresponding spectral and finite element methods. The deviations between the spectral and finite element solutions in the lower part turn out to be at least one order of magnitude larger than between the different approaches of computing the expectations operator in the upper part. Since we allowed significantly less degrees of freedom in the spectral methods compared to the finite element methods[20], the result might already suggest that the number of Chebyshev polynomials used in the spectral methods is too low to allow for a fitting approximation. The fact whether we use Gauss-Hermite quadrature or a discretization of the log LP process seems to affect the solution significantly less in comparison.

However, the absolute deviations of the policy function are not meaningful without providing a point of reference. Therefore, we will next examine some statistics regarding the goodness of fit of the approximations, before we analyze if the deviations encountered in the policy functions also yield different simulation results.

Euler Equation Residuals In figure 4.4 we first display the residuals of the functional equation $R_{GH}(\hat{g}_a, ., .) = 0$—evaluated on a significantly finer grid than used for the computation of the approximations—for all four approximation methods. [21] The errors displayed in figure 4.4 are

Figure 4.4: Euler Residuals



(a) Residuals: Chebyshev & Gauss-Hermite

(b) Residuals: Chebyshev & Rouwenhorst

(c) Residuals: Cubic Spline & Gauss-Hermite

(d) Residuals: Cubic Spline & Rouwenhorst

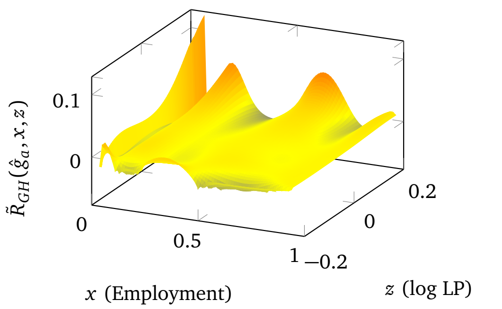
deviations in next period's discounted marginal value of a worker to the representative firm measured in consumption units. To allow for an easier interpretation of the magnitude, we also calculate interpretable Euler equation residuals similar to Christiano and Fisher (2000). The exact solution satisfies

$$C(g,x,z)^{-\eta} lhs(g,x,z) = C(g,x,z)^{-\eta} rhs(g,x,z), \quad \text{for all } (x,z) \in [0,1] \times \mathbb{R},$$

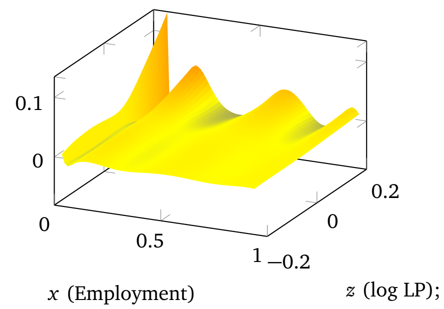
[20] Remember, we used only 35 Chebyshev polynomials, yet over 11000 piecewise bicubic polynomials, in the approximation for the case of computing the expectations operator by Gauss-Hermite quadrature.

[21] In simulations of the model's equilibrium outcomes we rely on the AR(1) process assumption for log LP. Since the two methods using the Rouwenhorst (1995) algorithm are initially only defined on the underlying finite state space $\mathscr{Z}$ for log LP, we need to use interpolation methods regarding the state of log LP in simulations. However, the residuals R_{RO} are only defined for states of log LP in $\mathscr{Z}$. Therefore, in order to consider the goodness of fit over the whole range of LP, we only examine the residuals of the functional equation $R_{GH} = 0$ for these two approximations, too.

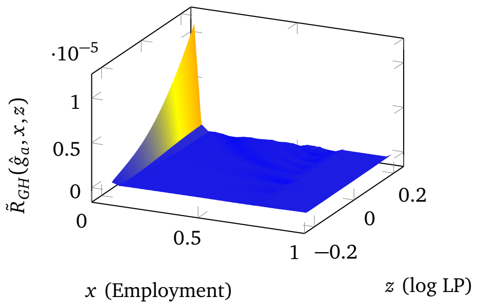
Figure 4.5: Interpretable Euler Residuals



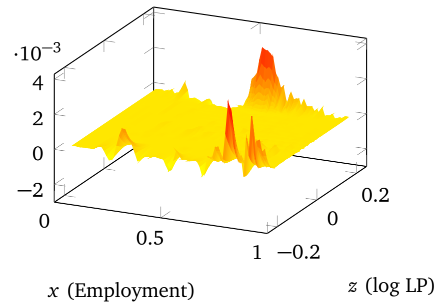
(a) Residuals: Chebyshev & Gauss-Hermite



(b) Residuals: Chebyshev & Rouwenhorst



(c) Residuals: Cubic Spline & Gauss-Hermite



(d) Residuals: Cubic Spline & Rouwenhorst

where C is the policy function of consumption as defined in equation (4.3.6)[22]. For any of the approximations $\hat{g}_a$, the equation will in general not hold exactly even when rhs is replaced by rhs_{GH}.[23] But, we can calculate $\tilde{C}$ so that

$$\tilde{C}(\hat{g}_a, x, z)^{-\eta} lhs(\hat{g}_a, x, z) = C(\hat{g}_a, x, z)^{-\eta} rhs_{GH}(\hat{g}_a, x, z).$$

Solving for $\tilde{C}$ then yields

$$\tilde{C}(\hat{g}_a, x, z) = C(\hat{g}_a, x, z) \left(\frac{rhs_{GH}(\hat{g}_a, x, z)}{lhs(\hat{g}_a, x, z)} \right)^{-\frac{1}{\eta}}$$

and one can interpret $\tilde{C}(\hat{g}_a, x, z)$ as the amount today's consumption would have to equal in order for the Euler equation to hold exactly, if all other variables including next period's consumption are computed from the obtained approximations.[24] The interpretable Euler residuals are then defined by

$$\tilde{R}_{GH}(\hat{g}_a, x, z) := \frac{\tilde{C}(\hat{g}_a, x, z)}{C(\hat{g}_a, x, z)} - 1 = \left(\frac{rhs_{GH}(\hat{g}_a, x, z)}{lhs(\hat{g}_a, x, z)} \right)^{-\frac{1}{\eta}} - 1$$

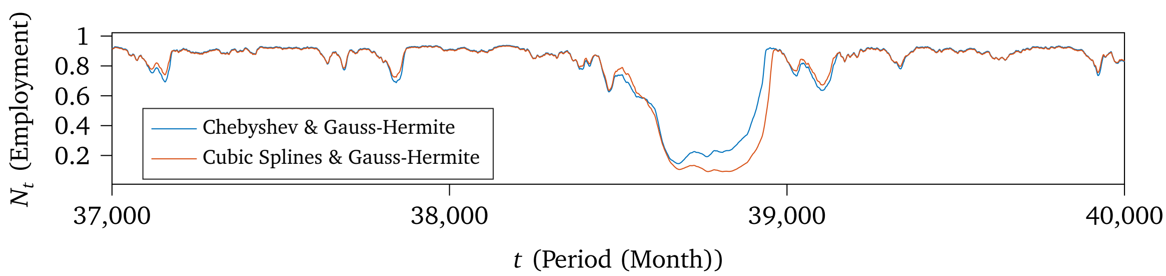
and can be interpreted as the fraction by which today's consumption would have to be raised compared to the value computed from the approximated policy function in order for the Euler equation to hold exactly. Figure 4.5 displays the interpretable Euler residuals computed again on a significantly finer grid than used in the computation of the approximations.[25] We can see that

[22]Note, that $C(g, x, z)$ cancels out on the right hand side.
[23]Except at the nodes in the collocation method.
[24]Since $C(\hat{g}_a, x, z)$ cancels out on the right hand side, all other variables excluding this period's consumption, but including next period's consumption, are computed from the approximations to the policy functions.
[25]Note that figure 4.5 displays the decimal values of $\tilde{R}_{GH}$, i.e. the interpretable Euler residuals are not in percentage points.

in particular our benchmark solution, the cubic spline paired with Gauss-Hermite quadrature, provides a sufficiently good approximation. The value for this period's consumption computed from the policy function would have to be changed even at most only about 0.001% in order for the Euler equation to hold exactly. For the cubic spline computed from a discretized Markov process governing LP, the interpretable Euler residuals rise somewhat higher at the borders for log LP due to extrapolation used for states of LP below and above −0.16 and 0.16, respectively. The solutions relying on Chebyshev polynomials produce significantly worse approximations, which affirms the conjecture that the degrees of freedom selected for the spectral methods are too low to render the approximations fitting.

Simulated Time Path Last, we want to examine whether the deviations between the approximations also manifest in different results in simulations of the model's equilibrium outcomes. We therefore computed a time path of equilibrium outcomes for a total of 1200000 time periods starting from the deterministic steady state where we used the same realization of pseudorandom iid$N(0,\sigma_\epsilon^2)$ distributed shocks to the AR(1) process governing log LP for all four approximation methods. In figure 4.6 we display the outcome for the time paths of employment during the

Figure 4.6: Comparison: Dynamics of Employment



periods 37000 to 40000, computed either by the spectral or finite element approximation, each with Gauss-Hermite quadrature. From this segment of the whole realization we can already observe that the time paths of employment show rare steep drops and while the time paths from the different approximations coincide for the most part, huge deviations during these drops become noticeable. To confirm this first impression, table 4.3 summarizes the maximal absolute deviations between the time paths of employment computed from the different approximations. First, we can see that for the solutions using a cubic spline, the fact whether the expectations

Table 4.3: Comparison: Maximal Deviations in Employment Time Paths

	Chebyshev&Gauss-Hermite	Chebyshev&Rouwenhorst	Cubic Spline&Gauss-Hermite	Cubic Spline&Rouwenhorst
Chebyshev&Gauss-Hermite	0	0.4015	0.4411	0.4400
Chebyshev&Rouwenhorst		0	0.4338	0.4327
Cubic Spline&Gauss-Hermite			0	0.0034
Cubic Spline&Rouwenhorst				0

operator in the functional equation is evaluated by Gauss-Hermite quadrature or discretization of the LP process a la Rouwenhorst does not change much for the simulation outcome; the computed time paths of employment do not deviate by more than 0.0034.[26] This is consistent to the previous result from figure 4.3b that the differences between these two approximations to the policy function are only small. For the case of Chebyshev polynomials serving as basis

[26] Since the solution relying on the discretized LP process was computed only on a smaller domain for log LP where extrapolation has to be used much more frequently in simulations, the conjecture that errors by extrapolation remain only small as long as the approximation is sufficiently accurate on its domain is additionally reinforced by this fact.

functions, the differences between the two approximations making use of different ways to compute the expectations operator, displayed in figure 4.3a, were one order of magnitude higher. Yet, these, on average, also seemingly not too large differences in the policy functions give rise to significant deviations between the respective simulated time paths. The employment rate deviates up to 0.4015 between the two solutions. Hence, the dynamics in the model seem sensitive to rather small differences in the policy function of $\hat{\xi}^f_t$. Further, the already discussed discrepancies between the spectral and finite element solutions result in huge deviations in the simulation.

Yet, the exact time path of equilibrium outcomes is only rarely of interest by itself. More common, properties of the variables' distribution in simulations are summarized by statistical measures. In order to analyze whether and in which way the differences in the time paths eventually translate into different results for such measures, table 4.4 shows the mean, the maximum and minimum value, certain quantiles as well as the standard deviation, the skewness and the kurtosis for the monthly employment rate. While the maximal deviations between the

Table 4.4: Comparison: Distribution of Monthly Employment Rates

	Chebyshev&Gauss-Hermite	Chebyshev&Rouwenhorst	Cubic Spline&Gauss-Hermite	Cubic Spline&Rouwenhorst
$\bar{N}$	0.8757	0.8768	0.8751	0.8753
$\max(N_t)$	0.9423	0.9426	0.9395	0.9394
$\min(N_t)$	0.0912	0.0601	0.0595	0.0596
$q_{0.01}(N_t)$	0.5481	0.5481	0.5164	0.5170
$q_{0.05}(N_t)$	0.6983	0.7117	0.7347	0.7354
$q_{0.1}(N_t)$	0.7854	0.7928	0.8068	0.8075
$q_{0.5}(N_t)$	0.9042	0.9038	0.8998	0.8998
$q_{0.9}(N_t)$	0.9259	0.9262	0.9215	0.9215
$q_{0.95}(N_t)$	0.9293	0.9295	0.9249	0.9249
$q_{0.99}(N_t)$	0.9340	0.9341	0.9299	0.9299
s_N	0.0817	0.0799	0.0773	0.0772
v_N	-3.3499	-3.5237	-3.9332	-3.9473
w_N	18.0385	20.0432	23.9280	24.0846

Notes: $\bar{N}$=average employment rate, $\max(N_t)$=maximal employment rate, $\min(N_t)$=minimal employment rate, $q_p(N_t)$ =p-quantile for cumulative distribution of employment rate, s_N, v_N, w_N=standard deviation, skewness, kurtosis of the cumulative distribution of employment rate. Statistics are computed from the outcome of a simulation of 1200000 (monthly) periods.

time paths of employment were comparably high—except between the two approximations based on cubic splines—the computed statistic indicators do not differ too much. This is in accordance to the fact that the employment rates coincide for most periods and seem to only deviate during the rare huge declines. Note already that the model gives rise to unemployment rates over 90%.

4.4.5 Approximation by Second Order Perturbation

Approximations, which rely on the basic idea of mean weighted residuals described in section 4.3.1, are constructed in such manner to provide a solution globally accurate on the chosen domain. The conditions (4.3.15), from which fitting parameter values in the approximation are derived, yield the errors in the Euler equation to vanish on average over the whole domain when weighted with the test functions. More concretely, the weighting by the test functions specifies, where on the domain residuals of the functional equation are more significant.[27] We will therefore also call these approximation methods global methods in the following. Differently, an approximation to the model's policy functions by perturbation, which relies solely on the policy functions derivatives in deterministic steady state, can in general only provide a locally,

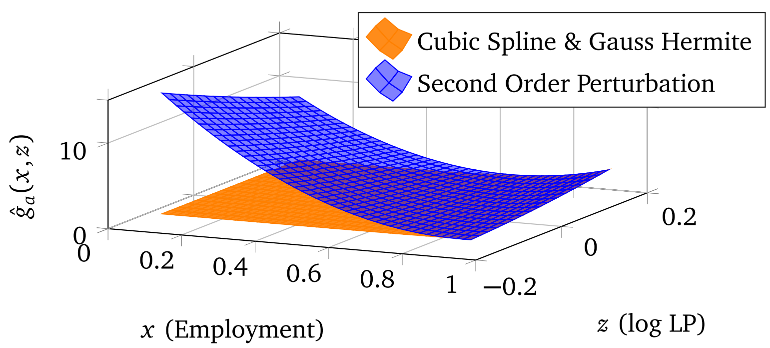
[27] For example, for the cubic spline collocation method with test functions equal to Dirac delta functions at the grid points, the residuals R_{GH} vanish exactly at all grid points over the domain, whereas no condition on the residuals in between is imposed.

around the steady state accurate solution to the model. The fact that employment drops to low levels far away from its steady state in the present model might therefore cause the perturbation solution to fail to adequately reproduce the model's dynamics.

We will use the solution based on a cubic spline paired with Gauss-Hermite quadrature as the reference solution for comparison in the following, since it proved to display a sufficiently good approximation in the discussion of the last subsection.

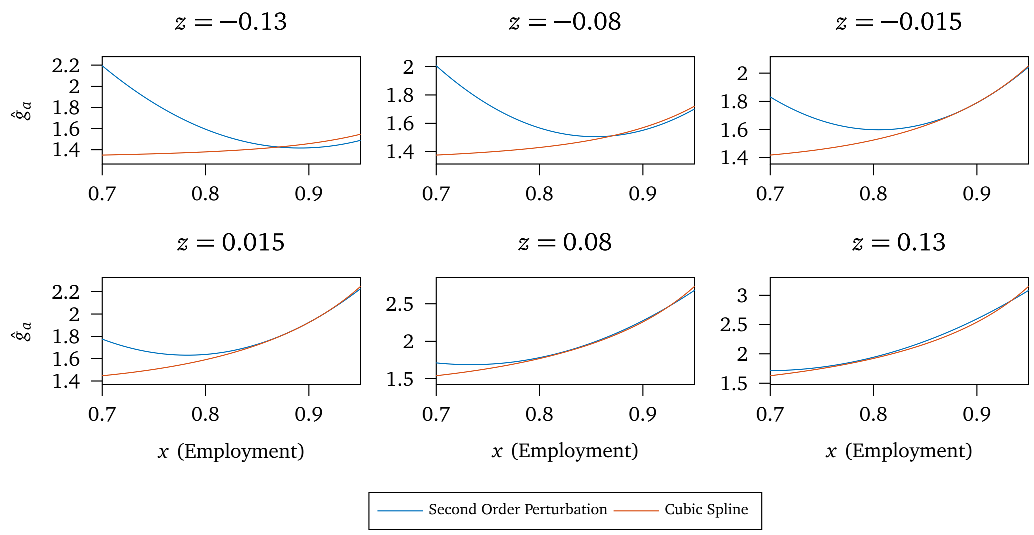
Approximation to the Policy Function We present the approximation to the policy function of $\hat{\xi}_t^f$ from a second order perturbation method in figure 4.7. Moreover, for more detailed

Figure 4.7: Policy Function obtained from Second Order Perturbation I



comparison we display cross-sections of the policy functions along different values of log LP in figure 4.8, namely for approximately the 99.9%, the 95%, the 66%, the 33%, the 5% and the 0.1% quantiles of log LP. As mentioned, an approximation by perturbation is expected to provide

Figure 4.8: Policy Function obtained from Second Order Perturbation II

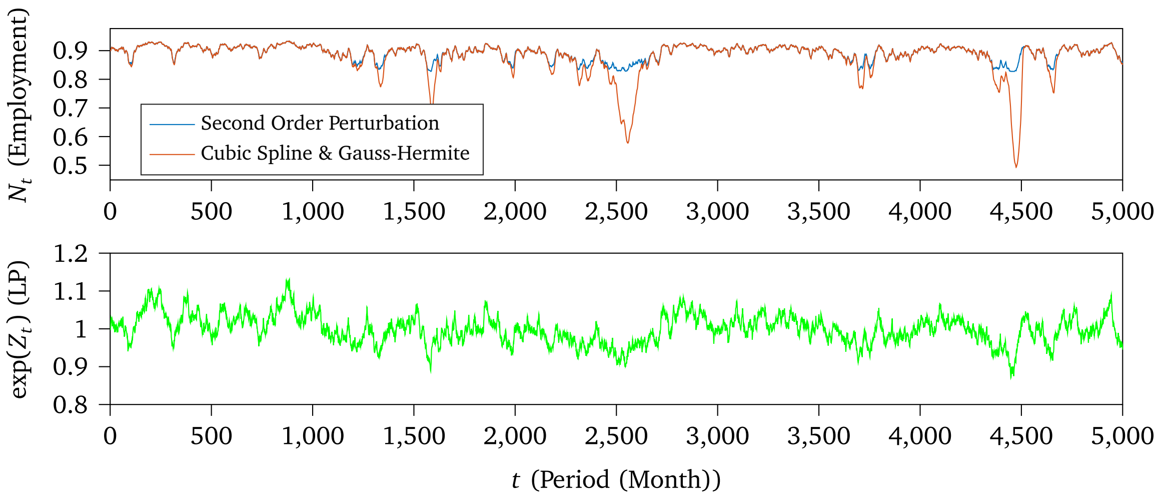


a good approximation locally in some area 'near' the steady state, but there is no guarantee for it delivering a suitable approximation farther away from the steady state. In the case at hand and with the global solution serving as reference, it becomes apparent that the perturbation solution gives a very poor approximation when employment declines, whereas the inaccuracies, although noticeable, remain only small for a broad range of log LP as long as employment stays near to its steady state value of 0.9.

Simulated Time Path Next, let us look at the consequences of these differences in the approximations when we use the perturbation solution to simulate the time series of the model's equilibrium outcomes from the same sample of pseudorandom shocks to log LP as before. It is important to note that—as in the case of the global solution—we only computed the outcomes for $\hat{\xi}^f_t$ from the second order approximation to the variable's policy function. All other period t variables and N_{t+1} were then derived from $\hat{\xi}^f_t$ (and the state variables) in the way described in section 4.2.6.[28]

The outcomes for employment in the first 5000 periods are displayed in figure 4.9. We observe

Figure 4.9: Dynamics of Employment: Second Order Perturbation vs. Cubic Spline

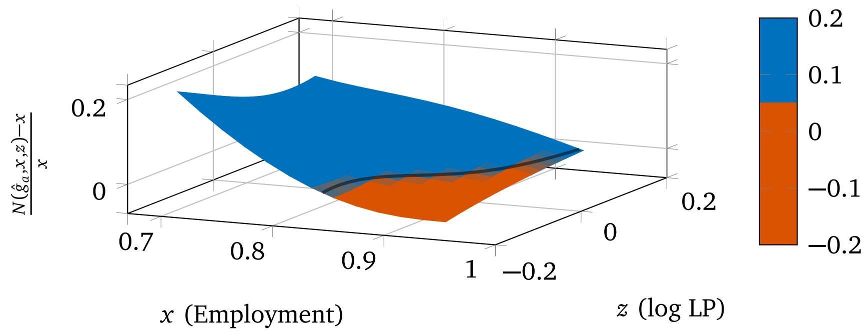


that the perturbation solution fails to adequately reproduce the occurrence of periods with very large unemployment rates in the model. The reason behind this result can immediately be deduced from the shape of the approximation to the policy function of $\hat{\xi}^f_t$ under a second order perturbation in figures 4.7 and 4.8. Contrary to the global solution, where $\hat{\xi}^f_t$ is throughout decreasing with the unemployment rate, the approximation by perturbation yields $\hat{\xi}^f_t$ to increase with the unemployment rate once employment falls below approximately 0.8. Hence, in the simulation using the perturbation solution, once the unemployment rate rises to ca. 20%, the present value $\hat{\xi}^f_t$ of a filled position in the next period from the firm's perspective will begin to increase, leading the representative firm to post more open vacancies and preventing the employment rate to fall any further. Consequently, we cannot expect to see unemployment rates above roughly 20% when the series of equilibrium outcomes is computed with the second order perturbation approximation. This is also illustrated in figure 4.10, which shows the growth rate of employment from period t into $t+1$ using the perturbation solution. Contrary to the result in figure 4.2d for the global solution, the lower bound for possible employment rates at approximately 0.8 is clearly visible. The poor approximation to the policy function of $\hat{\xi}^f_t$ for lower employment rates has serious implications for the dynamics of employment in simulations. The rare but steep declines in the unemployment rate cannot be replicated. The employment rate deviates up to 0.81 from the one computed by the cubic spline solution.

Yet, although all the global solution methods showed the huge declines in employment in simulations, the deviations during these periods in the time paths were also remarkably large. Nevertheless, these deviations had only comparable small impact if one is only interested in the distribution of employment in the simulation. This is different now. An illustration of the

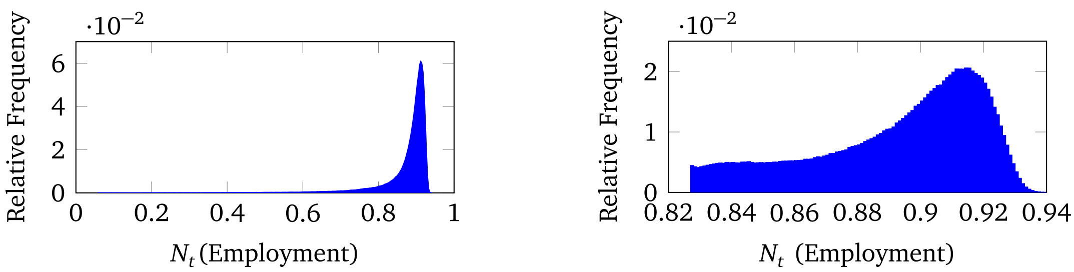
[28] If one, as commonly applied, computes all variables' outcomes from the second order approximations to their policy functions, larger deviations of LP and employment from steady state would at some point lead the model's dynamics to leave their basin of attraction under the second order approximation. The employment rate, and therefore also the other variables, would eventually tend to $\pm\infty$.

Figure 4.10: Second Order Perturbation: Relative Change in Employment



differences in the distribution of employment in the simulation is provided by the histograms in figure 4.11 and by the summary of statistical measures in table 4.5. Since the perturbation solution does not display the huge drops in employment, the minimum value as well as the quantiles for smaller probabilities greatly differ from the respective values obtained by the global solution methods. Yet, since these declines only rarely occur and since the perturbation solution resembles the global solution in the remaining periods, where employment is not too far from its steady state value, the average value, the maximal value and the quantiles for higher probabilities are similar. However, without accurately displaying the rare huge drops, the perturbation solution also fails to reproduce the volatility of employment in the model, the distribution is far less left skewed and the kurtosis is significantly smaller.

Figure 4.11: Distribution of Employment I: Second Order Perturbation vs. Cubic Spline



(a) Distribution of Employment computed by Cubic Spline

(b) Distribution of Employment computed by Second Order Perturbation

Table 4.5: Distribution of Employment II: Second Order Perturbation vs. Cubic Spline

	$\bar{N}$	$\max(N_t)$	$\min(N_t)$	$q_{0.01}$	$q_{0.05}$	$q_{0.1}$	$q_{0.5}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	s_N	v_N	w_N
Cubic Spline	0.8751	0.9395	0.0595	0.5164	0.7347	0.8068	0.8998	0.9215	0.9249	0.9299	0.0773	-3.9332	23.9280
Perturbation	0.8925	0.9396	0.8273	0.8293	0.8381	0.8482	0.9000	0.9215	0.9249	0.9298	0.0269	-0.7587	2.5413

Notes: $\bar{N}$=average employment rate, $\max(N_t)$=maximal employment rate, $\min(N_t)$=minimal employment rate, $q_p(N_t)$ =p-quantile for cumulative distribution of employment rate, s_N, v_N, w_N=standard deviation, skewness, kurtosis of the cumulative distribution of employment rate. Statistics are computed from the outcome of a simulation of 1200000 (monthly) periods.

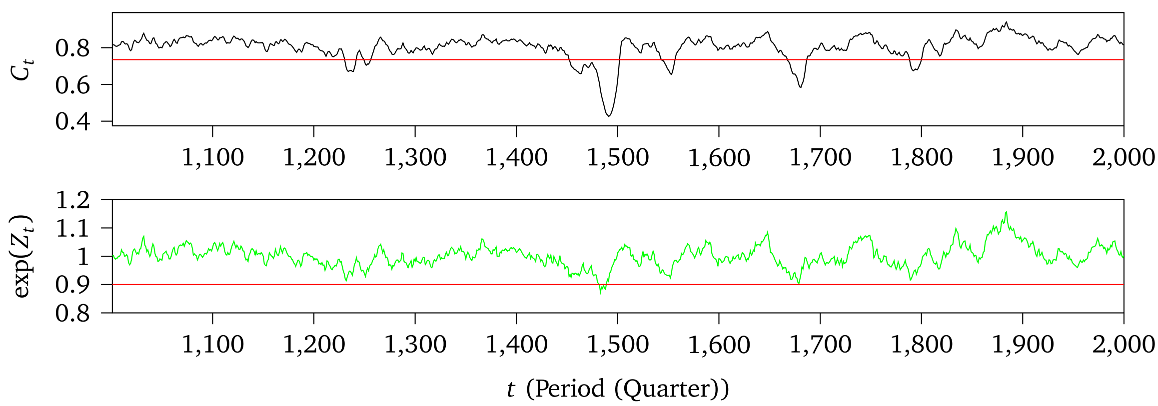
4.5 Endogenous Disasters, the Equity Premium and Second Moments of the Labor Market

In the previous section we observed that, if accurately solved, the model gives rise to periods of extremely high unemployment. In this chapter we will introduce a definition for the term 'disaster' and use this definition to compute disaster probabilities, disaster sizes and disaster durations in the model. Moreover, we will analyze whether the dynamics in the model help to replicate the high historical equity premium and the second moments of the labor market found in the data. All the upcoming results were computed from the solution using a cubic spline and Gauss-Hermite quadrature.

4.5.1 Endogenous Disasters in the Baseline Model

Before discussing the occurrence of disasters in the model, we first have to define the term for our purposes. Kuehn et al. (2015) compare disasters generated in the model to disasters found in data regarding several statistics. Following Barro and Ursua (2008) they use a peek-to-trough method to identify disasters from consumption and output data across different countries and apply the same method to simulations of the model's outcomes. Our main concern on the other hand will be to identify the reasons for the occurrence of disasters in the model and to subsequently check how the model's behavior changes to slight modifications. Since we are therefore not interested in possible resemblances of the disasters in the model to empirical patterns, we will employ a different approach to classify disasters in the model. We examine the same long time series of simulated equilibrium outcomes in the model for 1200000 periods, which we have already used in order to compare the different solution methods in section 4.4.4. We convert the sample for each variable to 400000 observations of quarterly averages and distinguish disasters from the resulting times series of consumption and output. We define a disaster to start in quarter t, if the quarterly average value of consumption, respectively output, falls below a given threshold fraction of the steady state value. The disaster ends in the period, where the value exceeds the threshold for the first time again, periods in between are called disaster periods. The size of the disaster is the fraction of decline from the steady state value to the lowest value during the disaster. The disaster probability measures the probability to fall from a non-disaster period into a disaster and is hence the number of disasters in the sample over the number of total non-disaster periods (with the selected threshold). In the segment of 1000 quarters from the time series of consumption pictured in the upper part of figure 4.12 and with a threshold fraction of 90% of the steady state value, there appear six disasters lasting for 11, 7, 49, 13, 19 and 12 quarters and which are of sizes 17.89%, 13.68%, 47.86%, 19.80%, 28.50% and 17.36% respectively according to this definition. There are 889 non-disaster periods, hence

Figure 4.12: Measurement of Disasters in Consumption



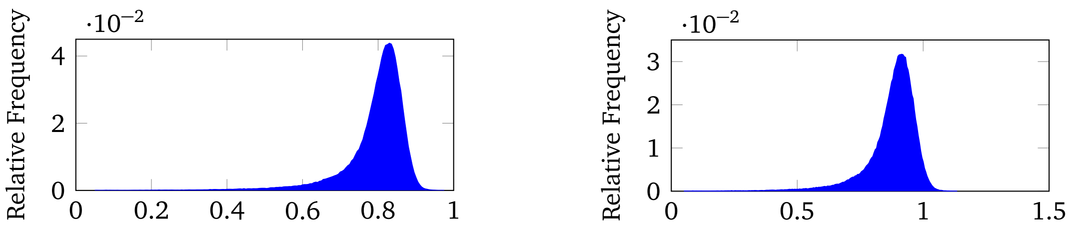
the disaster probability, measuring the relative frequency to enter a disaster from a non-disaster period is $\frac{6}{889} \approx 0.67\%$. The lower part of figure 4.12 displays the quarterly average labor productivity. We can already observe that the declines in consumption are far more pronounced than in LP, which gives a first indication that calling the disasters endogenous is somewhat justified.

We computed the number of disasters appearing, the total number of disaster periods, the disaster probability, the average duration of disasters and the average size of disasters for both consumption and output for different threshold fractions ranging from 90% to 30% from the complete sample. The results, together with the respective statistics for LP, are summarized in table 4.6. Further, the histograms for consumption and output are displayed in figure 4.13.

Table 4.6: Disasters in the Baseline Model

	number of disasters	total number of disaster periods	disaster probability	average disaster size	average disaster duration
			90% threshold		
C	3806	56934	1.11%	19.35%	14.96
Y	4801	64607	1.43%	18.76%	13.46
LP	663	2086	0.17%	11.05%	3.15
			85% threshold		
C	2166	35559	0.59%	25.67%	16.42
Y	2715	40355	0.75%	24.95%	14.86
LP	5	10	0.0013%	15.68%	2
			80% threshold		
C	1355	23183	0.36%	31.24%	17.11
Y	1711	26489	0.46%	30.33%	15.48
LP	0	0	0%	–	–
			75% threshold		
C	857	15881	0.22%	36.96%	18.53
Y	1007	17767	0.26%	36.54%	17.64
LP	0	0	0%	–	–
			70% threshold		
C	566	11243	0.15%	42.44%	19.86
Y	668	12472	0.17%	41.85%	18.67
LP	0	0	0%	–	–
			50% threshold		
C	124	2881	0.03%	61.59%	23.23
Y	142	3132	0.04%	61.07%	22.06
LP	0	0	0%	–	–
			30% threshold		
C	25	673	0.0063%	78.15%	26.92
Y	30	718	0.0075%	77.33%	23.93
LP	0	0%	–	–	0

Figure 4.13: Histograms for Consumption and Output in the Baseline Model



(a) Distribution of consumption in the baseline model

(b) Distribution of output in the baseline model

As a direct consequence of the already observed pattern of employment in the model exhibiting rare but huge declines, the same is true for consumption and output. For example, consumption in the model is less than 50% of its steady state value in 2881 of 400000 quarters in the simulation. These periods arise during 124 disasters lasting on average for approximately 23 periods. The probability of entering a period with consumption less than half of its steady state value is 0.03% conditional on the fact that consumption is more than half of its steady state value in the preceding period.

Epstein-Zin-Preferences As already mentioned in the calibration of the model, we also used an Epstein-Zin specification for the household's preferences as in Kuehn et al. (2012). The representative household's value function J^h is then defined in such way to satisfy

$$\begin{aligned} J^h(N_t, Z_t, S_t) = \quad & \max_{C_t, S_{t+1}} \left[(1-\beta) C_t^{1-\frac{1}{\psi}} + \beta \left(\mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{1-\gamma} \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\psi}}} \\ & \text{s.t.} \quad C_t \leq w_t N_t + d_t S_t - v_t (S_{t+1} - S_t) + b(1-N_t) - T_t, \\ & \text{and} \quad N_{t+1} = (1-\omega) N_t + \kappa_{w,t}(1-N_t), \end{aligned} \tag{4.5.1}$$

where $\gamma > 0, \gamma \neq 1$, defines the coefficient of relative risk aversion and $\psi > 0, \psi \neq 1$, the elasticity of intertemporal substitution and where we make explicit that the value function also depends on the exogenously determined states of LP. With the Epstein-Zin specification for the household's preferences the stochastic discount factor in the model changes as follows. First, with λ_t denoting the Lagrange multiplier of the budget constraint as before, the first order conditions for a bounded solution for the maximization problem on the right hand side are

$$\lambda_t = \frac{1}{1-\frac{1}{\psi}} [\ldots]^{\frac{1}{1-\frac{1}{\psi}}-1} (1-\beta) \left(1 - \frac{1}{\psi} \right) C_t^{-\frac{1}{\psi}} = (1-\beta) \left(J^h(N_t, Z_t, S_t) \right)^{\frac{1}{\psi}} C_t^{-\frac{1}{\psi}}$$

and

$$\begin{aligned} \lambda_t v_t = & \frac{1}{1-\frac{1}{\psi}} [\ldots]^{\frac{1}{1-\frac{1}{\psi}}-1} \beta \frac{1-\frac{1}{\psi}}{1-\gamma} \left(\mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{1-\gamma} \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}-1} \cdot \\ & \cdot \mathbb{E}_t \left[(1-\gamma) \left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{-\gamma} \frac{\partial J^h}{\partial S}(N_{t+1}, Z_{t+1}, S_{t+1}) \right] = \\ = & \left(J^h(N_t, Z_t, S_t) \right)^{\frac{1}{\psi}} \beta \left(\mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{1-\gamma} \right] \right)^{\frac{\gamma-\frac{1}{\psi}}{1-\gamma}} \cdot \\ & \cdot \mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{-\gamma} \frac{\partial J^h}{\partial S}(N_{t+1}, Z_{t+1}, S_{t+1}) \right]. \end{aligned}$$

Invoking the envelope theorem we further get

$$\frac{\partial J^h}{\partial S}(N_t, Z_t, S_t) = \lambda_t (d_t + v_t).$$

Plugging the equations for λ_t and $\frac{\partial J^h}{\partial S}$ into the Euler condition for the share price consequently yields

$$\begin{aligned} v_t = & \left(J^h(N_t, Z_t, S_t) \right)^{\frac{1}{\psi}} \beta \left(\mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{1-\gamma} \right] \right)^{\frac{\gamma-\frac{1}{\psi}}{1-\gamma}} \cdot \\ & \cdot \mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{-\gamma} \frac{\lambda_{t+1}}{\lambda_t} (d_{t+1} + v_{t+1}) \right] = \\ = & \left(J^h(N_t, Z_t, S_t) \right)^{\frac{1}{\psi}} \beta \left(\mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{1-\gamma} \right] \right)^{\frac{\gamma-\frac{1}{\psi}}{1-\gamma}} \cdot \\ & \cdot \mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{-\gamma} \left(\frac{J^h(N_{t+1}, Z_{t+1}, S_{t+1})}{J^h(N_t, Z_t, S_t)} \right)^{\frac{1}{\psi}} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} (d_{t+1} + v_{t+1}) \right] = \end{aligned}$$

$$
\begin{aligned}
&= \beta \left(\mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{1-\gamma} \right] \right)^{\frac{\gamma - \frac{1}{\psi}}{1-\gamma}} \mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{\frac{1}{\psi}-\gamma} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} (d_{t+1} + v_{t+1}) \right] \\
&= \mathbb{E}_t \left[\underbrace{\beta \left(\frac{J^h(N_{t+1}, Z_{t+1}, S_{t+1})}{\left(\mathbb{E}_t \left[(J^h(N_{t+1}, Z_{t+1}, S_{t+1}))^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}}} \right)^{\frac{1}{\psi}-\gamma} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}}}_{:= M^{EZ}_{t,t+1}} (d_{t+1} + v_{t+1}) \right],
\end{aligned}
$$

which differs from equation (4.2.7) only by the different stochastic discount factor $M^{EZ}_{t,t+1}$. If the parameter of relative risk aversion coincides with the reciprocal of the elasticity of intertemporal substitution, i.e. $\gamma = \frac{1}{\psi}$, the stochastic discount factor reduces to the standard case from the baseline model. Moreover, the envelope theorem also yields (when plugging in the dynamics for N_{t+1})

$$
\begin{aligned}
\frac{\partial J^h}{\partial N}(N_t, Z_t, S_t) &= \lambda_t (w_t - b) + \frac{1}{1-\frac{1}{\psi}} [\ldots]^{\frac{1}{1-\frac{1}{\psi}}-1} \beta \frac{1-\frac{1}{\psi}}{1-\gamma} \left(\mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{1-\gamma} \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}-1} \cdot \\
&\quad \cdot \mathbb{E}_t \left[(1-\gamma) \left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{-\gamma} \frac{\partial J^h}{\partial N}(N_{t+1}, Z_{t+1}, S_{t+1})(1 - \omega - \kappa_{w,t}) \right] = \\
&= \lambda_t (w_t - b) + (1 - \omega - \kappa_{w,t}) \left(J^h(N_t, Z_t, S_t) \right)^{\frac{1}{\psi}} \beta \left(\mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{1-\gamma} \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}-1} \cdot \\
&\quad \cdot \mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{-\gamma} \frac{\partial J^h}{\partial N}(N_{t+1}, Z_{t+1}, S_{t+1}) \right]
\end{aligned}
$$

so that $\xi^h_t := \frac{1}{\lambda_t} \frac{\partial J^h}{\partial N}(N_t, Z_t, S_t)$ now satisfies the recursion

$$
\begin{aligned}
\xi^h_t &= w_t - b + (1 - \omega - \kappa_{w,t}) \left(J^h(N_t, Z_t, S_t) \right)^{\frac{1}{\psi}} \beta \left(\mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{1-\gamma} \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}-1} \cdot \\
&\quad \cdot \mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{-\gamma} \frac{\lambda_{t+1}}{\lambda_t} \xi^h_{t+1} \right] = \\
&= w_t - b + (1 - \omega - \kappa_{w,t}) \left(J^h(N_t, Z_t, S_t) \right)^{\frac{1}{\psi}} \beta \left(\mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{1-\gamma} \right] \right)^{\frac{\gamma-\frac{1}{\psi}}{1-\gamma}} \cdot \\
&\quad \cdot \mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{-\gamma} \left(\frac{J^h(N_{t+1}, Z_{t+1}, S_{t+1})}{J^h(N_t, Z_t, S_t)} \right)^{\frac{1}{\psi}} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \xi^h_{t+1} \right] = \\
&= w_t - b + (1 - \omega - \kappa_{w,t}) \beta \left(\mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{1-\gamma} \right] \right)^{\frac{\gamma-\frac{1}{\psi}}{1-\gamma}} \cdot \\
&\quad \cdot \mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{\frac{1}{\psi}-\gamma} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \xi^h_{t+1} \right] = \\
&= w_t - b + (1 - \omega - \kappa_{w,t}) \mathbb{E}_t \left[\underbrace{\beta \left(\frac{J^h(N_{t+1}, Z_{t+1}, S_{t+1})}{\left(\mathbb{E}_t \left[(J^h(N_{t+1}, Z_{t+1}, S_{t+1}))^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}}} \right)^{\frac{1}{\psi}-\gamma} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}}}_{= M^{EZ}_{t,t+1}} \xi^h_{t+1} \right],
\end{aligned}
$$

which differs from equation (4.2.8) again only by the stochastic discount factor. Finally, adjusting the stochastic discount factor accordingly in the representative firm's objective function (4.2.14), yields the same conditions as in (4.2.15)-(4.2.21) with $\beta\frac{\lambda_{t+1}}{\lambda_t}$ replaced by $M^{EZ}_{t,t+1}$. Note further that the household's value function in equilibrium must be independent of the amount of outstanding shares S_t since neither total dividend payments less expenditures on shares, $d_tS_t - v_t(S_{t+1}-S_t) \underset{(4.2.12)}{=} \pi_t$, nor the firm value, $FV_t = S_{t+1}v_t = N_{t+1}\hat{\xi}^f_t$, depend on the amount of outstanding shares. Therefore, we may use the simplified notation of just $J^h(N_t, Z_t)$ instead.

Since the stochastic discount factor now contains the household's value function J^h, the model's equilibrium conditions can no longer be reduced to a single functional equation for the policy function g of the present value $\hat{\xi}^f$ of a worker to the firm in the next period. Instead, the policy function g of $\hat{\xi}^f$ and the household's value function J^h have to mutually solve the two functional equations

$$R_1(J^h, g, x, z) := lhs_1(J^h, g, x, z) - rhs_1(J^h, g, x, z) = 0,$$
$$R_2(J^h, g, x, z) := lhs_2(J^h, g, x, z) - rhs_2(J^h, g, x, z) = 0,$$

for all $x \in [0,1], z \in \mathbb{R}$, with

$$lhs_1(J^h, g, x, z) := J^h(x, z),$$
$$rhs_1(J^h, g, x, z) := \left[(1-\beta)C_t^{1-\frac{1}{\psi}} + \beta\left(\mathbb{E}\left[\left(J^h(N_{t+1}, Z_{t+1})\right)^{1-\gamma}\right]\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right]^{\frac{1}{1-\frac{1}{\psi}}},$$

and

$$lhs_2(J^h, g, x, z) := g(x, z),$$
$$rhs_2(J^h, g, x, z) := \mathbb{E}\left[\beta\left(\frac{J^h(N_{t+1}, Z_{t+1})}{\left(\mathbb{E}\left[(J^h(N_{t+1}, Z_{t+1}))^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}}\right)^{\frac{1}{\psi}-\gamma}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\cdot\right.$$
$$\left.\cdot\left(\exp(Z_{t+1}) - w_{t+1} + (1-\omega)g(N_{t+1}, Z_{t+1})\right)\right],$$

where C_t, N_{t+1}, Z_{t+1} and w_{t+1} in rhs_1 and rhs_2 are short for the expressions of the variables dependent on $x = N_t, z = Z_t$, $g(x,z) = \hat{\xi}^f_t$ and the innovation $\epsilon \sim N(0, \sigma^2_\epsilon)$ as before.[29]

In order to approximate both, the value function J^h as well as the policy function g, we adapt the mean weighted residual methods described in section 4.3 to the case of two functions mutually solving the two above functional equations. Since the procedure is completely analogous, we keep the explanation short. First, we replace all expectations appearing in rhs_1 and rhs_2 with corresponding Gauss-Hermite quadrature formulas with $n = 13$ nodes. We then choose degrees of freedom K_1 and K_2 and restrict the approximations to parameterized families

$$\{\hat{J}^h_{a_J} : X \to \mathbb{R} | a_J \in \mathbb{R}^{K_1}\} \quad \text{and} \quad \{\hat{g}_{a_g} : X \to \mathbb{R} | a_g \in \mathbb{R}^{K_2}\}.$$

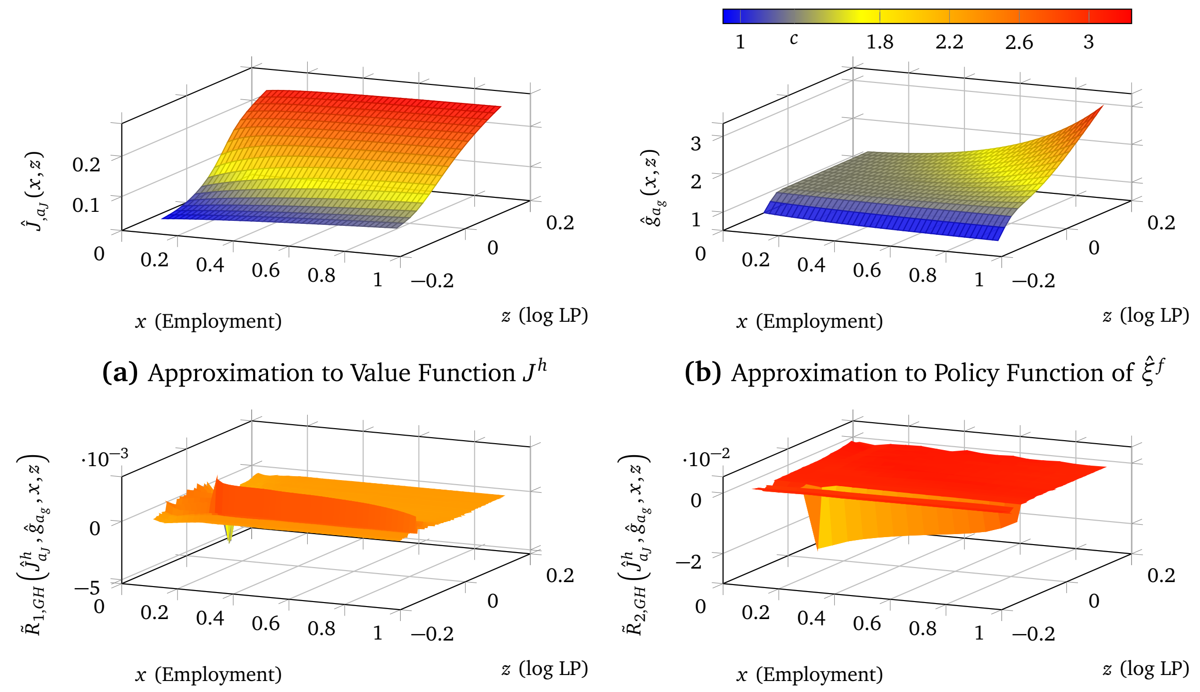
After choosing as many test functions $\Psi_{1,i}$ and $\Psi_{2,i}$ as degrees of freedom, the parameters in the

[29] For numerical reasons we use the following expression in the accompanying `Matlab` code:

$$rhs_1 = \left[(1-\beta)C_t^{1-\frac{1}{\psi}} + \beta\left(\left(\mathbb{E}\left[\left(J^h_{t+1}/J^h_t\right)^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}} J^h_t\right)^{1-\frac{1}{\psi}}\right]^{\frac{1}{1-\frac{1}{\psi}}}$$

$$rhs_2(J^h, g, x, z) := \mathbb{E}\left[\beta\left(\frac{J^h_{t+1}/J^h_t}{\left(\mathbb{E}\left[\left(J^h_{t+1}/J^h_t\right)^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}}\right)^{\frac{1}{\psi}-\gamma}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\left(\exp(Z_{t+1}) - w_{t+1} + (1-\omega)g(N_{t+1}, Z_{t+1})\right)\right].$$

Figure 4.14: Epstein-Zin Preferences: Approximation of Policy and Value Function



(a) Approximation to Value Function J^h

(b) Approximation to Policy Function of $\hat{\xi}^f$

(c) Interpretable Euler Residuals of R_1

(d) Interpretable Euler Residuals of R_2

approximations are determined as the solution to the system of equations

$$\int_{[\underline{x},\bar{x}]\times[\underline{z},\bar{z}]} R_1\left(\hat{J}^h_{a_J},\hat{g}_{a_g},x,z\right)\Psi_{1,i}(x,z)w(x,z)d(x,z)=0,\quad i=1,\dots,K_1,\text{ and}$$
$$\int_{[\underline{x},\bar{x}]\times[\underline{z},\bar{z}]} R_2\left(\hat{J}^h_{a_J},\hat{g}_{a_g},x,z\right)\Psi_{2,i}(x,z)w(x,z)d(x,z)=0,\quad i=1,\dots,K_2.$$

The obtained approximations by a finite element collocation method, where we set $\gamma=10$ and $\psi=1.5$ as in Bansal and Yaron (2004) and Kuehn et al. (2012), are pictured in the upper part of figure 4.14. Comparing to the results with additive time separable preferences in figure 4.1 shows that, although the shape of the policy function of $\hat{\xi}^f$ remains similar for the most part, there is one major difference. For values of log LP less than approximately -0.115 the present value of a worker in the next period to the firm now falls below the costs of posting a vacancy so that the non-negativity constraint on open vacancies becomes binding. The policy function of $\hat{\xi}^f$ apparently turns out to be non-smooth displaying a kink at the points where the amount of open vacancies falls to zero. In order to allow for the approximation to better copy this pattern, we used piecewise linear functions between the grid points $(d_x=50, d_z=43)$ instead of a cubic C^2-spline. However, computing an appropriate approximation turned out much more difficult nonetheless. In particular, the interpretable Euler residuals of the functional equation $R_2=0$ fall down to approximately -0.03 near the kink as shown in the lower right part of figure 4.14.[30] I.e. consumption would have to be decreased by approximately 3% in

[30] The interpretable Euler residuals for the functional equation $R_2=0$ are computed as before. For the functional equation $R_1=0$ one could introduce a similar definition in terms of the consumption good. However, since the weight, $1-\beta$, of C_t in rhs_1 is small, even comparable small absolute residuals of R_1 would imply that large changes in consumption are necessary for the functional equation to be satisfied exactly. Yet, the functional equation $R_1=0$ only contains the definition of the value function J^h which in turn is only needed to compute part of the stochastic discount factor, while all the other variables are derived from $\hat{\xi}^f_t$. In order to display inaccuracies in the part of the stochastic discount factor containing the value function, it is sufficient to consider relative deviations of J^h. We therefore set $\tilde{R}_1 := \frac{lhs_1}{rhs_1}-1$.

