THE ISOTROPY REPRESENTATIONS OF THE ROSENFELD PLANES

Dissertation

zur Erlangung des akademischen Grades
Dr. rer. nat.

eingereicht an der
Mathematisch-Naturwissenschaftlich-Technischen Fakultät
der Universität Augsburg

von

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Augsburg, Februar 2018

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Tag der mündlichen Prüfung: 20. 2. 2018

Dedicated to Jost-Hinrich Eschenburg

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Introduction

In addition to the infinite series of the classical symmetric spaces there are twelve of exceptional type. Among these twelve are four with the dimensions 16, 32, 64 and 128 which are called *Rosenfeld planes* after an idea from B. Rosenfeld (see [Fr2], [Be], [Ba]). The space with dimension 16 is the well known Cayley plane, a projective plane over the octonions. As a homogeneous symmetric space it is defined as $\mathbb{OP}^2 = F_4/Spin_9$. The 52-dimensional exceptional Lie group $F_4 = \mathrm{Aut}(H_3(\mathbb{O}))$ is the automorphism group of the Jordan algebra $H_3(\mathbb{O})$ (the hermitian 3×3 matrices over the octonions together with the Jordan product $A \circ B = \frac{1}{2}(AB + BA)$). $\mathbb{OP}^2$ can also be defined as the set $\{P \in H_3(\mathbb{O}) : P^2 = P, \text{trace } P = 1\}$. With this definition $\mathbb{OP}^2$ becomes an extrinsic symmetric space (or symmetric R-space) with equivariant isometric embedding in the 26-dimensional representation module $H_3(\mathbb{O})$ with fixed trace ($H_3(\mathbb{O})$ has dimension $27 = 3 \cdot 8 + 3$ and can be reduced by one dimension since the trace is an invariant of the automorphism group F_4, see [Fr1], [Es2]).

Rosenfeld's idea was to construct the other three spaces as projective planes over the algebras $\mathbb{O}\otimes\mathbb{C}$, $\mathbb{O}\otimes\mathbb{H}$ ($\mathbb{H}$ denoting the quaternions) and $\mathbb{O}\otimes\mathbb{O}$. But this does not work out properly. Still the idea seems to be partially correct, in a vague sense. Motivated by it we use the following notations for these symmetric spaces with the dimensions 32, 64 and 128: $(\mathbb{O}\otimes\mathbb{C})\mathbb{P}^2 := E_6/(Spin_{10} \cdot U_1)$, $(\mathbb{O}\otimes\mathbb{H})\mathbb{P}^2 := E_7/(Spin_{12} \cdot Sp_1)$, $(\mathbb{O}\otimes\mathbb{O})\mathbb{P}^2 := E_8/Spin_{16}$ with the exceptional Lie groups E_6, E_7, E_8 (with dimensions 78, 133 and 248) and the unitary resp. symplectic groups U_1, Sp_1.

If we would try to construct $(\mathbb{O}\otimes\mathbb{C})\mathbb{P}^2$ in a similar way as $\mathbb{OP}^2$ above (as a subset of $H_3(\mathbb{O}\otimes\mathbb{C})$), then the following problem arises: Every element in $\mathbb{O}\otimes\mathbb{C}$ can be written uniquely as $x+y\otimes i$ with $x, y \in \mathbb{O}$. First we have to define a conjugation for $\mathbb{O}\otimes\mathbb{C}$ for which the only reasonable choice is to claim the condition $\overline{y\otimes i} = \overline{y}\otimes\overline{i}$. Then the space of self-conjugated elements $\overline{x+y\otimes i} = \overline{x} + \overline{y\otimes i} = \overline{x} - \overline{y}\otimes i = x + y\otimes i$ has dimension 8 (since x is real and y an imaginary octonion in this case) and then $H_3(\mathbb{O}\otimes\mathbb{C})$ has dimension $72 = 3 \cdot 16 + 3 \cdot 8$. But it is known from representation theory that the fundamental irreducible representation modules for E_6 have dimensions 27, 78, 351 and 2925 (see [Ti]). Therefore there is no such embedding in the space of the hermitian matrices over $\mathbb{O}\otimes\mathbb{C}$. For the remaining two Rosenfeld planes we have the same difficulties. And in addition these two spaces are not even symmetric R-spaces after the results from Ferus (cf. [Fe] or also [BCO]). Since $(\mathbb{O}\otimes\mathbb{C})\mathbb{P}^2$ is hermitian symmetric, it is an extrinsic symmetric space and can be embedded in the Lie algebra $\mathfrak{e}_6$ of its own isometry group with the adjoint representation.

In the following approach an elementary construction of the Rosenfeld planes is given at the level of their Lie algebras. This construction is based on simple properties of the four division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$. The starting point is the known representation for $Spin_9$ operating on $\mathbb{O}^2$ (cf. [Es2], [Ha]), derived from a representation for the Clifford algebra Cl_8. Then generators for $Spin_{10}, Spin_{12}$ and $Spin_{16}$ can be found acting on $(\mathbb{O} \otimes \mathbb{C})^2$, $(\mathbb{O} \otimes \mathbb{H})^2$ and $(\mathbb{O} \otimes \mathbb{O})^2$ in a natural extension of $Spin_9$. These special representation modules have exactly the same dimensions as the Rosenfeld planes. Generally, all groups $Spin_{k+l}$ can be represented on $(\mathbb{K} \otimes \mathbb{L})^2$ where $k = \dim \mathbb{K}$ and $l = \dim \mathbb{L}$. In the special case $\mathbb{K} = \mathbb{O}$ we get the isotropy groups of