Table 4.7: Disasters in the Baseline Model with Epstein-Zin preferences

	number of disasters	total number of disaster periods	disaster probability	average disaster size	average disaster duration
90% threshold					
C	5694	67378	1.71%	22.19%	11.83
Y	6630	76551	2.05%	21.46%	11.55
LP	663	2086	0.17%	11.05%	3.15
85% threshold					
C	3559	43486	1.00%	29.43%	12.22
Y	4079	48793	1.16%	28.64%	11.96
LP	5	10	0.0013%	15.68%	2
80% threshold					
C	2402	30584	0.65%	36.31%	12.73
Y	2663	33709	0.73%	35.73%	12.66
LP	0	0	0%	–	–
75% threshold					
C	1680	22751	0.45%	42.87%	13.54
Y	1878	24730	0.50%	41.96%	13.17
LP	0	0	0%	–	–
70% threshold					
C	1221	17649	0.32%	49.22%	14.45
Y	1331	18905	0.35%	48.53%	14.20
LP	0	0	0%	–	–
50% threshold					
C	478	8388	0.12%	69.03%	17.55
Y	505	8729	0.13%	68.57%	17.29
LP	0	0	0%	–	–
30% threshold					
C	230	4166	0.0581%	81.88%	18.11
Y	232	4311	0.0586%	82.12%	18.58
LP	0	0%	–	–	0

order for the Euler equation to hold exactly at these points. Yet, larger residuals appear only around the kink and on the edges while the approximation seems better for the remaining parts. We could not increase accuracy of the approximation by adding nodes to the grid near the kink since the solution algorithm failed to find the zeros from the system of nonlinear equations defining the free parameter values in this case. Moreover, in long simulations of the model's equilibrium outcomes the unemployment rate falls even below the lower bound of 0.024 we used for this approximation. Using linear extrapolation the value of $\hat{\xi}^f$ declines too strongly in these periods leading the unemployment rate to eventually converge to zero. On the other hand, relying on cubic spline extrapolation the value of $\hat{\xi}^f$ declines too slowly so that the same is true for the amount of open vacancies and, given very low employment rates and LP, consumption may become even negative. We therefore artificially prevent the employment rate from falling below 0.024 in simulations. Again, we could not extend the approximation to cover even lower rates of employment since the system of nonlinear equations could not be solved anymore. Yet, we note that the upcoming results regarding the equity premium for Epstein-Zin preferences do not critically hinge on periods where LP is low or the unemployment rate is high.

The disaster statistics with Epstein-Zin preferences are summarized in table 4.7. Compared to the case of standard preferences, disasters appear more frequently and are even more drastic in size but do not last as long on average.

4.5.2 Equity Premium in the Baseline Model

The fact that the baseline model produces rare but severe economic downturns suggests that the model can potentially contribute to resolve the equity premium puzzle.[31] Rietz (1988),

[31] see Mehra and Prescott (1985).

Barro (2006) or Gourio (2012, 2013) are able to predict sizeable risk premia by introducing a possibility of economic disasters exogenously. The question therefore is, whether the baseline model with the endogenously arising rare but huge declines in consumption can also replicate the size of the historically observed U.S. risk premium summarized in table 4.8, taken from Mehra and Prescott (2003).

Table 4.8: U.S. equity premium in the data

real return on a market index (mean)	real return on a relatively riskless security (mean)	equity premium (mean)
8.06%	1.14%	6.92%

Source: Mehra and Prescott (2003), Table 1. For the time period 1889-2000.

Return on Equity The return on equity in the model is defined by

$$R^e_{t+1} = \frac{d_{t+1} + v_{t+1}}{d_t}.$$

We have already observed in equation (4.2.13) that the ex-dividend firm value equals the present value of future profits. Moreover, multiplying equation (4.2.19) with V_t and using $\mu_t V_t = 0$ yields

$$cV_t = \kappa_{f,t} V_t \hat{\xi}^f_t = M_t \hat{\xi}^f_t = (N_{t+1} - (1-\omega)N_t)\, \hat{\xi}^f_t$$

so that the firm's profits as defined in equation (4.2.11) equal

$$\pi_t = \exp(Z_t)N_t - w_t N_t - (N_{t+1} - (1-\omega)N_t)\, \hat{\xi}^f_t = N_t \left(\exp(Z_t) - w_t + (1-\omega)\hat{\xi}^f_t\right) - N_{t+1}\hat{\xi}^f_t. \tag{4.5.2}$$

Thus, the firm value can also be written as[32]

$$\begin{aligned} FV_t &= \mathbb{E}_t\left[\sum_{s=1}^{\infty} \beta^s \frac{\lambda_{t+s}}{\lambda_t} N_{t+s}\left(\exp(Z_{t+s}) - w_{t+s} + (1-\omega)\hat{\xi}^f_{t+s}\right)\right] - \mathbb{E}_t\left[\sum_{s=1}^{\infty} \beta^s \frac{\lambda_{t+s}}{\lambda_t} N_{t+s+1}\hat{\xi}^f_{t+s}\right] = \\ &= \mathbb{E}_t\left[\sum_{s=0}^{\infty} \beta^{s+1} \frac{\lambda_{t+s+1}}{\lambda_t} N_{t+s+1}\left(\exp(Z_{t+s+1}) - w_{t+s+1} + (1-\omega)\hat{\xi}^f_{t+s+1}\right)\right] \\ &\qquad - \mathbb{E}_t\left[\sum_{s=1}^{\infty} \beta^s \frac{\lambda_{t+s}}{\lambda_t} N_{t+s+1}\hat{\xi}^f_{t+s}\right] = \\ &= \mathbb{E}_t\left[\sum_{s=0}^{\infty} \beta^s \frac{\lambda_{t+s}}{\lambda_t} N_{t+s+1} \mathbb{E}_{t+s}\left[\beta \frac{\lambda_{t+s+1}}{\lambda_{t+s}}\left(\exp(Z_{t+s+1}) - w_{t+s+1} + (1-\omega)\hat{\xi}^f_{t+s+1}\right)\right]\right] \\ &\qquad - \mathbb{E}_t\left[\sum_{s=1}^{\infty} \beta^s \frac{\lambda_{t+s}}{\lambda_t} N_{t+s+1}\hat{\xi}^f_{t+s}\right] = \\ &\overset{(4.2.21)}{=} \mathbb{E}_t\left[\sum_{s=0}^{\infty} \beta^s \frac{\lambda_{t+s}}{\lambda_t} N_{t+s+1}\hat{\xi}^f_{t+s}\right] - \mathbb{E}_t\left[\sum_{s=1}^{\infty} \beta^s \frac{\lambda_{t+s}}{\lambda_t} N_{t+s+1}\hat{\xi}^f_{t+s}\right] = N_{t+1}\hat{\xi}^f_t. \end{aligned}$$

Using once more (4.5.2), it follows that the firm value at the beginning of period t, i.e. including period t's profits, is given by

$$\begin{aligned} FV_t^{\text{bop}} &:= \pi_t + FV_t = N_t\left(\exp(Z_t) - w_t + (1-\omega)\hat{\xi}^f_t\right) - N_{t+1}\hat{\xi}^f_t + N_{t+1}\hat{\xi}^f_t = \\ &= N_t\left(\exp(Z_t) - w_t + (1-\omega)\hat{\xi}^f_t\right). \end{aligned}$$

[32]The derivation applies analogously for Epstein-Zin preferences with $\beta\frac{\lambda_{t+1}}{\lambda_1}$ replaced by $M^{EZ}_{t,t+1}$.

The return on equity thus equals

$$
\begin{aligned}
R^e_{t+1} &= \frac{d_{t+1}+v_{t+1}}{d_t} = \frac{S_{t+1}d_{t+1}+S_{t+1}v_{t+1}}{S_{t+1}d_t} \overset{(4.2.12)}{=} \frac{\pi_{t+1}+(S_{t+2}-S_{t+1})v_{t+1}+S_{t+1}v_{t+1}}{S_{t+1}d_t} = \\
&= \frac{\pi_{t+1}+S_{t+2}v_{t+1}}{S_{t+1}d_t} = \frac{FV^{bop}_{t+1}}{FV_t} = \frac{\exp(Z_{t+1})-w_{t+1}+(1-\omega)\hat{\xi}^f_{t+1}}{\hat{\xi}^f_t}.
\end{aligned}
\tag{4.5.3}
$$

Risk Free Return The price v^f_t of an asset which guarantees with certainty one unit of the consumption good in the next period has to satisfy

$$v^f_t = \mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\cdot 1\right]$$

with standard preferences and

$$v^f_t = \mathbb{E}_t\left[M^{EZ}_{t,t+1}\cdot 1\right]$$

with Epstein-Zin preferences. The risk free return defined by $R^f_t := \frac{1}{v_f}$ can therefore be computed by

$$R^f_t = \frac{1}{\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\right]} \quad \text{or} \quad R^f_t = \frac{1}{\mathbb{E}_t\left[M^{EZ}_{t,t+1}\right]}$$

where we use Gauss-Hermite quadrature with 13 nodes to numerically compute the expectation.

Equity-Premium The equity premium in the model is computed as the average excess return on equity beyond the risk free rate, $R^e_{t+1} - R^f_t$.

Results Table 4.9 summarizes the average annualized return on equity, the average annualized risk free return as well as the average annualized equity premium in the baseline model. Although the baseline model generates huge declines in consumption indifferent of the fact

Table 4.9: Annualized Equity Premium in the Baseline Model

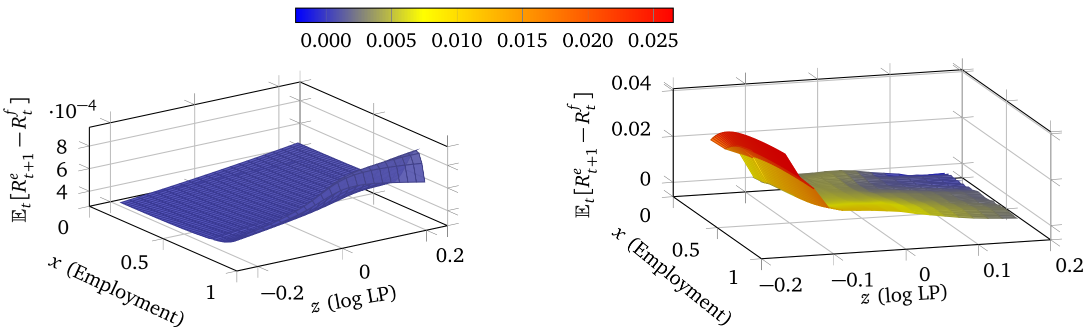
	$\eta = 2$	$\eta = 10$	EZ preferences (RRA=10, EIS=1.5)
$((\bar{R}^f)^{12}-1)100\%$	4.12%	2.02%	0.99%
$((\bar{R}^e)^{12}-1)100\%$	4.27%	2.85%	10.51%
EP	0.15%	0.81%	9.43%

Notes: $\bar{R}^f = \frac{1}{T-1}\sum_{t=1}^{T-1} R^f_t$=average monthly risk free return rate, $\bar{R}^e = \frac{1}{T-1} = \sum_{t=1}^{T-1} R^e_t$=average monthly return on equity, EP=$((1+\bar{R}^e-\bar{R}^f)^{12}-1)100\%$. All return rates are computed from the outcome of a simulation of 1200000 (monthly) periods.

whether additive time separable preferences or Epstein-Zin preferences are assumed, a sizeable equity premium results only with Epstein-Zin preferences even when the coefficient of relative risk aversion is the same. Under Epstein-Zin preferences, the average risk free rate is close to the value found in the data, while the return on equity and the equity premium even exceed their empirical counterparts by approximately 2.5 percentage points.

Two questions arise. In which regard does the model differ from a framework as in Rietz (1988) so that no significant equity premium is generated under standard preferences despite the fact that huge declines in consumption occur? And what changes once Epstein-Zin preferences are assumed?

Figure 4.15: Monthly Equity Premium in the Model



(a) Additive Time Separable CRRA ($\eta = 10$) **(b)** Epstein-Zin Preferences (RRA=10, EIS=1.5)

We show the monthly equity premium (as a decimal number), $\mathbb{E}_t[R^e_{t+1} - R^f_t]$, dependent on the state variables in figure 4.15. In the case of additive time separable preferences the equity premium is throughout only of magnitude 10^{-4}; only moderate increases for relative high rates of employment are visible. On the other hand, with Epstein-Zin preferences the expected excess of the (monthly) return on equity beyond the risk free rate is higher by at least an order of magnitude at around 0.8% for most states in the model. It moreover harshly increases even further at lower levels of LP where the non-negativity constraint on open vacancies is binding. Note however that these states where the equity premium rises up to 2% per month, i.e. approximately 26% per year, are not essential for the sizeable equity premium in the model with Epstein-Zin preferences. For example, computing the average return rates only from periods where the state variables are not too far from steady state, i.e. more specifically where employment is in $[0.8; 0.95]$ and log LP in $[-0.05; 0.05]$, yields the annualized average risk free rate, the annualized average return on equity as well as the average annualized equity premium to be $1.35\%, 11.55\%$ and 10.08%, respectively. Hence, although the results for Epstein-Zin preferences have to be considered with some caution due to substantially larger errors in the approximations in particular for lower LP and lower unemployment rates, there seems to be strong evidence for a sizeable equity premium which is mainly generated around the steady state, where the errors are only small, nonetheless.

In order to analyze why the (monthly) risk premium is mostly close to zero with standard preferences but rises to approximately 0.8% in most states with Epstein-Zin preferences, we consider the *risky* steady state to provide an example.[33] First, note that the equity premium equals

$$\mathbb{E}_t\left[R^e_{t+1} - R^f_t\right] = -\frac{\mathbf{Cov}\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\eta}, R^e_{t+1}\right]}{\mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\eta}\right]}$$

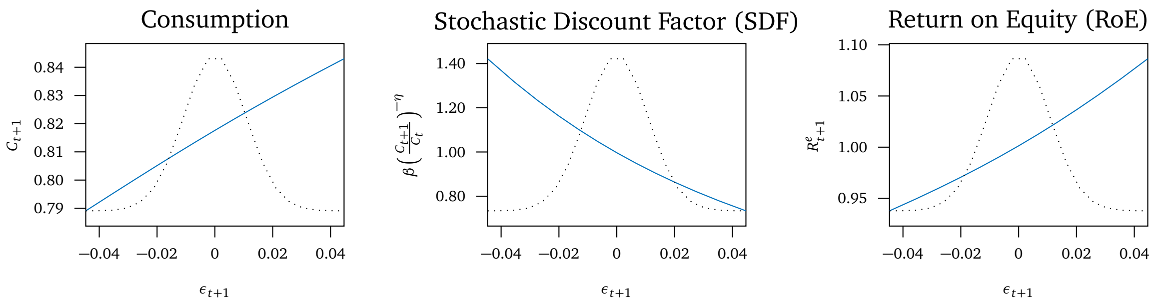
and

$$\mathbb{E}_t\left[R^e_{t+1} - R^f_t\right] = \frac{\mathbf{Cov}\left[M^{EZ}_{t,t+1}, R^e_{t+1}\right]}{\mathbb{E}_t\left[M^{EZ}_{t,t+1}\right]}$$

respectively. Figure 4.16 shows, for the case of standard preferences with $\eta = 10$, next period's consumption, the stochastic discount factor as well as the return on equity dependent on the realization of the $N(0, \sigma^2_\epsilon)$ distributed shock ϵ_{t+1} to LP when the economy is in its risky steady state in the current period. From period t's point of view, next period's employment rate is

[33] We define the risky steady state as the fix point of the approximation to the dynamic of employment in the model when LP is set to one. We consider the risky steady state as an example in order to abstract from effects induced by a change in the employment rate even when no shock hits the economy.

Figure 4.16: Intuition Equity Premium: Additive Time Separable CRRA ($\eta = 10$)



already determined and the only uncertainty regarding period $t+1$ lies in the level of LP. Consequently, output may only fluctuate proportionally to $\exp(\epsilon_{t+1})$ and since the amount of open vacancies is procyclical the effect on consumption C_{t+1} is even dampened. In particular, for the displayed range of ϵ_{t+1} the lottery over next period's consumption is limited to approximately 2.5% increases or decreases over this period's consumption level.[34] Compared to models where disasters in $t+1$ are the result of an exogenous, in period t yet to be observed, shock leading to a sudden and drastic decline in consumption from one period into the next one, the uncertainty about next period's consumption in the present model is only very limited. Severe declines in consumption may occur only gradually over a longer time span of decreasing employment. Although the model may exhibit much uncertainty regarding employment in the long run, there is no uncertainty regarding next period's employment rate. Assuming a relatively high coefficient of relative risk aversion, $\eta = 10$, fluctuations in the stochastic discount factor are amplified, yet its standard deviation is limited to approximately

$$\sqrt{\mathbf{Var}\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\eta}\right]} \approx 0.0574$$

while

$$\mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\eta}\right] \approx 0.9990.$$

Moreover, fluctuations in the return on equity are even smaller with a standard deviation of

$$\sqrt{\mathbf{Var}\left[R^e_{t+1}\right]} \approx 0.0127.$$

Consequently, even though the stochastic discount factor and the return on equity are almost perfectly negatively correlated with

$$\mathbf{Corr}\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\eta}, R^e_{t+1}\right] \approx -0.9953$$

the risk premium can account only to

$$\mathbb{E}_t\left[R^e_{t+1} - R^f_t\right] \approx \frac{0.9953 \cdot 0.0574 \cdot 0.0127}{0.9990} \approx 0.0007 = 0.07\%.$$

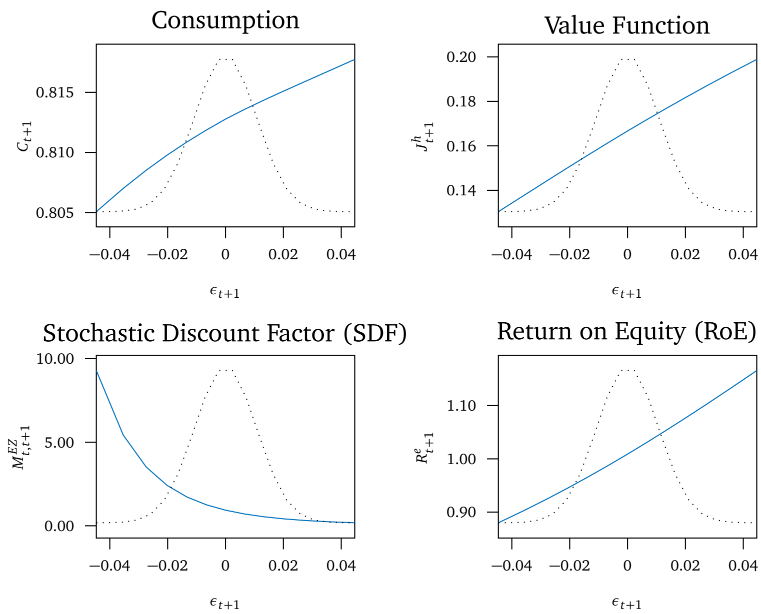
Hence, the annualized equity premium is only 0.84% in the risky steady state, similar to the average value from a long simulation of the model in table 4.9. We identify a lack of variability in next period's stochastic discount factor implied by a lack of variability in next period's

[34] The displayed range for ϵ_{t+1} includes all nodes appearing in an Gauss-Hermite quadrature with n=13 nodes and therefore all numerically relevant outcomes in the computation of the results.

consumption level as well as only small fluctuations in the return on equity as the main factors for the model not generating a substantial risk premium.

So what changes if Epstein-Zin preferences are assumed? Figure 4.16 displays next period's realizations for the same variables already discussed for the standard case, but also for the household's lifetime utility, dependent on the $N(0, \sigma_\epsilon^2)$ distributed shock ϵ_{t+1} to LP when the economy is again in the risky steady state in the current period. First, we observe that next

Figure 4.17: Intuition Equity Premium: Epstein-Zin Preferences (RRA=10, EIS=1.5)



period's consumption is now subject to even less fluctuations. Within the displayed range consumption may rise or fall only by approximately 0.6% compared to the current period. However, the lottery over next period's lifetime utility turns out to be much more risky with gains or losses by approximately 15%. Consequently, the stochastic discount factor, which now contains the factor

$$\left(\frac{J^h_{t+1}}{\left(\mathbb{E}_t \left[\left(J^h_{t+1} \right)^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}}} \right)^{\frac{1}{\psi} - \gamma}$$

regarding lotteries over next period's lifetime utility, underlies much stronger fluctuations. More concretely, now

$$\sqrt{\mathbf{Var}\left[M^{EZ}_{t,t+1} \right]} \approx 0.3572$$

is more than 6 times higher than in the case of standard preferences while

$$\mathbb{E}_t \left[M^{EZ}_{t,t+1} \right] \approx 0.9991$$

remains very similar. An ad hoc interpretation would be that although the gradually developing disasters in the model imply only small risk regarding consumption one period ahead, the probability of entering a disaster in the longer run may change significantly depending on the shock's realization. The household's lifetime utility is strongly influenced by the risk of an eventual downturn in the economy approaching. Further, the higher uncertainty regarding the stochastic discount factor also has a second effect. The fact that the firm's expected future profits are discounted more or less strongly in the next period also implies more fluctuations in the firm value and ultimately in the return on equity (see 4.5.3). The standard deviation

$$\sqrt{\mathbf{Var}\left[R^e_{t+1}\right]} \approx 0.0250$$

doubles. Although the stochastic discount factor and the return on equity are less negatively correlated with

$$\mathbf{Corr}\left[M^{EZ}_{t,t+1}, R^e_{t+1}\right] \approx -0.9519$$

the risk premium rises due to more volatility in both variables by an order of magnitude to

$$\mathbb{E}_t\left[R^e_{t+1} - R^f_t\right] \approx \frac{0.9519 \cdot 0.3572 \cdot 0.0250}{0.9991} \approx 0.0085 = 0.85\%.$$

This already corresponds to a large annualized equity premium of approximately 10.69%, even one percentage point above the average value from a long simulation given in table 4.9.

4.5.3 Second Moments of the Labor Market

Finally, we want to analyze the model's ability to replicate labor market moments. Shimer (2005) showed that in a textbook search and matching model, similar to the one considered here, unemployment, vacancies and the labor market tightness are far less volatile than in the data. The variables' standard deviations in simulations of the model are only approximately 10% of the values empirically observed. This gives rise to the so called Shimer puzzle. Hagedorn and Manovskii (2008) have argued that with a high value of unemployment activities and a low bargaining power of the household the model is consistent with the data.[35] We evaluate whether the model is able to generate the observed fluctuations.

First, table 4.10a, taken from Kuehn et al. (2012), provides some facts on the labor market moments. In order to evaluate the model's ability to replicate the second moments from the data, we simulated 5000 time series a 1666 (monthly) periods of the model's equilibrium outcomes starting in deterministic steady state. For each of the 5000 simulations, we throw away the first 1000 periods, convert the remaining observations to 222 quarterly averages and determine the moments from the cyclical components of the HP-filtered ($\lambda = 1600$) relative deviations from the mean. We report the average moments over the 5000 simulations along the 2.5% and 97.5% quantiles in tables 4.10b and 4.10c. It turns out that the volatility in the model is higher with Epstein-Zin preferences than with standard preferences. The average standard deviation of unemployment in the model among the 5000 simulations is slightly above the empirical counterpart with standard preferences but substantially higher with Epstein-Zin preferences. However, the empirical value lies between the 2.5% and 97.5% quantiles from the model in both cases. Yet, it should also be noted that the high volatility of unemployment in the model is primarily caused by infrequent but extreme deviations as opposed to more frequent but modestly sized deviations (the kurtosis of the cyclical components from the HP-filtered relative deviations from the mean is 8.67 and 13.09, respectively). It is questionable if this holds in this form for the empirical counterpart. For both preference structures, the average standard deviation of vacancies and the labor market tightness in the model are too low. With the exception of the standard deviation of vacancies for Epstein-Zin preferences, the empirical values even exceed the 97.5% quantiles in the model. Hence, although vacancies and the labor market tightness in the present model are more volatile than in Shimer (2005), who finds the standard deviations to be less than 10% of the value in the data, the volatility in the labor market model is still too

[35] Note however that Hagedorn and Manovskii (2008) set $b = 0.955$ even higher.

Table 4.10: Labor Market Moments in the Data and the Baseline Model

(a) Labor Market Moments in the Data

	U	V	θ	$p = \frac{Y}{N}$
s_x	0.119	0.134	0.255	0.012
$\frac{s_x}{s_p}$	9.917	11.167	21.250	1
r_x	0.902	0.922	0.889	0.761
		Cross Correlations		
U	1	-0.913	-0.801	-0.224
V		1	0.865	0.388
θ			1	0.299

Notes: U=unemployment, V=vacancies, $\theta = \frac{V}{U}$=labor market tightness, p=output per worker, s_x=standard deviation of variable X, r_x=first order autocorrelation of variable X. All moments are computed from the cyclical components of the HP-filtered ($\lambda = 1600$) relative deviations from the mean, i.e. as the cyclical component of the series $\frac{X_t - \bar{X}}{\bar{X}}$. The moments were computed from weekly series from January 1951 to June 2006 converted to 222 quarterly averages. Source: Kuehn et al. (2012), Table 3.

(b) Labor Market Moments in the Baseline Model with additive time separable preferences ($\eta = 2$)

	U	V	θ	$p = \frac{Y}{N}$
s_x	0.131 [0.071; 0.204]	0.088 [0.067; 0.117]	0.164 [0.115; 0.239]	0.016 [0.013; 0.019]
$\frac{s_x}{s_p}$	8.167 [4.653; 12.374]	5.501 [4.496; 6.924]	10.203 [7.754; 14.182]	1.000 [1.000; 1.000]
r_x	0.913 [0.837; 0.960]	0.724 [0.621; 0.810]	0.843 [0.775; 0.896]	0.774 [0.693; 0.840]
		Cross Correlations		
U		-0.572 [−0.716; −0.408]	-0.687 [−0.869; −0.405]	-0.541 [−0.789; −0.217]
V			0.889 [0.778; 0.949]	0.972 [0.900; 0.987]
θ				0.923 [0.740; 0.987]

(c) Labor Market Moments in the Baseline Model with Epstein-Zin preferences (RRA=10, EIS=1.5)

	U	V	θ	$p = \frac{Y}{N}$
s_x	0.178 [0.087; 0.310]	0.115 [0.077; 0.183]	0.186 [0.129; 0.275]	0.016 [0.013; 0.019]
$\frac{s_x}{s_p}$	11.092 [5.711; 18.918]	7.169 [5.054; 11.091]	11.580 [8.648; 16.386]	1.000 [1.000; 1.000]
r_x	0.900 [0.813; 0.957]	0.651 [0.520; 0.772]	0.813 [0.740; 0.872]	0.774 [0.693; 0.840]
		Cross Correlations		
U		-0.480 [−0.638; −0.347]	-0.626 [−0.839; −0.321]	-0.554 [−0.806; −0.204]
V			0.823 [0.642; 0.924]	0.913 [0.788; 0.951]
θ				0.963 [0.820; 0.997]

Notes: U=unemployment, V=vacancies, $\theta = \frac{V}{U}$=labor market tightness, p=output per worker, s_x=standard deviation of variable X, r_x=first order autocorrelation of variable X. All moments are computed from the cyclical components of the HP-filtered ($\lambda = 1600$) relative deviations from the mean, i.e. as the cyclical component of the series $\frac{X_t - \bar{X}}{\bar{X}}$. The table shows the average as well as the 2.5% and 97.5% quantiles from 5000 simulations of the model's equilibrium outcomes for 666 periods converted to 222 quarterly observations after 1000 throw-away periods.

low compared to the data. Moreover, the fact that the volatility in the model is mainly induced by rare outliers may further be in sharp contrast to the data. Finally, the model reproduces the negative correlation between the unemployment rate and vacancies from the Beveridge curve, yet also not in the magnitude found in the data. The empirical value lies below the 2.5% quantile from simulations for both preference structures.

4.6 Analysis of Endogenous Disasters in the Baseline Model

In this section, we want to find the key assumptions incorporated in the baseline model which prove essential for the occurrence of disasters and therefore drive the results in the labor market model. We will then check how the results change if these assumptions are modified in the extensions of the model.

4.6.1 Variants of the Baseline Model

In order to get a better understanding of which assumptions included in the baseline model lead to the periods of high unemployment and disasters in consumption and output, we want to analyze four basic variations of the model. In particular, we first consider a corresponding social planner's problem for which the unemployment rate proves to be stable around the steady state value. The baseline model differs from the social optimum by three distortions. We will then add these separately to the social optimal case. For each of the versions, we compare the results obtained by a global solution to the results from a second order perturbation solution and compute the disaster statistics. We want to single out, which of the three distortions is the main cause for the fact that the economy drifts to periods of huge unemployment which render the perturbation solution inadequate and imply disasters in consumption and output. When computing the global solution, we have stuck throughout to the method relying on cubic splines and Gauss-Hermite quadrature.

4.6.1.1 Social Planner

The first version of the baseline model is a corresponding social planner problem. We want to establish, whether the periods of high unemployment and low consumption also occur when social welfare is maximized or if they are the result of some distortions from social optimum in the baseline model.

Social Planner Problem Consider a social planner maximizing the representative household's lifetime utility (4.2.3). The representative household consists again of a unity mass of members. Each period t only the fraction N_t of the representative household's members can take part in the workforce, producing output according to the linear production technology (4.2.9). Members not capable of taking part in the production process, forming mass $U_t = 1 - N_t$, can be trained to eventually become productive in the next period. In order to train them, training vacancies V_t are necessary, which generate costs $c > 0$ lost from total output. The friction on the labor market implied by the fact that transitions from unemployment to taking part in production demand costly effort hence still persists. All of the unemployed members take part in training leading to a total of $M_t = M(U_t, V_t)$ newly skilled members in the next period. Moreover, each period the fraction $\omega \in (0,1)$ of the working members of the representative household loses their working ability. As in the baseline model, we can assume without any loss of generality that $U_t > 0$ for all $t \in \mathbb{N}$ and use the notation

$$\kappa_{w,t} := \frac{M_t}{U_t}$$

for the fraction of successful training of the unemployed household members and

$$\kappa_{f,t} := \begin{cases} \frac{M_t}{V_t}, & \text{if } V_t > 0 \\ 1, & \text{if } V_t = 0 \end{cases}$$

for the rate of successful training relative to training vacancies. Summing up, if we denote the social planner's value function by J^{soc}, we get

$$\begin{aligned} J^{soc}(N_t) = \quad & \max_{C_t, V_t} \quad \frac{C_t^{1-\eta} - 1}{1-\eta} + \beta \mathbb{E}_t \left[J^{soc}\left((1-\omega)N_t + M(1-N_t, V_t)\right)\right] \\ & \text{s.t.} \quad C_t \leq \exp(Z_t)N_t - cV_t, \\ & \qquad\; V_t \geq 0, \\ & \text{given} \quad N_t. \end{aligned}$$

Note that different from the baseline model, where it was assumed that neither the households nor the firms in the economy coordinate their job searching and recruiting decisions and are too small to have influence on average probabilities on their own, the social planner exploits the exact form of the matching process. The budget constraint has to hold with equality in optimum again. The KKT conditions for the maximization problem on the right hand side thus are

$$\lambda_t = C_t^{-\eta}, \tag{4.6.1}$$

$$\lambda_t c = \beta \mathbb{E}_t \left[\frac{\partial J^{soc}}{\partial N}(N_{t+1}) \frac{\partial M}{\partial V}(U_t, V_t) \right] + \lambda_t \mu_t, \tag{4.6.2}$$

$$V_t \geq 0, \tag{4.6.3}$$

$$\lambda_t \mu_t \geq 0, \tag{4.6.4}$$

$$\lambda_t \mu_t V_t = 0. \tag{4.6.5}$$

where λ_t is the Lagrange multiplier of the budget constraint and $\lambda_t \mu_t$ is the KKT multiplier for the non-negativity constraint.[36] We introduce the following notation in the same fashion as in the baseline model

$$\xi_t^{soc} := \frac{1}{\lambda_t} \frac{\partial J^{soc}}{\partial N}(N_t) \quad \text{and} \quad \hat{\xi}_t^{soc} := \mathbb{E}_t \left[\beta \frac{\lambda_{t+1}}{\lambda_t} \xi_{t+1}^{soc} \right].$$

Now, ξ_t^{soc} denotes this period's marginal social value of an additional worker measured in consumption units, while $\hat{\xi}_t^{soc}$ denotes the present value of a worker in the next period from the social planner's point of view. Invoking the envelope theorem we can derive

$$\frac{\partial J^{soc}}{\partial N}(N_t) = \lambda_t \exp(Z_t) + \beta \mathbb{E}_t \left[\frac{\partial J^{soc}}{\partial N}(N_{t+1}) \left(1 - \omega - \frac{\partial M}{\partial U}(U_t, V_t) \right) \right]$$

so that

$$\begin{aligned} \xi_t^{soc} &= \exp(Z_t) + \left(1 - \omega - \frac{\partial M}{\partial U}(U_t, V_t) \right) \mathbb{E}_t \left[\beta \frac{\lambda_{t+1}}{\lambda_t} \xi_{t+1}^{soc} \right] \\ &= \exp(Z_t) + \left(1 - \omega - \frac{\partial M}{\partial U}(U_t, V_t) \right) \hat{\xi}_t^{soc}. \end{aligned} \tag{4.6.6}$$

The social value of a working member is the sum of this period's labor productivity and the continuation value of expected discounted next period's social value weighted with the probability of him remaining operative less the change in matches induced by one less unemployed member entering training. Consequently, we can write

$$\hat{\xi}_t^{soc} = \mathbb{E}_t \left[\beta \frac{\lambda_{t+1}}{\lambda_t} \left(\exp(Z_{t+1}) + \left(1 - \omega - \frac{\partial M}{\partial U}(U_{t+1}, V_{t+1}) \right) \hat{\xi}_{t+1}^{soc} \right) \right], \tag{4.6.7}$$

and the second KKT condition (4.6.2) can equivalently be stated as

$$c = \frac{\partial M}{\partial V}(U_t, V_t) \mathbb{E}_t \left[\beta \frac{\lambda_{t+1}}{\lambda_t} \xi_{t+1}^{soc} \right] + \mu_t = \frac{\partial M}{\partial V}(U_t, V_t) \hat{\xi}_t^{soc} + \mu_t. \tag{4.6.8}$$

Further, since $\lambda_t > 0$ the KKT conditions (4.6.4) and (4.6.5) are equivalent to $\mu_t V_t = 0$ and $\mu_t \geq 0$. The interpretation of the optimality conditions is analogous to the baseline model. First, $\frac{\partial M}{\partial V} \hat{\xi}_t^{soc}$ is the social return of a marginal training vacancy, i.e. the amount of newly skilled workers in the next period generated by an additional marginal training vacancy times the present value of a worker in the next period. Equation (4.6.8) then states that in the case of a

[36] We can write the KKT multiplier in this form since $\lambda_t > 0$.

positive amount of training vacancies being posted, i.e. $\mu_t = 0$, training vacancies are posted up to the point, where the return of a training vacancy equals its cost. Else, the KKT multiplier μ_t measures the amount by which the costs c exceed the return.

Finally, note that

$$\begin{aligned}\frac{\partial M}{\partial U}(U_t,V_t) &= \frac{\left(U_t^\tau + V_t^\tau\right)^{\frac{1}{\tau}} V_t - U_t V_t \frac{1}{\tau}\left(U_t^\tau + V_t^\tau\right)^{\frac{1}{\tau}-1} \tau U_t^{\tau-1}}{(U_t^\tau + V_t^\tau)^{\frac{2}{\tau}}} = \frac{\left(U_t^\tau + V_t^\tau\right) V_t - U_t^\tau V_t}{(U_t^\tau + V_t^\tau)^{\frac{1}{\tau}+1}} = \\ &= \frac{V_t^{\tau+1}}{(U_t^\tau + V_t^\tau)^{\frac{\tau+1}{\tau}}} = \left(\frac{M_t}{U_t}\right)^{\tau+1} = \kappa_{w,t}^{\tau+1}.\end{aligned} \tag{4.6.9}$$

and for symmetry reasons

$$\frac{\partial M}{\partial V}(U_t,V_t) = \frac{U_t^{\tau+1}}{(U_t^\tau + V_t^\tau)^{\frac{\tau+1}{\tau}}} = \kappa_{f,t}^{\tau+1}. \tag{4.6.10}$$

Equilibrium Summing up, the equilibrium conditions for the social planner's problem are

$$U_t = 1 - N_t, \tag{4.6.11}$$

$$M_t = \frac{U_t V_t}{\left(U_t^\tau + V_t^\tau\right)^{\frac{1}{\tau}}}, \tag{4.6.12}$$

$$N_{t+1} = (1-\omega)N_t + M_t, \tag{4.6.13}$$

$$Y_t = \exp(Z_t)N_t, \tag{4.6.14}$$

$$Y_t = C_t + cV_t, \tag{4.6.15}$$

$$\lambda_t = C_t^{-\eta}, \tag{4.6.16}$$

$$c = \frac{\partial M}{\partial V}(U_t,V_t)\hat{\xi}_t^{soc} + \mu_t, \tag{4.6.17}$$

$$\mu_t \geq 0, \tag{4.6.18}$$

$$V_t \geq 0, \tag{4.6.19}$$

$$\mu_t V_t = 0, \tag{4.6.20}$$

$$\hat{\xi}_t^{soc} = \mathbb{E}_t\left[\beta \frac{\lambda_{t+1}}{\lambda_t}\left(\exp(Z_{t+1}) + \left(1 - \omega - \frac{\partial M}{\partial U}(U_{t+1},V_{t+1})\right)\hat{\xi}_{t+1}^{soc}\right)\right], \tag{4.6.21}$$

where $\frac{\partial M}{\partial U}$ and $\frac{\partial M}{\partial V}$ are determined by (4.6.9) and (4.6.10), respectively.

Similar to the baseline model, all other variables in the model can again be expressed by $\hat{\xi}_t^{soc}$ and the state variables. First, by plugging in (4.6.10) and using $\mu_t = 0$ in case of $V_t > 0$ from (4.6.20) we can rewrite (4.6.17) equivalently as

$$c = \begin{cases} \dfrac{U_t^{\tau+1}}{(U_t^\tau + V_t^\tau)^{\frac{\tau+1}{\tau}}}\hat{\xi}_t^{soc}, & \text{if } V_t > 0 \\ \hat{\xi}_t^{soc} + \mu_t, & \text{if } V_t = 0, \end{cases}$$

In the first case, $V_t > 0$, we get

$$c = \left(\frac{U_t}{\left(U_t^\tau + V_t^\tau\right)^{\frac{1}{\tau}}}\right)^{\tau+1} \hat{\xi}_t^{soc} \quad \Leftrightarrow \quad \left(U_t^\tau + V_t^\tau\right)^{\frac{1}{\tau}} = U_t\left(\frac{\hat{\xi}_t^{soc}}{c}\right)^{\frac{1}{\tau+1}}$$

$$\Leftrightarrow \quad V_t^\tau = U_t^\tau \left(\left(\frac{\hat{\xi}_t^{soc}}{c} \right)^{\frac{\tau}{\tau+1}} - 1 \right) \quad \Leftrightarrow \quad V_t = (1-N_t) \left(\left(\frac{\hat{\xi}_t^{soc}}{c} \right)^{\frac{\tau}{\tau+1}} - 1 \right)^{\frac{1}{\tau}}.$$

Hence, $V_t > 0$ is satisfied if and only if $\hat{\xi}_t^{soc} > c$. In the second case, i.e. $V_t = 0$, equation (4.6.17) yields $\mu_t = c - \hat{\xi}_t^{soc}$, so that (4.6.18) is satisfied if and only if $\hat{\xi}_t^{soc} \leq c$. Summing up, setting

$$V_t = \begin{cases} 0 & \text{if } \hat{\xi}_t^{soc} \leq c, \\ (1-N_t) \left(\left(\frac{\hat{\xi}_t^{soc}}{c} \right)^{\frac{\tau}{\tau+1}} - 1 \right)^{\frac{1}{\tau}} & \text{if } \hat{\xi}_t^{soc} > c, \end{cases} \quad \text{and} \quad \mu_t = \begin{cases} c - \hat{\xi}_t^{soc} & \text{if } \hat{\xi}_t^{soc} \leq c, \\ 0 & \text{if } \hat{\xi}_t^{soc} > c, \end{cases}.$$

the KKT conditions are met. The remaining variables are then easily derived from the equations (4.6.11)-(4.6.16).

Hence, the policy function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ for $\hat{\xi}_t^{soc}$ can again be characterized as the solution to a functional equation, namely

$$R(g,x,z) := lhs(g,x,z) - rhs(g,x,z) = 0 \text{ for all } x \in [0,1], z \in \mathbb{R}, \tag{4.6.22}$$

now with

$$lhs(g,x,z) := g(x,z) \tag{4.6.23}$$

and

$$rhs(g,x,z) := \mathbb{E}\left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\eta} \left(\exp(Z_{t+1}) + \left(1 - \omega - \left(\frac{M_{t+1}}{U_{t+1}} \right)^{\tau+1} \right) g(N_{t+1}, Z_{t+1}) \right) \right], \tag{4.6.24}$$

where $C_t, N_{t+1}, Z_{t+1}, C_{t+1}, U_{t+1}$ and M_{t+1} in rhs are short for

$$V_t \quad := V(g,x,z) := \begin{cases} 0, & \text{if } g(x,z) \leq c \\ (1-x) \left(\left(\frac{g(x,z)}{c} \right)^{\frac{\tau}{\tau+1}} - 1 \right)^{\frac{1}{\tau}}, & \text{if } g(x,z) > c \end{cases}, \tag{4.6.25}$$

$$C_t \quad := C(g,x,z) := \exp(z)x - cV(g,x,z), \tag{4.6.26}$$

$$M_t \quad := M(g,x,z) = \frac{(1-x)V(g,x,z)}{((1-x)^\tau + V(g,x,z)^\tau)^{\frac{1}{\tau}}}, \tag{4.6.27}$$

$$N_{t+1} := N(g,x,z) := (1-\omega)x + M(g,x,z), \tag{4.6.28}$$

$$Z_{t+1} := \rho z + \epsilon, \quad \epsilon \sim N(0, \sigma_\epsilon^2), \tag{4.6.29}$$

$$C_{t+1} := C(g, N(g,x,z), \rho z + \epsilon), \tag{4.6.30}$$

$$U_{t+1} := U(g,x,z) := 1 - N(g,x,z), \tag{4.6.31}$$

$$M_{t+1} := M(g, N(g,x,z), \rho z + \epsilon). \tag{4.6.32}$$

Calibration To allow comparison with the baseline model, all parameter values, including the parameters that were left free in the baseline model, remain the same and are repeated in table 4.11. Note however that this implies different steady state values for variables, for which the steady state values were fixed in the baseline model, i.e. for unemployment, for the rate of successful training in unemployed workers κ_w and for the rate of successful training relative to training vacancies κ_f.

Table 4.11: Social Planner: Calibration

Parameter	Value	Description
β	$0.99^{1/3}$	discount factor
η	2	relative risk aversion
ρ	$0.95^{1/3}$	Autocorrelation of LP shock
σ	0.0077	Standard deviation of innovations of LP shock
τ	1.2897	elasticity in matching function
ω	0.05	separation rate
c	1.3154	cost of posting an open vacancy

Steady State The deterministic steady state can be determined in a way similar to the baseline model. First, the steady state for log LP equals

$$Z = 0.$$

Next, we will derive an equation from which we can solve for the steady state value of employment. From the definition of unemployment, we get

$$U = 1 - N,$$

new matches in steady state must equal

$$M = \omega N,$$

and the number of training vacancies can be derived from (4.6.12) as

$$V = (1-N)\left(\left(\frac{1-N}{\omega N}\right)^{\tau} - 1\right)^{-\frac{1}{\tau}}.$$

Hence,

$$\frac{M}{V} = \frac{\omega N\left(\left(\frac{1-N}{\omega N}\right)^{\tau} - 1\right)^{\frac{1}{\tau}}}{1-N} = \left(1 - \left(\frac{\omega N}{1-N}\right)^{\tau}\right)^{\frac{1}{\tau}}.$$

Further, from (4.6.21), we get

$$\hat{\xi}^{soc} = \frac{\beta \exp(Z)}{1 - \beta\left(1 - \omega - \left(\frac{\omega N}{1-N}\right)^{\tau+1}\right)} \tag{4.6.33}$$

so that plugging the obtained expressions for $\frac{M}{V}$ and $\hat{\xi}^{soc}$ into (4.6.17) finally yields

$$\left(1 - \left(\frac{\omega N}{1-N}\right)^{\tau}\right)^{\frac{\tau+1}{\tau}} \frac{\beta \exp(Z)}{1 - \beta\left(1 - \omega - \left(\frac{\omega N}{1-N}\right)^{\tau+1}\right)} - c = 0.$$

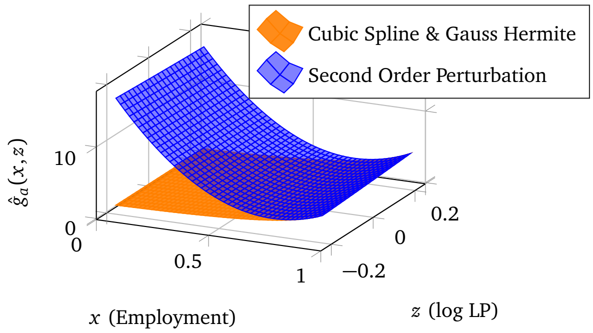
We solve this equation numerically for N and use the above expressions to derive U, M, V and $\hat{\xi}^{soc}$ afterwards. Finally, we have $C = N - cV$ and $\lambda = C^{-\eta}$.

The resulting steady state values for variables, for which the steady state was fixed in the baseline model, are presented in table 4.12.

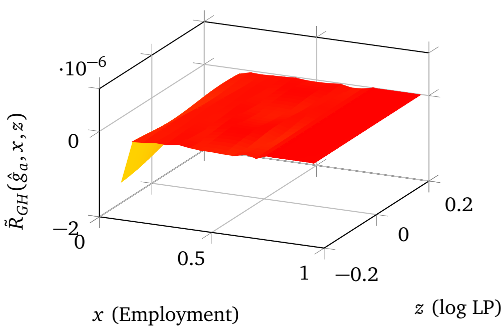
Table 4.12: Social Planner: Steady State Values

Variable	Value	Description
U	8.75%	Unemployment Rate
κ_f	0.6451	Ratio of successful training relative to vacancies
κ_w	0.5215	Fraction of skilless members being successfully trained

Figure 4.18: Social Planner: Approximation of Policy Function and Interpretable Euler Residuals



(a) Approximations to Policy Function of $\hat{\xi}^{soc}$

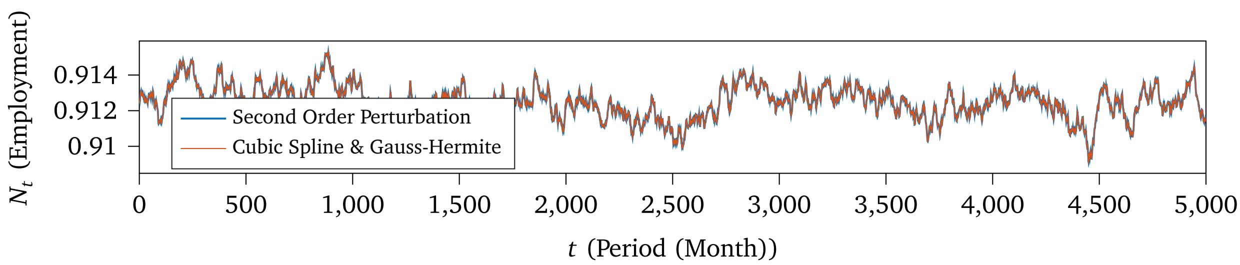


(b) Interpretable Euler Residuals for Cubic Spline Solution

Dynamics of Employment Figure 4.18a displays approximations to the policy function of $\hat{\xi}_t^{soc}$ computed either from the global finite element method using cubic splines or from a second order perturbation. Further, in order to establish the goodness of fit for the finite element solution, which we want to use as the reference solution, the interpretable Euler residuals are pictured in figure 4.18b. With interpretable Euler residuals that do not exceed $2 \cdot 10^{-6}$ in absolute value, the approximation seems sufficiently accurate for our purposes. The second order perturbation solution on the other hand again fails to provide a good approximation for employment rates below approximately 0.8.

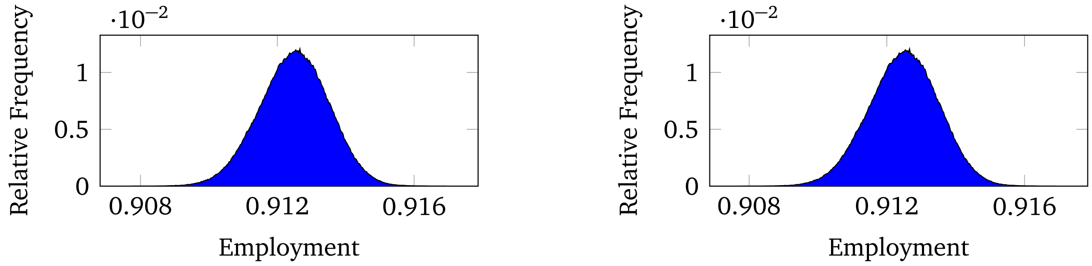
But do the differences in the approximations also transform into deviations, when simulating a time path of the model's equilibrium outcomes? In the baseline model such deviations only occurred, because the dynamics of the model led to periods of relative high unemployment where the second order approximation becomes inaccurate. We computed the series of equilibrium outcomes for 1200000 periods from the same sample of pseudorandom iidN$(0,\sigma_\epsilon^2)$ distributed shocks to log LP. In figure 4.19 we display the employment rate during the first 5000 periods for both the cubic spline and perturbation solution. Additionally, the resulting histograms for

Figure 4.19: Social Planner: Dynamics of Employment



the distribution of employment are shown in figure 4.20. When comparing to the dynamics of employment in the baseline model in figures 4.9 and 4.11, one can immediately observe two

Figure 4.20: Social Planner: Histograms for Distribution of Monthly Employment Rates



(a) Distribution of Employment computed by Cubic Spline

(b) Distribution of Employment computed by Second Order Perturbation

things. First, the periods of high unemployment are no more present in the social planner's solution. Second, as a consequence since employment stays close to its steady state value, the time paths computed by a global method and by perturbation do not differ anymore. The maximal absolute deviation between the two paths for all 1200000 periods is only $2.5 \cdot 10^{-5}$. Table 4.13 summarizes statistical measures for the distribution of employment in the simulation.[37] Both solution methods deliver the same results. The standard deviation of the monthly (non

Table 4.13: Social Planner: Statistic Measures for Distribution of Monthly Employment Rates

	$\bar{N}$	$\max(N_t)$	$\min(N_t)$	$q_{0.01}$	$q_{0.05}$	$q_{0.1}$	$q_{0.5}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	s_N	v_N	w_N
Cubic Spline	0.9125	0.9169	0.9077	0.9101	0.9108	0.9112	0.9125	0.9138	0.9141	0.9148	0.0010	-0.0935	2.9884
Perturbation	0.9125	0.9169	0.9078	0.9101	0.9108	0.9112	0.9125	0.9138	0.9141	0.9148	0.0010	-0.0930	2.9829

Notes: $\bar{N}$=average employment rate, $\max(N_t)$=maximal employment rate, $\min(N_t)$=minimal employment rate, $q_p(N_t)$ =p-quantile for cumulative distribution of employment rate, s_N, v_N, w_N=standard deviation, skewness, kurtosis of the cumulative distribution of employment rate. Statistics are computed from the outcome of a simulation of 1200000 (monthly) periods.

HP-filtered) employment rate declines by the factor 77 compared to the baseline model.

Endogenous Disasters? Examining the time path of employment already showed that employment stays throughout very close to its steady state value in social optimum. Output can therefore only fluctuate in the same magnitude as labor productivity and not decline as far as in the baseline model. Consequently, neither consumption can decline too much.[38] Nonetheless, we computed the number of disasters, the number of disaster periods, the disaster probability, average disaster size and disaster duration for both consumption and output for different threshold fractions. The results, along the respective outcomes for labor productivity, are summarized in table 4.14. We conclude that in the social planner's solution disasters only arise in the magnitude as directly induced by the exogenous process for LP. Declines by more than 20% from steady state did not appear at all in the simulation.

Second Moments of the Labor Market Table 4.15 displays the second moments in the social planner's problem. All volatilities drop substantially compared to the baseline model. Unemployment, vacancies and labor market tightness now fluctuate even less than labor productivity so that the variables' standard deviations found in the data are more than 25 times the standard deviations generated in the model. On the other hand, the negative correlation between unemployment and vacancies is closer to the value found in the data than it was for the baseline model. Yet, the empirical value is still below the 2.5% quantile computed from the 5000 simulations.

[37] Note that the steady state value of employment is 0.9125 here.

[38] Since employment stays stable, so must the amount of open vacancies and hence the cost inferred by them. Hence consumption should move similar to output.

Table 4.14: Endogenous Disasters in the Social Planner's Problem

	number of disasters	total number of disaster periods	disaster probability	average disaster size	average disaster duration
			90% threshold		
C	1207	4431	0.31%	11.32%	3.67
Y	792	2552	0.20%	11.10%	3.22
LP	663	2086	0.17%	11.05%	3.15
			85% threshold		
C	29	73	0.0073%	15.97%	2.52
Y	8	19	0.0020%	15.78%	2.38
LP	5	10	0.0013%	15.68%	2.00
			80% threshold		
C	0	0	0%	–	–
Y	0	0	0%	–	–
LP	0	0	0%	–	–

Table 4.15: Labor Market Moments in the Social Planner Model

	U	V	θ	$p = \frac{Y}{N}$
s_x	0.004 [0.003; 0.005]	0.005 [0.004; 0.006]	0.009 [0.007; 0.010]	0.016 [0.013; 0.019]
$\frac{s_x}{s_p}$	0.267 [0.259; 0.276]	0.298 [0.287; 0.311]	0.531 [0.524; 0.538]	1.000 [1.000; 1.000]
r_x	0.830 [0.767; 0.879]	0.630 [0.516; 0.731]	0.792 [0.717; 0.853]	0.774 [0.693; 0.840]
		Cross Correlations		
U		-0.762 [−0.823; −0.693]	-0.931 [−0.951; −0.905]	-0.910 [−0.936; −0.878]
V			0.946 [0.933; 0.958]	0.961 [0.952; 0.969]
θ				0.998 [0.998; 0.999]

Notes: U=unemployment, V=vacancies, $\theta = \frac{V}{U}$=labor market tightness, p=output per worker, s_x=standard deviation of variable X, r_x=first order autocorrelation of variable X. All moments are computed from the cyclical components of the HP-filtered ($\lambda = 1600$) relative deviations from the mean, i.e. as the cyclical component of the series $\frac{X_t - \bar{X}}{\bar{X}}$. The table shows the average as well as the 2.5% and 97.5% quantiles from 5000 simulations of the model's equilibrium outcomes for 666 periods converted to 222 quarterly observations after 1000 throw-away periods.

4.6.1.2 Distortions from Social Optimum in the Baseline Model

The baseline model differs from the social planner's problem regarding three aspects, which distort the amount of open vacancies posted from the social optimum. These three distortions ultimately manifest in the different equilibrium conditions

$$\begin{aligned} c &= \kappa_{f,t}\hat{\xi}_t^f + \mu_t, \\ \hat{\xi}_t^f &= \mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\left(\exp(Z_{t+1}) - w_{t+1} + (1-\omega)\hat{\xi}_{t+1}^f\right)\right], \end{aligned} \tag{4.6.34}$$

in the baseline model, whereas

$$\begin{aligned} c &= \frac{\partial M}{\partial V}(U_t, V_t)\hat{\xi}_t^{soc} + \mu_t, \\ \hat{\xi}_t^{soc} &= \mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\left(\exp(Z_{t+1}) + \left(1 - \omega - \frac{\partial M}{\partial U}(U_{t+1}, V_{t+1})\right)\hat{\xi}_{t+1}^{soc}\right)\right] \end{aligned} \tag{4.6.35}$$

in social optimum. The recursive formulation for the present value of a worker in the next period from the representative firm's and the social planner's view, respectively, can also be written as

$$\hat{\xi}_t^f = \mathbb{E}_t\left[\sum_{s=1}^{\infty}(1-\omega)^{s-1}\beta^s\frac{\lambda_{t+s}}{\lambda_t}\left(\exp(Z_{t+s}) - w_{t+s}\right)\right], \tag{4.6.36}$$

and

$$\hat{\xi}_t^{soc} = \mathbb{E}_t\left[\sum_{s=1}^{\infty}\left(\prod_{k=1}^{s-1}\left(1 - \omega - \frac{\partial M}{\partial U}(U_{t+k}, V_{t+k})\right)\right)\beta^s\frac{\lambda_{t+s}}{\lambda_t}\exp(Z_{t+s})\right], \tag{4.6.37}$$

if the series on the right hand side converge. We will now proceed to describe the three distortions from social optimum present in the baseline model and explain step by step how they manifest in the different equilibrium conditions stated above. Since the baseline model produces endogenous disasters, whereas this is not the case in the social planner's problem, the distortions must play a crucial role in this regard. We will then try to single out the main cause amongst them for the disasters in the baseline model.

The first distortion in the baseline model stems from the fact that the representative household regards the taxes T_t in his budget constraint (4.2.4) as exogenous, while in equilibrium the whole value $b(1-N_t)$ of unemployment activities is redistributed from taxes. As a consequence, the representative household attributes a period value of b to unemployment while there is no aggregate value of unemployment incorporated in the baseline model—apart from unemployed members serving as an essential input factor for generating job matches for the next period. Hence, any $b > 0$ will lower the total surplus attributed to a working member compared to the social value in (4.6.37). If we add the first distortion into the social planner's case by lowering the within period value from employment over unemployment by b each period too, the social value $\hat{\xi}_t^{soc}$ in (4.6.37) would accordingly change to

$$\hat{\xi}_t^{soc,b} = \mathbb{E}_t\left[\sum_{s=1}^{\infty}\left(\prod_{k=1}^{s-1}\left(1-\omega-\frac{\partial M}{\partial U}(U_{t+k},V_{t+k})\right)\right)\beta^s\frac{\lambda_{t+s}}{\lambda_t}(\exp(Z_{t+s})-b)\right]. \tag{4.6.38}$$

Moreover, as already noted in the discussion of the calibration of the model in subsection 4.4.1, part of the justification for the high value of $b = 0.85$ is the view that b should not be identified solely with unemployment benefits but rather also include other factors for which a worker wants to be compensated by the wage and which are not endogenously determined in the model. However, this interpretation hast two consequences. First, summing up all such factors for which a worker demands compensation in a fixed parameter also implies that the value may not change dependent on the state of the economy. Second, since all the value from unemployment in b is redistributed from taxes, the model ignores the positive effects on the household's utility from unemployment. Further, it also implicates that taxes exceed the household's wage income by a multitude (up to a factor of 16) in periods with high unemployment in the baseline model.

The second distortion from social optimum is caused by the fact that in a competitive equilibrium, as in the baseline model, the representative firm alone decides about the amount of open vacancies to post and bears all ensuing costs. Consequently, vacancies in 4.6.34 are posted up to the point, where the value of a filled position in the next period $\hat{\xi}_t^f$ from the firm's perspective times the probability of the vacancy being filled equals the cost of posting. On the other hand, for a social optimum as in (4.6.35) not the return of an open vacancy to the firm is decisive, but the total surplus generated. Using the fact that the firm receives a share of $1-\varphi$ from the total surplus in the baseline model, the second distortion can be implemented—next to the first one—into the social planner's case by changing (4.6.35) to

$$c = \frac{\partial M}{\partial V}(U_t, V_t)(1-\varphi)\hat{\xi}_t^{soc,b} + \mu_t \tag{4.6.39}$$

Since the first distortion—the household values unemployment by b—lowers the total surplus generated by a working member and in addition only the firm's share in it plays a role in the decision about posting open vacancies through the second distortion, the expected return of an open vacancy in (4.6.39) is, all things equal, smaller than in condition (4.6.35) for the social optimum. Hence, the first two distortions both decrease the amount of open vacancies in the economy.

Finally, the third distortion in the baseline model emerges from the assumption that neither households nor firms in the economy coordinate their decisions on the labor market by taking the complete matching function M into account. Both, the representative household and the representative firm, consider the average probabilities of finding a job and successfully filling a vacancy, respectively, as exogenously given. They have no influence on average probabilities on their own without coordinating. Since it proves more comprehensible, instead of showing how adding the third distortion into (4.6.39) and (4.6.38) finally yields the equilibrium conditions (4.6.34) and (4.6.36) from the baseline model, we will argue how removing the third distortion from the baseline model yields the conditions (4.6.39) and (4.6.38) including only the first two distortions. First, when deciding about the amount of open vacancies to post, the firm considers the average rate at which a vacancy can be filled to determine the benefit from posting a vacancy but neglects the influence of its own decision on that average probability. The socially relevant change in job matches in the economy induced by an additional marginal open vacancy is not determined by the average rate but the marginal change $\frac{\partial M}{\partial V}$. Correcting for this aspect would hence first yield

$$c = \frac{\partial M}{\partial V}(U_t, V_t)\hat{\xi}_t^f + \mu_t. \tag{4.6.40}$$

Since the average rate of successfully matched vacancies exceeds the marginal rate, too many open vacancies are posted. This reflects the fact that individual firms in the economy are assumed to ignore the side effect that posting open vacancies is making it harder for other firms to successfully recruit. Yet, condition (4.6.40) is still affected by the third distortion. The circumstance that the representative firm regards the probability $\kappa_{f,t}$ as exogenous also manifests in the value $\hat{\xi}_t^f$. The firm does not correct the continuation value in (4.2.20) for the marginal decrease in matches induced by one unemployed member less in the economy. Removing this effect of the distortion would yield

$$\xi_t^{f,b,\varphi} = \exp(Z_t) - w_t + \left(1 - \omega - \frac{\partial M}{\partial U}(U_t, V_t)\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^{f,b,\varphi}\right]. \tag{4.6.41}$$

This second effect of the distortion hence causes the representative firm to overestimate the value of a worker reflecting the fact that it is of no interest for individual firms in the economy how quickly a worker could find a job elsewhere. Consequently, the optimality condition (4.6.34) for open vacancies completely corrected for the third distortion is not (4.6.40), but

$$c = \frac{\partial M}{\partial V}(U_t, V_t)\underbrace{\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^{f,b,\varphi}\right]}_{:=\hat{\xi}_t^{f,b,\varphi}} + \mu_t = \frac{\partial M}{\partial V}(U_t, V_t)\hat{\xi}_t^{f,b,\varphi} + \mu_t. \tag{4.6.42}$$

To complete our goal to show that removing the third distortion from the baseline model yields the conditions (4.6.39) and (4.6.38), it only remains to be shown that $\hat{\xi}_t^{f,b,\varphi} = (1-\varphi)\hat{\xi}_t^{soc,b}$ at this point. This follows, if we observe that the representative household, too, was assumed to neglect side effects when deciding about the value of employment in (4.2.8). The continuation value is weighted with the probability of a working member remaining employed less the probability $\kappa_{w,t} = \frac{M_t}{U_t}$ for him finding a job for the next period elsewhere if he was unemployed. Yet, aggregate matches would only increase by the marginal matching rate $\frac{\partial M}{\partial U}$ with one marginal unemployed member more. With this adjustment in (4.2.8), the value of employment to the household would become

$$\xi_t^{h,b,\varphi} = w_t - b + \left(1 - \omega - \frac{\partial M}{\partial U}(U_t, V_t)\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^{h,b,\varphi}\right].$$

Since the average matching rate of unemployed members exceeds the marginal rate, the representative household underestimates the value of employment (over unemployment). This reflects the fact that workers ignore the effect that their employment makes it ceteris paribus easier for not yet employed members to find jobs. Or inversely stated, if a worker becomes unemployed and has to search for jobs too, the average probability for unemployed members to successfully find a job decreases. With the representative firm overestimating and the representative household underestimating the respective value they receive from an employment arrangement, the effect on the total surplus is not yet clear. In particular, with the discussed adjustments for completely removing the third distortion, the total surplus from an employment accounts to

$$\begin{aligned}\xi_t^{b,\varphi} &:= \xi_t^{h,b,\varphi} + \xi_t^{f,b,\varphi} = \exp(Z_t) - b + \left(1 - \omega - \frac{\partial M}{\partial U}(U_t, V_t)\right)\mathbb{E}_t\left[\beta \frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^{b,\varphi}\right] \\ &= \mathbb{E}_t\left[\sum_{s=0}^{\infty}\left(\prod_{k=0}^{s-1}\left(1 - \omega - \frac{\partial M}{\partial U}(U_{t+k}, V_{t+k})\right)\right)\beta^s \frac{\lambda_{t+s}}{\lambda_t}(\exp(Z_{t+s}) - b)\right],\end{aligned}$$

while the total surplus in the baseline model, according to (4.2.23) together with $\xi_{t+1}^h = \varphi\xi_{t+1}$ from the sharing rule, is

$$\begin{aligned}\xi_t &= \exp(Z_t) - b + \left(1 - \omega - \varphi\kappa_{w,t}\right)\mathbb{E}_t\left[\beta \frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}\right] \\ &= \mathbb{E}_t\left[\sum_{s=0}^{\infty}\left(\prod_{k=0}^{s-1}\left(1 - \omega - \varphi\frac{M_{t+k}}{U_{t+k}}\right)\right)\beta^s \frac{\lambda_{t+s}}{\lambda_t}(\exp(Z_{t+s}) - b)\right].\end{aligned}$$

Hence, whether the third distortion increases or decreases the total surplus attributed to an additional working member in the economy depends on whether $\varphi\frac{M_t}{U_t}$ exceeds $\frac{\partial M}{\partial U}$ or not. [39] Moreover, comparing to (4.6.38) shows $\xi_t^{b,\varphi} = \xi_t^{soc,b}$ so that

$$\hat{\xi}_t^{f,b,\varphi} = (1-\varphi)\hat{\xi}_t^{soc,b}.$$

Completely removing the third distortion from the baseline model hence yields (4.6.42) with $\hat{\xi}_t^{f,b,\varphi} = (1-\varphi)\hat{\xi}_t^{soc,b}$ which is equal to (4.6.39). The third distortion causes the representative firm to overestimate the probability at which marginal vacancies can be filled and moreover leads the firm to overestimate, while the household underestimates, the value of a worker. The effect on the total surplus is not clear so that the net effect of the third distortion on the amount of open vacancies is ambiguous.[40]

[39] Under the first Hosios (1990) condition $\varphi\frac{M_t}{U_t} = \frac{\partial M}{\partial U}(U_t, V_t)$ the effect of the firm overestimating and the household underestimating the value of a worker would cancel out. The total surplus in the baseline model would be equal to the total surplus from which the third distortion is removed, i.e. $\xi_t = \xi_t^{b,\varphi}$. Yet, with the matching function chosen, this condition can not be fulfilled.

[40] We already mentioned in footnote 39 that under the first Hosios condition $\varphi\frac{M_t}{U_t} = \frac{\partial M}{\partial U}(U_t, V_t)$ the effect of the firm overestimating and the household underestimating the value of a worker would cancel out so that $\xi_t = \xi_t^{b,\varphi}$. This would also imply $\hat{\xi}_t^f = (1-\varphi)\hat{\xi}_t = (1-\varphi)\hat{\xi}_t^{b,\varphi} = \hat{\xi}_t^{f,b,\varphi} = (1-\varphi)\hat{\xi}_t^{soc,b}$. Now, if additionally the second Hosios condition $(1-\varphi)\frac{M_t}{V_t} = \frac{\partial M}{\partial V}(U_t, V_t)$ was also met, the optimality condition (4.6.34) from the baseline model would already be equivalent to

$$c = \frac{M_t}{V_t}\hat{\xi}_t^f + \mu_t = \frac{1}{1-\varphi}\frac{\partial M}{\partial V}(U_t, V_t)(1-\varphi)\hat{\xi}_t^{soc,b} + \mu_t = \frac{\partial M}{\partial V}(U_t, V_t)\hat{\xi}_t^{soc,b} + \mu_t.$$

Under this condition, the effect of the firm overestimating the probability of an additional vacancy being filled caused by the third distortion and the effect of the second distortion, that only the firm's share in the total surplus is accounted for in its decision about the amount of open vacancies to be posted, exactly cancel out. Summing up, given both Hosios conditions the second and third distortion together would have zero net effect on the decisions in the economy. The baseline model would then differ from the social planner's problem only due the value b attributed by the household to unemployment—the first distortion. Yet, the Hosios conditions can not be fulfilled given the chosen functional form of the matching function.

Summing up the above description, the first distortion of the household appreciating the in aggregate non-existent unemployment value in b as well as the second distortion of only the firm's share in total surplus generated by a working member playing a role in the decision about the amount of open vacancies in a competitive equilibrium both lower the amount of vacancies and hence should favor the occurrence of disasters. Moreover, since b was set to a high value not too far from the productivity level in steady state, the effect of the first distortion should be rather high. On the other hand, the household's bargaining power was set to a relatively small value so that the firm receives most of the total surplus generated by a working member. Consequently, the impact of the second distortion can be expected to be rather small. The net effect of the third distortion—firms and households in the economy not coordinating in such way to exploit the whole form of the matching function optimally—is not clear. We therefore conjecture in particular the high value of b to be of importance for the formation of disasters in the baseline model. We will now proceed to separate the effects caused by the single distortions by adding them separately to the social planner's solution. Moreover, if our conjecture that the high value of b plays a crucial role for the model's dynamics turns out to be true, we may rethink the already mentioned assumptions which were made in the calibration of b.

4.6.1.3 Adding the first Distortion to the Social Planner's Problem

We will now add the first distortion from the baseline model to the social planner's case—unemployed members receive a value from unemployment of b consumption units per period, which in aggregate is completely redistributed from taxes that are considered exogenous. The equilibrium conditions are the same as for the social planner's problem only with $\hat{\xi}_t^{soc}$ replaced by $\hat{\xi}_t^{soc,b}$ from (4.6.38). For the sake of exposition, we repeat the derivations.

Social Planner Problem with first Distortion Essentially everything remains the same as in the social planner's problem from subsection 4.6.1.1. Only now unemployed members in the economy receive a value b from unemployment activities, which is completely financed by taxes T_t. Yet, the social planner does neglect the fact that $T_t = bU_t$ in his optimization problem. With $J^{soc,b}$ denoting the value function of the social planner including the first distortion, we get

$$
\begin{aligned}
J^{soc,b}(N_t) = \quad & \max_{C_t,V_t} && \frac{C_t^{1-\eta}-1}{1-\eta} + \beta \mathbb{E}_t\left[J^{soc,b}\left((1-\omega)N_t + M(1-N_t,V_t)\right)\right] \\
& \text{s.t.} && C_t \leq \exp(Z_t)N_t + b(1-N_t) - cV_t - T_t, \\
& && V_t \geq 0, \\
& \text{given} && N_t.
\end{aligned}
$$

The social planner considers taxes T_t as exogenous in his optimization. The KKT conditions remain as in (4.6.1)-(4.6.5), only now the value of an additional worker in this period measured in consumption units, i.e. $\xi_t^{soc,b} := \frac{1}{\lambda_t}\frac{\partial J^{soc,b}}{\partial N}(N_t)$, satisfies

$$
\xi_t^{soc,b} = \exp(Z_t) - b + \left(1 - \omega - \frac{\partial M}{\partial U}(U_t,V_t)\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^{soc,b}\right]. \tag{4.6.43}
$$

Consequently the expected discounted value from next period, $\hat{\xi}_t^{soc,b} := \mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^{soc,b}\right]$, meets

$$
\hat{\xi}_t^{soc,b} = \mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\left(\exp(Z_{t+1}) - b + \left(1-\omega-\frac{\partial M}{\partial U}(U_{t+1},V_{t+1})\right)\hat{\xi}_{t+1}^{soc,b}\right)\right],
$$

which is equivalent to (4.6.38).

Equilibrium Summing up, the optimality conditions for the social planner's problem including the first distortion are as in (4.6.11)-(4.6.21) only with $\hat{\xi}_t^{soc}$ replaced by $\hat{\xi}_t^{soc,b}$ as stated in the equation above.

Since the only difference from the original social planner's problem is the presence of the parameter b in the within period value of a worker in $\hat{\xi}^{soc,b}$ compared to (4.6.21), all the derivations, how the model's variables can be expressed dependent on $\hat{\xi}_t^{soc,b}$ and the state variables, remain valid. Hence, the policy function $g: [0,1] \times \mathbb{R} \to \mathbb{R}$ for $\hat{\xi}_t^{soc,b}$ can equivalently be characterized as the solution to the functional equation in (4.6.22)-(4.6.32) only with rhs in (4.6.24) adjusted to

$$rhs(g,x,z) := \mathbb{E}\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\eta}\left(\exp(Z_{t+1}) - b + \left(1-\omega-\left(\frac{M_{t+1}}{U_{t+1}}\right)^{\tau+1}\right)g(N_{t+1},Z_{t+1})\right)\right],$$

Calibration The calibration of all parameters remains the same as in the baseline model and the original social planner's problem, i.e. as in table 4.11 and with $b = 0.85$.

Steady State Again, since the only difference in the equilibrium conditions compared to the original social planner's problem is the presence of the parameter b in $\hat{\xi}^{soc,b}$, the steady state can be determined the same way only adjusting (4.6.33) to

$$\hat{\xi}^{soc,b} = \frac{\beta(\exp(Z) - b)}{1-\beta\left(1-\omega-\left(\frac{\omega N}{1-N}\right)^{\tau+1}\right)}. \tag{4.6.44}$$

The resulting steady state values for the variables, for which the steady state was fixed in the baseline model, are presented in table 4.16.

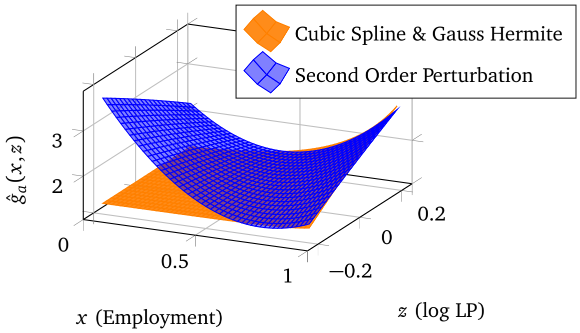
Table 4.16: Model with the first Distortion: Steady State Values

Variable	Value	Description
U	18.19%	Unemployment Rate
κ_f	0.8849	Ratio of successful training relative to vacancies
κ_w	0.2249	Fraction of skilless members being successfully trained

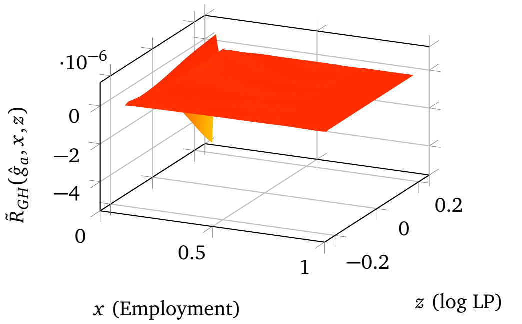
Dynamics of Employment As before, we computed an approximation to the policy function of $\hat{\xi}^{soc,b}$ with a global finite element method using cubic splines and with a second order perturbation method. First, figure 4.21a displays both approximations. Further, in order to establish the goodness of fit for the finite element solution, we computed the interpretable Euler residuals pictured in figure 4.21b. The interpretable Euler residuals of magnitude 10^{-6} suggest the approximation to be sufficiently accurate for our purposes. The same way as in the baseline model and the social planner's problem, the second order perturbation solution fails to provide a good approximation for lower employment rates. Yet, with an already lower steady state value of employment in the present version of the model, significant inaccuracies in the second order perturbation solution now appear only at even lower rates of employment (below approximately 0.7).

We want to analyze the dynamics of employment in the model and to observe whether and how the discrepancies in the approximations transform into deviations in simulations of the model's equilibrium outcomes. Therefore, we computed the series of equilibrium outcomes

Figure 4.21: Model with first Distortion: Approximation of Policy Function and Interpretable Euler Residuals

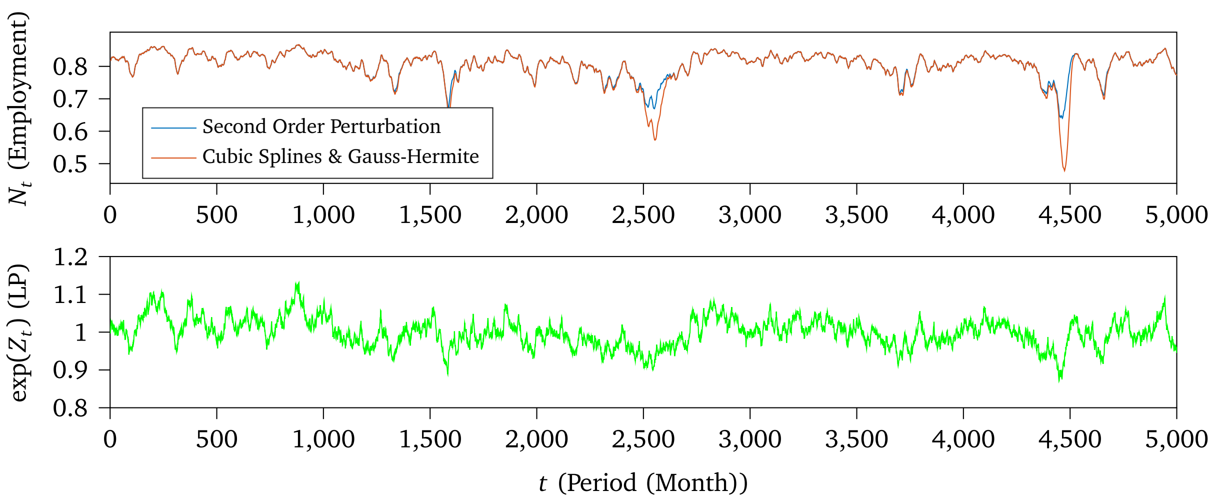


(a) Approximations to Policy Function of $\hat{\xi}^{soc,b}$



(b) Interpretable Euler Residuals for Cubic Spline Solution

Figure 4.22: Model with the first Distortion: Dynamics of Employment



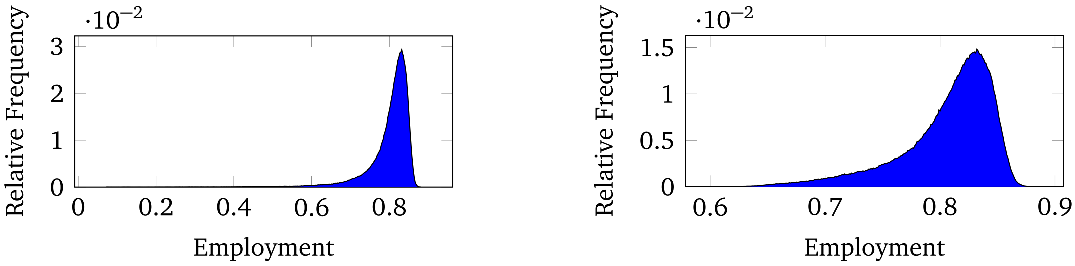
for 1200000 periods from the same sample of pseudorandom iid$N(0,\sigma_\epsilon^2)$ distributed shocks to log LP. In figure 4.22 we display the dynamics of employment in the first 5000 periods for both solutions. Additionally, the resulting histograms for the distribution of employment in the simulation are shown in figure 4.23, while table 4.17a summarizes some statistical properties of the distribution.[41] A comparison to the dynamics of employment in the social planner's case and in the baseline model shows that employment behaves similar as in the baseline model. If accurately solved by the global solution method, the model generates rare, huge drops in the employment rate. The second order perturbation solution cannot accurately reproduce this behaviour but shows high deviations in the time path of employment, up to 0.68 in absolute value, during these periods.

We further want to check how sensitive the dynamics of employment react to changes in the value of b. We therefore set $b = 0.75$ moderately lower and show the statistical measures for the resulting distribution of employment in a simulation from the same sample of pseudorandom shocks to log LP in table 4.17b.[42] With this modification, the dynamics of employment change substantially. Employment does not fall lower than 0.7424 anymore, the rare and huge declines disappear. Moreover, the perturbation solution already provides a much better approximation to the model's dynamics.

We conclude that once we assume that the social planner considers unemployed members

[41] Note that the steady state value of employment is 0.8181 here.
[42] Setting $b = 0.75$ results in a steady state value of 0.8612 for the employment rate.

Figure 4.23: Model with the first Distortion: Histograms for Distribution of Monthly Employment Rates



(a) Distribution of Employment computed by Cubic Spline

(b) Distribution of Employment computed by Second Order Perturbation

Table 4.17: Model with the first Distortion: Statistic Measures for Distribution of Monthly Employment Rates

(a) Value of Unemployment $b = 0.85$

	$\bar{N}$	$\max(N_t)$	$\min(N_t)$	$q_{0.01}$	$q_{0.05}$	$q_{0.1}$	$q_{0.5}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	s_N	v_N	w_N
Cubic Spline	0.8001	0.8804	0.0730	0.5278	0.6928	0.7397	0.8170	0.8469	0.8524	0.8612	0.0629	-3.6853	24.4154
Perturbation	0.8065	0.8797	0.6060	0.6729	0.7203	0.7489	0.8172	0.8468	0.8523	0.8610	0.0406	-1.3534	4.8395

(b) Value of Unemployment $b = 0.75$

	$\bar{N}$	$\max(N_t)$	$\min(N_t)$	$q_{0.01}$	$q_{0.05}$	$q_{0.1}$	$q_{0.5}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	s_N	v_N	w_N
Cubic Spline	0.8596	0.8907	0.7424	0.8251	0.8388	0.8447	0.8612	0.8724	0.8750	0.8793	0.0114	-1.0272	5.3433
Perturbation	0.8596	0.8904	0.7640	0.8259	0.8390	0.8448	0.8612	0.8724	0.8749	0.8792	0.0112	-0.9509	4.8037

Notes: $\bar{N}$=average employment rate, $\max(N_t)$=maximal employment rate, $\min(N_t)$=minimal employment rate, $q_p(N_t)$ =p-quantile for cumulative distribution of employment rate, s_N, v_N, w_N=standard deviation, skewness, kurtosis of the cumulative distribution of employment rate. Statistics are computed from the outcome of a simulation of 1200000 (monthly) periods.

in the economy to receive a value of $b = 0.85$ consumption units per period and ignores the fact that the value is redistributed from taxes, it would be 'socially optimal' for periods of high unemployment to arise. This makes sense. Note for example that the huge drop of employment appearing around period 4500 in figure 4.22 is caused by a decline in labor productivity to approximately 0.9. The social planner consequently expects a not working household member to contribute only moderately less to consumption with 0.85 units than a worker currently does. Although the difference between the productivity of a worker and the expected contribution of an unemployed member to consumption is predicted to increase again in the long run, there is only little incentive to train unemployed members in such periods. Consumption is already low and c consumption units can be saved for the moment if training is postponed into future periods when labor productivity is recovering. Yet, this result is highly sensitive to the value of b.

Endogenous Disasters? The discussion of the dynamics of employment already showed that the model will produce huge disasters in consumption and output for the case of $b = 0.85$. Table 4.18 confirms. In parenthesis are the results for $b = 0.75$. In accordance to the fact that huge declines in employment are highly sensitive to the value of b, the same holds for disasters in the model. While consumption is less than half of its steady state value during 1861 of the 400000 quarters in the simulation with $b = 0.85$, consumption does not decline by more than approximately 25% from its steady state at all with $b = 0.75$

Concluding, the high value of $b = 0.85$ seems to play a crucial role for generating the disasters in the model. Further, the behavior in the economy is highly sensitive to seemingly small changes in b. We will describe the economic intuition for this result in more detail in subsection 4.6.2.

Table 4.18: Endogenous Disasters in the Model with the first Distortion

	number of disasters	total number of disaster periods	disaster probability	average disaster size	average disaster duration
			90% threshold		
C	4373 (2487)	64733 (14328)	1.30% (0.64%)	18.06% (12.10%)	14.80 (5.76)
Y	4791 (2554)	65092 (13393)	1.43% (0.66%)	17.71% (12.04%)	13.59 (5.24)
LP	663	2086	0.17%	11.05%	3.15
			85% threshold		
C	2445 (331)	36354 (1569)	0.67% (0.08%)	23.81% (16.78%)	14.87 (4.74)
Y	2654 (336)	36773 (1409)	0.73% (0.08%)	23.48% (16.77%)	13.86 (4.19)
LP	5	10	0.0013%	15.68%	2.00
			80% threshold		
C	1401 (29)	21060 (137)	0.37% (0.0073%)	29.46% (21.66%)	15.03 (4.72)
Y	1538 (31)	21341 (122)	0.41% (0.0078%)	28.99% (21.57%)	13.88 (3.94)
LP	0	0	0%	–	–
			75% threshold		
C	792 (1)	13041 (5)	0.20% (0.0003%)	35.59% (26.88%)	16.47 (5.00)
Y	852 (2)	13229 (5)	0.22% (0.0005%)	35.22% (25.88%)	15.53 (2.50)
LP	0	0	0%	–	–
			70% threshold		
C	489 (0)	8490 (0)	0.12% (0%)	41.14% (–)	17.36 (–)
Y	528 (0)	8621 (0)	0.13% (0%)	40.68% (–)	16.33 (–)
LP	0	0	0%	–	–
			50% threshold		
C	91 (0)	1861 (0)	0.02% (0%)	61.03% (–)	20.45 (–)
Y	95 (0)	1890 (0)	0.02% (0%)	60.82%(–)	19.89 (–)
LP	0	0	0%	–	–
			30% threshold		
C	18 (0)	435 (0)	0.0045% (0%)	77.52% (–)	24.17 (–)
Y	19 (0)	440 (0)	0.0048% (0%)	77.28%(–)	23.16 (–)
LP	0	0	0%	–	–

Second Moments of the Labor Market Tables 4.19a and 4.19b show the second moments in the model with $b = 0.85$ and $b = 0.75$, respectively. Compared to the social planner's case, the standard deviations of unemployment, vacancies and the labor market tightness gain remarkably in size. As argued by Hagedorn and Manovskii (2008), increasing the value of unemployment activities in b yields higher fluctuations in all variables. Yet, even with b=0.85 the standard deviations found in the data lie well above the 97.5% quantiles from repeated simulations of the model. The correlation between unemployment and vacancies is negative in both cases and seems to decline with increasing b.

4.6.1.4 Adding the second Distortion to the Social Planner's Problem

We will next also introduce the second distortion separately to the model. The second distortion was caused by the fact that in a competitive equilibrium, in which the representative firm alone decides about the amount of open vacancies posted in the economy, not the total surplus generated by an additional worker is decisive, but only the share the firm receives. Different from the baseline model however, the first and third distortion remain removed. The representative household acknowledges in his evaluation that there is no aggregate benefit to him from unemployed members. Moreover, firms and households in the economy are assumed to coordinate in their decisions. Consequently, the whole endogenous nature of the probabilities of an open vacancy being filled and of finding a job in equilibrium is accounted for by the representative entities. Note that due to the low bargaining power of the representative household in the baseline model, it should be expected that the effect from the second distortion is rather small. Yet, we also check robustness by setting φ to a value that renders the size of the distortion from social optimum in steady state the same between the first and second distortion.

Table 4.19: Labor Market Moments in the Model with the first Distortion

(a) Value of Unemployment Activities $b = 0.85$

	U	V	θ	$p = \frac{Y}{N}$
s_x	0.068 $[0.041; 0.107]$	0.052 $[0.034; 0.077]$	0.087 $[0.060; 0.125]$	0.016 $[0.013; 0.019]$
$\frac{s_x}{s_p}$	4.195 $[2.742; 6.366]$	3.213 $[2.213; 4.708]$	5.384 $[4.062; 7.338]$	1.000 $[1.000; 1.000]$
r_x	0.930 $[0.886; 0.964]$	0.712 $[0.590; 0.822]$	0.866 $[0.808; 0.912]$	0.774 $[0.693; 0.840]$
		Cross Correlations		
U		-0.450 $[-0.543; -0.357]$	-0.793 $[-0.880; -0.598]$	-0.537 $[-0.723; -0.243]$
V			0.794$[0.682; 0.833]$	0.910 $[0.834; 0.953]$
θ				0.922 $[0.810; 0.968]$

(b) Value of Unemployment Activities $b = 0.75$

	U	V	θ	$p = \frac{Y}{N}$
s_x	0.028 $[0.020; 0.038]$	0.022 $[0.017; 0.027]$	0.041 $[0.032; 0.053]$	0.016 $[0.013; 0.019]$
$\frac{s_x}{s_p}$	1.715 $[1.392; 2.229]$	1.354 $[1.137; 1.647]$	2.573 $[2.229; 3.032]$	1.000 $[1.000; 1.000]$
r_x	0.888 $[0.841; 0.925]$	0.604$[0.498; 0.705]$	0.830 $[0.765; 0.882]$	0.774 $[0.693; 0.840]$
		Cross Correlations		
U		-0.514 $[-0.599; -0.426]$	-0.881 $[-0.915; -0.837]$	-0.781 $[-0.844; -0.696]$
V			0.842 $[0.823; 0.862]$	0.908 $[0.889; 0.920]$
θ				0.982 $[0.971; 0.989]$

Notes: U=unemployment, V=vacancies, $\theta = \frac{V}{U}$=labor market tightness, p=output per worker, s_x=standard deviation of variable X, r_x=first order autocorrelation of variable X. All moments are computed from the cyclical components of the HP-filtered ($\lambda = 1600$) relative deviations from the mean, i.e. as the cyclical component of the series $\frac{X_t - \bar{X}}{\bar{X}}$. The table shows the average as well as the 2.5% and 97.5% quantiles from 5000 simulations of the model's equilibrium outcomes for 666 periods converted to 222 quarterly observations after 1000 throw-away periods.

In order to derive the equilibrium conditions for the model that contains the second distortion only, it is sufficient to adjust the optimality condition (4.6.17) for the amount of open vacancies from the social planner's case to include $(1-\varphi)\hat{\xi}_t^{soc}$ instead of $\hat{\xi}_t^{soc}$. Equivalently, we can also consider the competitive equilibrium. For the sake of exposition we will repeat the derivations.

Representative Household As in the baseline model, the representative household chooses consumption and stock holdings to maximize (4.2.3). Yet, we assume the representative household to take into account that in aggregate the value of unemployment activities is paid by taxes, i.e. $T_t = b(1-N_t)$ in the budget constraints (4.2.4). Further, we assume households to coordinate in such way that they also consider side effects of their decisions on the labor market and therefore recognize

$$N_{t+1} = (1-\omega) + M(U_t, V_t).$$

If $J^{h,\varphi}(N_t, S_t)$ denotes the household's value function, then

$$\begin{aligned} J^{h,\varphi}(N_t, S_t) = \quad & \max_{C_t, S_{t+1}} \frac{C_t^{1-\eta} - 1}{1-\eta} + \beta \mathbb{E}_t[J^{h,\varphi}((1-\omega)N_t + M(1-N_t, V_t))] \\ & \text{s.t.} \quad C_t \leq w_t N_t + d_t S_t - v_t(S_{t+1} - S_t), \\ & \text{given} \quad N_t, S_t. \end{aligned}$$

The first order condition with respect to C_t for the maximization problem on the right hand side remains as in (4.2.6). Since we are not interested in asset prices, we skip the first order condition for S_{t+1}. The value of employment (over unemployment) to the household can be calculated via the envelope theorem as

$$\frac{\partial J^{h,\varphi}}{\partial N}(N_t, S_t) = \lambda_t w_t + \beta \mathbb{E}_t \left[\frac{\partial J^{h,\varphi}}{\partial N}(N_{t+1}) \left(1 - \omega - \frac{\partial M}{\partial U}(U_t, V_t) \right) \right],$$

and measured in consumption units $\xi_t^{h,\varphi} := \frac{1}{\lambda_t}\frac{\partial J^{h,\varphi}}{\partial N}(N_t, S_t)$ therefore becomes

$$\xi_t^{h,\varphi} = w_t + \left(1 - \omega - \frac{\partial M}{\partial U}(U_t, V_t)\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^{h,\varphi}\right]. \tag{4.6.45}$$

The interpretation is the same as in the baseline model. Yet, since the household now takes into account that in aggregate he gains no value from unemployed members, the instantaneous within period value of employment over unemployment is simply the wage payment. Moreover, since we assumed the households to coordinate, the weighing factor for the continuation value as of next period includes the marginal rather than the average change in job matches by an unemployed member.

Representative Firm The representative firm chooses the amount of open vacancies to post in order to maximize the beginning-of-period firm value FV_t^{bop} (4.2.14) subject to the production technology (4.2.9). But now, it also takes into account that employment in the economy evolves according to

$$N_{t+1} = (1 - \omega) + M(U_t, V_t).$$

Let $J^{f,\varphi}(N_t)$ denote the value function of the firm's maximization problem, then

$$\begin{aligned} J^{f,\varphi}(N_t) = \quad & \max_{V_t} \quad \exp(Z_t)N_t - w_t N_t - cV_t + \mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}J^{f,\varphi}((1-\omega)N_t + M(U_t, V_t))\right] \\ & \text{s.t.} \quad V_t \geq 0, \\ & \text{given} \;\; N_t. \end{aligned}$$

With μ_t denoting the KKT multiplier of the non-negativity constraint as before, the KKT conditions for the maximization problem on the right hand side are

$$c = \mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{\partial J^{f,\varphi}}{\partial N}(N_{t+1})\frac{\partial M}{\partial V}(U_t, V_t)\right] + \mu_t, \tag{4.6.46}$$

$$V_t \geq 0, \tag{4.6.47}$$

$$\mu_t \geq 0, \tag{4.6.48}$$

$$\mu_t V_t = 0. \tag{4.6.49}$$

Introducing the notation

$$\xi_t^{f,\varphi} := \frac{\partial J^{f,\varphi}}{\partial N}(N_t), \quad \text{and} \quad \hat{\xi}_t^{f,\varphi} := \mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^{f,\varphi}\right] = \mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{\partial J^{f,\varphi}}{\partial N}(N_{t+1})\right].$$

the first KKT condition can be reformulated as

$$c = \frac{\partial M}{\partial V}(U_t, V_t)\hat{\xi}_t^{f,\varphi} + \mu_t. \tag{4.6.50}$$

Applying again the envelope theorem, we can derive

$$\begin{aligned} \xi_t^{f,\varphi} &= \frac{\partial J^{f,\varphi}}{\partial N}(N_t) = \exp(Z_t) - w_t + \mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{\partial J^{f,\varphi}}{\partial N}(N_{t+1})\left(1 - \omega - \frac{\partial M}{\partial U}(U_t, V_t)\right)\right] = \\ &= \exp(Z_t) - w_t + \left(1 - \omega - \frac{\partial M}{\partial U}(U_t, V_t)\right)\hat{\xi}_t^{f,\varphi}, \end{aligned} \tag{4.6.51}$$

so that

$$\hat{\xi}_t^{f,\varphi} = \mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\left(\exp(Z_{t+1}) - w_{t+1} + \left(1-\omega-\frac{\partial M}{\partial U}(U_{t+1},V_{t+1})\right)\hat{\xi}_{t+1}^{f,\varphi}\right)\right].$$

Again, the interpretation remains the same as in the baseline model. Only now, the representative firm corrects the weighting factor for the continuation value of an employee in the next period for the marginal change in job matches induced by one less unemployed household member in the economy.

Wage Bargaining We employ the same sharing rule as in the baseline model for the total surplus $\xi_t^{\varphi} := \xi_t^{h,\varphi} + \xi_t^{f,\varphi}$ generated by a working member, i.e.

$$\xi_t^{h,\varphi} = \varphi\xi_t^{\varphi} \quad \text{and} \quad \xi_t^{f,\varphi} = (1-\varphi)\xi_t^{\varphi}.$$

From equations (4.6.45) and (4.6.51) we can derive the total surplus as

$$\begin{aligned}\xi_t^{\varphi} &= w_t + \exp(Z_t) - w_t + \left(1-\omega-\frac{\partial M}{\partial U}(U_t,V_t)\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\left(\xi_{t+1}^{h,\varphi}+\xi_{t+1}^{f,\varphi}\right)\right]\\ &= \exp(Z_t) + \left(1-\omega-\frac{\partial M}{\partial U}(U_t,V_t)\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^{\varphi}\right],\end{aligned}\tag{4.6.52}$$

so that the total surplus is the same as in (4.6.6) for the social planner, i.e. $\xi_t^{\varphi} = \xi_t^{soc}$. Yet, the sharing rule implies the firm to only receive the share of $1-\varphi$ from the total surplus, which equivalently requires the wage to account for φ times the labor productivity:

$$\begin{aligned}&\xi_t^{f,\varphi} = (1-\varphi)\xi_t^{\varphi}\\ \Leftrightarrow\quad &\exp(Z_t) - w_t + \left(1-\omega-\frac{\partial M}{\partial U}(U_t,V_t)\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^{\varphi,adj}\right] =\\ &\qquad = (1-\varphi)\left(\exp(Z_t) + \left(1-\omega-\frac{\partial M}{\partial U}(U_t,V_t)\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^{\varphi}\right]\right)\\ \Leftrightarrow\quad &\exp(Z_t) - w_t + \left(1-\omega-\frac{\partial M}{\partial U}(U_t,V_t)\right)(1-\varphi)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^{\varphi}\right] =\\ &\qquad = (1-\varphi)\left(\exp(Z_t) + \left(1-\omega-\frac{\partial M}{\partial U}(U_t,V_t)\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^{\varphi}\right]\right)\\ \Leftrightarrow\quad &\exp(Z_t) - w_t = (1-\varphi)\exp(Z_t)\\ \Leftrightarrow\quad &w_t = \varphi\exp(Z_t).\end{aligned}\tag{4.6.53}$$

Government sector The government runs a balanced budget, i.e. taxes equal the overall value of unemployment activities

$$T_t = b(1-N_t).$$

General Equilibrium The general equilibrium in the model is thus defined by the system of equations (4.6.11)-(4.6.21) with $\hat{\xi}_t^{soc}$ replaced by $\hat{\xi}_t^{f,\varphi}$ and the additional wage equation stated above. Since all equations remain the same, all the model's variables can be expressed dependent on $\hat{\xi}_t^{f,\varphi}$ and the state variables completely analogous to the social planner's case. Moreover the wage is completely determined by LP. Consequently, the policy function $g : [0,1]\times\mathbb{R}\to\mathbb{R}$ of $\hat{\xi}_t^{f,\varphi}$ can be characterized as the solution to the functional equation in (4.6.22)-(4.6.32) only with *rhs* adjusted to

$$rhs(g,x,z) := \mathbb{E}\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\eta}\left((1-\varphi)\exp(Z_{t+1}) + \left(1-\omega-\left(\frac{M_{t+1}}{U_{t+1}}\right)^{\tau+1}\right)g(N_{t+1},Z_{t+1})\right)\right].$$

Calibration The calibration of the model's parameters remains the same as in the baseline model, i.e. as in table 4.11 together with $\varphi = 0.052$.

Moreover, to check robustness, we also considered a second calibration of φ. We set φ in such way that the model yields the same steady state as in the model containing only the first distortion with $b = 0.85$, i.e. so that the extent of the distortion from social optimum in steady state is the same between the two cases. The within period value $\exp(Z) - b$ of a worker in the model containing the first distortion only is lowered by 85% in steady state compared to the within period value $\exp(Z)$ in the social planner's case. The same effect on the value of a worker that is decisive in the decision about posting open vacancies in the model, can therefore be achieved from the second distortion by setting $\varphi = 0.85$.

Steady State First, the wage equation simply yields $w = \varphi \exp(Z)$ so that the recursion for $\hat{\xi}_t^{f,\varphi}$ in deterministic steady state immediately shows

$$\hat{\xi}^{f,\varphi} = \frac{\beta(1-\varphi)\exp(Z)}{1-\beta\left(1-\omega-\left(\frac{\omega N}{1-N}\right)^{\tau+1}\right)}.$$ [43]

The remaining steady state values can then be computed the same way as in the social planner's case.

The steady state values for the variables, for which the steady state was fixed in the baseline model, are presented in table 4.20 for the case of $\varphi = 0.052$. The values are similar to the ones

Table 4.20: Model with second Distortion: Steady State Values ($\varphi = 0.052$)

Variable	Value	Description
U	8.87%	Unemployment Rate
κ_f	0.6523	Probability of filling an open vacancy
κ_w	0.5138	Probability of finding a job

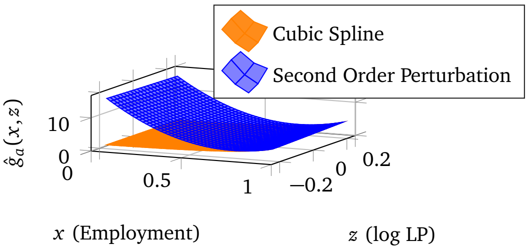
from the social planner's case in table 4.12 since the effect of the second distortion is only small if the value of φ is small. For the second case of $\varphi = 0.85$, the steady state is the same as in table 4.16 for the model containing the first distortion.

Dynamics of Employment Figure 4.24 shows, for both values assigned to φ, the resulting approximations to the policy function of $\hat{\xi}^{soc,\varphi}$ from a finite element method and from a second order perturbation. Additionally, the interpretable Euler residuals for the global method are provided. With interpretable Euler residuals of magnitude 10^{-5} and 10^{-6}, respectively, we consider the approximations by a cubic spline sufficiently accurate (the model's dynamics will not lead to states with very low employment where the Euler residuals are largest). As in the other models before, the perturbation solution again fails to provide a proper approximation to the policy function for lower employment rates.

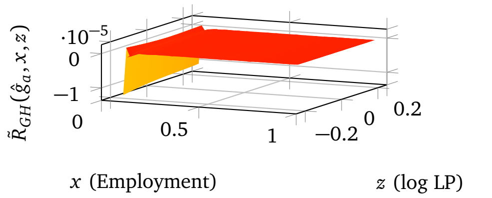
Figure 4.25 displays the time path of employment during the first 5000 periods in the simulation of the model's equilibrium outcomes for a total of 1200000 periods from the same sample of pseudorandom iid$N(0,\sigma_\epsilon^2)$ distributed shocks to log LP. Moreover, the histograms in figure 4.26 show the distribution of employment in the simulation and table 4.21 summarizes the statistical measures.

[43]Comparing to (4.6.44) for the model containing only the first distortion shows that setting $\varphi = b$ yields the same steady state (since $\exp(Z) = 1$) for the value of a worker that is decisive in the vacancy posting decision and therefore also for the remaining variables.

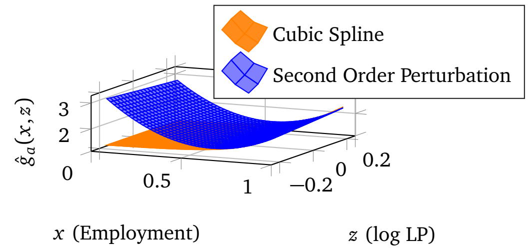
Figure 4.24: Model with second Distortion: Approximation of Policy Function and Interpretable Euler Residuals



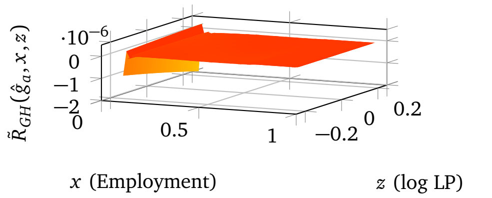
(a) Approximations to Policy Function of $\hat{\xi}^{f,\varphi}$ ($\varphi = 0.052$)



(b) Interpretable Euler Residuals for Cubic Spline Solution ($\varphi = 0.052$)

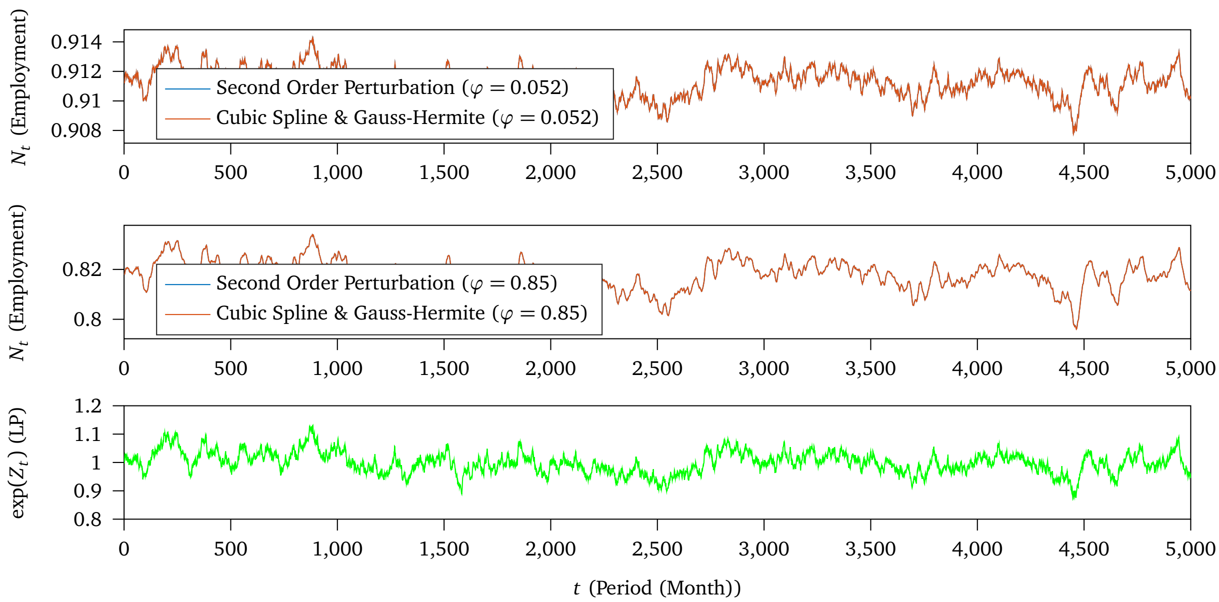


(c) Approximations to Policy Function of $\hat{\xi}^{f,\varphi}$ ($\varphi = 0.85$)



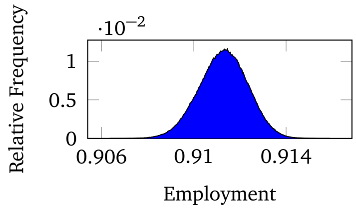
(d) Interpretable Euler Residuals for Cubic Spline Solution ($\varphi = 0.85$)

Figure 4.25: Model with second Distortion: Dynamics of Employment

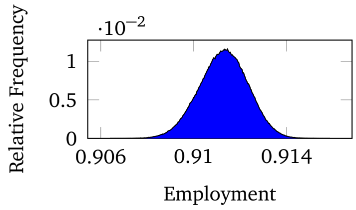


As expected, for $\varphi = 0.052$ the model's dynamics are very similar to the social planner's solution. Employment stays very close to its steady state value for the whole simulation and the solution by perturbation gives virtually the same results as the global solution. For $\varphi = 0.85$ the magnitude of distortion from social optimum in the deterministic steady state is the same as in the model including only the first distortion with $b = 0.85$. However, the effects on the dynamics in the model turn out completely different. Although employment becomes more volatile than in the case of $\varphi = 0.052$, the standard deviation is still only approximately 10% of the standard deviation in the model with $b = 0.85$. Moreover, there are no huge declines in employment, the model does not drift to states where the approximation by a second order perturbation method fails and consequently the perturbation solution reproduces the model's dynamics as well as the global solution. Finally, whereas a change in b from 0.85 to 0.75 already had major impact on the dynamics of employment, the model seems far less sensitive to changes in φ.

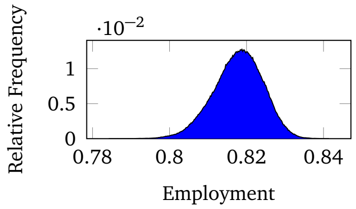
Figure 4.26: Model with second Distortion: Histogram for Distribution of Monthly Employment Rates



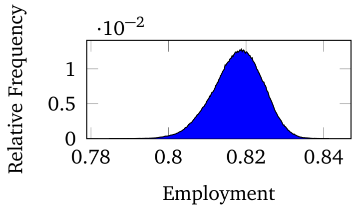
(a) Distribution of Employment computed by Cubic Spline ($\varphi = 0.052$)



(b) Distribution of Employment computed by Second Order Perturbation ($\varphi = 0.052$)



(c) Distribution of Employment computed by Cubic Spline ($\varphi = 0.85$)



(d) Distribution of Employment computed by Second Order Perturbation ($\varphi = 0.85$)

Table 4.21: Model with second Distortion: Statistic Measures for Distribution of Monthly Employment Rates

(a) Household's Bargaining Power $\varphi = 0.052$

	$\bar{N}$	$\max(N_t)$	$\min(N_t)$	$q_{0.01}$	$q_{0.05}$	$q_{0.1}$	$q_{0.5}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	s_N	v_N	w_N
Cubic Spline	0.9113	0.9159	0.9064	0.9088	0.9095	0.9099	0.9113	0.9126	0.9130	0.9136	0.0010	-0.0948	2.9888
Perturbation	0.9113	0.9159	0.9064	0.9088	0.9095	0.9099	0.9113	0.9126	0.9130	0.9136	0.0010	-0.0944	2.9834

(b) Household's Bargaining Power $\varphi = 0.85$

	$\bar{N}$	$\max(N_t)$	$\min(N_t)$	$q_{0.01}$	$q_{0.05}$	$q_{0.1}$	$q_{0.5}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	s_N	v_N	w_N
Cubic Spline	0.8179	0.8413	0.7843	0.8017	0.8068	0.8094	0.8182	0.8258	0.8278	0.8314	0.0064	-0.2957	3.1325
Perturbation	0.8179	0.8413	0.7847	0.8017	0.8068	0.8094	0.8182	0.8258	0.8278	0.8313	0.0064	-0.2924	3.1196

Notes: $\bar{N}$=average employment rate, $\max(N_t)$=maximal employment rate, $\min(N_t)$=minimal employment rate, $q_p(N_t)$ =p-quantile for cumulative distribution of employment rate, s_N, v_N, w_N=standard deviation, skewness, kurtosis of the cumulative distribution of employment rate. Statistics are computed from the outcome of a simulation of 1200000 (monthly) periods.

Endogenous Disasters? Table 4.22 summarizes the disaster statistics in the model. The results for $\varphi = 0.85$ are in brackets. With no steep drops in employment, disasters in consumption or output in the magnitude observed in the baseline model cannot arise. More precisely, we do not observe any declines below 80% of the steady state value in both consumption or output for $\varphi = 0.052$. With $\varphi = 0.85$ the model is further away from social optimum, yet consumption falls only once during the 400000 quarters below 80% of its steady state value while output does not at all. We conclude that the second distortion—introduced through the fact that in a competitive equilibrium only the firm's share in the surplus generated by a working household member plays a role in the decision about the amount of open vacancies in the economy—cannot generate disasters in the magnitude observed for the first distortion with $b = 0.85$.

Second Moments of the Labor Market Table 4.23 shows the second moments in the model with $\varphi = 0.052$ and $\varphi = 0.85$, respectively. Here too, for $\varphi = 0.052$ the results are very similar to the social planner problem. An increase of the household's bargaining power to $\varphi = 0.85$ yields the standard deviations to become twice as large, yet still only approximately $\frac{1}{6}$-th of the values found in the model including the first distortion with $b = 0.85$. As argued in Hagedorn and Manovskii (2008), modifications in b have significantly more impact on the second moments of the labor market (see table 4.19) than changes in the bargaining power φ.

Table 4.22: Endogenous Disasters in the Model with second Distortion

	number of disasters	total number of disaster periods	disaster probability	average disaster size	average disaster duration
			90% threshold		
C	1208 (1920)	4435 (8648)	0.31% (0.49%)	11.32% (11.53%)	3.67 (4.50)
Y	799 (1521)	2570 (6639)	0.20% (0.39%)	11.10% (11.44%)	3.22 (4.36)
LP	663	2086	0.17%	11.05%	3.15
			85% threshold		
C	29 (103)	74 (379)	0.0073% (0.0258%)	15.98% (16.21%)	2.55 (3.68)
Y	8 (67)	19 (225)	0.0020% (0.0168%)	15.80% (16.07%)	2.38 (3.36)
LP	5	10	0.0013%	15.68%	2.00
			80% threshold		
C	0 (1)	0 (1)	0% (0.0003%)	– (20.71%)	– (1.00)
Y	0 (0)	0 (0)	0% (0%)	– (–)	– (–)
LP	0	0	0%	–	–
			75% threshold		
C	0 (0)	0 (0)	0% (0%)	– (–)	– (–)
Y	0 (0)	0 (0)	0% (0%)	– (–)	– (–)
LP	0	0	0%	–	–

Table 4.23: Labor Market Moments in the Model with second Distortion

(a) Household's Bargaining Power $\varphi = 0.052$

	U	V	θ	$p = \frac{Y}{N}$
s_x	0.004 $[0.004; 0.005]$	0.005 $[0.004; 0.006]$	0.009 $[0.007; 0.010]$	0.016 $[0.013; 0.019]$
$\frac{s_x}{s_p}$	0.273 $[0.264; 0.282]$	0.298 $[0.287; 0.311]$	0.535 $[0.528; 0.542]$	1.000 $[1.000; 1.000]$
r_x	0.831 $[0.769; 0.880]$	0.626 $[0.511; 0.727]$	0.793 $[0.718; 0.853]$	0.774 $[0.693; 0.840]$
		Cross Correlations		
U		-0.755 $[-0.816; -0.684]$	-0.930 $[-0.950; -0.904]$	-0.908 $[-0.934; -0.876]$
V			0.943$[0.930; 0.955]$	0.959 $[0.950; 0.967]$
θ				0.998 $[0.998; 0.999]$

(b) Household's Bargaining Power $\varphi = 0.85$

	U	V	θ	$p = \frac{Y}{N}$
s_x	0.011 $[0.008; 0.014]$	0.009 $[0.007; 0.010]$	0.017 $[0.013; 0.021]$	0.016 $[0.013; 0.019]$
$\frac{s_x}{s_p}$	0.670 $[0.609; 0.738]$	0.552 $[0.512; 0.593]$	1.042 $[0.979; 1.109]$	1.000 $[1.000; 1.000]$
r_x	0.915 $[0.883; 0.940]$	0.672 $[0.578; 0.755]$	0.861 $[0.804; 0.903]$	0.774 $[0.693; 0.840]$
		Cross Correlations		
U		-0.459 $[-0.540; -0.376]$	-0.879 $[-0.912; -0.838]$	-0.706 $[-0.773; -0.629]$
V			0.823 $[0.812; 0.837]$	0.949 $[0.939; 0.958]$
θ				0.958$[0.948; 0.966]$

Notes: U=unemployment, V=vacancies, $\theta = \frac{V}{U}$=labor market tightness, p=output per worker, s_x=standard deviation of variable X, r_x=first order autocorrelation of variable X. All moments are computed from the cyclical components of the HP-filtered ($\lambda = 1600$) relative deviations from the mean, i.e. as the cyclical component of the series $\frac{X_t - \bar{X}}{\bar{X}}$. The table shows the average as well as the 2.5% and 97.5% quantiles of 5000 simulations of the model's equilibrium outcomes for 666 periods converted to 222 quarterly observations after 1000 throw-away periods.

4.6.1.5 Adding the first and second Distortion to the Social Planner's Problem

We have now separately introduced the first two distortions into the social planner's solution. Last, we also consider the model containing both distortions at the same time to see if important cross effects between the two distortions arise. Different from the baseline model however, firms and households in the economy still coordinate and take side effects of their decisions on average probabilities in the matching process into account. We can then also illustrate the effect of the third distortion in the baseline model by comparing the results to the model from this subsection.

In order to derive the equilibrium conditions which contain the first two distortions, it is sufficient to adjust the present value of a worker and the optimality condition for the amount of open vacancies from the social planner's problem to (4.6.41) and (4.6.42). For the sake of exposition we will repeat the derivations.

Representative Household The only difference from the previous model in section 4.6.1.4 regarding the representative household is the fact that he does no more acknowledge that $T_t = b(1-N_t)$. Consequently, the value of an employed (over an unemployed) member to the household in (4.6.45) changes to

$$\xi_t^{h,b,\varphi} = w_t - b + \left(1-\omega-\frac{\partial M}{\partial U}(U_t,V_t)\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^{h,b,\varphi}\right]. \tag{4.6.54}$$

Representative Firm On the representative firm's side nothing changes compared to the previous model in section 4.6.1.4. The KKT conditions for the optimization problem as well as the value of an employee to the firm remain the same, i.e. using the same notation

$$\xi_t^{f,b,\varphi} = \exp(Z_t) - w_t + \left(1-\omega-\frac{\partial M}{\partial U}(U_t,V_t)\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\hat{\xi}_t^{f,b,\varphi}\right]. \tag{4.6.55}$$

Wage Bargaining We employ the same sharing rule as in the baseline model for the total surplus $\xi_t^{b,\varphi} := \xi_t^{h,b,\varphi} + \xi_t^{f,b,\varphi}$ generated by a working member, i.e.

$$\xi_t^{h,b,\varphi} = \varphi\xi_t^{b,\varphi} \quad \text{and} \quad \xi_t^{f,b,\varphi} = (1-\varphi)\xi_t^{b,\varphi}.$$

From equations (4.6.54) and (4.6.55) we can derive the total surplus as

$$\begin{aligned}\xi_t^{b,\varphi} &= w_t - b + \exp(Z_t) - w_t + \left(1-\omega-\frac{\partial M}{\partial U}(U_t,V_t)\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\left(\xi_{t+1}^{h,b,\varphi}+\xi_{t+1}^{f,b,\varphi}\right)\right]\\ &= \exp(Z_t) - b + \left(1-\omega-\frac{\partial M}{\partial U}(U_t,V_t)\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^{b,\varphi}\right].\end{aligned} \tag{4.6.56}$$

Hence, the total surplus is the same as in (4.6.43) for the model including only the first distortion, i.e. $\xi_t^{b,\varphi} = \xi_t^{soc,b}$, while the second distortion additionally implies the firm to only receive the share of $1-\varphi$ from the total surplus so that $\hat{\xi}_t^{f,b,\varphi} = (1-\varphi)\hat{\xi}_t^{soc,b}$.

We can deduce the corresponding wage equation from

$$\begin{aligned} & \xi_t^{f,adj} = (1-\varphi)\xi_t^{adj}\\ \Leftrightarrow\quad & \exp(Z_t) - w_t + \left(1-\omega-\frac{\partial M}{\partial U}(U_t,V_t)\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^{f,adj}\right] =\\ & \qquad = (1-\varphi)\left(\exp(Z_t) - b + \left(1-\omega-\frac{\partial M}{\partial U}(U_t,V_t)\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^{adj}\right]\right)\\ \Leftrightarrow\quad & \exp(Z_t) - w_t + \left(1-\omega-\frac{\partial M}{\partial U}(U_t,V_t)\right)(1-\varphi)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^{adj}\right] =\\ & \qquad = (1-\varphi)\left(\exp(Z_t) - b + \left(1-\omega-\frac{\partial M}{\partial U}(U_t,V_t)\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^{adj}\right]\right)\\ \Leftrightarrow\quad & \exp(Z_t) - w_t = (1-\varphi)(\exp(Z_t) - b)\\ \Leftrightarrow\quad & w_t = \varphi\exp(Z_t) + (1-\varphi)b.\end{aligned}$$

The wage is hence determined by a weighted mean of the LP and the value of unemployment activities.

Government sector The government runs a balanced budget, i.e. taxes equal the overall value of unemployment activities

$$T_t = b(1-N_t).$$

General Equilibrium The general equilibrium is again determined by equations (4.6.11)-(4.6.21), only this time with $\hat{\xi}_t^{soc}$ replaced by $\hat{\xi}_t^{f,b,\varphi}$ where the wage is defined by the above stated equation. Consequently, the model's variables can be expressed dependent on $\hat{\xi}_t^{f,b,\varphi}$ and the state variables in the same way as before. Moreover the wage can easily be determined as the weighted mean of LP and the value of unemployment activities b. Hence, the policy function $g: [0,1] \times \mathbb{R} \to \mathbb{R}$ of $\hat{\xi}_t^{f,b,\varphi}$ can be characterized as the solution to the functional equation in (4.6.22)-(4.6.32) only with rhs in (4.6.24) adjusted to

$$rhs(g,x,z) := \mathbb{E}\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\eta}\left(\exp(Z_{t+1}) - w_{t+1} + \left(1-\omega-\left(\frac{M_{t+1}}{U_{t+1}}\right)^{\tau+1}\right)g(N_{t+1}, Z_{t+1})\right)\right].$$

Calibration The calibration of the model's parameters remains the same as in the baseline model and the previous two variations of the model, i.e. as in table 4.11 together with $\varphi = 0.052$ and $b = 0.85$.

Steady State Since the wage equation simply implies

$$w = \varphi \exp(Z) + (1-\varphi)b,$$

the value of a worker to the firm in deterministic steady state must satisfy

$$\hat{\xi}^{f,b,\varphi} = \frac{\beta(1-\varphi)(\exp(Z)-b)}{1-\beta\left(1-\omega-\left(\frac{\omega N}{1-N}\right)^{\tau+1}\right)}.$$

The remaining steady state values can then be computed in the same way as before.

The steady state values for the variables, for which these were fixed in the baseline model, are presented in table 4.24. The values differ only slightly from the steady state values in table 4.16 for the model including the first distortion only, since the additional effect of the second distortion in the present model is only small due to the low value assigned to φ.

Table 4.24: Model including the first two Distortions: Steady State Values

Variable	Value	Description
U	18.85%	Unemployment Rate
κ_f	0.8913	Probability of filling an open vacancy
κ_w	0.2153	Probability of finding a job

A comparison between the steady states in the baseline model and the present model reveals the net effect of the third distortion, at least for the deterministic steady state in the economy. First, the present value of a worker in the next period to the firm in the baseline model, i.e. $\hat{\xi}^f$, exceeds the corresponding value $\hat{\xi}^{f,b,\varphi}$ for the model at hand. This is the case for the following reason. While the value is determined by

$$\hat{\xi}^f = \frac{\beta(1-\varphi)(\exp(Z)-b)}{1-\beta(1-\omega-\varphi\frac{M}{U})},$$

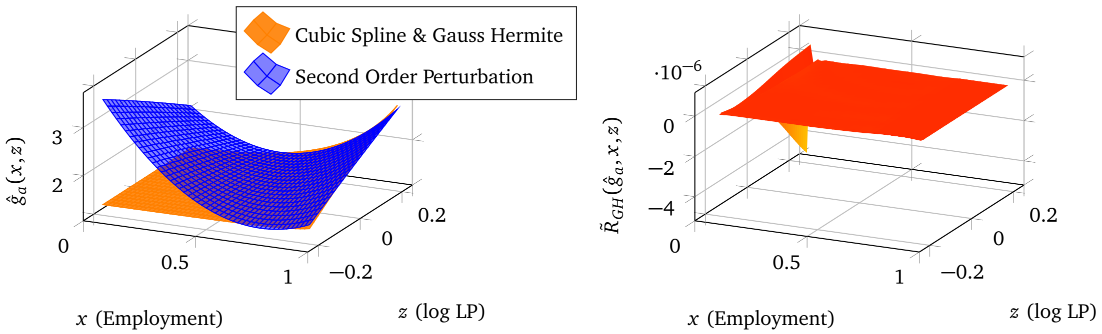
in the baseline model (see (4.4.2)), removing the third distortion results in

$$\hat{\xi}^{f,b,\varphi} = \frac{\beta(1-\varphi)(\exp(Z)-b)}{1-\beta\left(1-\omega-\left(\frac{M}{U}\right)^{\tau+1}\right)} = \frac{\beta(1-\varphi)(1-b)}{1-\beta\left(1-\omega-\frac{\partial M}{\partial U}(U,V)\right)}.$$

We already argued in section 4.6.1.2 that the third distortion causes the representative firm to overestimate and the representative household to underestimates the respective value of a worker to them. Both effects would cancel out in the total surplus in case of the first Hosios condition, $\frac{\partial M}{\partial U} = \varphi\frac{M}{U}$, holding. Yet here, with the low bargaining power of the household, the effect concerning the firm dominates, i.e. $\frac{\partial M}{\partial U} > \varphi\frac{M}{U}$. Hence, once the first two distortions are in the model, the third one has a positive effect on the total surplus attributed to a worker and therefore on the firm's share in it in steady state. Moreover, the firm additionally overestimates the probability at which a vacancy can be filled under the third distortion so that the positive effect on the amount of vacancies in the economy is even further reinforced.

Dynamics of Employment We show the approximations to the policy function of $\hat{\xi}_t^{f,b,\varphi}$ from a global finite element method and from a second order perturbation method in figure 4.27a. The interpretable Euler residuals for the global solution are displayed in figure 4.27b. The Euler

Figure 4.27: Model including the first two Distortions: Approximation of Policy Function and Interpretable Euler Residuals



(a) Approximations to Policy Function of $\hat{\xi}^{f,adj}$

(b) Interpretable Euler Residuals for Cubic Spline Solution

residuals do not exceed $4\cdot 10^{-6}$ in absolute value so that we consider the approximation by the cubic spline sufficiently accurate.

Figure 4.28 pictures the outcome for the series of employment during the first 5000 periods in the simulation of the model's equilibrium outcomes for 1200000 periods from the same sample of pseudorandom iidN(0,σ_ϵ^2) distributed shocks to log LP, while figure 4.29 displays the histograms for the ensuing distribution. Moreover, some statistical measures for the distribution of employment in the simulation are summarized in table 4.25. Compared to the model including only the first distortion in section 4.6.1.3, the distribution of employment is slightly

Table 4.25: Model including the first two Distortions: Statistic Measures for Distribution of Monthly Employment Rates

	$\bar{N}$	$\max(N_t)$	$\min(N_t)$	$q_{0.01}$	$q_{0.05}$	$q_{0.1}$	$q_{0.5}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	s_N	v_N	w_N
Cubic Spline	0.7907	0.8779	0.0573	0.4853	0.6666	0.7217	0.8102	0.8426	0.8485	0.8578	0.0700	-3.4369	20.8303
Perturbation	0.7989	0.8774	0.6009	0.6616	0.7062	0.7355	0.8104	0.8425	0.8484	0.8576	0.0435	-1.2549	4.3736

Notes: $\bar{N}$=average employment rate, $\max(N_t)$=maximal employment rate, $\min(N_t)$=minimal employment rate, $q_p(N_t)$ =p-quantile for cumulative distribution of employment rate, s_N, v_N, w_N=standard deviation, skewness, kurtosis of the cumulative distribution of employment rate. Statistics are computed from the outcome of a simulation of 1200000 (monthly) periods.

Figure 4.28: Model including the first two Distortions: Dynamics of Employment

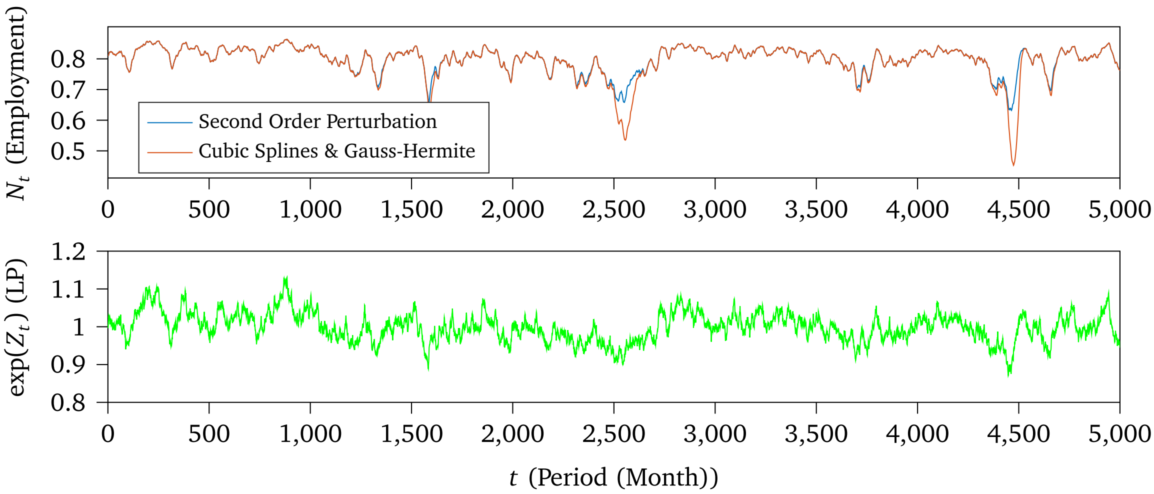
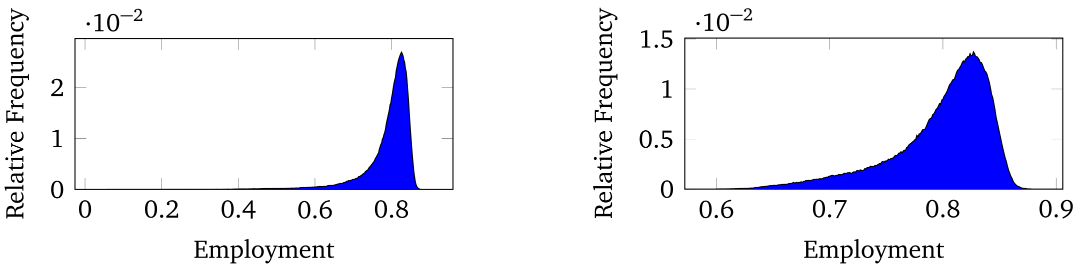


Figure 4.29: Model including the first two Distortions: Histograms for Distribution of Monthly Employment Rates



(a) Distribution of Employment computed by Cubic Splines

(b) Distribution of Employment computed by Second Order Perturbation

shifted to the left due to the additional negative effect on vacancies by the firm receiving only a share of $1 - \varphi$ of the total surplus generated by a working member in the economy. Apart from that, adding the second distortion does not change much for the dynamics in the model once the first distortion is already included, i.e. no significant cross effects between the two distortions can be identified.

Moreover, we can also identify the effect of the third distortion by comparing the results to the ones from the baseline model shown in figures 4.9, 4.11 and table 4.5. As already mentioned, the third distortion has a positive net effect on the amount of open vacancies in the economy's deterministic steady state and therefore leads to a higher steady state level of employment in the baseline model. Yet, the minimum employment rate in the simulation is similar to the one in the present model so that introducing the third distortion yields even higher relative drops in the economy's employment. The distribution of employment becomes slightly more volatile and left skewed with an even higher kurtosis.

Endogenous Disasters? The similarity of the dynamics of employment in the present model to the model with only the first distortion implies that the disaster statistics should also be similar. This is confirmed by the results in table 4.26.

Second Moments of the Labor Market Table 4.27 summarizes the second moments for the labor market. Again, the results are almost the same as in table 4.19a for the model containing only the first distortion. Hence, with the small value assigned to φ no important cross effects between the two distortions become apparent. Moreover, a comparison with the results in table

Table 4.26: Endogenous Disasters in the Model including the first two Distortions

	number of disasters	total number of disaster periods	disaster probability	average disaster size	average disaster duration
			90% threshold		
C	4403	71133	1.34%	18.81%	16.16
Y	4832	71386	1.47%	18.41%	14.77
LP	663	2086	0.17%	11.05%	3.15
			85% threshold		
C	2589	42208	0.72%	24.56%	16.30
Y	2823	42548	0.79%	24.15%	15.07
LP	5	10	0.0013%	15.68%	2.00
			80% threshold		
C	1607	25988	0.43%	29.88%	16.17
Y	1732	26268	0.46%	29.55%	15.17
LP	0	0	0%	–	–
			75% threshold		
C	922	16493	0.24%	36.17%	17.89
Y	991	16699	0.26%	35.76%	16.85
LP	0	0	0%	–	–
			70% threshold		
C	599	11141	0.15%	41.48%	18.60
Y	637	11289	0.16%	41.14%	17.72
LP	0	0	0%	–	–
			50% threshold		
C	120	2618	0.03%	61.01%	21.82
Y	126	2637	0.03%	60.77%	20.93
LP	0	0	0%	–	–
			30% threshold		
C	25	608	0.0063%	77.48%	24.32
Y	26	615	0.0065%	77.36%	23.65
LP	0	0	0%	–	–

Table 4.27: Labor Market Moments the Model including the first two Distortions

	U	V	θ	$p=\frac{Y}{N}$
s_x	0.068 [0.042; 0.104]	0.053 [0.035; 0.079]	0.089 [0.061; 0.129]	0.016 [0.013; 0.019]
$\frac{s_x}{s_p}$	4.218 [2.797; 6.180]	3.313 [2.286; 4.767]	5.524 [4.163; 7.538]	1.000 [1.000; 1.000]
r_x	0.933 [0.891; 0.964]	0.723 [0.602; 0.828]	0.870 [0.813; 0.915]	0.774 [0.693; 0.840]
		Cross Correlations		
U		-0.450 [−0.542; −0.357]	-0.787 [−0.878; −0.590]	-0.513 [−0.707; −0.228]
V			0.792 [0.677; 0.833]	0.915 [0.833; 0.959]
θ				0.912 [0.786; 0.964]

Notes: U=unemployment, V=vacancies, $\theta=\frac{V}{U}$=labor market tightness, p=output per worker, s_x=standard deviation of variable X, r_x=first order autocorrelation of variable X. All moments are computed from the cyclical components of the HP-filtered ($\lambda = 1600$) relative deviations from the mean, i.e. as the cyclical component of the series $\frac{X_t-\bar{X}}{\bar{X}}$. The table shows the average as well as the 2.5% and 97.5% quantiles from 5000 simulations of the model's equilibrium outcomes for 666 periods converted to 222 quarterly observations after 1000 throw-away periods.

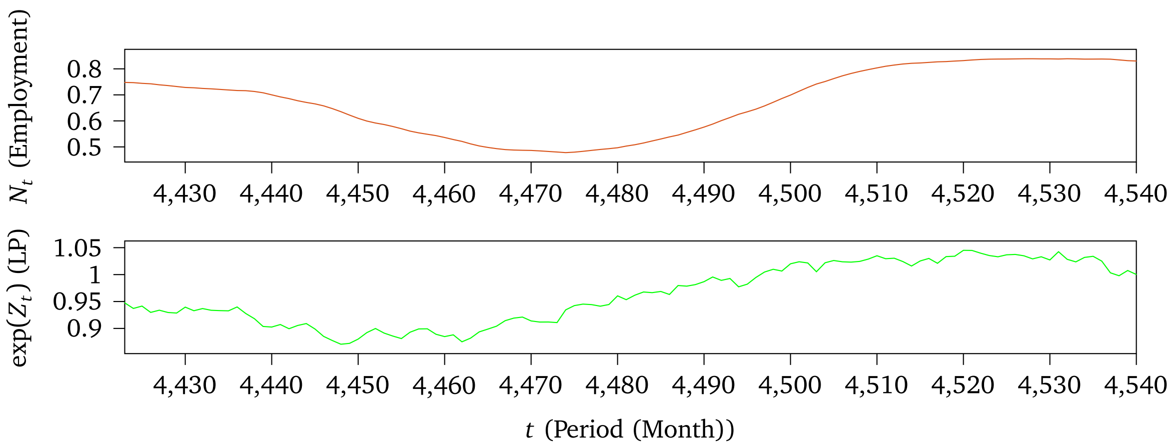
4.10b for the baseline model shows the effect of the third distortion on the second moments. When the third distortion is added to the model, the volatility in all three variables considered increases. The average standard deviations of unemployment and labor market tightness in the baseline model are almost double the values from the present model. Further, the negative correlation between unemployment and vacancies is also closer to the value found in the data.

4.6.2 Comparison and Intuition

We have now separated the effects of the single distortions which set apart the baseline model from the social planner's solution. The first distortion, introduced through the fact that the representative household does not take into account that unemployment activities summarized in b yield no aggregate value to him, has major impact on the model's dynamics. It proved crucial for generating periods of huge unemployment rates in the economy which result in disasters in both consumption and output. Moreover, the dynamics in the economy appear highly sensitive to seemingly small changes in the period value b attributed to unemployment activities. On the other hand, the second distortion—only the firm's share in the total surplus generated by an employment is decisive for the amount of open vacancies in a competitive equilibrium—had only minor impact on the model's dynamics. Even if the size of the distortion is adjusted in such way to yield the same steady state as in the model with the first distortion, the model shows no steep drops in the employment rate. The third distortion—firms and households in the economy do not coordinate in such way to account for marginal changes instead of only average probabilities in the job matching process—results in a higher steady state level of employment, while employment drops as low as in the model excluding the distortion. Consequently, the labor market becomes more volatile when the third distortion is added. Before we proceed to endogenize some of the assumptions made in order to calibrate b at the high value and analyze how these changes alter the economy's behavior, we first want to provide some further intuition for the observed results.

Elasticity of Realized Matches with Respect to Present Value of a Worker Let us first examine the effect in the model including only the first distortion. Figure 4.30 shows the employment rate between periods 4420 and 4540 in the simulation of the equilibrium outcomes for a value of unemployment activities equal to $b = 0.85$. The huge drop in the employment rate, to below

Figure 4.30: Drop in Employment for the Model including the first Distortion with b=0.85



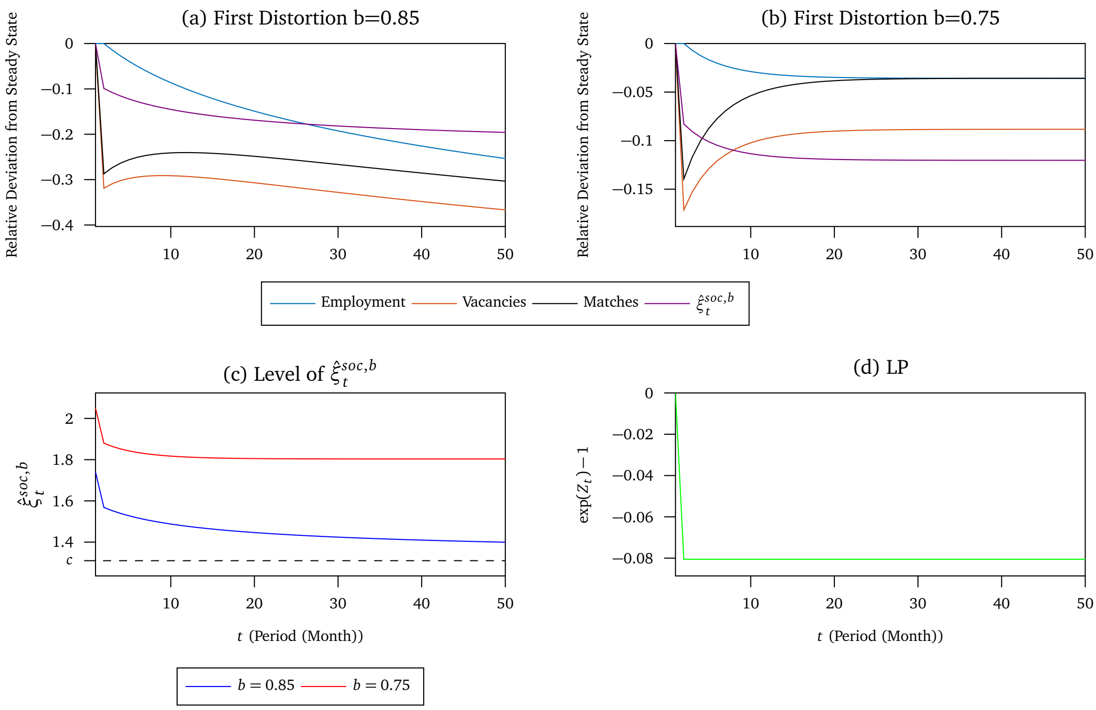
0.5, is triggered by a series of (mainly) negative shocks to log LP lowering labor productivity to below 0.95 for approximately 50 consecutive periods. We simplify the model's dynamics during these periods by considering the following (unrealistic) scenario, which should nonetheless render the crucial mechanism leading to the huge drop. We suppose the economy to start in the steady state in period $t = 1$. Instead of a series of negative shocks affecting labor productivity, we assume log LP to fall at once by two unconditional standard deviations in period $t = 2$ and to remain at the value for the following 50 periods, i.e.

$$Z_t = -2\frac{\sigma_\epsilon}{\sqrt{1-\rho^2}}, \quad t \geq 2.$$ [44]

Figure 4.31 shows the resulting reactions for the employment rate, open vacancies, matches and the present value $\hat{\xi}^{soc,b}$ of a working member in the next period with $b = 0.85$ and $b = 0.75$, respectively. We can immediately observe the substantial differences in the impact on the employment rate. While the employment rate stabilizes after a few periods after dropping only by approximately 3% in the case of $b = 0.75$, the employment rate has already fallen by

[44] This implies labor productivity to fall approximately to $\exp(Z_2) \approx 0.92 > b$.

Figure 4.31: First Distortion: Effects of Drop in Labor Productivity



approximately 25% after 50 periods with $b = 0.85$ and even keeps dropping (and in fact will converge to 0 as long as labor productivity does not recover).

Now why is this the case? First, the drop in labor productivity in period $t = 2$ lowers the expectation about future within period value from an employed over an unemployed member, i.e. of $\exp(Z_{t+s}) - b, s \geq 1$, in

$$\hat{\xi}_t^{soc,b} = \mathbb{E}_t \left[\sum_{s=1}^{\infty} \left(\prod_{k=1}^{s-1} \left(1 - \omega - \frac{\partial M}{\partial U}(U_{t+k}, V_{t+k}) \right) \right) \beta^s \frac{\lambda_{t+s}}{\lambda_t} (\exp(Z_{t+s}) - b) \right],$$

so that the present value of a worker (over an unemployed member) in the next period declines. Consequently, the equilibrium condition for the amount of open vacancies, i.e.

$$c = \frac{\partial M}{\partial V}(U_t, V_t)\hat{\xi}_t^{soc,b} \tag{4.6.57}$$

demands the marginal rate at which a vacancy can be filled, i.e. $\frac{\partial M}{\partial V}(U_t, V_t)$, to rise. With the unemployment rate fixed in $t = 2$ the amount of open vacancies posted and ergo the realized matches have to decline. Now, the first important difference between the two cases shows in the magnitude of the effects in period $t = 2$. For $b = 0.85$ the present value of a worker $\hat{\xi}_t^{soc,b}$ falls by approximately 10%, while it declines only moderately less by 8% for $b = 0.75$. However, the impact on the amount of realized matches differs significantly with declines of approximately 29% and 14%, respectively. Hence, the amount of realized matches seems to react noticeably more sensitive to changes in $\hat{\xi}^{soc,b}$ in the case of $b = 0.85$ than for $b = 0.75$. As already shown, the optimality condition (4.6.57) yields (since the non-negativity constraint on V_t remains non-binding)

$$V_t = U_t \left(\left(\frac{\hat{\xi}_t^{soc,b}}{c} \right)^{\frac{\tau}{\tau+1}} - 1 \right)^{\frac{1}{\tau}},$$

so that

$$M_t = \frac{U_t V_t}{\left(U_t^\tau + V_t^\tau\right)^{\frac{1}{\tau}}} = \frac{V_t}{\left(1+\left(\frac{V_t}{U_t}\right)^\tau\right)^{\frac{1}{\tau}}} = \frac{U_t\left(\left(\frac{\hat{\xi}_t^{soc,b}}{c}\right)^{\frac{\tau}{\tau+1}}-1\right)^{\frac{1}{\tau}}}{\left(1+\left(\frac{\hat{\xi}_t^{soc,b}}{c}\right)^{\frac{\tau}{\tau+1}}-1\right)^{\frac{1}{\tau}}} = U_t\left(1-\left(\frac{\hat{\xi}_t^{soc,b}}{c}\right)^{-\frac{\tau}{\tau+1}}\right)^{\frac{1}{\tau}}. \tag{4.6.58}$$

Therefore, the elasticity of realized matches M_t with respect to $\hat{\xi}_t^{soc,b}$ given U_t (fixed in t=2) turns out to be

$$\epsilon_{M_t,\hat{\xi}_t^{soc,b}} = U_t\frac{1}{\tau}\left(1-\left(\frac{\hat{\xi}_t^{soc,b}}{c}\right)^{-\frac{\tau}{\tau+1}}\right)^{\frac{1}{\tau}-1}\frac{\tau}{\tau+1}\left(\frac{\hat{\xi}_t^{soc,b}}{c}\right)^{-\frac{\tau}{\tau+1}-1}\frac{1}{c}\frac{\hat{\xi}_t^{soc,b}}{M_t} =$$

$$= \frac{1}{\tau+1}\frac{\left(\frac{\hat{\xi}_t^{soc,b}}{c}\right)^{-\frac{\tau}{\tau+1}}}{1-\left(\frac{\hat{\xi}_t^{soc,b}}{c}\right)^{-\frac{\tau}{\tau+1}}} = \frac{1}{\tau+1}\frac{1}{\left(\frac{\hat{\xi}_t^{soc,b}}{c}\right)^{\frac{\tau}{\tau+1}}-1}.$$ [45]

As a consequence, the amount of realized matches reacts more sensitive to changes in $\hat{\xi}_t^{soc,b}$ the closer the present value $\hat{\xi}_t^{soc,b}$ of a worker in the next period is to the costs c of posting an open vacancy. Now, a higher value of b implies $\hat{\xi}_t^{soc,b}$ to be already lower in steady state and consequently yields a higher elasticity $\epsilon_{M_t,\hat{\xi}_t^{soc,b}}$ (given the same costs c).[46] In particular, the elasticity turns out to be approximately 2.5 for $b = 0.85$ and 1.5 for $b = 0.75$ in steady state. This explains why the seemingly small difference in the relative decline of $\hat{\xi}_t^{soc,b}$ leads to substantial different declines in the amount of matches in period $t = 2$.

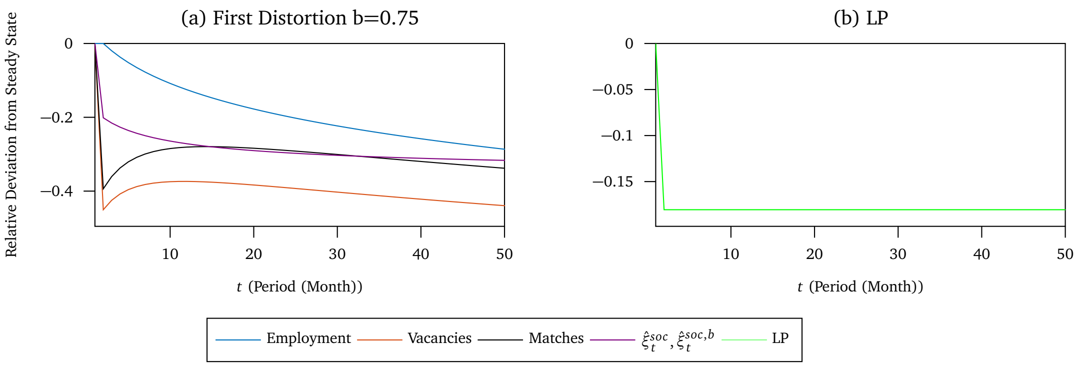
Now, with no further changes to LP for the remaining periods, the further progression in the economy is induced solely by the dynamics of employment from this point forward. The lower amount of M_t in period $t = 2$ yields the unemployment rate to rise in t=3 in both cases. An inclining unemployment rate has three effects on the amount of realized job matches in the economy. First, more unemployed members enter the matching process. Second, other things equal, the marginal rate $\frac{\partial M}{\partial V}$ for an open vacancy to be filled increases[47] which exercises a positive effect on the expected return of an open vacancy and therefore on the number of vacancies. Yet third, consumption is decreasing and future within period value generated by an employed member in $\hat{\xi}_t^{soc,b}$ is discounted more so that the present value of a worker continues to fall further—even though expectations about future labor productivity do not change—which exercises a negative effect on the number of vacancies. The first two effects yield the amount of successful job matches to increase with the unemployment rate and are rendered in the first factor in the right hand side of (4.6.58), while the third effect causes a decline in realized job matches and is contained in the second factor in brackets in the right hand side of (4.6.58). Moreover, (4.6.58) shows that the positive effect becomes smaller, while the negative effect increases the closer the present value $\hat{\xi}_t^{soc,b}$ of a worker in the next period falls to the costs c of posting vacancies. We can see from figure 4.31 that the positive effect is dominating in the next periods and the amount of realized job matches rises again compared to period t=2 for both values of b. Nonetheless, with $\hat{\xi}_t^{soc,b}$ closer to c in the case of $b = 0.85$, the positive net

[45] Equivalently, in the baseline model $V_t = U_t\left(\left(\hat{\xi}_t^f/c\right)^\tau - 1\right)^{\frac{1}{\tau}}$ implies $\epsilon_{M_t,\hat{\xi}_t^f} = \frac{1}{\left(\left(\hat{\xi}_t^f/c\right)^\tau-1\right)}$.

[46] Note that the calibration strategy in the baseline model did not fix c but rather implied $\frac{\hat{\xi}^f}{c} = \frac{1}{\kappa_f} = \frac{1}{0.71}$ to be fixed in steady state.

[47] Since $\frac{\partial M}{\partial V \partial U} > 0$.

Figure 4.32: First Distortion: Effects of Drop in Labor Productivity II



effect on matches turns out much weaker in comparison. As long as the decline in matches still exceeds the decline in the employment rate, exogenous separations exceed the number of realized job matches and employment continues to fall. While in the case of $b = 0.75$ matches eventually recover enough after some more periods to stop the economy's employment to fall any further, this is different with $b = 0.85$. Matches recover too weakly, unemployment increases faster, which further lowers $\hat{\xi}_t^{soc,b}$ and increases the elasticity $\epsilon_{M_t, \hat{\xi}_t^{soc,b}}$ until the effect of $\hat{\xi}_t^{soc,b}$ on matches eventually becomes dominant. Matches start to decline again, unemployment is not stopped from falling any further, $\hat{\xi}_t^{soc,b}$ decreases even more and the economy enters a downward spiral.

We conclude from this scenario that once negative shocks to LP render $\hat{\xi}_t^{soc,b}$ to fall close enough to c in order for the high elasticity $\epsilon_{M_t, \hat{\xi}_t^{soc,b}}$ to play a dominant role in the reaction of matches during rising unemployment, the economy wont stabilize until LP and consequently $\hat{\xi}_t^{soc,b}$ recover. Now, $\hat{\xi}_t^{soc,b}$ depends on the expectation about future excess of labor productivity net value of unemployment activities, i.e. $\exp(Z_{t+s}) - b$. Hence, in order for $\hat{\xi}_t^{soc,b}$ to decline to approximately the same level with $b = 0.75$ as it is the case for $b = 0.85$ in the scenario pictured in figure 4.31, LP would have to decline by 0.1 more. We therefore repeat the scenario for $b = 0.75$ in figure 4.32, where we now assume log LP to jump in period $t = 2$ to

$$Z_t = \ln\left(\exp\left(-2\frac{\sigma_\epsilon}{\sqrt{1-\rho^2}}\right) - 0.1\right), \quad t \geq 2.$$[48]

Employment does not stabilize after some periods anymore, but has declined by over 25% after 50 periods and keeps dropping until LP recovers. Hence, disasters in the model with the first distortion and $b = 0.75$ would arise in the same fashion, if LP would only fall low enough. Yet, while a decline in log LP below $-2\frac{\sigma_\epsilon}{\sqrt{1-\rho^2}}$ appears with a probability of approximately 2.5%, an additional drop as low as in the second scenario becomes disproportionately less likely.[49] In consequence, the formation of disasters seems to depend sensitively on the value of the parameter b in the results in table 4.18.

We have identified the present value of a worker in the next period falling close to the costs of posting an open vacancy for lower (but plausible) states of labor productivity as a first important factor for the occurrence of disasters in the model. The elasticity of realized matches with regard to $\hat{\xi}_t^{soc,b}$ becomes large and amplifies declines in the present value for realized matches and—even more importantly—prevents the economy from stabilizing during periods of rising unemployment. Following the calibration strategy for the model with the first distortion, where c was fixed, a higher value of unemployment activities implicates a lower steady state value

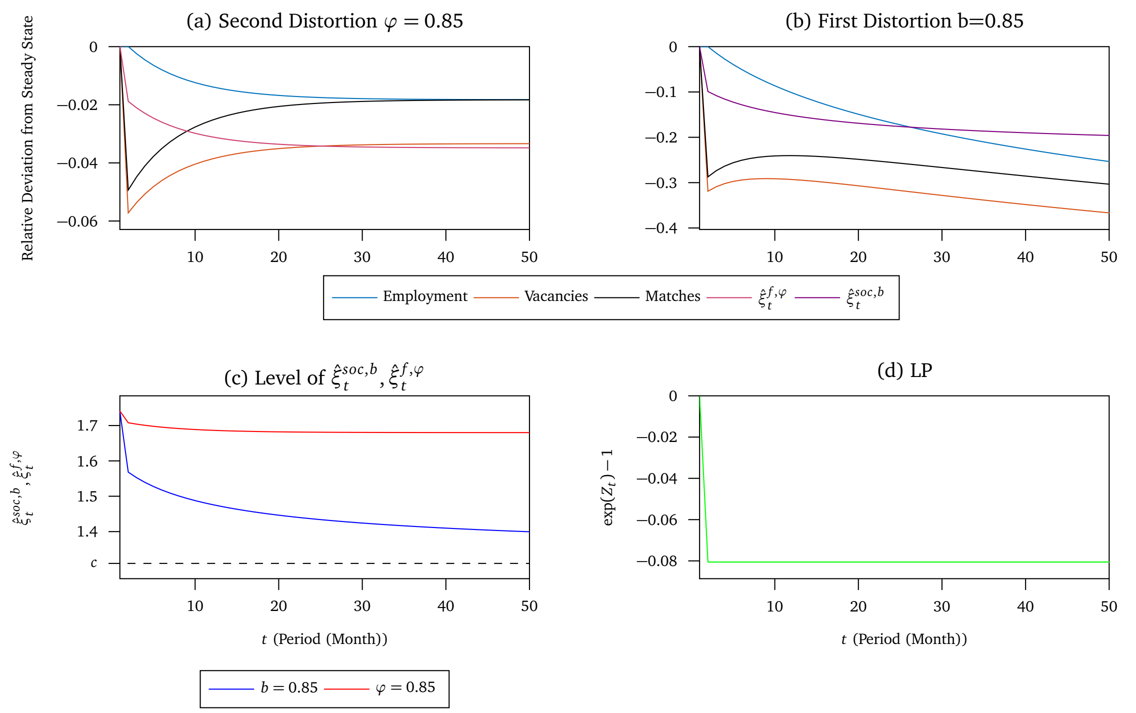
[48] This implies labor productivity to fall approximately to $\exp(Z_2) \approx 0.82 > b = 0.75$.

[49] Log LP would also have to stay at the low level for some periods in order for disasters to arise.

for the present value of a worker so that the condition will be satisfied more quickly. However, following the initial calibration strategy from the baseline model from which the costs c were computed, the value of b is not important. The average probability κ_f that a vacancy can be filled was set to 0.71 in steady state so that the ratio $\frac{\hat{\xi}^f}{c} = \frac{1}{\kappa_f}$ is fixed independently of b. A higher value of b in this calibration strategy would only imply a lower cost parameter c. Nonetheless, an already high elasticity of realized job matches with respect to the present value of a worker can not be the only factor in generating disasters in the model. A high value of the household's bargaining power of $\varphi = 0.85$ in the model containing only the second distortion from section 4.6.1.4 yields the same steady state and therefore the same high elasticity as it is the case for b=0.85. Yet, employment did not fall by more than approximately 4% in the simulation results in table 4.21. So why is this the case?

Relative Changes in the Present Value of a Worker Loosely speaking, if realized job matches react highly sensitive to changes in the present value of a worker, mechanisms that render relative changes in the present value of a worker more or less pronounced in response to fluctuations in productivity should affect the model's dynamics significantly. In order to get an intuition about the economy's dynamics in the model containing only the second distortion with $\varphi = 0.85$, we consider the same artificial scenario of 50 periods, where log LP drops to $Z_2 = -2\frac{\sigma_\epsilon}{\sqrt{1-\rho^2}}$ in the second period and remains at the level for the remaining periods, in figure 4.33. With

Figure 4.33: Second Distortion: Effects of Drop in Labor Productivity



only the second distortion in the model, the employment rate falls by approximately 2% before stabilizing after some periods. Compared to the model including only the first distortion the most important difference can already be identified in the immediate impact of the drop in LP in period $t = 2$ on the present value of a worker. While $\hat{\xi}_t^{soc,b}$ falls by 10% due to the shock to LP, $\hat{\xi}_t^{f,\varphi}$ does only by 2%. Now,

$$\hat{\xi}_t^{soc,b} = \mathbb{E}_t\left[\sum_{s=1}^{\infty}\left(\prod_{k=1}^{s-1}\left(1-\omega-\frac{\partial M}{\partial U}(U_{t+k},V_{t+k})\right)\right)\beta^s\frac{\lambda_{t+s}}{\lambda_t}(\exp(Z_{t+s})-b)\right],$$

and

$$\hat{\xi}_t^{f,\varphi} = \mathbb{E}_t\left[\sum_{s=1}^{\infty}\left(\prod_{k=1}^{s-1}\left(1-\omega-\frac{\partial M}{\partial U}(U_{t+k},V_{t+k})\right)\right)\beta^s\frac{\lambda_{t+s}}{\lambda_t}(\exp(Z_{t+s})-w_{t+s})\right] =$$
$$\stackrel{(4.6.53)}{=} \mathbb{E}_t\left[\sum_{s=1}^{\infty}\left(\prod_{k=1}^{s-1}\left(1-\omega-\frac{\partial M}{\partial U}(U_{t+k},V_{t+k})\right)\right)\beta^s\frac{\lambda_{t+s}}{\lambda_t}(1-\varphi)\exp(Z_{t+s})\right].$$

In the model including the first distortion a decline in labor productivity by approximately 8% from $\exp(Z_1)=1$ to $\exp(Z_2)\approx 0.92$ yields the within period value attributed to an employed over an unemployed member to fall from $\exp(Z_1)-b=0.15$ to $\exp(Z_2)-b\approx 0.07$ by over 50%. On the other hand, in the model containing only the second distortion the firm's profit after wage payments from a worker drop from $(1-\varphi)\exp(Z_1)=0.15$ to $(1-\varphi)\exp(Z_2)\approx 0.138$ only by 8%. Similar, until LP recovers the expectation about future within period value from a worker declines significantly more in the case with $b=0.85$ compared to the case with $\varphi=0.85$. As a consequence, $\hat{\xi}_t^{soc,b}$ is much more affected than $\hat{\xi}_t^{f,\varphi}$ by the sudden decline in LP, although the effect on both values is smaller than on the corresponding within period value in $t=2$, since LP is expected to eventually recover in the long run. For the model with the first distortion the economy enters a downward spiral as laid out in the preceding paragraph. This is different for the model including only the second distortion. The elasticity of matches with respect to the present value of a worker is the same in the steady state for both scenarios. Therefore, in the model with the second distortion only, the (much smaller) decline of $\hat{\xi}_t^{f,\varphi}$ also triggers a relative drop of matches almost three times as large. However, as can be seen in the lower left part of figure 4.33, the decline of $\hat{\xi}_t^{f,\varphi}$ relative to the costs c of posting vacancies remains moderate so that $\epsilon_{M_t,\hat{\xi}_t^{f,\varphi}}$ does not increase by as much as in the case of the first distortion. In the subsequent periods with unemployment rising, the input of unemployed members in the matching process increases, the marginal vacancy filling rate $\frac{\partial M}{\partial V}$ ceteris paribus also increases, while $\hat{\xi}_t^{f,\varphi}$ continues to decline. However, with $\epsilon_{M_t,\hat{\xi}_t^{f,\varphi}}$ not having gained too much, the first two effects yield matches to recover enough in order for the economy to eventually stabilize.

In the model containing the first distortion only, the value the social planner attributes each period to unemployed members is fixed by b. Contrarily, in the model containing only the second distortion the wage, $w_t=\varphi\exp(Z_t)$, firms have to pay partly absorbs declines in LP. Since the within period value from a worker, i.e. $\exp(Z_t)-b$ or $\exp(Z_t)-w_t$, respectively, is small to begin with in both cases, the fact whether part of a decline in labor productivity is absorbed or not yields highly different relative declines which manifest in $\hat{\xi}_t^{soc,b}$ and $\hat{\xi}_t^{f,\varphi}$. The high elasticity of matches additionally magnifies these differences for the further progress of employment.

Disasters in the Baseline Model To sum up, there are three important factors in the baseline model favoring the occurrence of disasters. First, the high and fixed value of unemployment activities in b renders the wage

$$w_t = \varphi\exp(Z_t)+(1-\varphi)b+\varphi\frac{V_t}{1-N_t}c.$$

close to labor productivity in steady state but also inflexible to fluctuations in productivity. Small changes in LP implicate high relative variations in the within period value $\exp(Z_t)$—w_t of a worker to the firm, which manifest in high relative changes in $\hat{\xi}_t^f$. Second, the calibration strategy in the baseline model implied a ratio $\frac{\hat{\xi}^f}{c}=\frac{1}{\kappa_f}=\frac{1}{0.71}$ and therefore an elasticity $\epsilon_{M_t,\hat{\xi}_t^f}=\frac{1}{\left(\hat{\xi}_t^f/c\right)^{\tau}-1}\approx 1.8$ in steady state so that high relative declines in the present value of a worker to the firm will entail even larger relative declines in realized job matches. Third and particularly important, with unemployment rising in the following periods, the present value of a worker to the firm will decline further even without labor productivity continuing to decline—due to higher discounting of future profits obtained from a worker. Although more unemployed members are searching for jobs and vacancies are more likely to be filled, the now even higher elasticity with respect to $\hat{\xi}_t^f$ implicates that realized matches will not recover sufficiently in the subsequent periods. On the contrary, realized matches eventually begin to decline even more so that unemployment is not stopped from increasing and the economy is destined to enter a downward spiral until productivity sufficiently recovers.

Modifications to the model, which imply the wage to absorb declines in LP only slightly better, should already have substantial effects on the relative declines of newly created jobs and therefore significantly alter the model's dynamics. Hence, the already mentioned assumption which were made in order to justify the high value of b become particularly important at this point. First, summarizing the complete period value of unemployment activities into the fixed parameter b implies that the value remains fixed over the business cycle. A worker's reservation wage does not adjust indifferent from the fact how much unemployment rises and how low consumption falls. However, this changes once leisure is introduced into the utility function in such way that the marginal rate of substitution with the consumption good is not constant. During a recession with decreasing consumption and increasing free time the value of unemployment from leisure measured in units of the consumption good for which a worker demands compensation will decline. An equivalent argument might be made for the value of unemployment due to home work if the MRS between the home produced good and the market good is not constant. Second, the baseline model neglected all positive effects from unemployment on the household's utility. This also changes, if we add leisure and home production to the household's utility.

4.7 Extensions

In this section we consider several extensions of the baseline model. In particular, we will endogenize the household's utility drawn from free time of unemployed members, introduce variable working hours for the household's working members, introduce variable search effort for the household's unemployed members and endogenize home production. We will analyze for each variation of the model, how the dynamics of employment change in simulations and if a solution by perturbation[50] can reproduce the results in an accurate way. We present the disaster statistics as well as the second moments of the labor market. Finally, we also summarize the implications for the equity premium in the models with standard preferences and under Epstein-Zin preferences.

4.7.1 Leisure in Utility

We will first endogenize the household's utility from leisure when unemployed in the spirit of Merz (1995).

Search and Matching The matching process between unemployed members and open vacancies posted by the representative firm (4.2.1) as well as the dynamics of employment in the economy (4.2.2) remain the same as in the baseline model. In particular, we still assume all unemployed members to enter the matching process with fixed search intensity.

[50] Note that we still only compute the variables' values as far as needed from the second order perturbation approximation in the following and derive all other variables' outcomes as will be described.

Representative Household Different from the baseline model, the representative household's aggregate utility drawn from leisure of his unemployed members or equivalently disutility from work of his employed members[51] is explicitly contained in his utility function, i.e. the representative household is now assumed to maximize lifetime utility

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s\left(\frac{C_{t+s}^{1-\eta}-1}{1-\eta}-\frac{\nu_0}{1+\nu_1}N_{t+s}^{1+\nu_1}\right)\right], \quad \eta,\nu_0,\nu_1>0,\eta\neq 1.$$

In this model we will consider working hours to be fixed. Moreover, the chosen utility function implies the representative household's marginal disutility from a working member to be increasing with the employment rate or, identically, the marginal utility from an unemployed member due to leisure to be decreasing with the unemployment rate. With fixed working hours this can equivalently be interpreted as increasing marginal disutility from total working hours or decreasing marginal utility from total leisure of the household's members. The representative household maximizes his lifetime utility under the budget constraints (4.2.4) and the dynamics (4.2.5) of employed members, taking the probability of an unemployed member finding a job as exogenous. With J^h as before denoting the value function, we now have

$$\begin{aligned}
J^h(N_t,S_t)= \quad & \max_{C_t,S_{t+1}} && \frac{C_t^{1-\eta}-1}{1-\eta}-\frac{\nu_0}{1+\nu_1}N_t^{1+\nu_1}+\beta\mathbb{E}_t[J^h((1-\omega)N_t+\kappa_{w,t}(1-N_t),S_{t+1})]\\
& \text{s.t.} && C_t\leq w_tN_t+d_tS_t-v_t(S_{t+1}-S_t)+b(1-N_t)-T_t,\\
& \text{given} && N_t,S_t.
\end{aligned}$$

The household's optimality conditions (4.2.6) and (4.2.7) for the maximization problem on the right-hand side remain the same. However, the value of an employed over an unemployed member from the household's perspective now explicitly contains the utility from leisure over work effort by unemployed members included in the utility function, i.e.

$$\frac{\partial J^h}{\partial N}(N_t)=\lambda_t(w_t-b)-\nu_0N_t^{\nu_1}+\beta\mathbb{E}_t\left[\frac{\partial J^h}{\partial N}(N_{t+1})(1-\omega-\kappa_{w,t})\right].$$

With the established notation of $\xi_t^h := \frac{1}{\lambda_t}\frac{\partial J^h}{\partial N}$ denoting the value of an employed (over an unemployed) member to the representative household measured in consumption units, we now get

$$\xi_t^h=w_t-b-\frac{\nu_0N_t^{\nu_1}}{\lambda_t}+(1-\omega-\kappa_{w,t})\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^h\right] \tag{4.7.1}$$

The interpretation for (4.7.1) is the same as for (4.2.8) in the baseline model with the only difference that the fixed value b is replaced by

$$z_t := b+\frac{\nu_0N_t^{\nu_1}}{\lambda_t}.$$

In the baseline model the parameter b contained the total per period value of all unemployment activities, while b now only measures the period value of unemployment except for leisure and the value due to leisure (over work effort) is explicitly contained in the term $\frac{\nu_0N_t^{\nu_1}}{\lambda_t}$.

[51] We use 'disutility from work effort of employed household members' and 'utility from leisure (over work effort) of unemployed members' synonymously.

Representative Firm Nothing changes from the side of the representative firm. The KKT conditions are as in (4.2.16)-(4.2.19). The marginal value of a worker to the representative firm ξ_t^f as well as the present value of a worker in the next period to the firm $\hat{\xi}_t^f$ satisfy (4.2.20) and (4.2.21), respectively.

Wage Bargaining As in the baseline model, the wage is determined in such way to maximize (4.2.22), which was shown to imply the following sharing rule in total surplus $\xi_t := \xi_t^h + \xi_t^f$ generated by a working member in the economy

$$\xi_t^h = \varphi \xi_t \quad \text{and} \quad \xi_t^f = (1-\varphi)\xi_t.$$

Observing that the only difference to the baseline model lies in the value ξ_t^h now containing $b + \frac{\nu_0 N_t^{\nu_1}}{\lambda_t}$ as the within period value of an unemployed member instead of only b, we obtain the wage equation (4.2.26) only with b replaced by $b + \frac{\nu_0 N_t^{\nu_1}}{\lambda_t}$, i.e.

$$w_t = \varphi \exp(Z_t) + (1-\varphi)\left(b + \frac{\nu_0 N_t^{\nu_1}}{\lambda_t}\right) + \varphi \frac{V_t}{1-N_t} c. \tag{4.7.2}$$

Government Sector The government runs a balanced budget and we assume, as before, the value of all unemployment activities contained in the parameter b to be just redistributed from taxes, i.e.

$$T_t = b(1-N_t).$$

Consequently, the only aggregate value from unemployment in the economy stems from leisure over work effort in the household's utility.

General Equilibrium The only thing changed compared to the baseline model is the fact that a worker's reservation wage now explicitly contains the endogenously determined foregone utility from leisure, or suffered disutility from work effort, for which he demands compensation. Hence, the general equilibrium is determined by equations (4.2.28)-(4.2.41) only with the wage equation (4.2.36) replaced by (4.7.2). As a consequence, all of the model's variables can again be determined dependent on $\hat{\xi}_t^f$ and the state variables in the exact same way. The policy function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ of the present value of a worker in the next period to the representative firm is the solution to the functional equation (4.3.1)-(4.3.11) only now with

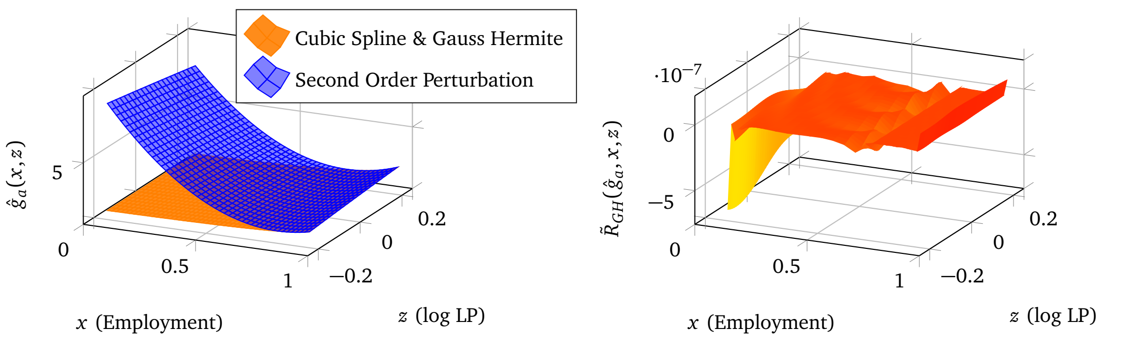
$$w_t := w(g,x,z) = \varphi \exp(z) + (1-\varphi)\left(b + \frac{\nu_0 x^{\nu_1}}{C(g,x,z)^{-\eta}}\right) + \varphi \frac{V(g,x,z)}{1-x} c$$

in rhs.

Calibration The calibration of the model's parameters follows the calibration in the baseline model. More specifically, all steady state values and all calibrated parameter values in table 4.1 except for the value of b remain the same. The definition of the parameter b is different now. In the baseline model b denoted the total period value of all unemployment activities which pin down a worker's reservation wage. In the present model the value of unemployment obtained by leisure is incorporated endogenously through the utility function and the parameter b captures only the value from remaining factors. We assume the total period value of unemployment activities in steady state to remain at the same level as in the baseline model, i.e.

$$z = b + \frac{\nu_0 N^{\nu_1}}{\lambda} = 0.85. \tag{4.7.3}$$

Figure 4.34: Leisure in Utility: Approximation of Policy Function and Interpretable Euler Residuals



(a) Approximations to Policy Function of $\hat{\xi}^f$

(b) Interpretable Euler Residuals for Cubic Spline Solution

This also guarantees that the numeric steady state values for all other variables as well as the free parameter values do not change. We set $\nu_1 = 2$. At this point, we are now free to choose any allocation of the total period value of unemployment between unemployment activities still captured by b and the value of leisure in $\frac{\nu_0 N^{\nu_1}}{\lambda}$. This will be different in the upcoming models. In foresight, we choose the value of unemployment due to leisure to be $\frac{1}{3}$ and accordingly set $b = 0.85 - \frac{1}{3}$. Finally, we compute ν_0 for $\frac{\nu_0 N^{\nu_1}}{\lambda} = \frac{1}{3}$ to hold.

We also employed two other calibrations, in which we set $\nu_1 = 0.8$ as in Merz (1995) and $b = 0.65$ or $b = 0.42$. We will only report the disaster statistics for these cases in the following.

Steady State The remaining variables' steady state values and the free parameter values can be computed the same way as in the baseline model, only with b replaced by $b + \frac{\nu_0 N^{\nu_1}}{\lambda}$. I.e. (4.4.2) now becomes

$$\hat{\xi}^f = \frac{\beta(1-\varphi)(\exp(Z) - b - \frac{\nu_0 N^{\nu_1}}{\lambda})}{1-\beta(1-\omega-\varphi\kappa_f)} = \frac{\beta(1-\varphi)(\exp(Z)-0.85)}{1-\beta(1-\omega-\varphi\kappa_f)},$$

the costs of posting a vacancy are

$$c = \frac{\beta\kappa_f(1-\varphi)(\exp(Z) - b - \frac{\nu_0 N^{\nu_1}}{\lambda})}{1-\beta(1-\omega-\varphi\kappa_f)} = \frac{\beta\kappa_f(1-\varphi)(\exp(Z)-0.85)}{1-\beta(1-\omega-\varphi\kappa_f)} \tag{4.7.4}$$

and the wage is determined by

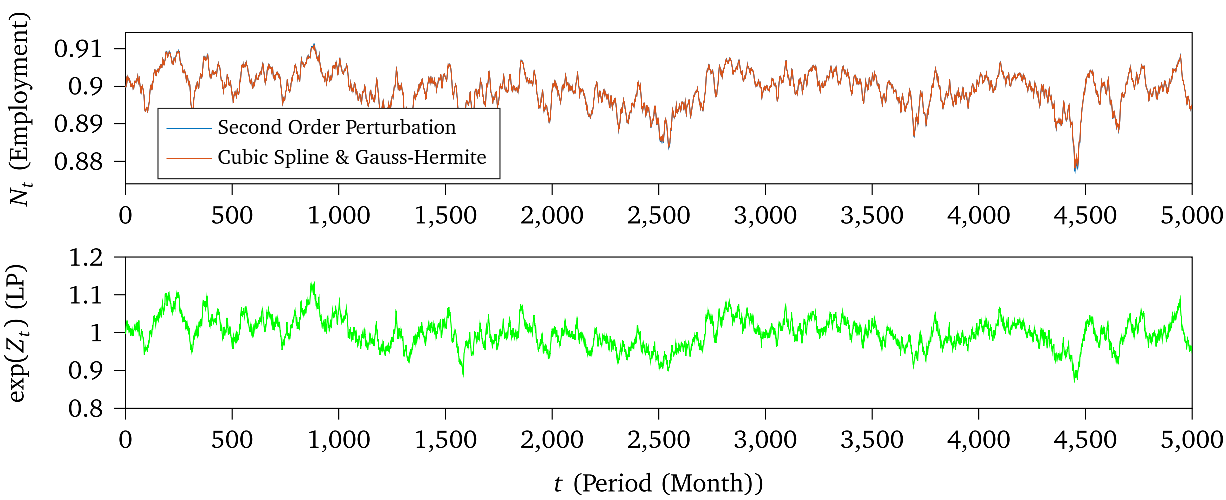
$$w = \varphi\exp(Z) + (1-\varphi)\left(b + \frac{\nu_0 N^{\nu_1}}{\lambda}\right) + \varphi\frac{V}{1-N}c = \varphi\exp(Z) + (1-\varphi)0.85 + \varphi\frac{V}{1-N}c.$$

More importantly, all the resulting numeric steady state values and free parameter values remain the same as in the baseline model.

Dynamics of Employment We present approximations to the policy function of $\hat{\xi}^f_t$, computed either from the finite element method or from a second order perturbation approach, in figure 4.34a. Interpretable Euler residuals of magnitude 10^{-7} for the finite element solution in figure 4.34b suggest that the global solution is sufficiently accurate. On the other hand, it is obvious that the second order perturbation solution again fails to provide a correct approximation to the policy function of $\hat{\xi}^f_t$ for higher rates of unemployment.

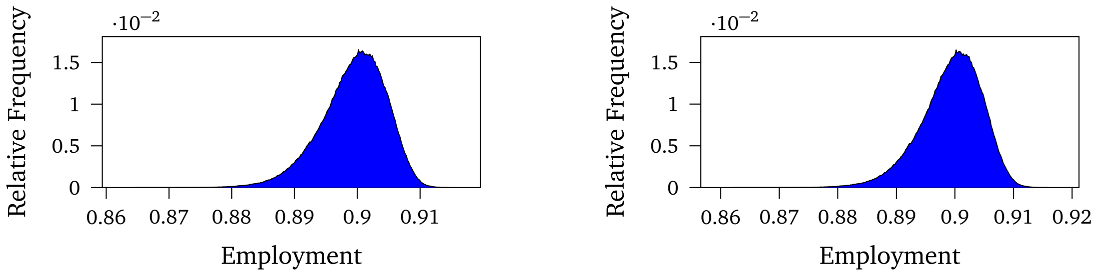
Once more, we computed the series of equilibrium outcomes for 1200000 periods from the same sample of pseudorandom iid$N(0,\sigma^2_\epsilon)$ distributed shocks to log LP. In figure 4.35 we display

Figure 4.35: Leisure in Utility: Dynamics of Employment



the dynamics of employment in the first 5000 periods, computed by both the cubic spline and perturbation solution. Additionally, the resulting histograms for the distribution of employment in the simulation are shown in figure 4.36, while table 4.28 summarizes some statistical measures of the distribution. As expected, once the utility drawn from leisure of unemployed members is

Figure 4.36: Leisure in Utility: Histograms for Distribution of Monthly Employment Rates



(a) Distribution of Employment computed by Cubic Splines

(b) Distribution of Employment computed by Second Order Perturbation

Table 4.28: Leisure in Utility: Statistic Measures for Distribution of Monthly Employment Rates

	$\bar{N}$	$\max(N_t)$	$\min(N_t)$	$q_{0.01}$	$q_{0.05}$	$q_{0.1}$	$q_{0.5}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	s_N	v_N	w_N
Cubic Spline	0.8994	0.9144	0.8645	0.8850	0.8901	0.8925	0.9000	0.9056	0.9069	0.9091	0.0052	-0.6384	3.6479
Perturbation	0.8994	0.9156	0.8620	0.8847	0.8900	0.8924	0.9000	0.9056	0.9070	0.9093	0.0053	-0.6626	3.8008

Notes: $\bar{N}$=average employment rate, $\max(N_t)$=maximal employment rate, $\min(N_t)$=minimal employment rate, $q_p(N_t)$ =p-quantile for cumulative distribution of employment rate, s_N, v_N, w_N=standard deviation, skewness, kurtosis of the cumulative distribution of employment rate. Statistics are computed from the outcome of a simulation of 1200000 (monthly) periods.

endogenized in the assumed form, the periods of extremely high unemployment disappear. The employment rate does not fall below 0.86 anymore. This is in sharp contrast to the baseline model where employment declined to approximately 0.06. Moreover, the standard deviation of employment drops by the factor 15. With only moderate fluctuations around the steady state, the fact that the approximation to the policy function of $\hat{\xi}_t^f$ by a second order perturbation fails for lower rates of employment has no consequences for the simulation outcomes. The perturbation solution gives an accurate view of employment's dynamics in the model, deviating even at most only by 0.0025 from the global solution in the whole simulation.

Table 4.29: Disasters in the Model with Leisure in Utility

	number of disasters	total number of disaster periods	disaster probability	average disaster size	average disaster duration
			90% threshold		
C	1449 (2292, 1288)	6265 (13400, 4980)	0.37% (0.59%, 0.33%)	11.48%(12.14%, 11.35%)	4.32 (5.85, 3.87)
Y	1393 (3008, 983)	5741 (16534, 3591)	0.35% (0.78%, 0.25%)	11.47%(12.22%, 11.28%)	4.12 (5.50, 3.65)
LP	663	2086	0.1667%	11.05%	3.15
			85% threshold		
C	59 (320, 39)	209 (1491, 113)	0.0148% (0.08%, 0.0098%)	16.18% (16.77%, 15.95%)	3.54 (4.66, 2.90)
Y	61 (473, 23)	178 (2077, 57))	0.0153% (0.12%, 0.0058%)	16.02% (16.82%, 15.91%)	2.92 (4.39, 2.48)
LP	5	10	0.0013%	15.68%	2.00
			80% threshold		
C	1 (27, 0)	1 (112, 0)	0.0003% (0.0068%, 0%)	20.03%(21.45%, –)	1 (4.15, –)
Y	0 (42, 0)	0 (168, 0)	0% (0.0105%, 0%)	– (21.64%, –)	– (4, –)
LP	0	0	0%	–	–
			75% threshold		
C	0 (1, 0)	0 (4, 0)	0% (0.0003%, 0%)	– (25.52%, –)	– (4, –)
Y	0 (1, 0)	0 (4, 0)	0% (0.0003%, 0%)	– (26.46%, –)	– (4, –)
LP	0	0	0%	–	–
			70% threshold		
C	0 (0, 0)	0 (0, 0)	0% (0%, 0%)	– (–, –)	– (–, –)
Y	0 (0, 0)	0 (0, 0)	0% (0%, 0%)	– (–, –)	– (–, –)
LP	0	0	0%	–	–

Endogenous Disasters? Table 4.29 shows the disaster statistics for the model with leisure in the utility. In parenthesis are the results for the alternative calibrations with $\nu_1 = 0.8$ and $b = 0.65$ or $b = 0.42$. Compared to the baseline model disasters become much less likely for all of the calibrations. Declines by over 30% from steady state disappear completely. Moreover, as should be expected, the more of the total period value of unemployment activities is still captured in the fixed parameter b and the less is attributed to the variable value from leisure, the more likely can disasters in the model's economy arise. For our standard calibration consumption did fall only once during the 400000 quarters in the simulation below a threshold of 80% of its steady state value, while output did not at all. For comparison, in the baseline model consumption and output were less than 30% of the steady state value during approximately 700 periods. Endogenizing the value of unemployment from leisure changes the disaster statistics in the model substantially.

Second Moments of the Labor Market Table 4.30 summarizes the second moments for the labor market. We only show the results for the calibration with $\nu_1 = 2$ and $b = 0.85 - \frac{1}{3}$. Since employment is far more stable in the model with leisure in the utility compared to the baseline model, the labor market variables fluctuate significantly less. The average standard deviations are only approximately $\frac{1}{5}$-th of the values obtained in the baseline model. The correlation between unemployment and vacancies becomes more negative, yet the value found in the data is still lower than the 2.5% quantile from the simulations.

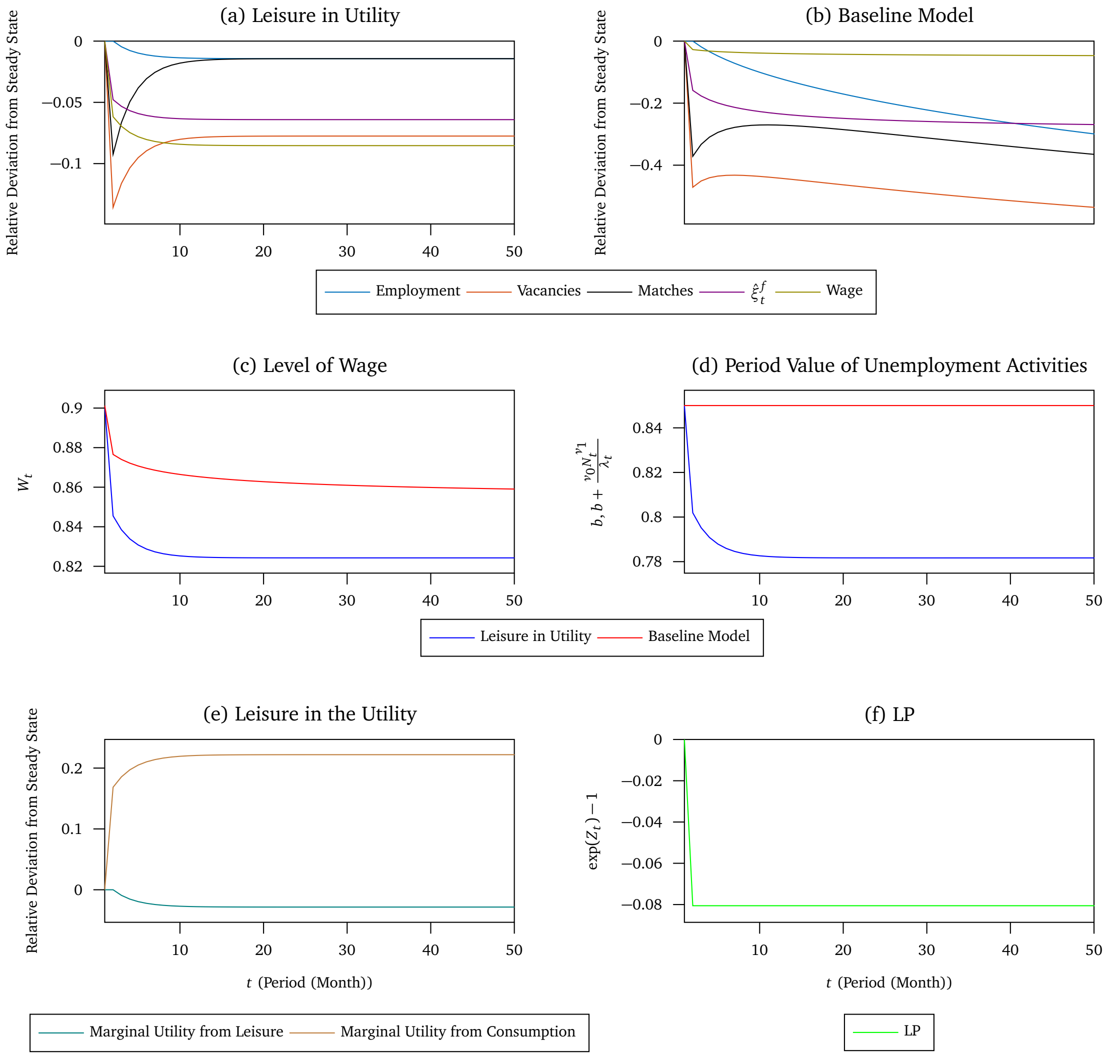
Comparison and Intuition We show the reaction to the scenario of the economy starting in steady state, log LP dropping at once by two unconditional standard deviations in the second period and remaining at the level for the remaining periods in figure 4.37 for the model with leisure in the utility and the baseline model. The baseline model enters a downward spiral, whereas with endogenous utility from leisure the employment rate stabilizes quickly after dropping only by approximately 1.5%. The explanation provided in section 4.6.2 for the different reactions between the models including only the first and second distortion, respectively, carries over. The impact of the drop in labor productivity in period $t = 2$ differs substantially. With leisure in the utility the present value of a worker to the firm declines by less than 5%, while it declines by over 15% in the baseline model. The already high elasticity of vacancies and

Table 4.30: Labor Market Moments in the Model with Leisure in Utility

	U	V	θ	$p = \frac{Y}{N}$
s_x	0.019 [0.014; 0.024]	0.017 [0.014; 0.020]	0.032 [0.026; 0.040]	0.016 [0.013; 0.019]
$\frac{s_x}{s_p}$	1.159 [0.992; 1.363]	1.059 [0.951; 1.171]	2.008 [1.799; 2.219]	1.000 [1.000; 1.000]
r_x	0.857 [0.801; 0.900]	0.633 [0.523; 0.730]	0.817 [0.748; 0.870]	0.774 [0.693; 0.840]
		Cross Correlations		
U		-0.693 [−0.765; −0.615]	-0.918 [−0.941; −0.889]	-0.856 [−0.895; −0.809]
V			0.916 [0.898; 0.934]	0.954 [0.944; 0.963]
θ				0.990 [0.986; 0.993]

Notes: U=unemployment, V=vacancies, $\theta = \frac{V}{U}$=labor market tightness, p=output per worker, s_x=standard deviation of variable X, r_x=first order autocorrelation of variable X. All moments are computed from the cyclical components of the HP-filtered ($\lambda = 1600$) relative deviations from the mean, i.e. as the cyclical component of the series $\frac{X_t - \bar{X}}{\bar{X}}$. The table shows the average as well as the 2.5% and 97.5% quantilies from 5000 simulations of the model's equilibrium outcomes for 666 periods converted to 222 quarterly observations after 1000 throw-away periods.

Figure 4.37: Leisure in Utility: Effects of Drop in Labor Productivity



matches with respect to $\hat{\xi}_t^f$ yields the amount of realized matches to decrease by less than 10% in the first case, whereas matches fall by over 37% in the baseline model. The different reaction of $\hat{\xi}_t^f$ between the two models is amplified for the amount of matches. In the present model matches do not decline by too much. The growing unemployment rate in the subsequent periods increases the probability for the firm to fill a vacancy and provides more unemployed members entering the matching process. Matches partially recover despite the further decreasing $\hat{\xi}_t^f$ due to higher discounting. Since the elasticity of matches with respect to $\hat{\xi}_t^f$ becomes not as high as in the baseline model, where $\hat{\xi}_t^f$ dropped substantially more, matches eventually recover enough to stop the unemployment rate from falling further.

Now,

$$\hat{\xi}_t^f = \mathbb{E}_t\left[\sum_{s=1}^{\infty}(1-\omega)^{s-1}\beta^s \frac{\lambda_{t+s}}{\lambda_t}\left(\exp(Z_{t+s})-w_{t+s}\right)\right],$$

is the sum of expected discounted future labor productivity net wage costs from an employee to the firm. Expectations about future labor productivity are the same between the models. Yet, in the model with leisure in the utility the wage falls by approximately 6% in the second period from $w_1 \approx 0.9$ to $w_2 \approx 0.845$, whereas the wage declines by less than 2% from $w_1 \approx 0.9$ to $w_2 \approx 0.877$ in the baseline model. The seemingly small differences in the decline of the wage between the two models however lead to substantial different relative declines in the excess of labor productivity net wage costs from $\exp(Z_1)-w_1 \approx 0.1$ to $\exp(Z_2)-w_2 \approx 0.075$ by 25% in the first and to $\exp(Z_2)-w_2 \approx 0.043$ by 57% in the second case. Similar, the expectation about future labor productivity net wage costs declines, in percentage points, substantially less in the model with leisure in the utility so that $\hat{\xi}_t^f$ falls less. The wage in the present model is determined by

$$w_t = \varphi\exp(Z_t)+(1-\varphi)\left(b+\frac{\nu_0 N_t^{\nu_1}}{\lambda_t}\right)+\varphi\frac{V_t}{1-N_t}c.$$

In period $t=2$ the marginal disutility from work or utility from leisure of unemployed members, i.e. $\nu_0 N_t^{\nu_1}$, is fixed, but the marginal utility of consumption λ_t is increasing strongly due to less output and consumption in response to the negative shock to labor productivity. The total period value of unemployment activities $b+\frac{\nu_0 N_t^{\nu_1}}{\lambda_t}$ and therefore the workers' reservation wage starts to decline. The effect persists in the following periods. Unemployment is increasing while consumption is decreasing so that the MRS between the two variables and consequently the reservation wage further declines, whereas it was fixed to stay throughout at 0.85 in the baseline model. Hence, with leisure in the utility function, the wage is firstly absorbing more of the decline of LP in $t=2$, but also adjusts better to the rising unemployment in subsequent periods.

Remarks In their appendix Kuehn et al. (2015) also consider an extension of the baseline model with leisure in the utility. However, they arrive at a contradictory conclusion. They note that the model with leisure in utility produces results similar to the ones in the baseline model. The probability of disasters is only moderately lower, while the average size of disasters even increases. Yet, their results are not so much a contradiction to ours, when taking a closer look. They assume the representative household's within period utility to be

$$\log(C_t+hU_t), \quad h>0.$$

This utility function yields a constant marginal rate of substitution of h between leisure from unemployed members and consumption. In particular, instead of (4.7.1) the value of an employed over an unemployed member to the household would become

$$\xi_t^h = w_t-(b+h)+\left(1-\omega-\kappa_{w,t}\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^h\right]$$

so that the wage equation (4.7.2) would change to

$$w_t = \varphi\exp(Z_t)+(1-\varphi)(b+h)+\varphi\frac{V_t}{1-N_t}c.$$

Assuming the total value of unemployment activities to stay at the level from the baseline model implies $b+h=0.85$. Consequently, apart from a different stochastic discount factor nothing changes compared to the baseline model. Most important, the value of unemployment due to leisure measured in consumption units is again fixed by h and will not decline during recessions. The mechanism rendering the wage to become more flexible caused by a declining MRS is absent.

In the models prior to the present one there was no aggregate value in the economy from unemployment. It was assumed throughout that all the value from unemployment activities was redistributed from taxes. Disasters were consequently measured solely in output or consumption. Yet, in the present model leisure from unemployed members increases the household's lifetime utility. Measuring disasters only in output and consumption does not take this effect into account. To some extent decreasing consumption could also be the result of a shift to drawing utility from leisure during periods of lower productivity. Therefore, we could adjust the definition of disasters to consider the achieved within period utility instead. For example, one could transform the total within period utility from consumption and leisure into a consumption equivalent and consider declines in the consumption equivalent. I.e. for each period t we compute $C_{eq,t}$ so that

$$\frac{C_{eq,t}^{1-\eta}-1}{1-\eta}=\frac{C_t^{1-\eta}-1}{1-\eta}-\frac{\nu_0}{1+\nu_1}N_t^{1+\nu_1},$$

which yields

$$C_{eq,t}=\left((1-\eta)\left(\frac{C_t^{1-\eta}-1}{1-\eta}-\frac{\nu_0}{1+\nu_1}N_t^{1+\nu_1}\right)+1\right)^{\frac{1}{1-\eta}}.$$

We then simply adopt the definition of disasters in consumption C_t for $C_{eq,t}$. The results are summarized in table 4.31. In parentheses are the results for the alternative calibrations with $\nu_1=0.8$ and $b=0.65$ or $b=0.42$. In terms of the consumption equivalent, no declines by

Table 4.31: Disasters in Consumption Equivalent in the Model with Leisure in Utility

	number of disasters	total number of disaster periods	disaster probability	average disaster size	average disaster duration
			90% threshold		
C_{eq}	650 (1146, 253)	2413 (6244, 781)	0.16% (0.29%, 0.06%)	11.17% (11.76%, 10.86%)	3.71 (5.45, 3.09)
LP	663	2086	0.1667%	11.05%	3.15
			85% threshold		
C_{eq}	9 (83, 1)	23 (390, 1)	0.0023% (0.02%, 0.0003%)	15.86% (16.56%, 15.18%)	2.56 (4.70, 1.00)
LP	5	10	0.0013%	15.68%	2.00
			80% threshold		
C_{eq}	0 (4, 0)	0 (12, 0)	0% (0.001%, 0%)	– (20.83%, –)	– (3, –)
LP	0	0	0%	–	–
			75% threshold		
C_{eq}	0 (0, 0)	0 (0, 0)	0% (0%, 0%)	– (–, –)	– (–, –)
LP	0	0	0%	–	–

more than 25% from its steady state value can be observed during the 400000 quarterly periods in the simulation. For our standard calibration, the consumption equivalent did only decline during 23 periods below 85% of its steady state value.

4.7.2 Endogenous Hours

In the previous model, working hours of the household's employed members were fixed. It was then assumed that the marginal disutility from work effort was increasing in total working hours and therefore in the employment rate. Moreover, the marginal disutility could react only delayed in the period following a shock to the economy once the employment rate changes. In this subsection we will introduce one more building block into the labor market model. We will allow for variable working hours in a way similar to Andolfatto (1996). The marginal disutility the household faces from work effort will be increasing in the working hours per worker but not in the employment rate. Dependent on the reaction of working hours, the marginal disutility will already react in the period a shock hits the economy. Moreover, adjustments in working hours may dampen or amplify declines in output and consumption.

Search and Matching The matching process between unemployed members and open vacancies posted by the representative firm (4.2.1) and the dynamics of employment in the economy (4.2.2) remain unchanged from the baseline model. In particular, we still assume all the unemployed members to enter the matching process with fixed search intensity.

Representative Household The representative household draws utility from consumption, but faces disutility from the hours (measured as fraction of total time endowment) $h_t \in [0,1]$ worked by employed members. More concretely, we assume the representative household's lifetime utility to be

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s\left(\frac{C_{t+s}^{1-\eta}-1}{1-\eta}-N_{t+s}\frac{\nu_0}{1+\nu_1}h_{t+s}^{1+\nu_1}\right)\right], \quad \eta,\nu_0,\nu_1>0,\eta\neq 1.$$

Different from the model with fixed hours in the previous subsection, the marginal disutility of work to the representative household is no more increasing in the employment rate, but only in the individual hours worked by employed members. The budget constraint in period t is

$$C_t+v_t(S_{t+1}-S_t)\leq w_tN_t\frac{h_t}{h}+b(1-N_t)+d_tS_t-T_t,$$

where h is the steady state value of the fraction of hours spent working by the employed members. We introduce the normalization in order for w_t to denote—in the same way as in the baseline model—the wage rate per steady state working hours. Employment from the household's perspective evolves as in (4.2.5), where the household considers $\kappa_{w,t}$ as exogenous. The representative household chooses consumption and share holdings. He does not decide about the hours employed members work, but they will be the outcome of a bargaining process with the representative firm. With $J^h(N_t,S_t)$ as the household's value function, we can write

$$\begin{aligned}
J^h(N_t,S_t)= \quad & \max_{C_t,S_{t+1}} \quad \frac{C_t^{1-\eta}-1}{1-\eta}-N_t\frac{\nu_0}{1+\nu_1}h_t^{1+\nu_1}+\beta\mathbb{E}_t\left[J^h\left((1-\omega)N_t+\kappa_{w,t}(1-N_t),S_{t+1}\right)\right]\\
& \text{s.t.} \quad C_t\leq w_tN_t\frac{h_t}{h}+d_tS_t-v_t(S_{t+1}-S_t)+b(1-N_t)-T_t,\\
& \text{given} \quad N_t,S_t.
\end{aligned}$$

The household's optimality conditions (4.2.6) and (4.2.7) for the maximization problem on the right-hand side remain the same. The value of an employed over an unemployed member from the household's perspective now becomes

$$\frac{\partial J^h}{\partial N}(N_t,S_t)=\lambda_t\left(w_t\frac{h_t}{h}-b\right)-\frac{\nu_0}{1+\nu_1}h_t^{1+\nu_1}+\beta\mathbb{E}_t\left[\frac{\partial J^h}{\partial N}(N_{t+1})\left(1-\omega-\kappa_{w,t}\right)\right].$$

Hence, measured in consumption units by $\xi_t^h:=\frac{1}{\lambda_t}\frac{\partial J^h}{\partial N}(N_t,S_t)$, we get

$$\xi_t^h=w_t\frac{h_t}{h}-\left(b+\frac{\nu_0}{1+\nu_1}\frac{h_t^{1+\nu_1}}{\lambda_t}\right)+\left(1-\omega-\kappa_{w,t}\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^h\right]. \tag{4.7.5}$$

The interpretation remains analogously to the one of equation (4.7.1) in the previous model. The total wage payment an employed member receives in the period now becomes $w_t \frac{h_t}{h}$, while the total period value of unemployment activities (over work effort) becomes

$$z_t = b + \frac{\nu_0}{1+\nu_1} \frac{h_t^{1+\nu_1}}{\lambda_t}$$

with the last term representing the value from leisure. Yet, as already mentioned and different from the model with leisure in the utility from the previous subsection, the value of unemployment due to leisure is no more increasing in the employment rate but only in individual working hours of a worker.

Representative Firm On the representative firm's side everything remains essentially the same. With endogenous working hours h_t, total output is now determined by

$$Y_t = \exp(Z_t) N_t \frac{h_t}{h},$$

where we again introduce the normalization of labor productivity with the steady state value h in order for total output in steady state to remain at the same level as before. Let $J^f(N_t)$ denote the value function of the firm's maximization problem, then

$$\begin{aligned} J^f(N_t) = \quad & \max_{V_t} && \exp(Z_t) N_t \frac{h_t}{h} - w_t N_t \frac{h_t}{h} - cV_t + \mathbb{E}_t\left[\beta \frac{\lambda_{t+1}}{\lambda_t} J^f\left((1-\omega)N_t + \kappa_{f,t} V_t\right)\right] \\ & \text{s.t.} && V_t \geq 0, \\ & \text{given} && N_t. \end{aligned}$$

The KKT conditions for the maximization problem on the right hand side remain as in (4.2.16)-(4.2.19), only now we have to adjust the value of a worker to the representative firm to

$$\xi_t^f := \frac{\partial J^f}{\partial N}(N_t) = \exp(Z_t)\frac{h_t}{h} - w_t \frac{h_t}{h} + (1-\omega)\mathbb{E}_t\left[\beta \frac{\lambda_{t+1}}{\lambda_t} \xi_{t+1}^f\right]. \tag{4.7.6}$$

Consequently, the present value of a worker in the next period from the firm's perspective becomes

$$\hat{\xi}_t^f := \mathbb{E}_t\left[\beta \frac{\lambda_{t+1}}{\lambda_t} \xi_{t+1}^f\right] = \mathbb{E}_t\left[\beta \frac{\lambda_{t+1}}{\lambda_t}\left(\exp(Z_{t+1})\frac{h_{t+1}}{h} - w_{t+1}\frac{h_{t+1}}{h} + (1-\omega)\hat{\xi}_{t+1}^f\right)\right]. \tag{4.7.7}$$

Hours and Wage Bargaining The hours worked and the wage are determined jointly by the representative household and the representative firm through a Nash bargain maximizing the weighted geometric mean of the respective shares in total surplus created by an additional worker in the economy, i.e. by the solution to

$$\max_{w_t, h_t} \quad (\xi_t^h)^{\varphi} (\xi_t^f)^{1-\varphi} \quad \text{with} \quad 0 \leq h_t \leq 1. \tag{4.7.8}$$

First, using (4.7.5) and (4.7.6) the total surplus $\xi_t := \xi_t^h + \xi_t^f$ generated by a worker is

$$\xi_t = \exp(Z_t)\frac{h_t}{h} - b - \frac{\nu_0}{1+\nu_1}\frac{h_t^{1+\nu_1}}{\lambda_t} - \kappa_{w,t}\mathbb{E}_t\left[\beta \frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^h\right] + (1-\omega)\mathbb{E}_t\left[\beta \frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}\right].$$

Since the total surplus is again independent of period t's wage, maximizing (4.7.8) is equivalent to first solving for working hours that maximize the total surplus, then determining the shares

ξ_t^h and ξ_t^f which maximize the geometric mean given the amount of total surplus and finally establishing the wage which implicates this sharing rule.

Working hours maximizing the total surplus must be strictly positive, since the marginal product of labor is always strictly positive while the marginal disutility of working hours is zero in $h_t = 0$. Hence, the KKT conditions are

$$\nu_0 h_t^{\nu_1} = \frac{\exp(Z_t)}{h}\lambda_t - \mu_t^w, \quad 0 \leq h_t \leq 1, \quad \mu_t^w \geq 0 \quad \text{and} \quad \mu_t^w(1-h_t) = 0. \tag{4.7.9}$$

Note that the period value of unemployment due to leisure (over work effort) therefore satisfies

$$\frac{\nu_0}{1+\nu_1}\frac{h_t^{1+\nu_1}}{\lambda_t} = \frac{\exp(Z_t)}{1+\nu_1}\frac{h_t}{h} - \mu_t^w \frac{h_t}{(1+\nu_1)\lambda_t}. \tag{4.7.10}$$

Moreover, the sharing rule for the total surplus which maximizes the geometric mean in 4.7.8 was shown to be

$$\xi_t^h = \varphi \xi_t \quad \text{and} \quad \xi_t^f = (1-\varphi)\xi_t.$$

Proceeding along the same lines as in the baseline model, we hence can determine the wage by

$$\begin{aligned}
&\xi_t^h = \varphi\xi_t = \varphi(\xi_t^h + \xi_t^f)\\
\Leftrightarrow &(1-\varphi)\xi_t^h = \varphi\xi_t^f\\
\Leftrightarrow &(1-\varphi)\left(w_t\frac{h_t}{h} - b - \frac{\nu_0}{1+\nu_1}\frac{h_t^{1+\nu_1}}{\lambda_t} + \left(1-\omega-\kappa_{w,t}\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^h\right]\right)\\
&\quad = \varphi\left(\exp(Z_t)\frac{h_t}{h} - w_t\frac{h_t}{h} + (1-\omega)\hat{\xi}_t^f\right)\\
\Leftrightarrow &(1-\varphi)\left(w_t\frac{h_t}{h} - b - \frac{\nu_0}{1+\nu_1}\frac{h_t^{1+\nu_1}}{\lambda_t}\right) + \varphi\left(1-\omega-\kappa_{w,t}\right)\hat{\xi}_t^f\\
&\quad = \varphi\left(\exp(Z_t)\frac{h_t}{h} - w_t\frac{h_t}{h} + (1-\omega)\hat{\xi}_t^f\right)\\
\Leftrightarrow &w_t\frac{h_t}{h} = \varphi\exp(Z_t)\frac{h_t}{h} + (1-\varphi)\left(b + \frac{\nu_0}{1+\nu_1}\frac{h_t^{1+\nu_1}}{\lambda_t}\right) + \varphi\kappa_{w,t}\hat{\xi}_t^f.
\end{aligned} \tag{4.7.11}$$

As in (4.2.25) for the baseline model, we can use $\kappa_{w,t}\xi_t^f = c\frac{V_t}{1-N_t}$ to write

$$w_t\frac{h_t}{h} = \varphi\exp(Z_t)\frac{h_t}{h} + (1-\varphi)\left(b + \frac{\nu_0}{1+\nu_1}\frac{h_t^{1+\nu_1}}{\lambda_t}\right) + \varphi c\frac{V_t}{1-N_t}.$$

Government Sector The government runs a balanced budget and we assume the value of all unemployment activities contained in the parameter b to be just redistributed from taxes, i.e.

$$T_t = b(1-N_t).$$

Consequently, aside from leisure in the household's utility unemployment yields no aggregate value in the economy.

General Equilibrium The system of equations defining the general equilibrium is similar to the baseline model. Working hours are determined by the additional equations (4.7.9) and the wage equation is adjusted to include the endogenous value from leisure (over work) of unemployed members:

$$U_t = 1 - N_t, \tag{4.7.12}$$

$$M_t = \frac{U_t V_t}{\left(U_t^\tau + V_t^\tau\right)^{\frac{1}{\tau}}}, \tag{4.7.13}$$

$$\kappa_{f,t} = \begin{cases} \frac{M_t}{V_t}, & \text{if } V_t > 0, \\ 1, & \text{if } V_t = 0, \end{cases} \tag{4.7.14}$$

$$\kappa_{w,t} = \frac{M_t}{U_t}, \tag{4.7.15}$$

$$N_{t+1} = (1-\omega)N_t + M_t, \tag{4.7.16}$$

$$Y_t = \exp(Z_t) N_t \frac{h_t}{h}, \tag{4.7.17}$$

$$Y_t = C_t + cVt, \tag{4.7.18}$$

$$\lambda_t = C_t^{-\eta}, \tag{4.7.19}$$

$$\nu_0 h_t^{\nu_1} = \frac{\exp(Z_t)}{h}\lambda_t - \mu_t^w, \quad 0 \le h_t \le 1, \quad \mu_t^w \ge 0, \quad \mu_t^w(1-h_t) = 0, \tag{4.7.20}$$

$$w_t \frac{h_t}{h} = \varphi \exp(Z_t)\frac{h_t}{h} + (1-\varphi)\left(b + \frac{\nu_0}{1+\nu_1}\frac{h_t^{1+\nu_1}}{\lambda_t}\right) + \varphi c \frac{V_t}{1-N_t}, \tag{4.7.21}$$

$$c = \kappa_{f,t}\hat{\xi}_t^f + \mu_t^f, \tag{4.7.22}$$

$$\mu_t^f \ge 0, \tag{4.7.23}$$

$$V_t \ge 0, \tag{4.7.24}$$

$$\mu_t^f V_t = 0, \tag{4.7.25}$$

$$\hat{\xi}_t^f = \mathbb{E}_t\left[\beta \frac{\lambda_{t+1}}{\lambda_t}\left(\exp(Z_{t+1})\frac{h_{t+1}}{h} - w_{t+1}\frac{h_{t+1}}{h} + (1-\omega)\,\hat{\xi}_{t+1}^f\right)\right]. \tag{4.7.26}$$

There is one major difference from the previous models however. We can now analytically solve for all the other period t variables in the model (and N_{t+1}) only dependent on $\hat{\xi}_t^f$ and, e.g., λ_t next to the state variables. Specifically, since nothing has changed in the search and matching process and the firm's decision problem compared to the baseline model, V_t, μ_t, M_t, $\kappa_{w,t}$ and $\kappa_{f,t}$ can be computed the same way as before. Subsequently, one can determine working hours from (4.7.20) by

$$h_t = \min\left\{\left(\frac{\exp(Z_t)\lambda_t}{\nu_0 h}\right)^{\frac{1}{\nu_1}}, 1\right\} \quad \text{and} \quad \mu_t^w = \frac{\exp(Z_t)}{h}\lambda_t - \nu_0 h_t^{\nu_1},$$

so that output Y_t and consumption C_t can then be derived from (4.7.17) and (4.7.18). Finally w_t can be computed from (4.7.21).

This way the model's remaining variables are pinned down dependent on λ_t, $\hat{\xi}_t^f$ and the state variables in such way that all equilibrium conditions except (4.7.19) and (4.7.26) hold.

Solution Method We reduced the system of equations determining the model's equilibrium to two equations in two variables. Hence, the conditions give rise to two functional equations the policy functions of $\hat{\xi}_t^f$ and λ_t have to mutually solve. We can easily adjust the solution methods presented in section 4.3 to the present case.

More specifically let

$$g_\lambda : [0,1] \times \mathbb{R} \to \mathbb{R} \quad \text{and} \quad g_{\hat{\xi}^f} : [0,1] \times \mathbb{R} \to \mathbb{R}$$

denote the (time invariant) policy functions for the marginal utility of consumption and the present value of a worker in the next period to the firm, i.e.

$$\lambda_t = g_\lambda(N_t, Z_t) \quad \text{and} \quad \hat{\xi}_t^f = g_{\hat{\xi}^f}(N_t, Z_t).$$

Then according to the preceding paragraph, g_λ and $g_{\hat{\xi}^f}$ have to mutually solve

$$\begin{aligned} R_1(g_\lambda, g_{\hat{\xi}^f}, x, z) &:= lhs_1(g_\lambda, g_{\hat{\xi}^f}, x, z) - rhs_1(g_\lambda, g_{\hat{\xi}^f}, x, z) = 0, \\ R_2(g_\lambda, g_{\hat{\xi}^f}, x, z) &:= lhs_2(g_\lambda, g_{\hat{\xi}^f}, x, z) - rhs_2(g_\lambda, g_{\hat{\xi}^f}, x, z) = 0, \end{aligned} \tag{4.7.27}$$

for all $x \in [0,1], z \in \mathbb{R}$, with

$$\begin{aligned} lhs_1(g_\lambda, g_{\hat{\xi}^f}, x, z) &:= g_\lambda(x, z), \\ rhs_1(g_\lambda, g_{\hat{\xi}^f}, x, z) &:= C_t^{-\eta}, \end{aligned} \tag{4.7.28}$$

and

$$\begin{aligned} lhs_2(g_\lambda, g_{\hat{\xi}^f}, x, z) &:= g_{\hat{\xi}^f}(x, z), \\ rhs_2(g_\lambda, g_{\hat{\xi}^f}, x, z) &:= \mathbb{E}\left[\beta \frac{g_\lambda(N_{t+1}, Z_{t+1})}{g_\lambda(x, z)} \left(\frac{h_{t+1}}{h}(\exp(Z_{t+1}) - w_{t+1}) + (1-\omega) g_{\hat{\xi}^f}(N_{t+1}, Z_{t+1})\right)\right] \end{aligned} \tag{4.7.29}$$

where $C_t, N_{t+1}, Z_{t+1}, h_{t+1}$ and w_{t+1} in rhs_1 and rhs_2 are short for the respective expressions of the variables dependent on $x = N_t, z = Z_t, g_{\hat{\xi}^f}(x,z) = \hat{\xi}_t^f, g_\lambda(x,z) = \lambda_t$ and the innovation $\epsilon \sim N(0, \sigma_\epsilon^2)$ as described.

In order to approximate the policy functions, we again adapt the mean weighted residual methods from section 4.3 to the case of two policy functions solving the two above stated functional equations. We do this analogous to the discussion in section 4.5.1 for the case of Epstein-Zin preferences in the baseline model, where the household's value function needed to be approximated next to the policy function of $\hat{\xi}_t^f$.

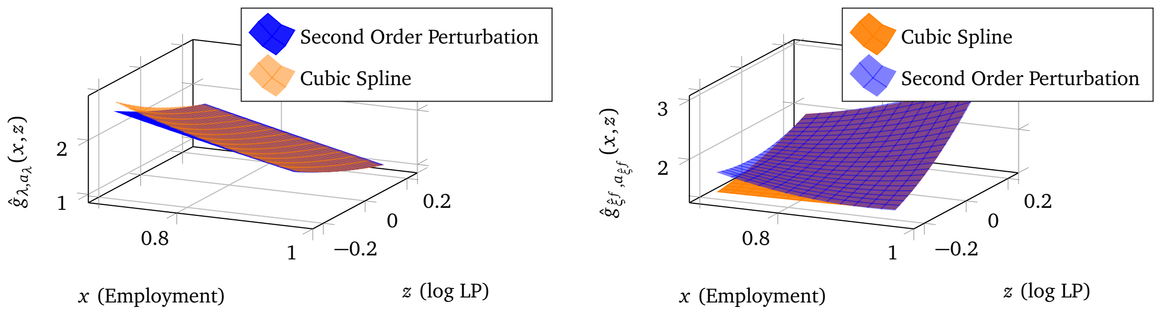
Calibration The calibration of the model's parameters follows the calibration in the baseline model. All fixed steady state values and calibrated parameters except for b remain as in table 4.1. We stick to $\nu_1 = 2$ equal to the previous model. Moreover, the total period value of unemployment activities (over work effort) to the representative household in steady state is set to remain at 0.85, i.e.

$$z = b + \frac{\nu_0}{1+\nu_1} \frac{h^{1+\nu_1}}{\lambda} = 0.85.$$[52]

Different from the previous model, the value of unemployment to the representative household due to leisure is already pinned down by the calibration of ν_1 so that no more allocation to b and

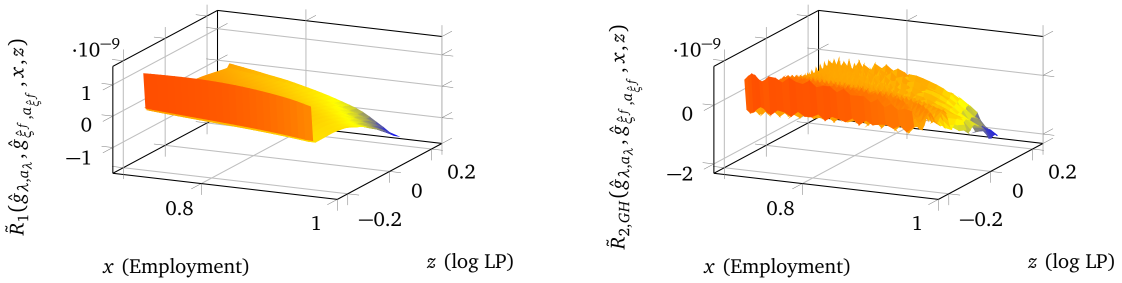
[52]Note that although employed members now only work a fraction of total time, the productivity was adjusted in such way that the output from a worker remains at the same level in steady state. The interpretation that the total period value of unemployment activities equals 85% of a worker's output in steady state hence carries over.

Figure 4.38: Endogenous Hours: Approximation of Policy Function



(a) Approximations to Policy Function of λ **(b)** Approximations to Policy Function of $\hat{\xi}^f$

Figure 4.39: Endogenous Hours: Interpretable Euler Residuals



(a) Interpretable Euler Residuals of $R_{1,GH}$ **(b)** Interpretable Euler Residuals of $R_{2,GH}$

$\frac{\nu_0}{1+\nu_1}\frac{h^{1+\nu_1}}{\lambda}$ must be chosen. It follows from (4.7.10) that the value of unemployment attributed to leisure must satisfy

$$\frac{\nu_0}{1+\nu_1}\frac{h^{1+\nu_1}}{\lambda} = \frac{\exp(Z)}{1+\nu_1}.$$

The fact that we set $\nu_1 = 2$ hence yields the value of leisure to be $\frac{1}{3}$ and consequently $b = 0.85 - \frac{1}{3}$. In foresight, we had already chosen the same allocation for the value in the previous model with leisure in the utility. We set working hours in steady state to $h = 0.33$ and determine the parameter ν_0 from (4.7.20).

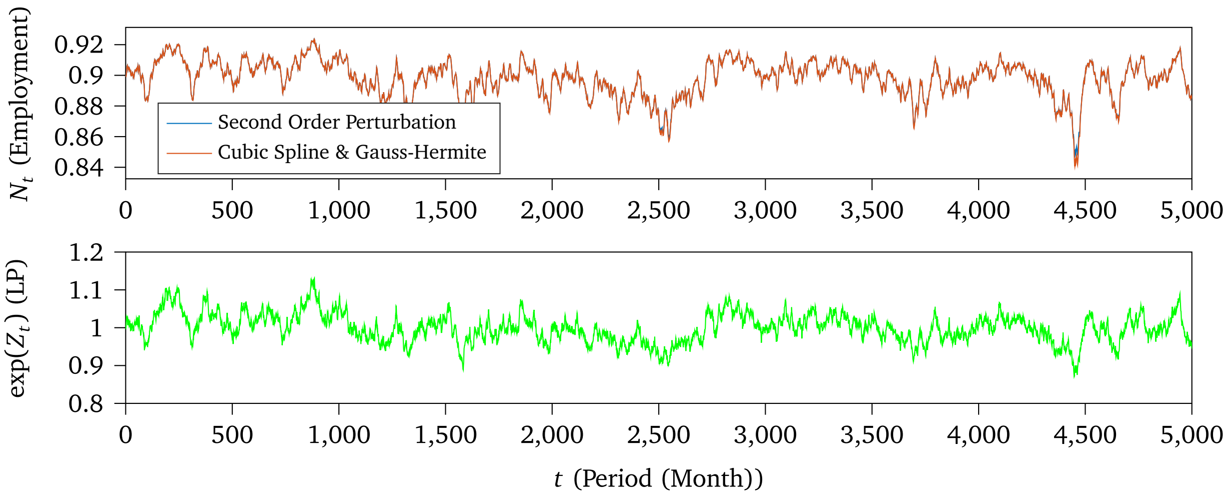
Finally, the chosen calibration together with the normalization of the wage rate in the household's budget constraint and of productivity in the production technology guarantees that all numeric steady state values as well as the free parameter values remain the same as in the baseline model.

Steady State The steady state can be computed the same way as in the baseline model only with b replaced by $b + \frac{\nu_0}{1+\nu_1}\frac{h^{1+\nu_1}}{\lambda} = 0.85$. The parameter ν_0 is given by

$$\nu_0 = \frac{\exp(Z)\lambda}{h^{\nu_1+1}}.$$

Dynamics of Employment Figure 4.38 pictures the obtained approximations to the policy functions of λ_t and $\hat{\xi}^f_t$ by a cubic spline and by a second order perturbation. Since it proved sufficient for simulations, we solved for the global solution only on the smaller domain $[\underline{x}, \bar{x}] \times [\underline{z}, \bar{z}] = [0.7, 0.97] \times [-0.21, 0.21]$ with $d_x = 61$ and $d_z = 85$ non-equidistant grid points. The interpretable Euler residuals of the cubic spline solution for both functional equations defined

Figure 4.40: Endogenous Hours: Dynamics of Employment



in (4.7.27)—where the expectation in rhs_2 is replaced by the Gauss-Hermite quadrature—are shown in figure 4.39. For the functional equation $R_2 = 0$ the interpretable Euler residuals remain as defined in section 4.4.3, while they are computed analogously for the functional equation $R_1 = 0$ as follows. We compute $\tilde{C}_1$ such way that

$$\hat{g}_{\lambda,a_\lambda}(x,z) = \tilde{C}_1^{-\eta}(\hat{g}_{\lambda,a_\lambda}, \hat{g}_{\hat{\xi}^f,a_{\hat{\xi}^f}}, x, z),$$

which yields

$$\tilde{C}_1(\hat{g}_{\lambda,a_\lambda}, \hat{g}_{\hat{\xi}^f,a_{\hat{\xi}^f}}, x, z) = \hat{g}_{\lambda,a_\lambda}^{-\frac{1}{\eta}}(x,z).$$

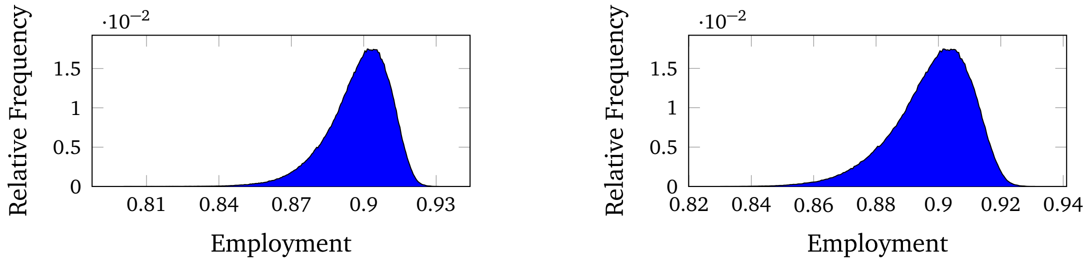
The interpretable Euler residuals $\tilde{R}_1$ are then defined by

$$\tilde{R}_1 := \frac{\tilde{C}_1(\hat{g}_{\lambda,a_\lambda}, \hat{g}_{\hat{\xi}^f,a_{\hat{\xi}^f}}, x, z)}{C(\hat{g}_{\lambda,a_\lambda}, \hat{g}_{\hat{\xi}^f,a_{\hat{\xi}^f}}, x, z)} - 1 = \left(\frac{\hat{g}_{\lambda,a_\lambda}(x,z)}{C(\hat{g}_{\lambda,a_\lambda}, \hat{g}_{\hat{\xi}^f,a_{\hat{\xi}^f}}, x, z)}\right)^{-\frac{1}{\eta}} - 1 =$$
$$= \left(\frac{lhs_1(\hat{g}_{\lambda,a_\lambda}, \hat{g}_{\hat{\xi}^f,a_{\hat{\xi}^f}}, x, z)}{rhs_1(\hat{g}_{\lambda,a_\lambda}, \hat{g}_{\hat{\xi}^f,a_{\hat{\xi}^f}}, x, z)}\right)^{-\frac{1}{\eta}} - 1.$$

A size of 10^{-9} for the interpretable Euler residuals in both functional equations indicates that the global approximations are sufficiently accurate. The perturbation solution is close to the global solution on the pictured domain, yet deviations between the two methods for lower rates of employment already become visible.

We computed the series of equilibrium outcomes for 1200000 periods from the same sample of pseudorandom iidN$(0,\sigma_\epsilon^2)$ distributed shocks to log LP. In figure 4.40 we show the employment rate in the first 5000 periods, when we use either the global or the perturbation solution. Additionally, the resulting histograms for the distribution of employment in the simulation are displayed in figure 4.41, while table 4.32 summarizes some statistical measures of the distribution. Compared to the baseline model the periods of extremely high unemployment disappear. Unemployment does not rise above 20% in the simulation. However, this is higher than for the model with leisure in the utility but fixed working hours, where unemployment did not rise above 13.5%. Further, the volatility of employment increases compared to the previous model with a standard deviation approximately 2.5 times as large, but still only approximately 17% of the standard deviation obtained in the baseline model.

Figure 4.41: Endogenous Hours: Histograms for Distribution of Monthly Employment Rates



(a) Distribution of Employment computed by Cubic Spline

(b) Distribution of Employment computed by Second Order Perturbation

Table 4.32: Endogenous Hours: Statistic Measures for Distribution of Monthly Employment Rates

	$\bar{N}$	$\max(N_t)$	$\min(N_t)$	$q_{0.01}$	$q_{0.05}$	$q_{0.1}$	$q_{0.5}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	s_N	v_N	w_N
Cubic Spline	0.8981	0.9307	0.8009	0.8597	0.8742	0.8809	0.9001	0.9126	0.9154	0.9198	0.0128	-0.9372	4.4356
Perturbation	0.8982	0.9306	0.8301	0.8618	0.8747	0.8811	0.9001	0.9126	0.9154	0.9198	0.0125	-0.7978	3.7492

Notes: $\bar{N}$=average employment rate, $\max(N_t)$=maximal employment rate, $\min(N_t)$=minimal employment rate, $q_p(N_t)$ =p-quantile for cumulative distribution of employment rate, s_N, v_N, w_N=standard deviation, skewness, kurtosis of the cumulative distribution of employment rate. Statistics are computed from the outcome of a simulation of 1200000 (monthly) periods.

With no too huge deviations of the employment rate from steady state during the simulation, the perturbation solution gives an accurate view of the model's dynamics for the most part. It fails to reproduce the highest unemployment rate as well as the skewness and kurtosis of employment's distribution in the simulation, yet the standard deviation and most quantiles differ only marginally.[53]

Endogenous Disasters? We computed the number of disasters, the number of disaster periods as well as the disaster probability, average disaster size and disaster duration for both consumption and output for different threshold fractions. Moreover, we also computed the disaster statistics for a consumption equivalent analogously to the one introduced in the preceding model, i.e. we compute $C_{eq,t}$ satisfying

$$\frac{C_{eq,t}^{1-\eta}-1}{1-\eta} = \frac{C_t^{1-\eta}-1}{1-\eta} - N_t \frac{\nu_0}{1+\nu_1} h_t^{1+\nu_1},$$

which yields

$$C_{eq,t} = \left((1-\eta) \left(\frac{C_t^{1-\eta}-1}{1-\eta} - N_t \frac{\nu_0}{1+\nu_1} h_t^{1+\nu_1} \right) + 1 \right)^{\frac{1}{1-\eta}}.$$

The results, along the respective outcomes for labor productivity, are summarized in table 4.33. Neither output nor consumption did fall below 80% of its steady state value in the simulation. In particular, for consumption declines by 10% or 15% from its steady state value did occur only more rarely than equivalent declines in the exogenous LP process. Moreover, compared to the model with leisure in the utility but fixed working hours from the previous section, declines in both variables become less frequent despite the fact that unemployment in the present model does rise somewhat higher. Variable working hours hence tend to additionally dampen the occurrence of disasters in the model.

[53] Note however that for the case that all variables' outcomes are computed from the second order approximations to their policy functions, larger declines of LP still lead the model's dynamics to leave their basin of attraction under the second order approximation. The outcomes would at some point tend to $\pm\infty$.

Table 4.33: Disasters in the Model with Endogenous Hours

	number of disasters	total number of disaster periods	disaster probability	average disaster size	average disaster duration
			90% threshold		
C	343	1312	0.0860%	11.11%	3.83
Y	761	2854	0.1916%	11.26%	3.75
C_{eq}	264	1047	0.0662%	11.07%	3.97
LP	663	2086	0.1667%	11.05%	3.15
			85% threshold		
C	3	9	0.0008%	15.91%	3.00
Y	18	51	0.0045%	16.00%	2.83
C_{eq}	3	8	0.0008%	15.49%	2.67
LP	5	10	0.0013%	15.68%	2.00
			80% threshold		
C	0	0	0%	–	–
Y	0	0	0%	–	–
C_{eq}	0	0	0%	–	–
LP	0	0	0%	–	–

Second Moments of the Labor Market The second moments for the labor market are shown in table 4.34. The average standard deviations of unemployment, vacancies and the labor market tightness found from repeated simulations of the present model are more than double the values found in the model with leisure in the utility but fixed working hours. Yet, while introducing variable working hours increases the volatility on the labor market, the standard deviations still lie substantially below the values from the baseline model.

Table 4.34: Labor Market Moments in the Model with Endogenous Hours

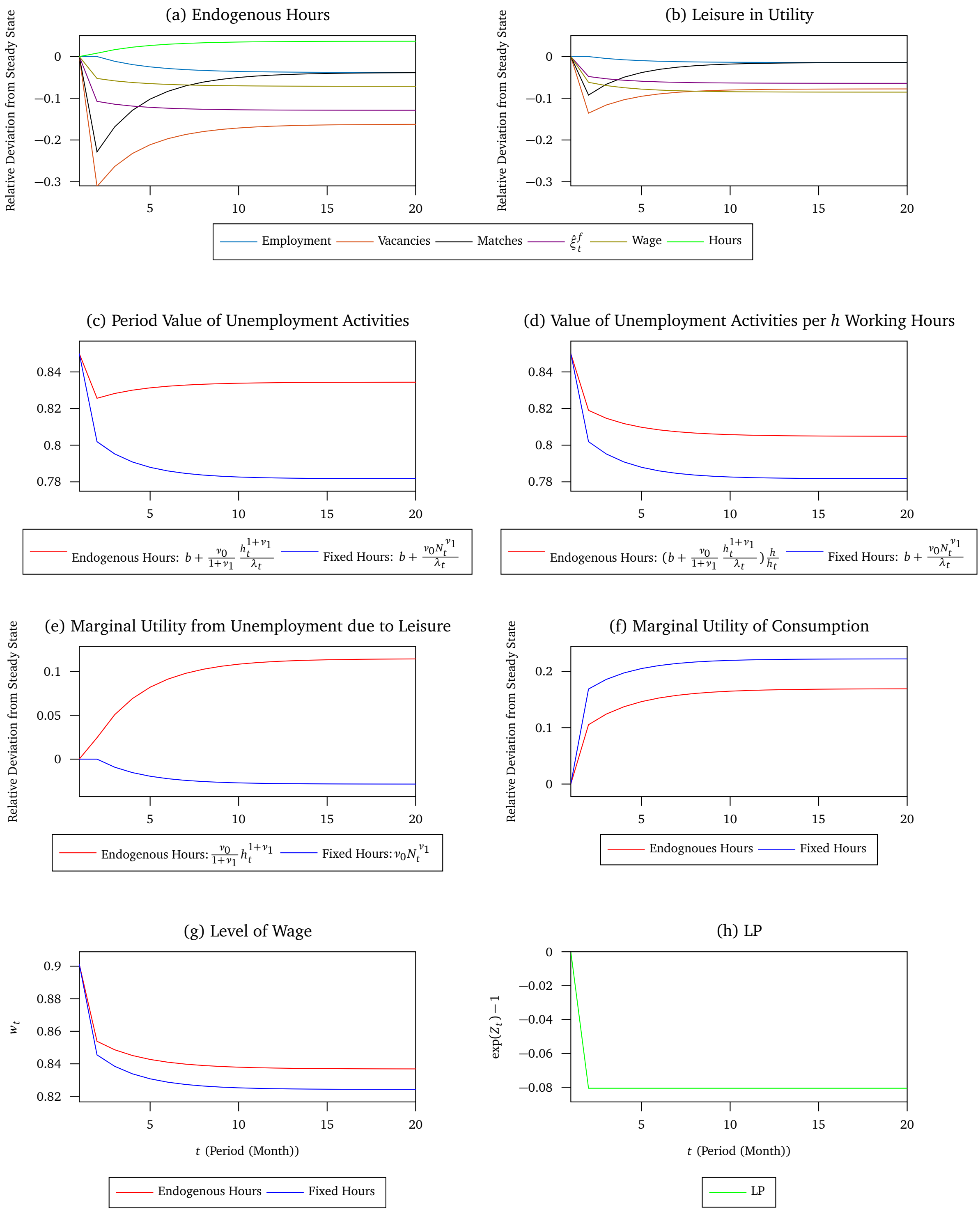
	U	V	θ	$p = \frac{Y}{N}$
s_x	0.046 [0.034; 0.060]	0.041 [0.035; 0.049]	0.077 [0.062; 0.095]	0.011 [0.009; 0.013]
$\frac{s_x}{s_p}$	4.084 [3.263; 5.147]	3.698 [3.364; 4.096]	6.889 [6.106; 7.798]	1.000 [1.000; 1.000]
r_x	0.853 [0.791; 0.901]	0.609 [0.492; 0.713]	0.804 [0.731; 0.861]	0.696 [0.591; 0.784]
		Cross Correlations		
U		-0.651 [−0.743; −0.552]	-0.871 [−0.916; −0.802]	-0.742 [−0.825; −0.635]
V			0.913 [0.883; 0.940]	0.986 [0.985; 0.989]
θ				0.966 [0.941; 0.981]

Notes: U=unemployment, V=vacancies, $\theta = \frac{V}{U}$=labor market tightness, p=output per worker, s_x=standard deviation of variable X, r_x=first order autocorrelation of variable X. All moments are computed from the cyclical components of the HP-filtered ($\lambda = 1600$) relative deviations from the mean, i.e. as the cyclical component of the series $\frac{X_t - \bar{X}}{\bar{X}}$. The table shows the average as well as the 2.5% and 97.5% quantiles from 5000 simulations of the model's equilibrium outcomes for 666 periods converted to 222 quarterly observations after 1000 throw-away periods.

Comparison and Intuition Figure 4.42 shows the reactions in the model to a sudden and lasting drop in log LP by two unconditional standard deviations starting from steady state. With endogenous working hours the immediate impact of the shock on the value of a worker to the firm $\hat{\xi}_t^f$ turns out to be stronger. This manifests even more clearly in the reaction of open vacancies and job matches due to the high elasticities of these variables. Nonetheless, the economy stabilizes in the subsequent periods in the same way. Unemployment declines by approximately 4% compared to only about 1.4% in the case of fixed working hours. We will now try to lay out the reasons for the differently sized impact of the shock to LP.

In the present model working hours start to increase following the shock. Equation (4.7.20) shows that a decline in LP has two opposing effects on optimal working hours. On the one hand side output per hour decreases, but at the same time an increasing marginal utility of consumption raises its value. The second, positive effect is dominating the first, negative one and working hours increase in response to the shock. Increasing working hours then yield three effects on the present value of a worker in the next period to the firm which we want to illustrate

Figure 4.42: Endogenous Hours: Effects of Drop in Labor Productivity



separately. First, different from the previous model, there now is a second channel through which the effect of a decline in productivity on $\hat{\xi}_t^f$ can be dampened next to the wage absorbing the decline to some extent. We can write $\hat{\xi}_t^f$ as

$$\hat{\xi}_t^f = \mathbb{E}_t\left[\sum_{s=1}^{\infty}(1-\omega)^{s-1}\beta^s\frac{\lambda_{t+s}}{\lambda_t}\frac{h_{t+s}}{h}\left(\exp(Z_{t+s})-w_{t+s}\right)\right],$$

where we have to keep in mind that the introduced normalization implies that $\exp(Z_t)$ and w_t

denote the labor productivity and the wage rate per h working hours, respectively.[54] Hence, increasing working hours yield a higher value of a worker to the firm as long as productivity net wage costs per hour remain positive. The second and third effect of increasing working hours on $\hat{\xi}_t^f$ appear through the wage equation. The wage is determined by

$$w_t \frac{h_t}{h} = \varphi \exp(Z_t) \frac{h_t}{h} + (1-\varphi)\left(b + \frac{\nu_0}{1+\nu_1} \frac{h_t^{1+\nu_1}}{\lambda_t}\right) + \varphi c \frac{V_t}{1-N_t},$$

where the total period value of unemployment activities (in consumption units)

$$b + \frac{\nu_0}{1+\nu_1} \frac{h_t^{1+\nu_1}}{\lambda_t}$$

defines a worker's reservation demand for his total wage income $w_t \frac{h_t}{h}$ in the period. Since hours are increasing, the utility from leisure (over work effort) of unemployed members $\frac{\nu_0}{1+\nu_1} h_t^{1+\nu_1}$ does the same. However, the marginal utility of consumption is also increasing. In the period of the shock the second effect prevails and the total period value of unemployment activities in consumption units decreases, before partially climbing back up during the subsequent periods. The third and last effect of increasing working hours manifests through the fact that the reservation demand for the hourly wage w_t does not depend on the whole period value of unemployment activities, but on the value per h working hours, i.e.

$$\left(b + \frac{\nu_0}{1+\nu_1} \frac{h_t^{1+\nu_1}}{\lambda_t}\right) \frac{h}{h_t}.$$

On top of the total period value of unemployment activities falling moderately below its initial value, the fraction per working hour is decreasing even more.[55] In particular, the net effect of increasing working hours on the hourly reservation wage can be deduced from (4.7.10) as

$$\left(b + \frac{\nu_0}{1+\nu_1} \frac{h_t^{1+\nu_1}}{\lambda_t}\right) \frac{h}{h_t} = b \frac{h}{h_t} + \frac{\exp(Z_t)}{1+\nu_1}. \tag{4.7.30}$$

Since the period value b of unemployment activities next to leisure is fixed, it is declining when converted to the value per working hour. Moreover, the value of unemployment due to leisure $\frac{\nu_0}{1+\nu_1} \frac{h_t^{1+\nu_1}}{\lambda_t}$ converted to the ratio per working hour moves proportional to LP. However, the value of unemployment activities declines still less than in the model with fixed hours even when converted to the value per working hour. Consequently, the hourly wage rate w_t drops also less. We want to stress again that the seemingly small differences in the decline of w_t imply highly different relative declines in productivity net wage costs of a worker. With endogenous hours the hourly wage falls to approximately 0.837 in the long run so that output less wage per h working hours falls from approximately 0.1 to 0.083 by 17%.[56] Even when considering the total output less total wage costs of a worker, i.e. $\frac{h_t}{h}(\exp(Z_t) - w_t)$, the increase in working hours by approximately 3.5% dampens the decline only slightly to 14%. With fixed hours on the other hand, the wage declines slightly more to approximately 0.825 so that $\exp(Z_t) - w_t$ drops only to 0.095 by 5%.

Summing up, the effects of increasing working hours are as following. First, there is a direct positive effect on the value of a worker to the firm as long as productivity per hour exceeds the

[54]For simplification, we will also call w_t simply the hourly wage rate without explicitly mentioning 'per h working hours' in the following.

[55]The same is also true for the last term in the wage equation.

[56]Remember that labor productivity falls to approximately 0.92.

wage per hour. Second, the utility from leisure over work effort by unemployed members in the period is increasing. Yet, when measured in consumption units the effect of an increasing marginal utility of consumption dominates moderately. Third, the value of unemployment activities per working hour is decreasing. The second effect implies the hourly wage to fall less compared to the case of fixed hours despite the third effect. The higher wage has a comparatively strong impact on the relative decline of productivity less wage costs per working hour which also maintains when considering total output less total wage costs. This leads $\hat{\xi}_t^f$ to decline more following the shock to LP with the laid out consequences for the unemployment rate.

However, consumption drops less than in the case of fixed working hours despite unemployment rising higher. This is explained mainly by the fact that employed members are working more hours, but also by the fact that less output is wasted on vacancy posting.

4.7.3 Endogenous Hours and Search Intensity

In this subsection we will extend the labor market model with one further building block. When unemployed, the household's members now have to actively decide about their job seeking effort in a fashion similar to Andolfatto (1996).

Search and Matching Different from the previous models unemployed members may now adjust their job searching effort which has influence on their probability to find jobs. We denote the time fraction unemployed members decide to spend seeking job offers by $e_t \in [0,1]$. From the firm's perspective nothing changes and V_t denotes the amount of open vacancies posted as before. The amount of resulting job matches is then given by

$$M_t := \min\{M\left(\tfrac{e_t}{e}U_t, V_t\right), U_t\}, \tag{4.7.31}$$

where the functional form of M remains as defined in (4.2.1). The parameter e denotes the steady state value of e_t and serves as normalization to ease comparison with the baseline model. Since the functional form of M only guarantees $M(\frac{e_t}{e}U_t, V_t) \leq \min\{\frac{e_t}{e}U_t, V_t\}$, the amount of matches is additionally bounded by U_t to ensure that it does not exceed the mass of unemployed members no matter how much search activities eventually exceed their steady state level. Yet, to simplify the upcoming analysis, we will assume $M_t = M\left(\frac{e_t}{e}U_t, V_t\right)$ in the following without any further theoretical justification and only check afterward in the numerical solution that the obtained policy function of matches never exceeds the unemployment rate.

We stick to the assumption of the constant fraction ω of workers separating from their job each period so that the dynamics of employment remain as in (4.2.2). As before, we can assume without any loss of generality that $U_t > 0$ for all $t \in \mathbb{N}$. Yet, it might now be the case that the representative household decides to not search for jobs at all in some period, i.e. it might be the case that $e_t = 0$. We therefore define the average probability for an unemployed member to find a job per (relative) time unit spent searching by

$$\kappa_{w,t} := \frac{M_t}{\frac{e_t}{e}U_t} = \frac{V_t}{\left(\left(\frac{e_t}{e}U_t\right)^\tau + V_t^\tau\right)^{\frac{1}{\tau}}}, \text{ if } e_t > 0,$$

and set

$$\kappa_{w,t} := \begin{cases} 1, & \text{if } e_t = 0, V_t > 0; \\ 0, & \text{if } e_t = 0, V_t = 0. \end{cases}$$

Analogously, from the representative firm's perspective the average probability of filling an open vacancy is

$$\kappa_{f,t} := \frac{M_t}{V_t} = \frac{\frac{e_t}{e}U_t}{\left(\left(\frac{e_t}{e}U_t\right)^\tau + V_t^\tau\right)^{\frac{1}{\tau}}}, \text{ if } V_t > 0,$$

and we set

$$\kappa_{f,t} := \begin{cases} 1, & \text{if } V_t = 0, e_t > 0; \\ 0, & \text{if } V_t = 0, e_t = 0. \end{cases}$$

Note that $\kappa_{w,t}$ is continuous in $(0,0)$ along the e_t-axis but not along the V_t-axis and vice versa for $\kappa_{f,t}$.

Representative Household The representative household draws utility from consumption, but faces disutility from the hours (measured as fraction of total time) $h_t \in [0,1]$ worked by employed members as well as from time e_t spent searching for jobs by unemployed members. More concretely, we assume the representative household's lifetime utility to be

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s\left(\frac{C_{t+s}^{1-\eta}-1}{1-\eta} - N_{t+s}\frac{\nu_0}{1+\nu_1}h_{t+s}^{1+\nu_1} - U_{t+s}\frac{\gamma_0}{1+\gamma_1}e_{t+s}^{1+\gamma_1}\right)\right], \quad \eta, \nu_0, \nu_1, \gamma_0, \gamma_1 > 0, \eta \neq 1.$$

The budget constraint in period t is

$$C_t + v_t(S_{t+1} - S_{t+s}) \leq w_t N_t \frac{h_t}{h} + b(1-N_t) + d_t S_t - T_t,$$

and from the household's perspective the mass of employed members develops according to

$$N_{t+1} = (1-\omega)N_t + \kappa_{w,t}\frac{e_t}{e}U_t,$$

where the household considers $\kappa_{w,t}$ as exogenous. The representative household chooses consumption, share holdings and the time unemployed members spend searching for jobs. He does not decide about the hours employed members work, but they will still be the outcome of a bargaining process with the representative firm. With $J^h(N_t, S_t)$ as the household's value function, we can write

$$\begin{aligned} J^h(N_t,S_t) = \quad & \max_{C_t, e_t S_{t+1}} \quad \frac{C_t^{1-\eta}-1}{1-\eta} - N_t\frac{\nu_0}{1+\nu_1}h_t^{1+\nu_1} - U_t\frac{\gamma_0}{1+\gamma_1}e_t^{1+\gamma_1} + \\ & \qquad + \beta\mathbb{E}_t\left[J^h\left((1-\omega)N_t + \kappa_{w,t}\frac{e_t}{e}(1-N_t), S_{t+1}\right)\right] \\ & \text{s.t.} \quad C_t \leq w_t N_t \frac{h_t}{h} + d_t S_t - v_t(S_{t+1} - S_t) + b(1-N_t) - T_t, \\ & \text{and} \quad 0 \leq e_t \leq 1, \\ & \text{given} \quad N_t, S_t. \end{aligned}$$

In addition to conditions (4.2.6) and (4.2.7) which remain as before, the following KKT conditions determine the search intensity e_t of unemployed household members

$$\gamma_0 e_t^{\gamma_1} = \frac{\kappa_{w,t}}{e}\beta\mathbb{E}_t\left[\frac{\partial J^h}{\partial N}(N_{t+1}, S_{t+1})\right] + \mu_{1,t}^h - \mu_{2,t}^h, \tag{4.7.32}$$

$$0 \leq e_t \leq 1, \tag{4.7.33}$$

$$\mu_{1,t}^h \geq 0, \quad \mu_{2,t}^h \geq 0, \tag{4.7.34}$$

$$\mu_{1,t}^h e_t = 0, \quad \mu_{2,t}(1-e_t) = 0, \tag{4.7.35}$$

where $U_t\mu_{1,t}^h$ and $U_t\mu_{2,t}^h$ are the KKT multipliers on the constraint of e_t.[57]

[57]We can write the KKT multipliers this way, since $U_t > 0$.

First, if the probability $\kappa_{w,t}$ of finding a job and the expected value $E_t\left[\frac{\partial J^h}{\partial N}(N_{t+1},S_{t+1})\right]$ of a marginal worker in the next period to the representative household both are strictly positive, so must be the time e_t unemployed members search for jobs (otherwise the left hand side of (4.7.32) would equal zero, but the right hand side would be strictly positive). On the other hand, if $\kappa_{w,t}=0$, then $e_t=0$ (if $\kappa_{w,t}=0$ and $e_t>0$ then $\mu^h_{1,t}=0$ for (4.7.35) to hold and (4.7.32) can not be fulfilled with $\mu^h_{2,t}\geq 0$), i.e. it can not be optimal for the representative household to search a strictly positive amount of time for jobs, if there is no chance of finding one. Summing up, as long as the value of a worker to the household remains strictly positive (which is the case), hours spent for job searching will vanish if and only if $\kappa_{w,t}=0$, i.e. if and only if the firm posts no vacancies. The KKT conditions (4.7.32)-(4.7.35) are thus already equivalent to

$$\gamma_0 e_t^{\gamma_1}=\frac{\kappa_{w,t}}{e}\beta\mathbb{E}_t\left[\frac{\partial J^h}{\partial N}(N_{t+1},S_{t+1})\right]-\mu^h_{2,t},, \tag{4.7.36}$$

$$0\leq e_t\leq 1, \tag{4.7.37}$$

$$\mu^h_{2,t}\geq 0, \tag{4.7.38}$$

$$\mu_{2,t}(1-e_t)=0. \tag{4.7.39}$$

The first term on the right hand side of equation (4.7.36) is the expected return from unemployed household members investing an additional marginal time unit into job searching, i.e. the average amount of new employments realized from the increase in search intensity times the expected marginal utility gained from an additional employed member in the next period. The optimality condition (4.7.36) states that unemployed household members invest time into job searching activities up to the point, where this expected return equals the marginal disutility from an additional marginal time unit spent searching for jobs, eventually capped by the upper bound of available total time.

Proceeding along the lines from the previous sections, we can derive

$$\frac{\partial J^h}{\partial N}(N_t,S_t)=\lambda_t\left(w_t\frac{h_t}{h}-b\right)-\left(\frac{\nu_0}{1+\nu_1}h_t^{1+\nu_1}-\frac{\gamma_0}{1+\gamma_1}e_t^{1+\gamma_1}\right)+ \\ +\beta\mathbb{E}_t\left[\frac{\partial J^h}{\partial N}(N_{t+1})\left(1-\omega-\kappa_{w,t}\frac{e_t}{e}\right)\right].$$

Hence, the value of an employed (over an unemployed) member to the representative household measured in consumption units, $\xi^h_t:=\frac{1}{\lambda_t}\frac{\partial J^h}{\partial N}(N_t,S_t)$, is

$$\xi^h_t=w_t\frac{h_t}{h}-\left(b+\frac{\nu_0}{1+\nu_1}\frac{h_t^{1+\nu_1}}{\lambda_t}-\frac{\gamma_0}{1+\gamma_1}\frac{e_t^{1+\gamma_1}}{\lambda_t}\right)+\left(1-\omega-\kappa_{w,t}\frac{e_t}{e}\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi^h_{t+1}\right]. \tag{4.7.40}$$

The interpretation is analogous to the previous models. Only now, the period value of unemployment activities (over work effort) to the household

$$z_t:=b+\frac{\nu_0}{1+\nu_1}\frac{h_t^{1+\nu_1}}{\lambda_t}-\frac{\gamma_0}{1+\gamma_1}\frac{e_t^{1+\gamma_1}}{\lambda_t}$$

explicitly contains the endogenously determined disutility from searching for jobs if unemployed.

Note that with the introduced notation, equations (4.7.36)-(4.7.39) can also be written as

$$\gamma_0 e_t^{\gamma_1}=\frac{\kappa_{w,t}}{e}\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi^h_{t+1}\right]\lambda_t-\mu^h_{2,t},\quad 0\leq e_t\leq 1,\quad \mu^h_{2,t}\geq 0,\quad \mu^h_{2,t}(1-e_t)=0. \tag{4.7.41}$$

Representative Firm On the representative firm's side everything remains the same as in the previous model. The KKT conditions are as in (4.2.16)-(4.2.19) and the value of a worker to the representative firm is given by (4.7.6), while the present value of a worker in the next period from the firm's perspective remains as in (4.7.7).

Hours and Wage Bargaining The same way as in the previous model, the hours worked and the wage are determined jointly by the representative household and the representative firm through a Nash bargain maximizing (4.7.8). Hence, since the only difference in ξ_t^h in the present model in (4.7.40) compared to the previous model in (4.7.5) is that $\frac{\nu_0}{1+\nu_1}\frac{h_t^{1+\nu_1}}{\lambda_t}$ is now replaced by $\frac{\nu_0}{1+\nu_1}\frac{h_t^{1+\nu_1}}{\lambda_t} - \frac{\gamma_0}{1+\gamma_1}\frac{e_t^{1+\gamma_1}}{\lambda_t}$ and $\kappa_{w,t}$ by $\frac{e_t}{e}\kappa_{w,t}$, while ξ_t^f remains unchanged, the KKT conditions for working hours (4.7.9) stay the same and the wage equation (4.7.11) becomes

$$w_t\frac{h_t}{h} = \varphi\exp(Z_t)\frac{h_t}{h} + (1-\varphi)\left(b + \frac{\nu_0}{1+\nu_1}\frac{h_t^{1+\nu_1}}{\lambda_t} - \frac{\gamma_0}{1+\gamma_1}\frac{e_t^{1+\gamma_1}}{\lambda_t}\right) + \varphi\kappa_{w,t}\frac{e_t}{e}\hat{\xi}_t^f.$$

Note that the fixed ratio between the shares ξ_t^h and ξ_t^f in total surplus, i.e. $\xi_t^h = \frac{\varphi}{1-\varphi}\xi_t^f$, lets us rewrite (4.7.41) as

$$\gamma_0 e_t^{\gamma_1} = \frac{\varphi}{1-\varphi}\frac{\kappa_{w,t}}{e}\hat{\xi}_t^f\lambda_t - \mu_{2,t}^h, \quad 0 \le e_t \le 1, \quad \mu_{2,t}^h \ge 0, \quad \mu_{2,t}^h(1-e_t) = 0. \tag{4.7.42}$$

Government Sector As before, the government runs a balanced budget and we assume the value of all unemployment activities contained in the parameter b to be just redistributed from taxes, i.e.

$$T_t = b(1-N_t).$$

General Equilibrium The system of equations determining the general equilibrium is similar to the previous model. The additional equations (4.7.42) determine the search effort by unemployed members in the economy. Further, the wage equation is adjusted to take the disutility from search effort in the period value of unemployment activities into account. The equilibrium in the model's economy is thus defined by:

$$U_t = 1 - N_t, \tag{4.7.43}$$

$$M_t = \begin{cases} \frac{\frac{e_t}{e}U_tV_t}{\left(\left(\frac{e_t}{e}U_t\right)^{\tau} + V_t^{\tau}\right)^{\frac{1}{\tau}}}, & \text{if } (e_t, V_t) \neq (0,0) \\ 0, & \text{if } (e_t, V_t) = (0,0) \end{cases} \tag{4.7.44}$$

$$\kappa_{f,t} = \begin{cases} \frac{M_t}{V_t}, & \text{if } V_t > 0, \\ 1, & \text{if } V_t = 0, e_t > 0, \\ 0, & \text{if } V_t = 0, e_t = 0, \end{cases} \tag{4.7.45}$$

$$\kappa_{w,t} = \begin{cases} \frac{M_t}{\frac{e_t}{e}U_t}, & \text{if } e_t > 0, \\ 1, & \text{if } e_t = 0, V_t > 0, \\ 0, & \text{if } e_t = 0, V_t = 0, \end{cases} \tag{4.7.46}$$

$$N_{t+1} = (1-\omega)N_t + M_t, \tag{4.7.47}$$

$$Y_t = \exp(Z_t)N_t\frac{h_t}{h}, \tag{4.7.48}$$

$$Y_t = C_t + cV_t, \tag{4.7.49}$$

$$\lambda_t = C_t^{-\eta}, \tag{4.7.50}$$

$$\gamma_0 e_t^{\gamma_1} = \frac{\varphi}{1-\varphi}\frac{\kappa_{w,t}}{e}\hat{\xi}_t^f \lambda_t - \mu_{2,t}^h, \quad 0 \le e_t \le 1, \quad \mu_{2,t}^h \ge 0, \quad \mu_{2,t}^h(1-e_t) = 0, \tag{4.7.51}$$

$$\nu_0 h_t^{\nu_1} = \frac{\exp(Z_t)}{h}\lambda_t - \mu_t^w, \quad 0 \le h_t \le 1, \quad \mu_t^w \ge 0, \quad \mu_t^w(1-h_t) = 0, \tag{4.7.52}$$

$$w_t \frac{h_t}{h} = \varphi \exp(Z_t)\frac{h_t}{h} + (1-\varphi)\left(b + \frac{\nu_0}{1+\nu_1}\frac{h_t^{1+\nu_1}}{\lambda_t} - \frac{\gamma_0}{1+\gamma_1}\frac{e_t^{1+\gamma_1}}{\lambda_t}\right) + \varphi\kappa_{w,t}\frac{e_t}{e}\hat{\xi}_t^f, \tag{4.7.53}$$

$$c = \kappa_{f,t}\hat{\xi}_t^f + \mu_t^f, \tag{4.7.54}$$

$$\mu_t^f \ge 0, \tag{4.7.55}$$

$$V_t \ge 0, \tag{4.7.56}$$

$$\mu_t^f V_t = 0, \tag{4.7.57}$$

$$\hat{\xi}_t^f = \mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\left(\exp(Z_{t+1})\frac{h_{t+1}}{h} - w_{t+1}\frac{h_{t+1}}{h} + (1-\omega)\,\hat{\xi}_{t+1}^f\right)\right], \tag{4.7.58}$$

We can again solve analytically for all the other period t variables in the model (and N_{t+1}) dependent on λ_t, $\hat{\xi}_t^f$ and the state variables. However, there is an important assumption we must make in the following. It would now always be mutually optimal for neither the unemployed members to search for jobs nor for the firm to post any open vacancies, since given one party chooses zero, the average matching probability for the other party is zero no matter the other party's choice. We will exclude this case in the following and only assume zero vacancies and search effort in case no other equilibrium exists.

First, consider the case of $V_t > 0$ and $\mu_t^f = 0$. We already showed that $e_t > 0$ must hold in this case (from (4.7.51), since $\kappa_{w,t} > 0$). Further, plugging (4.7.45) into (4.7.54) now yields

$$c = \frac{\frac{e_t}{e}(1-N_t)}{\left(\left(\frac{e_t}{e}(1-N_t)\right)^\tau + V_t^\tau\right)^{\frac{1}{\tau}}}\hat{\xi}_t^f \Leftrightarrow \left(\frac{e_t}{e}(1-N_t)\right)^\tau + V_t^\tau = \left(\frac{e_t}{e}(1-N_t)\right)^\tau\left(\frac{\hat{\xi}_t^f}{c}\right)^\tau$$

$$\Leftrightarrow V_t = \frac{e_t}{e}(1-N_t)\left(\left(\frac{\hat{\xi}_t^f}{c}\right)^\tau - 1\right)^{\frac{1}{\tau}} \Leftrightarrow \frac{V_t}{\frac{e_t}{e}(1-N_t)} = \left(\left(\frac{\hat{\xi}_t^f}{c}\right)^\tau - 1\right)^{\frac{1}{\tau}}.$$

Using $\hat{\xi}_t^f = \frac{c}{\kappa_{f,t}}$ from (4.7.54), since $\kappa_{f,t} > 0$ and $\mu_t^f = 0$, we can write the first equation in (4.7.51) equivalently as

$$\gamma_0 e_t^{\gamma_1} = \frac{\varphi}{1-\varphi}\frac{c}{e}\frac{\kappa_{w,t}}{\kappa_{f,t}}\lambda_t - \mu_{2,t}^h = \frac{\varphi}{1-\varphi}\frac{c}{e}\frac{V_t}{\frac{e_t}{e}(1-N_t)}\lambda_t - \mu_{2,t}^h = \frac{\varphi}{1-\varphi}\frac{c}{e}\left(\left(\frac{\hat{\xi}_t^f}{c}\right)^\tau - 1\right)^{\frac{1}{\tau}}\lambda_t - \mu_{2,t}^h,$$

so that the KKT conditions (4.7.51) are met if and only if

$$e_t = \min\left\{\left(\frac{\varphi}{1-\varphi}\frac{c}{\gamma_0 e}\left(\left(\frac{\hat{\xi}_t^f}{c}\right)^\tau - 1\right)^{\frac{1}{\tau}}\lambda_t\right)^{\frac{1}{\gamma_1}}, 1\right\} \quad \text{and}$$

$$\mu_{2,t}^h = \frac{\varphi}{1-\varphi}\frac{c}{e}\left(\left(\frac{\hat{\xi}_t^f}{c}\right)^\tau - 1\right)^{\frac{1}{\tau}}\lambda_t - \gamma_0 e_t^{\gamma_1}.$$

Then vacancies are given by

$$V_t = \frac{1}{e}(1-N_t)\min\left\{\left(\frac{\varphi}{1-\varphi}\frac{c}{\gamma_0 e}\left(\left(\frac{\hat{\xi}^f_t}{c}\right)^{\tau}-1\right)^{\frac{1}{\tau}}\lambda_t\right)^{\frac{1}{\gamma_1}}, 1\right\}\left(\left(\frac{\hat{\xi}^f_t}{c}\right)^{\tau}-1\right)^{\frac{1}{\tau}}.$$

Finally $V_t > 0$ and $e_t > 0$ are satisfied if and only if $\hat{\xi}^f_t > c$.

Let us now consider the second case, where $V_t = 0$. We already showed that also $e_t = 0$ in this case. From (4.7.45) and (4.7.46) it follows that $\kappa_{w,t} = 0$ and $\kappa_{f,t} = 0$ so that (4.7.54) is met for $\mu^f_t = c \geq 0$. As we already mentioned, this always yields a possible equilibrium indifferent of the value of $\hat{\xi}^f_t$. Expressed in words, the difference to the previous models is as follows. In the previous models, where unemployed members were always passively searching for jobs and unemployment was guaranteed to be strictly positive, the probability for the 'first' marginal vacancy posted in the economy being filled was 1. Hence, if $\hat{\xi}^f_t$ exceeded the cost c of posting a vacancy, it could not be optimal for the firm to post no vacancies at all, since starting from zero open vacancies a marginal additional vacancy would be filled with probability 1 and the return would therefore exceed the cost of posting. This is different now, if the representative household on the other side too puts no effort into searching for jobs. Then both, the representative firm as well as the representative household would act mutually optimal given the other's choice, if neither the firm posts vacancies nor the household searches jobs. Nonetheless, we will assume the representative household and firm to coordinate in such way that we suppose this case only, if the first case is not possible, i.e. if and only if $\hat{\xi}^f_t \leq c$.

Specifying e_t and V_t the way derived above, equations (4.7.51) and (4.7.54)-(4.7.57) are always met. We can then proceed to derive h_t from (4.7.52) by

$$h_t = \min\left\{\left(\frac{\exp(Z_t)\lambda_t}{\nu_0 h}\right)^{\frac{1}{\nu_1}}, 1\right\} \quad \text{and} \quad \mu^w_t = \frac{\exp(Z_t)}{h}\lambda_t - \nu_0 h_t^{\nu_1},$$

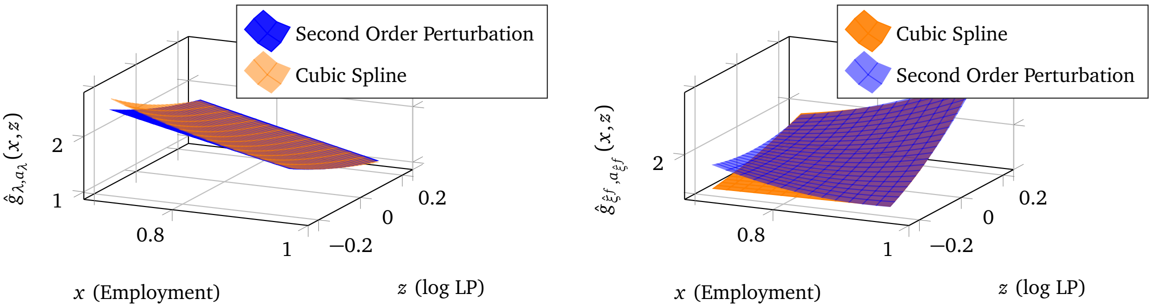
and M_t, $\kappa_{f,t}$, $\kappa_{w,t}$, N_{t+1}, Y_t, C_t and w_t arise successively from (4.7.44), (4.7.45), (4.7.46), (4.7.47), (4.7.48), (4.7.49) and (4.7.53). Summing up, by substituting the model's variables with the derived expressions in $\hat{\xi}^f_t$, λ_t and the state variables, the system of equations defining the model's equilibrium reduces to the two equations (4.7.50) and (4.7.58) in two (not predetermined) variables.

Solution Method As in the previous model, we reduced the system of equations determining the model's equilibrium to two equations in two variables. Hence, the policy functions for the marginal utility of consumption $g_\lambda : [0,1] \times \mathbb{R} \to \mathbb{R}$ and next period's discounted marginal value of a worker to the firm $g_{\hat{\xi}^f} : [0,1] \times \mathbb{R} \to \mathbb{R}$ have to mutually solve the functional equations (4.7.27)-(4.7.29), where the definitions of $C_t, N_{t+1}, Z_{t+1}, h_{t+1}$ and w_{t+1} dependent on $x = N_t, z = Z_t, g_{\hat{\xi}^f}(x,z) = \hat{\xi}^f_t, g_\lambda(x,z) = \lambda_t$ and the innovation $\epsilon \sim N(0,\sigma^2_\epsilon)$ are adjusted according to the preceding paragraph. The mean weighted residual methods can thus be applied as before.

Calibration The calibration of the model's parameters follows the calibration in the baseline model. All steady state values and calibrated parameters except for b remain as in table 4.1. We stick to $\nu_1 = 2$ and assume $\gamma_1 = \nu_1$. Moreover, the total period value of unemployment activities to the representative household is set to remain at 0.85 in steady state, i.e.

$$z = b + \frac{\nu_0}{1+\nu_1}\frac{h^{1+\nu_1}}{\lambda} - \frac{\gamma_0}{1+\gamma_1}\frac{e^{1+\gamma_1}}{\lambda} = 0.85.$$ [58]

Figure 4.43: Endogenous Hours and Search Intensity: Approximation of Policy Function



(a) Approximations to Policy Function of λ **(b)** Approximations to Policy Function of $\hat{\xi}^f$

The value of unemployment to the representative household from leisure and the costs of unemployment from job search, both measured in consumption units, are already pinned down by the chosen values for ν_1 and γ_1. First, as in the previous model (4.7.52) determines the value of unemployment due to leisure by $\frac{\nu_0}{1+\nu_1}\frac{h^{1+\nu_1}}{\lambda} = \frac{\exp(Z)}{1+\nu_1} = \frac{1}{3}$. Further, (4.7.54) in (4.7.51) yields

$$\frac{\gamma_0 e_1^\gamma}{\lambda} = \frac{\varphi}{1-\varphi}\frac{c}{e}\frac{\kappa_w}{\kappa_f}$$

so that the cost of unemployment due to job searching activities is pinned down by

$$\frac{\gamma_0}{1+\gamma_1}\frac{e^{1+\gamma_1}}{\lambda} = \frac{\varphi}{1-\varphi}\frac{c}{1+\gamma_1}\frac{\kappa_w}{\kappa_f} \approx 0.0152.^{59}$$

Consequently the value of unemployment activities still contained in b becomes $b \approx 0.85 - \frac{1}{3} + 0.0152 \approx 0.5319$. As before, we set working hours in steady state to $h = 0.33$. We follow Andolfatto (1996) in setting the steady state value of hours spent searching for jobs by unemployed members to $e = \frac{1}{2}h$. We then determine the parameters ν_0 and γ_0 for (4.7.52) and (4.7.51) to hold.

Finally, the chosen calibration together with the normalization of search effort in the matching function, of the wage rate in the household's budget constraint and of productivity in the production technology guarantees that all numeric steady state values and free parameter values remain the same as in the baseline model.

Steady State The steady state can be computed the same way as in the baseline model only now with b replaced by $b + \frac{\nu_0}{1+\nu_1}\frac{h^{1+\nu_1}}{\lambda} - \frac{\gamma_0}{1+\gamma_1}\frac{e^{1+\gamma_1}}{\lambda} = 0.85$. The parameters γ_0 and ν_0 are given by

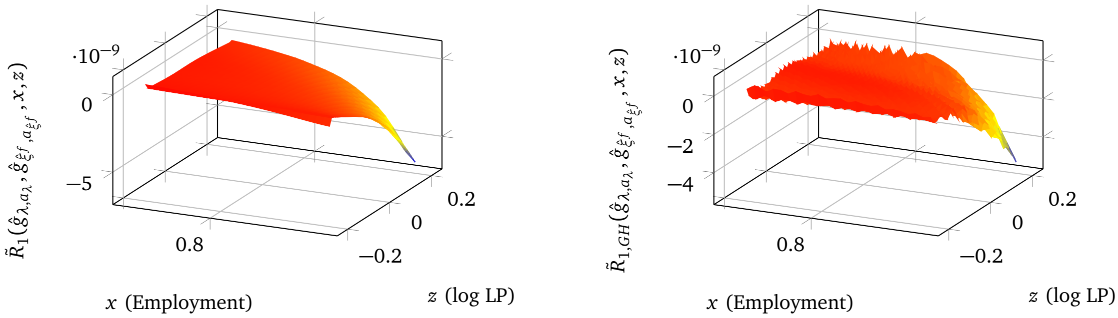
$$\gamma_0 = \frac{\varphi}{1-\varphi}\frac{c}{e^{\gamma_1+1}}\frac{\kappa_w}{\kappa_f}\lambda \quad \text{and} \quad \nu_0 = \frac{\exp(Z)\lambda}{h^{\nu_1+1}}.$$

Dynamics of Employment We present approximations to the policy functions of λ_t and $\hat{\xi}^f_t$, computed either from the finite element method or from a second order perturbation approach, in figure 4.43. We again solved for the global approximation only on the smaller, but still sufficient, domain $[\underline{x},\bar{x}] \times [\underline{z},\bar{z}] = [0.7, 0.97] \times [-0.21, 0.21]$ with $d_x = 61$ and $d_z = 85$ non-equidistant grid points. The interpretable Euler residuals of the finite element solution for both

[58]Note that although employed members now only work a fraction of total time, the productivity was adjusted in such way that the output from a worker remains at the same level in steady state. The interpretation that the total period value of unemployment activities equals 85% of a worker's output in steady state carries over.

[59]The parameter c, although not calibrated, is already determined as in equation (4.7.4).

Figure 4.44: Endogenous Hours and Search Intensity: Interpretable Euler Residuals



(a) Interpretable Euler Residuals of $R_{1,GH}$ **(b)** Interpretable Euler Residuals of $R_{2,GH}$

Figure 4.45: Endogenous Hours and Search Intensity: Dynamics of Employment

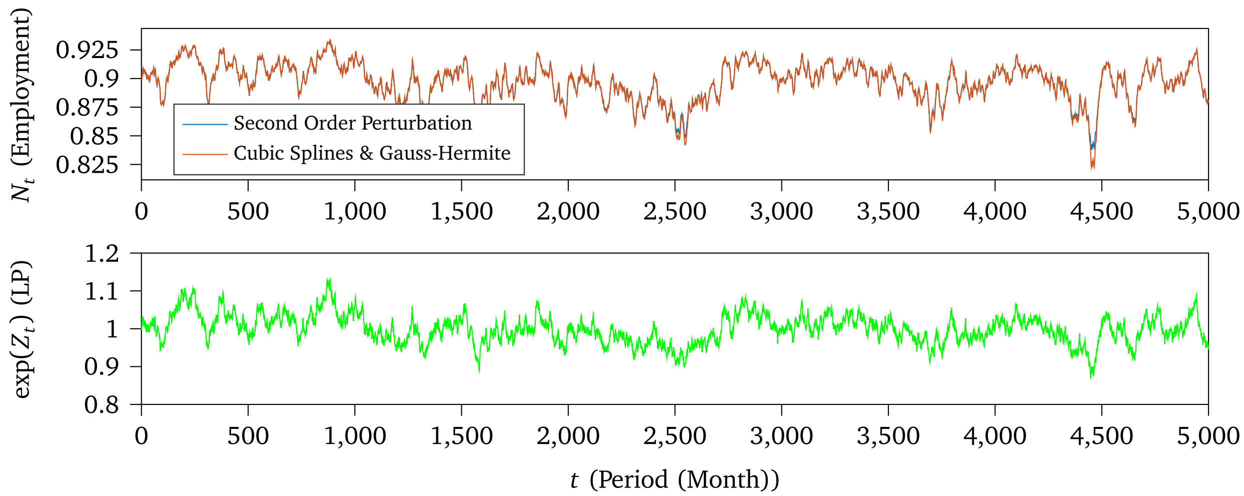
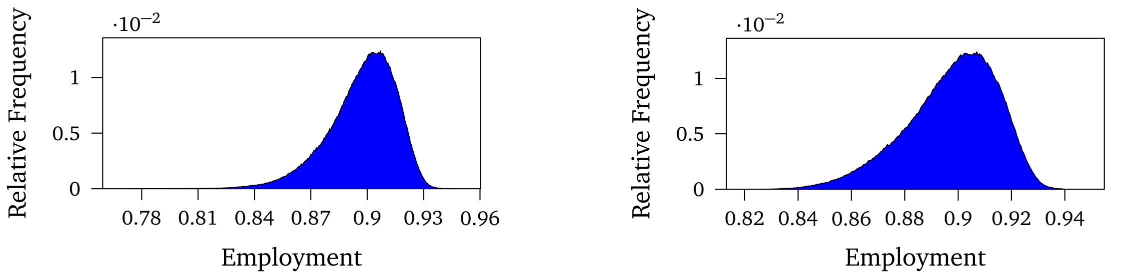


Figure 4.46: Endogenous Hours and Search Intensity: Histograms for Distribution of Monthly Employment Rates



(a) Distribution of Employment computed by Cubic Splines **(b)** Distribution of Employment computed by Second Order Perturbation

functional equations are shown in figure 4.44. The interpretable Euler residuals do not exceed 10^{-9} in magnitude so that the global solution should again be accurate enough for the upcoming results. Deviations of the approximation by a second order perturbation from the global solution appear for lower rates of employment.

Figure 4.45 displays the employment rate during the first 5000 periods in the simulation of the model's equilibrium outcomes from the same sample of pseudorandom iid$N(0,\sigma_\epsilon^2)$ distributed shocks to log LP. The distribution of employment in the whole simulation for 1200000 periods is illustrated by the histograms in figure 4.46 and by the statistical indicators summarized in table 4.35. First, no huge drops in the employment rate occurred in the simulation. The

Table 4.35: Endogenous Hours and Search Intensity: Statistical Measures for Monthly Employment Rates

	$\bar{N}$	$\max(N_t)$	$\min(N_t)$	$q_{0.01}$	$q_{0.05}$	$q_{0.1}$	$q_{0.5}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	s_N	v_N	w_N
Cubic Spline	0.8975	0.9431	0.7764	0.8451	0.8641	0.8732	0.9001	0.9180	0.9219	0.9282	0.0179	-0.8443	4.0276
Perturbation	0.8978	0.9429	0.8251	0.8518	0.8660	0.8740	0.9001	0.9180	0.9218	0.9280	0.0170	-0.6130	3.1210

Notes: $\bar{N}$=average employment rate, $\max(N_t)$=maximal employment rate, $\min(N_t)$=minimal employment rate, $q_p(N_t)$ =p-quantile for cumulative distribution of employment rate, s_N, v_N, w_N=standard deviation, skewness, kurtosis of the cumulative distribution of employment rate. Statistics are computed from the outcome of a simulation of 1200000 (monthly) periods.

unemployment rate did not rise above 22.5% in the simulation. However, employment declines slightly more than in the model with endogenous hours but exogenous search intensity from the previous subsection. The standard deviation of employment increases by the factor 1.4, the distribution is a little less left skewed with a lightly smaller kurtosis.The solution by perturbation deviates up to 0.05 from the global solution, fails to reproduce the maximal unemployment rate as well as the skewness and kurtosis of employment's distribution, while the standard deviation is similar.[60]

Endogenous Disasters? We computed the number of disasters, the number of disaster periods as well as the disaster probability, average disaster size and disaster duration for consumption, output and a consumption equivalent for different threshold fractions. The consumption equivalent $C_{eq,t}$ is now defined through

$$C_{eq,t} = \left((1-\eta)\left(\frac{C_t^{1-\eta}-1}{1-\eta} - N_t\frac{\nu_0}{1+\nu_1}h_t^{1+\nu_1} - U_t\frac{\gamma_0}{1+\gamma_1}e_t^{1+\gamma_1}\right)+1\right)^{\frac{1}{1-\eta}}.$$

The results, along the respective values for labor productivity, are summarized in table 4.36. Compared to the previous model extensions, disasters appear more frequently than in the model

Table 4.36: Disasters in the Model with Endogenous Hours and Search Intensity

	number of disasters	total number of disaster periods	disaster probability	average disaster size	average disaster duration
			90% threshold		
C	558	2521	0.1404%	11.28%	4.52
Y	1091	4576	0.2759%	11.46%	4.19
C_{eq}	448	2052	0.1126%	11.25%	4.58
LP	663	2086	0.1667%	11.05%	3.15
			85% threshold		
C	13	48	0.0033%	16.03%	3.69
Y	45	139	0.0113%	16.04%	3.09
C_{eq}	12	32	0.0030%	15.72%	2.67
LP	5	10	0.0013%	15.68%	2.00
			80% threshold		
C	0	0	0%	–	–
Y	0	0	0%	–	–
C_{eq}	0	0	0%	–	–
LP	0	0	0%	–	–

with endogenous working hours but fixed search effort discussed in section 4.7.2, yet still substantially less frequently than in the model with leisure in the utility but fixed working hours from section 4.7.1.

[60]Note however that for the case that all variables' outcomes are computed from the second order approximations to their policy functions, larger declines of LP still lead the model's dynamics to leave their basin of attraction under the second order approximation so that the outcomes would at some point tend to $\pm\infty$.

Second Moments of the Labor Market Table 4.37 summarizes the second moments of the labor market. Unemployment and the labor market tightness become more volatile than in the model with endogenous hours but fixed search intensity, while the standard deviation of open vacancies remains almost the same. However, the fluctuations remain substantially below the values found in the data and also substantially below the values from simulations of the baseline model.

Table 4.37: Labor Market Moments in the Model with Endogenous Hours and Search Intensity

	U	V	θ	$p = \frac{Y}{N}$
s_x	0.063 [0.049; 0.080]	0.046 [0.039; 0.054]	0.096 [0.077; 0.119]	0.011 [0.009; 0.013]
$\frac{s_x}{s_p}$	5.888 [4.921; 7.049]	4.320 [3.838; 4.839]	8.989 [7.976; 10.154]	1.000 [1.000; 1.000]
r_x	0.858 [0.799; 0.904]	0.572 [0.455; 0.678]	0.811 [0.740; 0.867]	0.656 [0.542; 0.754]
		Cross Correlations		
U		-0.594 [−0.690; −0.496]	-0.862 [−0.914; −0.782]	-0.678 [−0.783; −0.550]
V			0.864 [0.832; 0.897]	0.986 [0.981; 0.993]
θ				0.930 [0.886; 0.961]

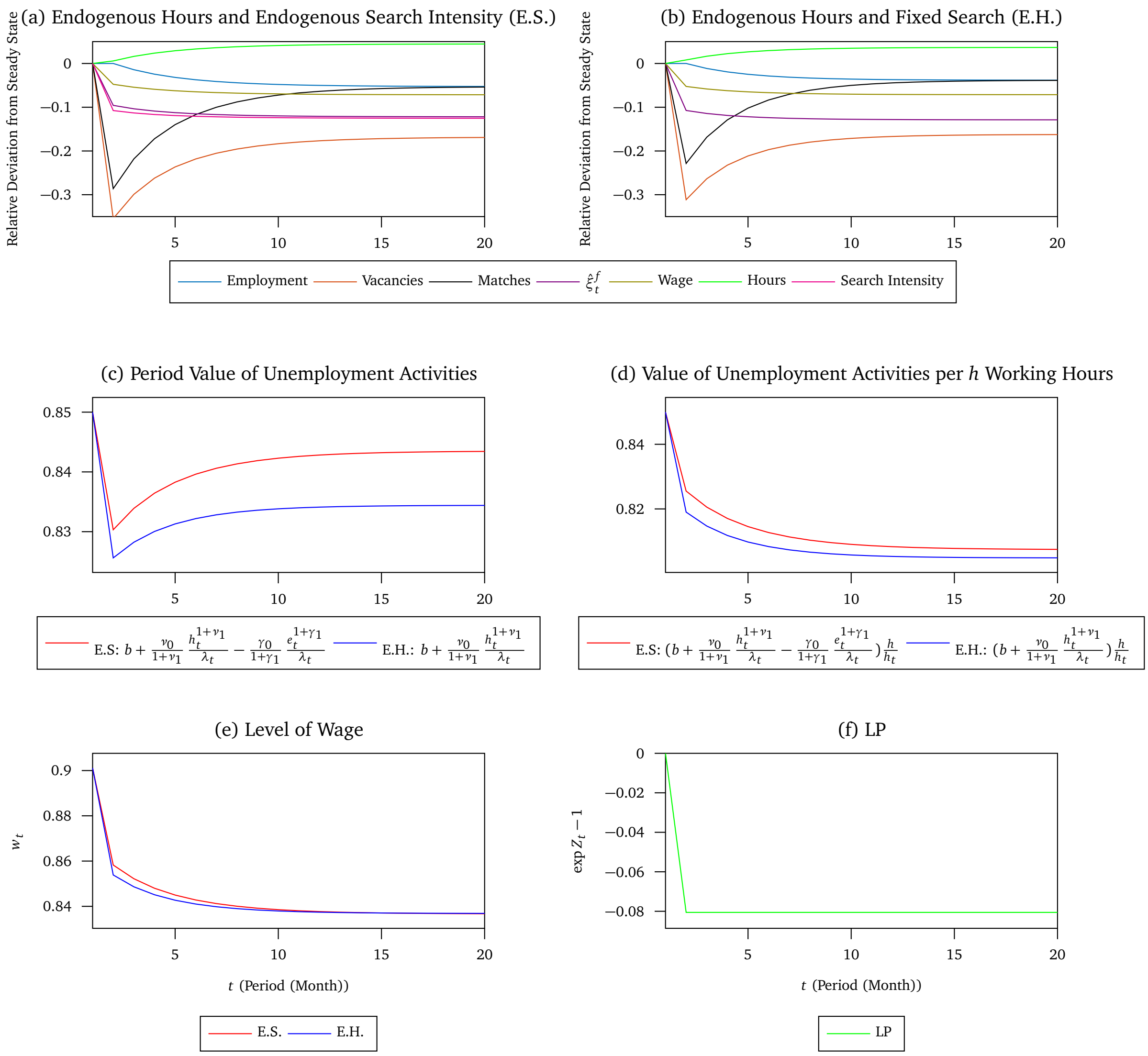
Notes: U=unemployment, V=vacancies, $\theta = \frac{V}{U}$=labor market tightness, p=output per worker, s_x=standard deviation of variable X, r_x=first order autocorrelation of variable X. All moments are computed from the cyclical components of the HP-filtered ($\lambda = 1600$) relative deviations from the mean, i.e. as the cyclical component of the series $\frac{X_t - \bar{X}}{\bar{X}}$. The table shows the average as well as the 2.5% and 97.5% quantiles of 5000 simulations of the model's equilibrium outcomes for 666 periods converted to 222 quarterly observations after 1000 throw-away periods.

Comparison and Intuition Figure 4.47 shows the response in the model to a sudden and permanent drop in log LP by two unconditional standard deviations starting from steady state. The induced decline in $\hat{\xi}_t^f$ following the shock is similar to the model with endogenous hours but fixed search intensity from section 4.7.2. Vacancies and realized matches decline moderately more and employment stabilizes at a slightly lower rate after some periods. We will try to lay out the effects of variable search effort in the following.

First, the effort unemployed members put into job searching is declining in response to the shock. The interpretation for the reaction can be deduced from (4.7.41). Expectations about the value ξ_{t+1}^h of employment from the household's perspective are declining, since the total surplus in the economy from a worker is declining at a lower productivity level. Moreover, with the firm posting less vacancies, the probability $\kappa_{w,t}$ for the household's search efforts turning out to be successful is also decreasing. However, on the other hand the marginal utility of consumption will increase due to less consumption after the shock. The first two effects must be dominating and unemployed members spend less time searching for jobs. As a result, the disutility $\frac{\gamma_0}{1+\gamma_1} e_t^{1+\gamma_1}$ the representative household faces from job searching activities of an unemployed member is decreasing and even more so when measured in consumption units, i.e. $\frac{\gamma_0}{1+\gamma_1} \frac{e_t^{1+\gamma_1}}{\lambda_t}$. While the cost of unemployment from job searching was contained in the fixed parameter b before, it is now decreasing so that the total period value of unemployment activities declines less in the present model. Yet, when converted to the value per h working hours the difference becomes small so that the hourly wage rate differs only negligible between the models and the same is true for the present value of a worker to the firm $\hat{\xi}_t^f$. However, with less search effort of unemployed members the average probability at which the firm can fill a vacancy is lower and leads the firm to post fewer vacancies nonetheless. With less search effort and less vacancies fewer matches can be realized and the unemployment rate becomes slightly higher than in the model with fixed search effort.

In consequence, drops in output and consumption slightly regain in size compared to the model with endogenous hours but fixed search effort. Yet, increasing working hours still significantly dampen the drops in comparison to the model with leisure in utility but fixed working hours.

Figure 4.47: Endogenous Hours and Search Intensity: Effects of Drop in Labor Productivity



4.7.4 Endogenous Hours and Home Production

In the previous extensions of the model, the value of unemployment to the household from home production was still assumed to be contained in the fixed parameter b and effects on the household's utility were neglected. We analyze the consequences if we endogenize home production in this subsection.

Search and Matching We return to the assumption of unemployed members passively searching for jobs. The matching process (4.2.1) and the dynamics of unemployment (4.2.2) as well as the average probabilities of finding a job $\kappa_{w,t}$ and filling an open vacancy $\kappa_{f,t}$ are the same as in the baseline model.

Representative Household Unemployed members of the representative household now may decide about hours l_t spent in home production.[61] Following Kuehn et al. (2015), who note that Aguiar et al. (2013) find no evidence for shocks to home production, we assume the home production good $C_{h,t}$ to be produced with a non-stochastic and linear home production

[61] It is assumed that all unemployed members can take part in home production while also searching for jobs.

technology, i.e.

$$C_{h,t} := XU_t l_t, \quad X > 0.$$

The representative household draws utility from a composite consumption good C_t consisting of the market good $C_{m,t}$ and the home produced good $C_{h,t}$, where

$$C_t = \left(aC_{m,t}^e + (1-a)C_{h,t}^e\right)^{\frac{1}{e}}, \quad e \in (0,1), a \in (0,1).$$

Moreover, the representative household faces disutility from working hours of the employed members in market production as well as from working hours of unemployed members in home production. Total lifetime utility is assumed to be

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s\left(\frac{C_{t+s}^{1-\eta}-1}{1-\eta} - N_{t+s}\frac{\nu_0}{1+\nu_1}h_{t+s}^{1+\nu_1} - U_{t+s}\frac{\gamma_0}{1+\gamma_1}l_{t+s}^{1+\gamma_1}\right)\right], \quad \eta, \nu_0, \nu_1, \gamma_0, \gamma_1 > 0, \eta \neq 1.$$

The budget constraint in period t remains as before

$$C_{m,t} + v_t(S_{t+1} - S_t) \leq w_t N_t \frac{h_t}{h} + b(1-N_t) + d_t S_t - T_t,$$

and the mass of unemployed members from the household's perspective evolves as in (4.2.5), where the household considers $\kappa_{w,t}$ as exogenous. The representative household chooses consumption, share holdings and the time unemployed members spend working at home. He does not decide about the hours employed members work, but they will still be subject to a bargaining process with the firm. With $J^h(N_t, S_t)$ as the household's value function, we can write

$$\begin{aligned}
J^h(N_t,S_t) = \quad & \max_{C_t, l_t S_{t+1}} \quad \frac{C_t^{1-\eta}-1}{1-\eta} - N_t\frac{\nu_0}{1+\nu_1}h_t^{1+\nu_1} - U_t\frac{\gamma_0}{1+\gamma_1}l_t^{1+\gamma_1} + \\
& \qquad\qquad + \beta\mathbb{E}_t\left[J^h\left((1-\omega)N_t + \kappa_{w,t}(1-N_t), S_{t+1}\right)\right] \\
& \text{s.t.} \qquad C_{m,t} \leq w_t N_t \frac{h_t}{h} + d_t S_t - v_t(S_{t+1} - S_t) + b(1-N_t) - T_t, \\
& \text{and} \qquad C_{h,t} = X(1-N_t)l_t, \\
& \text{and} \qquad C_t = \left(aC_{m,t}^e + (1-a)C_{h,t}^e\right)^{\frac{1}{e}}, \\
& \text{and} \qquad 0 \leq l_t \leq 1, \\
& \text{given} \quad N_t, S_t.
\end{aligned}$$

Since the marginal product from a working hour in home production as well as the marginal utility drawn from the home production good are always strictly positive, while the marginal disutility of a working hour in home production is zero in $l_t = 0$, the optimal amount of l_t must always be strictly positive. Exploiting that the budget constraint holds with equality in optimum and plugging the remaining constraints into the objective function, the KKT conditions for the maximization problem on the right hand side thus are

$$\lambda_t = aC_t^{-\eta}\left(\frac{C_t}{C_{m,t}}\right)^{1-e}, \tag{4.7.59}$$

$$\gamma_0 l_t^{\gamma_1} = (1-a)C_t^{-\eta}\left(\frac{C_t}{C_{h,t}}\right)^{1-e} X - \mu_t^h = \frac{1-a}{a}\lambda_t\left(\frac{C_{m,t}}{C_{h,t}}\right)^{1-e} X - \mu_t^h, \tag{4.7.60}$$

$$l_t \leq 1,\ \mu_t^h \geq 0,\ \mu_t^h(1-l_t) = 0, \tag{4.7.61}$$

where the condition for asset prices remains as in (4.2.7) and where λ_t denotes the Lagrange multiplier on the budget constraint and $U_t\mu_t^h$ the KKT multiplier on the constraint on working hours in home production.[62] The value of an employed (over an unemployed) member to the representative household now becomes

$$\frac{\partial J^h}{\partial N}(N_t,S_t)=\lambda_t\left(w_t\frac{h_t}{h}-b\right)-\left(\frac{\nu_0}{1+\nu_1}h_t^{1+\nu_1}+(1-a)C_t^{-\eta}\left(\frac{C_t}{C_{h,t}}\right)^{1-e}Xl_t-\frac{\gamma_0}{1+\gamma_1}l_t^{1+\gamma_1}\right)+ \\ +\beta\mathbb{E}_t\left[\frac{\partial J^h}{\partial N}(N_{t+1})\left(1-\omega-\kappa_{w,t}\right)\right].$$

In market consumption units, $\xi_t^h:=\frac{1}{\lambda_t}\frac{\partial J^h}{\partial N}(N_t,S_t)$, we get

$$\xi_t^h=w_t\frac{h_t}{h}-\left(b+\frac{\nu_0}{1+\nu_1}\frac{h_t^{1+\nu_1}}{\lambda_t}+\left(\frac{1-a}{a}\left(\frac{C_{m,t}}{C_{h,t}}\right)^{1-e}Xl_t-\frac{\gamma_0}{1+\gamma_1}\frac{l_t^{1+\gamma_1}}{\lambda_t}\right)\right)+ \\ +\left(1-\omega-\kappa_{w,t}\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^h\right].$$

The interpretation for the value of an employed (over an unemployed) member to the representative household remains analogous to the previous models. Only now, the total period value of unemployment activities turns out to be

$$z_t=b+\frac{\nu_0}{1+\nu_1}\frac{h_t^{1+\nu_1}}{\lambda_t}+\left(\frac{1-a}{a}\left(\frac{C_{m,t}}{C_{h,t}}\right)^{1-e}Xl_t-\frac{\gamma_0}{1+\gamma_1}\frac{l_t^{1+\gamma_1}}{\lambda_t}\right).$$

Next to the fixed value in b, the value due to leisure over work effort in market production in the second term as well as the value due to home production in brackets are explicitly contained. The value of unemployment due to home production arises endogenously as the excess of marginal utility derived from the output of the home produced good over disutility from working hours in home production by an unemployed member—everything converted to market consumption units, i.e.

$$\frac{1-a}{a}\left(\frac{C_{m,t}}{C_{h,t}}\right)^{1-e}Xl_t-\frac{\gamma_0}{1+\gamma_1}\frac{l_t^{1+\gamma_1}}{\lambda_t}.$$

Using (4.7.60) we can simplify the value of unemployment due to home production either to

$$\frac{1-a}{a}\left(\frac{C_{m,t}}{C_{h,t}}\right)^{1-e}Xl_t-\frac{\gamma_0}{1+\gamma_1}\frac{l_t^{1+\gamma_1}}{\lambda_t}=\frac{\gamma_0 l_t^{1+\gamma_1}}{\lambda_t}+\frac{\mu_t^h l_t}{\lambda_t}-\frac{\gamma_0}{1+\gamma_1}\frac{l_t^{1+\gamma_1}}{\lambda_t}=\frac{\gamma_0\gamma_1}{1+\gamma_1}\frac{l_t^{1+\gamma_1}}{\lambda_t}+\frac{\mu_t^h l_t}{\lambda_t}$$

or equivalently to

$$\frac{1-a}{a}\left(\frac{C_{m,t}}{C_{h,t}}\right)^{1-e}Xl_t-\frac{\gamma_0}{1+\gamma_1}\frac{l_t^{1+\gamma_1}}{\lambda_t}=\frac{1-a}{a}\left(\frac{C_{m,t}}{C_{h,t}}\right)^{1-e}\frac{\gamma_1}{1+\gamma_1}Xl_t+\frac{\mu_t^h l_t}{(1+\gamma_1)\lambda_t}.$$

The second equation states that as long as the constraint on l_t remains non-binding, the value of unemployment due to home production is the optimal output Xl_t of the home produced good by an unemployed member adjusted for the disutility faced from work effort in the production process with the factor $\frac{\gamma_1}{1+\gamma_1}$ and converted to units of the market good with the marginal rate of substitution. The value must of course be positive, since the household acts optimal. We can then rewrite the recursive equation for ξ_t^h as

$$\xi_t^h=w_t\frac{h_t}{h}-\left(b+\frac{\nu_0}{1+\nu_1}\frac{h_t^{1+\nu 1}}{\lambda_t}+\frac{\gamma_0\gamma_1}{1+\gamma_1}\frac{l_t^{1+\gamma_1}}{\lambda_t}+\frac{\mu_t^h l_t}{\lambda_t}\right)+\left(1-\omega-\kappa_{w,t}\right)\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\xi_{t+1}^h\right]. \tag{4.7.62}$$

[62] We can write the KKT multiplier this way, since $U_t>0$.

Representative Firm Nothing changes on the side of the representative firm from the previous model. The KKT conditions are as in (4.2.16)-(4.2.19). The marginal value of a worker to the representative firm ξ_t^f as well as next period's discounted marginal value of an employee to the firm $\hat{\xi}_t^f$ satisfy (4.7.6) and (4.7.7), respectively.

Hours and Wage Bargaining Working hours in market production and the wage are determined jointly by the representative household and the representative firm through the maximization of (4.7.8). Proceeding the same way as for the previous models, working hours in market production must satisfy (4.7.9) while the wage equation now becomes

$$w_t \frac{h_t}{h} = \varphi \exp(Z_t)\frac{h_t}{h} + (1-\varphi)\left(b + \frac{\nu_0}{1+\nu_1}\frac{h_t^{1+\nu_1}}{\lambda_t} + \frac{\gamma_0\gamma_1}{1+\gamma_1}\frac{l_t^{1+\gamma_1}}{\lambda_t} + \frac{\mu_t^h l_t}{\lambda_t}\right) + \varphi \frac{V_t}{1-N_t}c. \quad (4.7.63)$$

Government Sector As before, the government runs a balanced budget and we assume the value of all unemployment activities still contained in the parameter b to be just redistributed from taxes, i.e. $T_t = b(1-N_t)$.

General Equilibrium The system of equations determining the general equilibrium remains similar to section 4.7.2, only now augmented by equations (4.7.60) and (4.7.61) for working hours in home production by unemployed members and by the wage equation explicitly taking into account the value of unemployment to the household from home production:

$$U_t = 1 - N_t, \quad (4.7.64)$$

$$M_t = \frac{U_t V_t}{\left(U_t^\tau + V_t^\tau\right)^{\frac{1}{\tau}}}, \quad (4.7.65)$$

$$\kappa_{f,t} = \begin{cases} \dfrac{M_t}{V_t}, & \text{if } V_t > 0, \\ 1, & \text{if } V_t = 0, \end{cases} \quad (4.7.66)$$

$$\kappa_{w,t} = \frac{M_t}{U_t}, \quad (4.7.67)$$

$$N_{t+1} = (1-\omega)N_t + M_t, \quad (4.7.68)$$

$$Y_t = \exp(Z_t) N_t \frac{h_t}{h}, \quad (4.7.69)$$

$$Y_t = C_{m,t} + cVt, \quad (4.7.70)$$

$$C_{h,t} = XU_t l_t, \quad (4.7.71)$$

$$C_t = \left(aC_{m,t}^e + (1-a)C_{h,t}^e\right)^{\frac{1}{e}}, \quad (4.7.72)$$

$$\lambda_t = aC_t^{-\eta}\left(\frac{C_t}{C_{m,t}}\right)^{1-e}, \quad (4.7.73)$$

$$\gamma_0 l_t^{\gamma_1} = \frac{1-a}{a}\lambda_t \left(\frac{C_{m,t}}{C_{h,t}}\right)^{1-e} X - \mu_t^h, \quad l_t \le 1, \quad \mu_t^h \ge 0, \quad \mu_t^h(1-l_t) = 0, \quad (4.7.74)$$

$$\nu_0 h_t^{\nu_1} = \frac{\exp(Z_t)}{h}\lambda_t - \mu_t^w, \quad 0 \le h_t \le 1, \quad \mu_t^w \ge 0, \quad \mu_t^w(1-h_t) = 0, \quad (4.7.75)$$

$$w_t\frac{h_t}{h}=\varphi\exp(Z_t)\frac{h_t}{h}+(1-\varphi)\left(b+\frac{\nu_0}{1+\nu_1}\frac{h_t^{1+\nu_1}}{\lambda_t}+\frac{\gamma_0\gamma_1}{1+\gamma_1}\frac{l_t^{1+\gamma_1}}{\lambda_t}+\frac{\mu_t^h l_t}{\lambda_t}\right)+\varphi\frac{V_t}{1-N_t}c, \tag{4.7.76}$$

$$c=\kappa_{f,t}\hat{\xi}_t^f+\mu_t^f, \tag{4.7.77}$$

$$\mu_t^f\geq 0, \tag{4.7.78}$$

$$V_t\geq 0, \tag{4.7.79}$$

$$\mu_t^f V_t=0, \tag{4.7.80}$$

$$\hat{\xi}_t^f=\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\left(\exp(Z_{t+1})\frac{h_{t+1}}{h}-w_{t+1}\frac{h_{t+1}}{h}+(1-\omega)\,\hat{\xi}_{t+1}^f\right)\right], \tag{4.7.81}$$

As in the previous model, we can analytically solve for all the other period t variables in the model (and N_{t+1}) dependent on $\hat{\xi}_t^f$ and λ_t next to the state variables. Specifically, since the search and matching process and the firm's decision problem are the same as in the baseline model, V_t, μ_t, M_t, $\kappa_{w,t}$ and $\kappa_{f,t}$ can be computed the same way. Moreover, one can determine market working hours from (4.7.75) by

$$h_t=\min\left\{\left(\frac{\exp(Z_t)\lambda_t}{\nu_0 h}\right)^{\frac{1}{\nu_1}},1\right\}\quad\text{and}\quad \mu_t^w=\frac{\exp(Z_t)}{h}\lambda_t-\nu_0 h_t^{\nu_1},$$

so that market production Y_t and market consumption $C_{m,t}$ can then be derived from (4.7.69) and (4.7.70). After obtaining these values, plugging (4.7.71) into (4.7.74) yields

$$\gamma_0 l_t^{\gamma_1}=\frac{1-a}{a}\lambda_t\left(\frac{C_{m,t}}{XU_tl_t}\right)^{1-e}X-\mu_t^h,\quad 0\leq l_t\leq 1,\quad \mu_t^h\geq 0,\quad \mu_t^h(1-l_t)=0,$$

which we can solve for hours worked in home production by

$$l_t=\min\left\{\left(\frac{1-a}{a}\left(\frac{C_{m,t}}{U_t}\right)^{1-e}\frac{X^e}{\gamma_0}\lambda_t\right)^{\frac{1}{1+\gamma_1-e}},1\right\}\quad\text{and}\quad \mu_t^h=\frac{1-a}{a}\lambda_t\left(\frac{C_{m,t}}{XU_tl_t}\right)^{1-e}X-\gamma_0 l_t^{\gamma_1}.$$

The amount of the home produced good is then pinned down by (4.7.71) and the composite consumption good by (4.7.72). Finally, w_t can be computed from (4.7.76).

This way, the model's remaining variables are determined dependent on λ_t and $\hat{\xi}_t^f$ in such way that all equilibrium conditions except for (4.7.73) and (4.7.81) hold.

Solution Method The policy functions $g_\lambda:[0,1]\times\mathbb{R}\to\mathbb{R}$ and $g_{\hat{\xi}^f}:[0,1]\times\mathbb{R}\to\mathbb{R}$ for λ_t and $\hat{\xi}_t^f$, respectively, are characterized as the solution to the functional equations (4.7.27)-(4.7.29), now with

$$rhs_1(g_\lambda,g_{\hat{\xi}^f},x,z):=aC_t^{-\eta}\left(\frac{C_t}{C_{m,t}}\right)^{1-e}, \tag{4.7.82}$$

and where the definitions of $C_{m,t},C_t,N_{t+1},Z_{t+1},h_{t+1}$ and w_{t+1} in rhs_1 and rhs_2 are adjusted accordingly to the expressions of the respective variables dependent on $x=N_t, z=Z_t, g_{\hat{\xi}^f}(x,z)=\hat{\xi}_t^f, g_\lambda(x,z)=\lambda_t$, and the $N(0,\sigma_\epsilon^2)$-distributed innovation ϵ, from the preceding paragraph. The mean weighted residual methods can be applied as before.

Calibration The calibration of the model's parameters follows the calibration in the baseline model. Specifically, all the steady state values and calibrated parameters except for b remain as in table 4.1. We follow Kuehn et al. (2015) in calibrating the additional parameters stemming from endogenous home production. We set $e = 0.85$, $a = 0.8$ and determine X in such way that output per hour in home production from an unemployed member equals output per hour in market production from an employed member in steady state, i.e. $X = \frac{\exp(Z)}{h}$. We stick to $\nu_1 = 2$ and set $\gamma_1 = \nu_1$. Moreover, the total period value of unemployment to the representative household in steady state is set to remain at 0.85, i.e.

$$b + \frac{\nu_0}{1+\nu_1}\frac{h^{1+\nu_1}}{\lambda} + \frac{\gamma_0\gamma_1}{1+\gamma_1}\frac{l^{1+\gamma_1}}{\lambda} = 0.85.\quad (4.7.83)$$

[63]

As in the already discussed models with endogenous working hours, using (4.7.75), the value of unemployment from leisure (over market working effort) to the representative household in steady state can again be written as $\frac{\nu_0}{1+\nu_1}\frac{h^{1+\nu_1}}{\lambda} = \frac{\exp(Z)}{1+\nu_1}$, and is thus pinned down to be $\frac{1}{3}$. Nonetheless, without further assumptions we can now freely allocate the remaining value of $0.85-\frac{1}{3}$ to home production or other benefits of unemployment contained in the parameter b. We choose $b = 0.4$ as in Shimer (2005) and consequently attribute a value of $0.85-\frac{1}{3}-0.4 \approx 0.117$ to home production. This allocation implies, as will be derived in the next paragraph, that working hours in home production equal $l \approx 0.1497$, i.e. unemployed members spend approximately half the time of market working hours in home production. The parameters γ_0 and ν_0 are computed from (4.7.74) and (4.7.75).

Finally, the chosen calibration together with the normalization of the wage rate in the household's budget constraint and of productivity in the market production technology guarantees that all numeric steady state values as well as the free parameter values remain the same as in the baseline model. The additionally calibrated parameters and steady state values are summarized in table 4.38.

Table 4.38: Hours and Home Production: Calibration

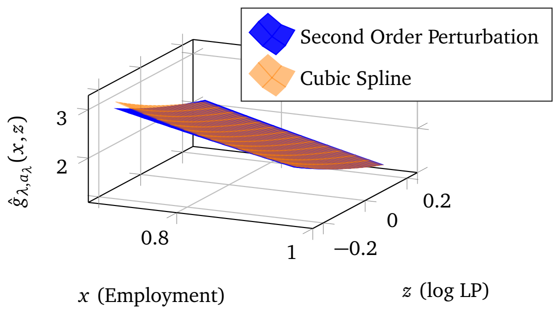
Variable	Value	Description
h	0.33	market working hours

Parameter	Value	Description
ν_1	2	disutility market working hours
γ_1	2	disutility home working hours
b	0.4	value of unemployment benefits
a	0.8	weight of market consumption in consumption bundle
e	0.85	parameter determining elasticity of substitution between market and home production good

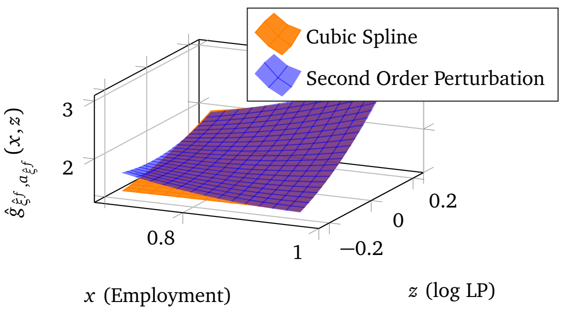
Steady State The steady state values for all variables already included in the baseline model can be computed the same way as in section 4.4.2, only with b replaced by $b + \frac{\nu_0}{1+\nu_1}\frac{h^{1+\nu_1}}{\lambda} + \frac{\gamma_0\gamma_1}{1+\gamma_1}\frac{l^{1+\gamma_1}}{\lambda} = 0.85$ in the respective equations. Working hours in home production can then be derived as

[63] Note that although employed members now only work a fraction of total time, the productivity was adjusted in such way that the output from a worker remains at the same level in steady state. The interpretation that the total period value of unemployment activities equals 85% of a worker's output in steady state hence carries over.

Figure 4.48: Endogenous Hours and Home Production: Approximation of Policy Function



(a) Approximations to Policy Function of λ



(b) Approximations to Policy Function of $\hat{\xi}^f$

follows. First, the value of unemployment due to home production can be transformed with (4.7.74) and (4.7.71) to

$$\frac{\gamma_0\gamma_1}{1+\gamma_1}\frac{l^{1+\gamma_1}}{\lambda} = \frac{\gamma_1}{1+\gamma_1}\frac{1-a}{a}\left(\frac{C_m}{C_h}\right)^{1-e} Xl = \frac{\gamma_1}{1+\gamma_1}\frac{1-a}{a}\left(\frac{C_m}{XUl}\right)^{1-e} Xl =$$
$$= \frac{\gamma_1}{1+\gamma_1}\frac{1-a}{a}\left(\frac{C_m}{U}\right)^{1-e}(Xl)^e$$

Hence, for (4.7.83) to hold, we can write

$$\frac{\gamma_1}{1+\gamma_1}\frac{1-a}{a}\left(\frac{C_m}{U}\right)^{1-e}(Xl)^e = 0.85 - b - \frac{\exp(Z)}{1+\nu_1},$$

which we can solve for working hours in home production by

$$l = \frac{1}{X}\left(\frac{1+\gamma_1}{\gamma_1}\frac{a}{1-a}\left(\frac{U}{C_m}\right)^{1-e}\left(0.85 - b - \frac{\exp(Z)}{1+\nu_1}\right)\right)^{\frac{1}{e}}.$$

The steady state values of C_h, C and λ immediately follow from the home production technology (4.7.71), the definition of the composite good (4.7.72) and (4.7.73). The parameters γ_0 and ν_0 can be derived from (4.7.74) and (4.7.75) by

$$\gamma_0 = (1-a)C^{-\eta}\frac{C}{C_h}^{1-e}\frac{X}{l^{\gamma_1}} \quad \text{and} \quad \nu_0 = \frac{\exp(Z)\lambda}{h^{\nu_1+1}}.$$

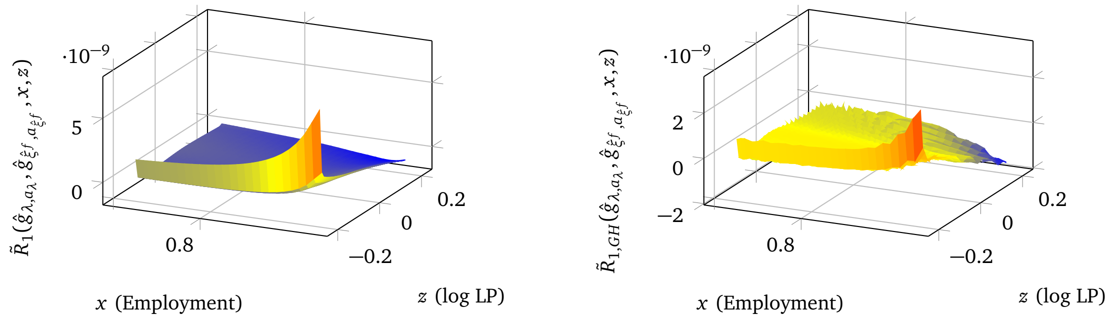
Dynamics of Employment Figure 4.48 displays approximations to the policy functions of λ_t and $\hat{\xi}^f_t$ from a finite element method and from a second order perturbation. We computed the global approximation only on the smaller, but for simulations sufficient, domain $[\underline{x},\bar{x}]\times[\underline{z},\bar{z}] = [0.7, 0.97]\times[-0.21, 0.21]$ with $d_x = 61$ and $d_z = 85$ non-equidistant grid points. The interpretable Euler residuals for the finite element solution in both functional equations that define the policy functions are shown in figure 4.49. In order to compute the interpretable Euler residuals for the functional equation $R_1 = 0$ in the present case, we first set $\tilde{C}_1$ such way that

$$\hat{g}_{\lambda,a_\lambda}(x,z) = a\tilde{C}_1^{-\eta}(\hat{g}_{\lambda,a_\lambda},\hat{g}_{\hat{\xi}^f,a_{\hat{\xi}^f}},x,z)\left(\frac{\tilde{C}_1(\hat{g}_{\lambda,a_\lambda},\hat{g}_{\hat{\xi}^f,a_{\hat{\xi}^f}},x,z)}{C_m(\hat{g}_{\lambda,a_\lambda},\hat{g}_{\hat{\xi}^f,a_{\hat{\xi}^f}},x,z)}\right)^{1-e},$$

which yields

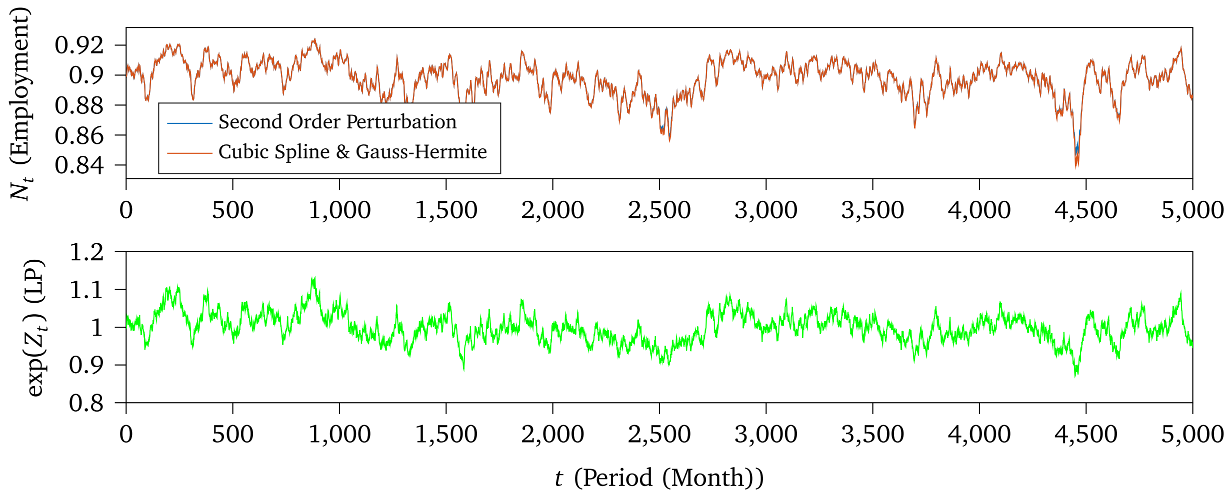
$$\tilde{C}_1(\hat{g}_{\lambda,a_\lambda},\hat{g}_{\hat{\xi}^f,a_{\hat{\xi}^f}},x,z) = \left(\frac{\hat{g}_{\lambda,a_\lambda}(x,z)}{aC_m^{e-1}(\hat{g}_{\lambda,a_\lambda},\hat{g}_{\hat{\xi}^f,a_{\hat{\xi}^f}},x,z)}\right)^{\frac{1}{1-e-\eta}}$$

Figure 4.49: Endogenous Hours and Home Production: Interpretable Euler Residuals



(a) Interpretable Euler Residuals of $R_{1,GH}$ **(b)** Interpretable Euler Residuals of $R_{2,GH}$

Figure 4.50: Endogenous Hours and Home Production: Dynamics of Employment



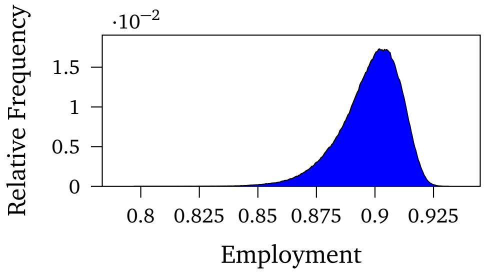
The interpretable Euler residuals $\tilde{R}_{1,GH}$ are then defined by

$$\tilde{R}_{1,GH} := \frac{\tilde{C}_1(\hat{g}_{\lambda,a_\lambda}, \hat{g}_{\hat{\xi}f,a_{\hat{\xi}f}}, x, z)}{C(\hat{g}_{\lambda,a_\lambda}, \hat{g}_{\hat{\xi}f,a_{\hat{\xi}f}}, x, z)} - 1 = \left(\frac{\hat{g}_{\lambda,a_\lambda}(x,z)}{aC^{-\eta}(\hat{g}_{\lambda,a_\lambda}, \hat{g}_{\hat{\xi}f,a_{\hat{\xi}f}}, x, z)\left(\frac{C(\hat{g}_{\lambda,a_\lambda}, \hat{g}_{\hat{\xi}f,a_{\hat{\xi}f}}, x,z)}{C_m(\hat{g}_{\lambda,a_\lambda}, \hat{g}_{\hat{\xi}f,a_{\hat{\xi}f}}, x,z)}\right)^{1-e}} \right)^{\frac{1}{1-e-\eta}} - 1$$
$$= \left(\frac{lhs_1(\hat{g}_{\lambda,a_\lambda}, \hat{g}_{\hat{\xi}f,a_{\hat{\xi}f}}, x, z)}{rhs_1(\hat{g}_{\lambda,a_\lambda}, \hat{g}_{\hat{\xi}f,a_{\hat{\xi}f}}, x, z)} \right)^{\frac{1}{1-e-\eta}} - 1.$$

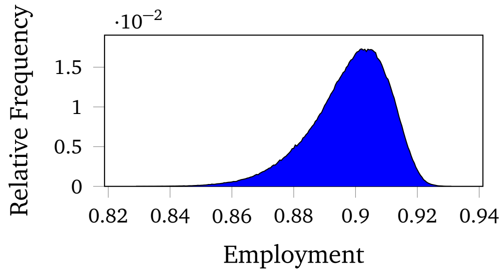
The magnitude of 10^{-9} for the Euler residuals in both of the functional equations indicates that the global approximations provide a satisfactory accuracy. On the other hand, the approximations by a second order perturbation again show increasing deviations from the global solution for lower rates of employment.

Figure 4.50 displays the outcome in the series of employment for the first 5000 periods in the simulation of the model's equilibrium outcomes using the same sample of pseudorandom iidN$(0,\sigma_\epsilon^2)$ distributed shocks to log LP. Additionally, the distribution of employment in the simulation is summarized by the histograms in figure 4.51 and by the statistical measures in table 4.39. The results turn out very similar to the results in the model with endogenous hours (see table 4.32). Hence, adding home production to the model changes the distribution of employment only marginally.

Figure 4.51: Endogenous Hours and Home Production: Histograms for Distribution of Monthly Employment Rates



(a) Distribution of Employment computed by Cubic Splines



(b) Distribution of Employment computed by Second Order Perturbation

Table 4.39: Endogenous Hours and Search Intensity: Statistic Measures for Distribution of Monthly Employment Rates

	$\bar{N}$	$\max(N_t)$	$\min(N_t)$	$q_{0.01}$	$q_{0.05}$	$q_{0.1}$	$q_{0.5}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	s_N	v_N	w_N
Cubic Spline	0.8981	0.9311	0.7971	0.8594	0.8740	0.8808	0.9000	0.9128	0.9156	0.9201	0.0130	-0.9343	4.4566
Perturbation	0.8982	0.9309	0.8294	0.8619	0.8746	0.8810	0.9000	0.9128	0.9155	0.9200	0.0126	-0.7815	3.7107

Notes: $\bar{N}$=average employment rate, $\max(N_t)$=maximal employment rate, $\min(N_t)$=minimal employment rate, $q_p(N_t)$ =p-quantile for cumulative distribution of employment rate, s_N, v_N, w_N=standard deviation, skewness, kurtosis of the cumulative distribution of employment rate. Statistics are computed from the outcome of a simulation of 1200000 (monthly) periods.

Endogenous Disasters? Table 4.40 summarizes the number of disasters, the number of disaster periods as well as the disaster probability, average disaster size and disaster duration for the consumption bundle, for market output and for market consumption. Note however that

Table 4.40: Disasters in the Model with Endogenous Hours and Home Production

	number of disasters	total number of disaster periods	disaster probability	average disaster size	average disaster duration
			90% threshold		
C	189	709	0.0473%	10.99%	3.75
Y	978	3965	0.2469%	11.45%	4.05
C_m	520	2194	0.1307%	11.27%	4.22
C_{eq}	135	544	0.0338%	10.99%	4.03
LP	663	2086	0.1667%	11.05%	3.15
			85% threshold		
C	1	1	0.0003%	15.49%	1
Y	38	121	0.0095%	16.10%	3.18
C_m	15	43	0.0038%	15.92%	2.87
C_{eq}	1	1	0.0003%	15.02%	1
LP	5	10	0.0013%	15.68%	2.00
			80% threshold		
C	0	0	0%	–	–
Y	0	0	0%	–	–
C_m	0	0	0%	–	–
C_{eq}	0	0	0%	–	–
LP	0	0	0%	–	–

measuring disasters by declines of market output or market consumption now also neglects potential increases in the home produced good during periods of higher unemployment. Since productivity in the home production technology is deterministic, it might even be reasonable to substitute production of the market good to some extent with the home produced good during periods where productivity in the market technology becomes low. Moreover, we also computed the disaster statistics for a consumption equivalent, which also takes the household's disutility from work effort in both market and home production into account, i.e. we compute

$C_{eq,t}$ satisfying

$$\frac{C_{eq,t}^{1-\eta}-1}{1-\eta}=\frac{C_t^{1-\eta}-1}{1-\eta}-N_t\frac{\nu_0}{1+\nu_1}h_t^{1+\nu_1}-U_t\frac{\gamma_0}{1+\gamma_1}l_t^{1+\gamma_1},$$

which yields

$$C_{eq,t}=\left((1-\eta)\left(\frac{C_t^{1-\eta}-1}{1-\eta}-N_t\frac{\nu_0}{1+\nu_1}h_t^{1+\nu_1}-U_t\frac{\gamma_0}{1+\gamma_1}l_t^{1+\gamma_1}\right)+1\right)^{\frac{1}{1-\eta}}.$$

There are no declines by more than 20% from steady state in neither one of the variables. Moreover, the consumption bundle declined only once during the 400000 periods in the simulation below 85% of its steady state value. Declines in the consumption bundle or the consumption equivalent appear even significantly less frequently than declines in the exogenous LP.

Second Moments of the Labor Market We show the second moments of the labor market in table 4.41. The results are again very similar to the results in table 4.34 for the model with

Table 4.41: Labor Market Moments and in the Model with Endogenous Hours and Home Production

	U	V	θ	$p=\frac{Y}{N}$
s_x	0.047 $[0.035;0.061]$	0.042 $[0.036;0.050]$	0.079 $[0.063;0.097]$	0.012 $[0.010;0.014]$
$\frac{s_x}{s_p}$	3.958 $[3.196;4.939]$	3.612 $[3.312;3.980]$	6.693 $[5.999;7.487]$	1.000 $[1.000;1.000]$
r_x	0.849 $[0.787;0.898]$	0.597 $[0.478;0.704]$	0.798 $[0.724;0.858]$	0.707 $[0.606;0.792]$
		Cross Correlations		
U		-0.644 $[-0.738;-0.543]$	-0.868 $[-0.914;-0.797]$	-0.766 $[-0.839;-0.671]$
V			0.911 $[0.881;0.939]$	0.980 $[0.977;0.983]$
θ				0.975 $[0.958;0.986]$

Notes: U=unemployment, V=vacancies, $\theta=\frac{V}{U}$=labor market tightness, p=output per worker, s_x=standard deviation of variable X, r_x=first order autocorrelation of variable X. All moments are computed from the cyclical components of the HP-filtered ($\lambda=1600$) relative deviations from the mean, i.e. as the cyclical component of the series $\frac{X_t-\bar{X}}{\bar{X}}$. The table shows the average as well as the 2.5% and 97.5% quantiles from 5000 simulations of the model's equilibrium outcomes for 666 periods converted to 222 quarterly observations after 1000 throw-away periods.

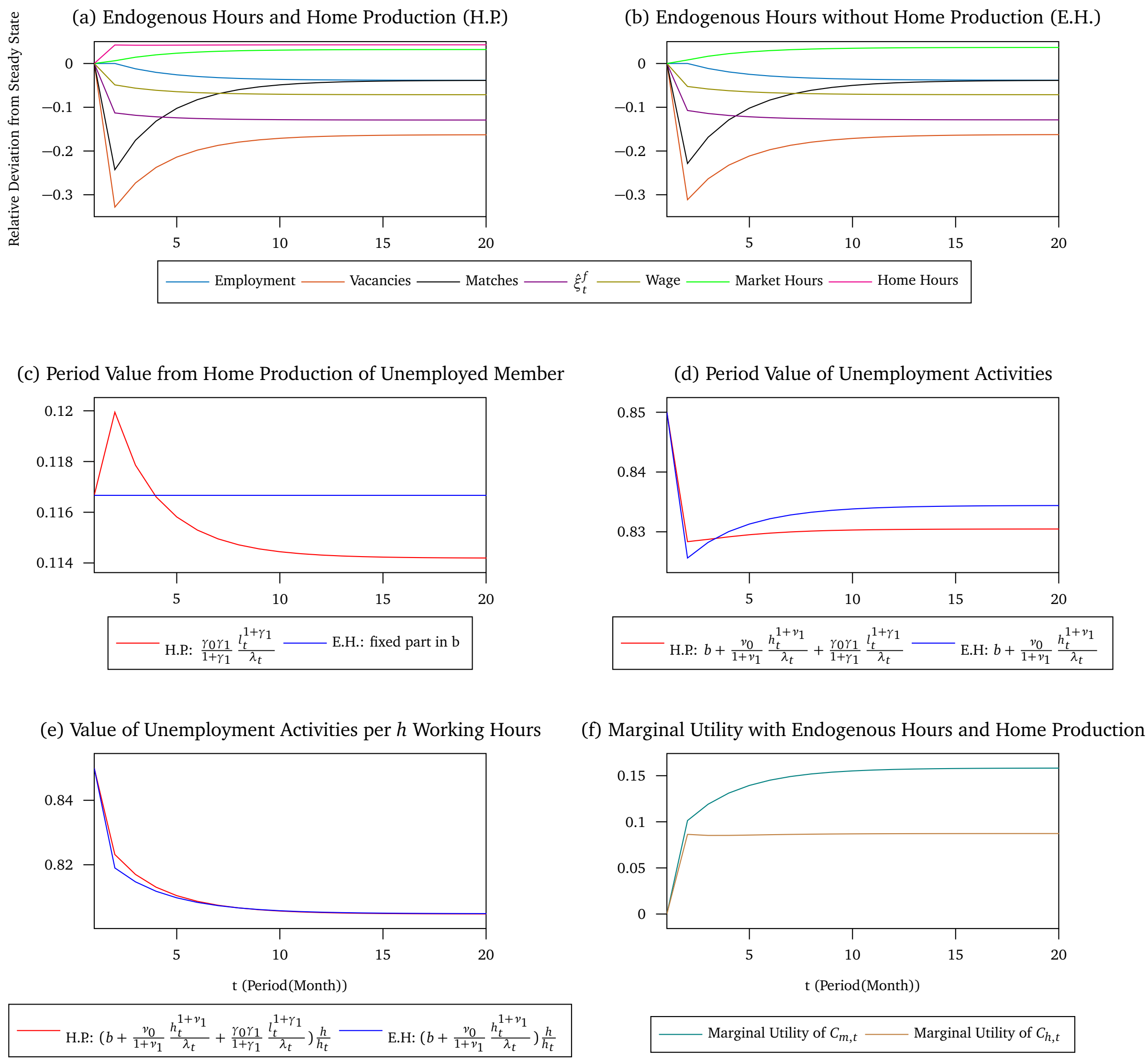
endogenous hours but without home production. The standard deviations are noticeably lower than in the baseline model and the values found in data all lie above the 97.5% quantiles from repeated simulations of the model.

Comparison and Intuition In order to provide some intuition about the effects in the economy during a scenario, which led to a huge decline in the employment rate in the baseline model, we show the reactions to a sudden and lasting drop in log LP by two unconditional standard deviations from steady state in figure 4.52. Most importantly, we conclude that the employment rate stabilizes after some periods, dropping only by approximately 4%. The size of all effects in the model following the shock to LP is very similar to the model with endogenous hours but without home production from section 4.7.2. We proceed to describe the implications of the now endogenous home production in the following.

First, in the period the shock hits the economy, the household's unemployed members increase the hours worked in home production. The reason can be identified from the first equality in the optimality condition in (4.7.60). The shock lowers output and consumption of the market good implying ceteris paribus an increase in the marginal utility of the home produced good on the right hand side.[64] Hence, with a fixed amount of unemployed members taking part in home production, the hours l_t must increase in $t=2$. In the subsequent periods growing

[64] Since $\frac{\partial u}{\partial C_{m,t}C_{h,t}}=a(1-a)(1-\eta-e)C_t^{-\eta-1}\left(\frac{C_t}{C_{m,t}}\right)^{1-e}\left(\frac{C_t}{C_{h,t}}\right)^{1-e}<0$, where u denotes the household's within period utility.

Figure 4.52: Endogenous Home Production: Effects of Drop in Labor Productivity



unemployment yields market consumption to decline even further, but also allows for more output of the home produced good with the same amount of time spent working in home production by the unemployed members. The effects of decreasing market and increasing home production due to higher unemployment on the marginal utility of $C_{h,t}$ are opposite, but turn out to have almost zero net effect after the second period. Consequently, working hours in home production remain essentially constant from there on.

The implications for the period value of unemployment activities are then as follows. The value of unemployment stemming from home production and measured in market consumption units is now endogenously determined by the excess of the value from consumption of the output over the cost of effort put into production of the home good, i.e. (as long as the constraint on l_t is not binding) by

$$\frac{1}{\lambda_t}\left((1-a)C_t^{-\eta}\left(\frac{C_t}{C_{h,t}}\right)^{1-e}Xl_t-\frac{\gamma_0}{1+\gamma_1}l_t^{1+\gamma_1}\right).$$

Using (4.7.74), one can also rewrite the above stated value of home production as

$$\frac{1}{\lambda_t}\left((1-a)C_t^{-\eta}\left(\frac{C_t}{C_{h,t}}\right)^{1-e}Xl_t-\frac{\gamma_0}{1+\gamma_1}l_t^{1+\gamma_1}\right)=\frac{\gamma_1}{1+\gamma_1}\frac{1-a}{a}\left(\frac{C_{m,t}}{C_{h,t}}\right)^{1-e}Xl_t.$$

As already described, in the period of the negative shock to LP, the increasing marginal utility of the home produced good renders it optimal for unemployed members to invest more time into home production. Consequently, the output in home production from an unemployed member, i.e. the last term on the right hand side, is increasing. But at the same time the MRS of the market good with the home production good is decreasing. Nonetheless, it turns out that the first effect dominates and the value of home production in units of the market consumption good increases in the period of the shock. However, since the MRS between the two goods continues to decrease while working hours in home production remain essentially unchanged in the periods following the shock, the effect eventually turns. The value of unemployment due to home production in units of the market consumption good begins to decline and eventually falls below the steady state value. Consequently, compared to the previous models the total period value of unemployment activities declines less at first, but decreases more in the following periods. However, the differences are only small. Moreover, working hours in market production do not increase as much as in the previous models, since part of the market good is substituted by the home production good. Once the total period value of unemployment is converted to the value per working hour, the differences between the models almost vanish. This implies that the hourly wage reacts almost the same and so does $\hat{\xi}_t^f$ as well as vacancies and realized matches.

Finally, during recessions with increasing unemployment, the decline of market consumption in the consumption bundle can now partly be compensated by an increase in the home production good. Thus, if disasters are measured in the consumption bundle, they become even less frequent and less pronounced.

4.7.5 Equity Premium in the Extensions

Lastly, we also want to present the implications of the introduced extensions for the return rates in the labor market model. The baseline model, at least when combined with Epstein-Zin preferences, generated a sizeable equity premium. We check whether the extended models are still capable to reproduce this result despite the fact that disasters in the size observed in the baseline model disappear.

Epstein-Zin Preferences We also consider the case of Epstein-Zin preferences for all of the model extensions. We therefore generalize the household's value function from (4.5.1) in the baseline model to

$$\begin{aligned}\tilde{J}^h(N_t, Z_t, S_t) = \quad & \max_{C_t, S_{t+1}} \left[(1-\beta) u_t^{1-\frac{1}{\psi}} + \beta \left(\mathbb{E}_t \left[\left(\tilde{J}^h (N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{1-\gamma} \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\psi}}} \\ & \text{s.t.} \quad C_t \leq \dots,\end{aligned}$$

where u_t denotes the within period utility function, which will be specified immediately. We thereby restrict ourselves to the case where u_t is strictly positive in order to guarantee that $\tilde{J}^h$ as stated above is well defined. Following e.g. Rudebusch and Swanson (2012), we define

$$J^h(N_t, Z_t, S_t) := \left(\tilde{J}^h(N_t, Z_t, S_t) \right)^{1-\frac{1}{\psi}}$$

so that the recursive formulation for the household's value function can equivalently be written either as

$$\begin{aligned}J^h(N_t, Z_t, S_t) = \quad & \max_{C_t, S_{t+1}} \quad (1-\beta) u_t^{1-\frac{1}{\psi}} + \beta \left(\mathbb{E}_t \left[\left(J^h (N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{\frac{1-\gamma}{1-\frac{1}{\psi}}} \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \\ & \text{s.t.} \quad C_t \leq \dots,\end{aligned}$$

in case of $\psi > 1$ or with max replaced by min in case of $\psi \in (0,1)$. To allow for a more convenient notation, we further introduce the parameter

$$\alpha := 1 - \frac{1-\gamma}{1-\frac{1}{\psi}}$$

and write

$$\begin{aligned} J^h(N_t, Z_t, S_t) = \quad & \max_{C_t, S_{t+1}} / \min \quad (1-\beta) u_t^{1-\frac{1}{\psi}} + \beta \left(\mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{1-\alpha} \right] \right)^{\frac{1}{1-\alpha}} \\ & \text{s.t.} \qquad C_t \leq \ldots, \end{aligned}$$

We stick to a coefficient of relative risk aversion of $\gamma = 10$ and an elasticity of intertemporal substitution of $\psi = 1.5$ as in Bansal and Yaron (2004) and Kuehn et al. (2012) so that $\alpha = 28$. Moreover, we choose the within period utility functions for the model extensions as summarized in table 4.42. Note however that the premise that u_t must be strictly positive may be violated by

Table 4.42: Overview: Within Period Utility Functions

Model	Within Period Utility
Leisure in Utility	$u_t = \left(\frac{C_t^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} - \frac{\nu_0}{1+\nu_1} N_t^{1+\nu_1} \right)^{\frac{1}{1-\frac{1}{\psi}}}$
Endogenous Hours	$u_t = \left(\frac{C_t^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} - N_t \frac{\nu_0}{1+\nu_1} h_t^{1+\nu_1} \right)^{\frac{1}{1-\frac{1}{\psi}}}$
Endogenous Hours and Search Intensity	$u_t = \left(\frac{C_t^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} - N_t \frac{\nu_0}{1+\nu_1} h_t^{1+\nu_1} - U_t \frac{\gamma_0}{1+\gamma_1} e_t^{1+\gamma_1} \right)^{\frac{1}{1-\frac{1}{\psi}}}$
Endogenous Hours and Home Production	$u_t = \left(\frac{C_t^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} - N_t \frac{\nu_0}{1+\nu_1} h_t^{1+\nu_1} - U_t \frac{\gamma_0}{1+\gamma_1} l_t^{1+\gamma_1} \right)^{\frac{1}{1-\frac{1}{\psi}}}$

the chosen specifications (u_t may not even be well-defined for arbitrary values of $\psi \in \mathbb{R}_{>0} \setminus \{1\}$). We therefore follow Rudebusch and Swanson (2008) and assume that consumption does not fall below some subsistence level which guarantees positivity of the terms in brackets and therefore of u_t. Yet, we do not impose this additional restriction for the optimality conditions but only check afterwards whether the assumption is fulfilled by the obtained approximations.[65]

Following the explanations from section 4.5.1 for the baseline model, the Epstein-Zin specification for the household's preferences yields the same equilibrium conditions as with standard preferences only with an adjusted stochastic discount factor. The stochastic discount factor can be derived as follows. Let $(1-\beta)\lambda_t$ denote the Lagrange multiplier for the household's budget constraint, then, for instance, the first order condition for S_{t+1} becomes

$$\begin{aligned} (1-\beta)\lambda_t v_t = \beta \frac{1}{1-\alpha} & \left(\mathbb{E}_t \left[\left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{1-\alpha} \right] \right)^{\frac{1}{1-\alpha}-1} \cdot \\ & \cdot \mathbb{E}_t \left[(1-\alpha) \left(J^h(N_{t+1}, Z_{t+1}, S_{t+1}) \right)^{-\alpha} \frac{\partial J^h}{\partial S}(N_{t+1}, Z_{t+1}, S_{t+1}) \right] = \\ = \mathbb{E}_t & \left[\beta \left(\frac{J^h(N_{t+1}, Z_{t+1}, S_{t+1})}{\left(\mathbb{E}_t \left[(J^h(N_{t+1}, Z_{t+1}, S_{t+1}))^{1-\alpha} \right] \right)^{\frac{1}{1-\alpha}}} \right)^{-\alpha} \frac{\partial J^h}{\partial S}(N_{t+1}, Z_{t+1}, S_{t+1}) \right]. \end{aligned}$$

[65] More specifically, we check whether the condition is fulfilled on the domain the approximation was computed on, which is throughout broad enough to never be left in simulations.

Since the envelope theorem further yields

$$\frac{\partial J^h}{\partial S}(N_t, Z_t, S_t) = (1-\beta)\lambda_t(v_t + d_t)$$

the Euler condition for share prices now reads

$$v_t = \mathbb{E}_t\left[\beta\left(\frac{J^h(N_{t+1}, Z_{t+1}, S_{t+1})}{\left(\mathbb{E}_t\left[(J^h(N_{t+1}, Z_{t+1}, S_{t+1}))^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}}}\right)^{-\alpha}\frac{\lambda_{t+1}}{\lambda_t}(d_{t+1} + v_{t+1})\right].$$

Noting that the household's value function must be independent of the number of outstanding shares in equilibrium (neither total dividend payments less expenditures on shares, $d_t S_t - v_t(S_{t+1} - S_t) \underset{(4.2.12)}{=} \pi_t$, nor the firm value, $FV_t = S_{t+1}v_t = N_{t+1}\hat{\xi}^f_t$, depend on the amount of outstanding shares), the stochastic discount factor is hence defined by

$$M^{EZ}_{t,t+1} = \beta\left(\frac{J^h(N_{t+1}, Z_{t+1})}{\left(\mathbb{E}_t\left[(J^h(N_{t+1}, Z_{t+1}))^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}}}\right)^{-\alpha}\frac{\lambda_{t+1}}{\lambda_t},$$

where

$$\lambda_t = C_t^{-\frac{1}{\psi}}$$

for the model's variants without home production and

$$\lambda_t = aC_t^{-\frac{1}{\psi}}\left(\frac{C_t}{C_{m,t}}\right)^{1-e}$$

for the version with endogenous home production.[66][67]

Return on Equity For the model with leisure in the utility but fixed working hours the return on equity remains as in equation (4.5.3). Observe further that the only changes in $\hat{\xi}^f_t$ in equation (4.7.7) for variable working hours compared to (4.2.21) for fixed working hours is the fact that $\exp(Z_t)$ and w_t are replaced by $\exp(Z_t)\frac{h_t}{h}$ and $w_t\frac{h_t}{h}$, respectively. Hence, proceeding analogously as in (4.5.3) implies that the return on equity becomes

$$R^e_{t+1} = \frac{\exp(Z_{t+1})\frac{h_{t+1}}{h} - w_{t+1}\frac{h_{t+1}}{h} + (1-\omega)\hat{\xi}^f_{t+1}}{\hat{\xi}^f_t}$$

for the models from subsections 4.7.2, 4.7.3 and 4.7.4 as well as for their variations with Epstein-Zin preferences.

[66] Although the Lagrange multiplier on the household's budget constraint would be $(1-\beta)\lambda_t$, the factor $1-\beta$ only serves as scaling for the household's lifetime utility and therefore cancels out in all equilibrium conditions as well as in the stochastic discount factor.

[67] For numerical reasons we again use the following representation in the accompanying `Matlab` code

$$M^{EZ}_{t,t+1} = \left(\frac{J^h_{t+1}/J^h_t}{\left(\mathbb{E}_t\left[(J^h_{t+1}/J^h_t)^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}}}\right)^{-\alpha}\frac{\lambda_{t+1}}{\lambda_t}.$$

Risk Free Return We compute the risk free rate under standard or Epstein-Zin preferences, i.e.

$$R_t^f = \frac{1}{\mathbb{E}_t\left[\beta \frac{\lambda_{t+1}}{\lambda_t}\right]} \quad \text{or} \quad R_t^f = \frac{1}{\mathbb{E}_t\left[M_{t,t+1}^{EZ}\right]},$$

again by Gauss-Hermite quadrature with 13 nodes.

Equity-Premium The equity premium in the model is computed as the average excess return on equity beyond the risk free rate, $R_{t+1}^e - R_t^f$.

Results We summarize the average annualized return rates found in the model extensions in table 4.43. First and most importantly, even when Epstein-Zin preferences are assumed, neither

Table 4.43: Annualized Equity Premium in the Model Extensions

	Leisure in Utility	Endogenous Hours	Endogenous Hours and Search Intensity	Endogenous Hours and Home Production
		$\eta = 2$		
$((\bar{R}^f)^{12}-1)100\%$	3.98	4.05	4.06	4.06
$((\bar{R}^e)^{12}-1)100\%$	4.07	4.19	4.18	4.20
EP	0.09	0.13	0.11	0.13
		EZ Preferences (RRA=10, EIS=1.5)		
$((\bar{R}^f)^{12}-1)100\%$	4.04	4.02	4.02	4.02
$((\bar{R}^e)^{12}-1)100\%$	4.36	4.49	4.51	4.43
EP	0.31	0.46	0.46	0.40

Notes: $\bar{R}^f = \frac{1}{T-1}\sum_{t=1}^{T-1} R_t^f$=average monthly risk free return rate, $\bar{R}^e = \frac{1}{T-1} = \sum_{t=1}^{T-1} R_t^e$=average monthly return on equity, EP=$((1+\bar{R}^e-\bar{R}^f)^{12}-1)100\%$. All return rates are computed from the outcome of a simulation of 1200000 (monthly) periods.

of the extensions can generate a sizeable equity premium. Compared to table 4.8, the equity premia in the extensions turn out lower by an order of magnitude than what can be observed in the data. The risk free rate in the model extensions is almost four times as large as the empirical counterpart.

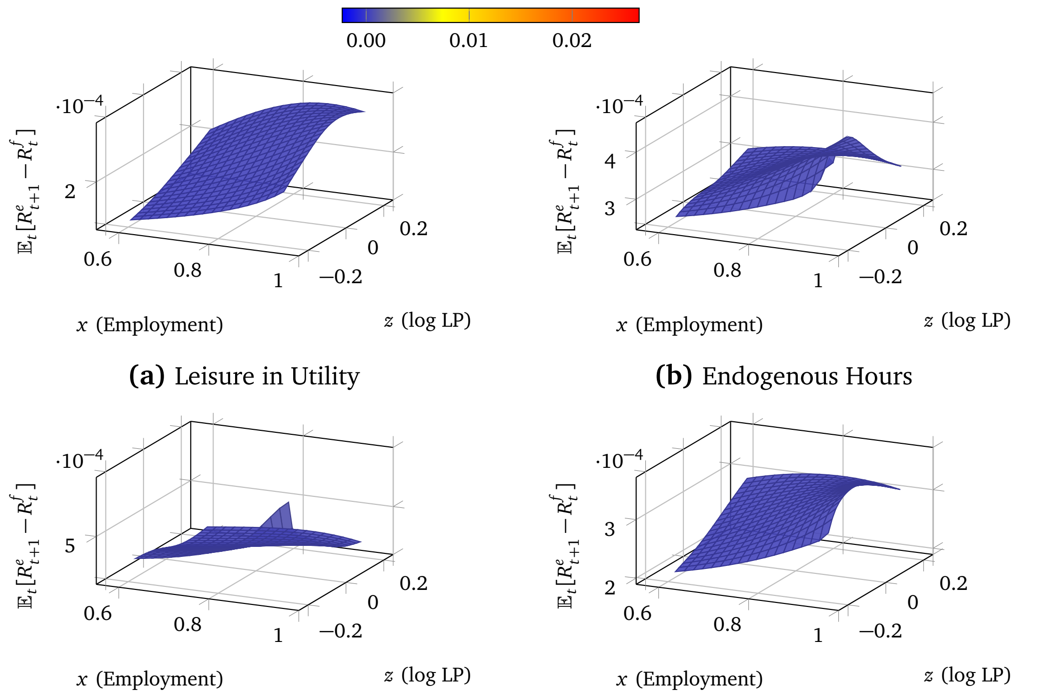
Figure 4.53 shows the monthly equity premium (as a decimal number) in the model's extensions with Epstein-Zin preferences. Compared to the baseline model in figure 4.15b, the equity premium turns out lower by an order of magnitude around the steady state. To provide some reasoning, table 4.44 additionally summarizes the variability in the lottery over the stochastic discount factor and over the return on equity when the economy is in the risky steady state. The baseline model, giving rise to extreme disasters, generated a highly risky lottery regarding

Table 4.44: Intuition Equity Premium: Model Extensions with Epstein-Zin Preferences

	$\mathbb{E}_t\left[M_{t,t+1}^{EZ}\right]$	$\sqrt{\mathbf{Var}\left[M_{t,t+1}^{EZ}\right]}$	$\sqrt{\mathbf{Var}\left[R_{t+1}^{e}\right]}$	$\mathbf{Corr}\left[M_{t,t+1}^{EZ}, R_{t+1}^{e}\right]$	$\mathbb{E}_t\left[R_{t+1}^e - R_t^f\right]$
Baseline	0.9991	0.3572	0.0250	-0.9519	0.85%
Leisure in Utility	0.9967	0.0171	0.0152	-0.9994	0.03%
Endogenous Hours	0.9967	0.0198	0.0196	-0.9992	0.04%
Endogenous Hours and Search Intensity	0.9967	0.0220	0.0179	-0.9992	0.04%
Endogenous Hours and Home Production	0.9967	0.0183	0.0183	-0.9994	0.03%

next period's lifetime utility and therefore a highly variable stochastic discount factor. Although disasters in the model extensions with Epstein-Zin preferences appear somewhat more frequently than in the discussed counterparts with standard preferences, they are substantially less frequent and less pronounced than in the baseline model. The standard deviation of the stochastic discount factor declines by a factor of approximately 30 and the model can no longer produce a considerable equity premium.

Figure 4.53: Monthly Equity Premium in the Model Extensions with Epstein-Zin Preferences



(a) Leisure in Utility

(b) Endogenous Hours

(c) Endogenous Hours and Search Intensity

(d) Endogenous Hours and Home Production

4.8 Conclusion

Employing the calibration strategy proposed by Hagedorn and Manovskii (2008) in a standard search an matching model significantly increases the volatility of the labor market compared to the results reported by Shimer (2005) for the standard calibration. However, the higher volatility of the labor market is primarily caused by infrequent but extreme declines in the employment rate as opposed to more frequent but modestly sized deviations. Consequently, in order to appropriately display the model's dynamics, globally accurate approximation methods must be used. A second order perturbation solution can not adequately replicate these results.

The fact that the model produces rare but severe economic downturns suggests that the model can potentially contribute to the strand of literature which explains a sizeable equity premium through the possibility of economic disasters. Yet, with standard additive time separable preferences this is not the case. Economic downturns may occur only gradually over a longer time span of decreasing employment, whereas uncertainty about consumption one period ahead is only very limited. Consequently, the volatility of the stochastic discount factor in the model remains too low to produce an equity premium of the empirically observed size. When Epstein-Zin preferences are introduced into the model, the equity premium rises substantially. In this case, the stochastic discount factor also includes the lottery over the household's lifetime utility in the next period which exhibits much higher risk. Dependent on the realization of the shock to labor productivity, the prospects about future consumption in the longer run may vary drastically. The volatility of the stochastic discount factor and therefore the equity premium in the model increase significantly.

We found the interaction of three factors essential for the occurrence of steep drops in the employment rate. First, a high and fixed period value of unemployment activities implies a high and fixed reservation wage a worker always demands. The wage is already close to labor productivity in steady state so that small fluctuations in productivity which are not likewise absorbed by adjustments of the wage imply high relative changes in the excess of labor productivity over the wage and consequently in the value of a worker to the firm. Second, the elasticities of open vacancies and job matches become higher the closer the present value of a worker to the firm falls to the costs of posting open vacancies. Third, during periods of rising unemployment the present value of a worker to the firm will decline even further due to stronger discounting of future profits, realized job matches react even more sensible and continue to fall even more, unemployment will rise further and so on until labor productivity sufficiently recovers.

The high and fixed period value of unemployment activities plays a particularly important role for the mechanism. A worker's reservation wage does not adjust indifferent from the fact how much unemployment rises and how low consumption falls. However, this changes once leisure is introduced into the utility function in such way that the marginal rate of substitution with the consumption good is not constant. During a recession with decreasing consumption and increasing aggregate free time, the value of unemployment from leisure measured in units of the consumption good for which a worker demands compensation declines. The wage can adjust more, the value of a worker to the firm does not fall as much and extreme declines in the employment rate do not occur anymore. The result maintains if working hours and job search effort are determined endogenously. Further, working hours in the model are increasing during a recession so that declines in output and consumption are dampened even more. Moreover, consumption of the market good is partially substituted with the home produced good once home production is added to the model.

Since neither of the model's extensions displays huge drops in the employment rate, the volatility of the labor market decreases substantially compared to the baseline model. Moreover, the extensions can no longer generate a sizeable equity premium indifferent from the fact whether standard preferences or Epstein-Zin preferences are assumed. The ability of the search and matching model calibrated in a fashion similar to Hagedorn and Manovskii (2008) to explain the volatility of the labor market as well as a sizeable equity premium in a plausible way critically hinges on one's view regarding the following two points: the credibility that disasters in the size and with the frequency observed for the baseline model may appear (and were also responsible for the volatility observed on the labor market) and the plausibility that high wage demands—as the main factor triggering the huge declines—persist during such extreme recessions. Such high and inflexible wage demands cannot be explained anymore in the framework of the model extensions but would need alternative justification. Note that even if the workers' reservation wage is understood to be always high and to not adjust according to the state of the economy because a constant MRS of h between leisure and consumption is assumed by specifying the within period utility to $\log(C_t + hU_t), h > 0$, where $b + h = 0.85$ as in the appendix of Kuehn et al. (2015), it is questionable if a sizeable equity premium can be reproduced even when using Epstein-Zin preferences. The constant MRS (if not set too low) implies that the effect of huge drops in the employment rate on the household's lifetime utility should be significantly dampened compared to the baseline model. Consequently, the volatility of the stochastic discount factor should drop considerably under Epstein-Zin preferences and the equity premium should be noticeably reduced.

Appendix

A Documentation of Computer Programs

We provide a short documentation for the accompanying `Matlab` programs used to derive the numerical results. The programs rely on the `Matlab` version of CoRRAM in order to compute the perturbation solution. We illustrate the routines for the baseline model contained in the subfolder `Benchmark`; the code for the remaining models is analogous.[68]

The script `Benchmark.m` replicates all the results described in the paper for the baseline model in the following way.

Computing/Loading Approximation First, the script computes a second order perturbation solution from the equilibrium conditions in (4.2.28)-(4.2.41) provided in `Benchmark_Eqs.m` by executing the routines from CoRRAM. It then loads the solution for the free parameters for each of the four global approximation approaches described in section 4.3. More specifically, the parameters for the Chebyshev-Galerkin method, which we characterized as the solution to the system of nonlinear equations (4.3.24), are determined as the fixed point of the input matrix `A` in `galSys2.m`. They are loaded from `lsg_benchmark_cheb2.mat` and checked. In the same way, the parameters for the Chebyshev-Galerkin method with discretized labor productivity from (4.3.28) are pinned down as the fixed point of the input vector `a` in `galSysDeg.m`. They are loaded from `lsg_benchmark_cheb1.mat`. For the finite element methods, the solutions for the free parameters from (4.3.29) and (4.3.30) are the zeros in the input variables `xinodes` and `xinodesvec` to `colFEMSysSplines2d.m` and `colFEMSysSplines.m`, respectively. The functional rhs_{GH} defined in (4.3.16) is computed by `getrhs2.m` for the spectral method and by `getrhsFEMSplines2d.m` for the finite element method. For discretized labor productivity the functional rhs_{RO} is evaluated in `getrhs.m` and `getrhsFEMSplines.m`.

Plots and Euler Residuals After loading and checking the free parameters for the different approximations, `Benchmark.m` reproduces the various plots, i.e. the approximated policy functions, the Euler residuals etc.

Second Moments The script invokes `Simulate_Benchmark_mom2.m` in order to compute the second moments from repeated simulations using the samples of shocks in `eps_array3.mat`. The results are written to `Momente.txt`.

Dynamics, Disaster Statistics and Return Rates Last, a long time path of equilibrium outcomes in the model is simulated from the sample of shocks in `eps_array2.mat` by calling the script `Simulate_Benchmark.m`. The results in tables 4.3, 4.4 and 4.5 can be found in

[68] The included `README` file specifies in which subfolder the respective programs for the other model variants can be found.

Vergleich_Dynamik.txt, the average return rates in Equity.txt and the disaster statistics in Desaster.txt.

For the models where additional intuition for the equity premium is provided, as displayed in figures 4.16 and 4.17 for the baseline model or in table 4.44 for the various extensions, the results are computed in IntPremium.m and saved to EquityVergleich.txt.

Intuition: Effects of Drop in LP The scripts expl1d_resp.m, expl1d_resp.m, expl2d_resp.m, comp_endutil.m, comp_endhours.m, comp_endsearch.m and comp_endhome.m contained in the subfolder Comparison&Intuition generate the plots pictured in figures 4.31, 4.33, 4.37, 4.42, 4.47 and 4.52.

Chapter 5

Conclusion

The thesis contains three essays commonly concerned with replicating the high equity premium found in the data. The first two papers are dedicated to the approach of altering the standard preference structure by using generalized recursive preferences of the class introduced by Epstein and Zin (1989). The third paper considers a standard search and matching labor market model that, as Kuehn et al. (2012, 2015) claim, can endogenously explain severe economic disasters and can therefore also help to replicate the equity premium in a fashion originally introduced by Rietz (1988).

In "Applied Macroeconomic Analysis with Epstein-Zin Utility" (2014a) we summarize the construction of EZ's generalized recursive utility adding illustrative examples. We outline the concept of temporal lotteries describing infinite probability trees over consumption and how this concept naturally leads to a recursive utility representation which allows the disentanglement of the RRA from the EIS. We show how EZ utility directly lends itself for dynamical programming. Further, we discuss the stochastic discount factor under EZ preferences and the influence of the RRA and the EIS on its volatility under a second order perturbation solution. We find that the stochastic discount factor shows a stronger negative comovement with the return on equity for smaller EIS given a fixed coefficient of RRA. The EZ representation offers a channel for the replication of the empirical equity premium: one may impose a strong enough aversion to nonsmooth consumption without having to set the risk aversion parameter unreasonably high. The paper is accompanied by a flexible `Maple-Matlab` perturbation toolbox.

The paper "Epstein-Zin Utility, Asset Prices, and the Business Cycle Revisited" (2014b) analyzes to what extent a number of currently prevalent building blocks of DSGE models are helpful in trying to replicate characteristic empirical figures of the German real economy within an EZ framework. We thereby target both, classical RBC statistics as well as asset pricing figures. We find that within the EZ utility representation, the frictionless model already yields simulation results in good accordance with the German empirical data. Amongst the considered labor market frictions, allowing for real wage stickiness leads to the most remarkable improvement in fit. Compared to the results reported in Heer and Maußner (2013) for standard preferences, the additional flexibility of the EZ framework seems to allow for significant improvements of the models' fit to empirical characteristics. The degree of additional flexibility, however, primarily hinges on the allowed magnitude of deviation from the standard case and therefore to the degree of deviation from non-indifference towards the timing of uncertainty resolution. We were very liberal in setting the EIS substantially lower than the reciprocal of the RRA. Although this is in accordance with empirical findings (see e.g. Hall (1988)), we note that the literature does not provide a consensus on the extent to which such non-indifference towards the timing of uncertainty resolution can be considered plausible.

Finally, in "Search Frictions in the Labor Market and Endogenous Economic Disasters" (2017) I study a standard search and matching labor market model calibrated in the fashion proposed by Hagedorn and Manovskii (2008) where the household's bargaining power is low but the period value from unemployment is high. Kuehn et al. (2012, 2015) argue that the model provides an endogenous mechanism leading to drastic economic downturns and can replicate a high equity premium as found in the data. The fact that the household's period value from unemployment activities is high and fixed implies that the workers' reservation wage does not adjust over the business cycle no matter how far the situation in the economy worsens. The assumption proves crucial for the mechanism leading to severe disasters in the model. Once the assumption is relaxed, e.g. by endogenously deriving the value of unemployment from leisure from a utility function with a non constant marginal rate of substitution between leisure and consumption, the dynamics of the model change substantially. Most importantly, the equity premium in the model is again by an order of magnitude lower than in the data. Moreover, the standard deviations from simulations of the labor market model also fall significantly below the empirically observed values.

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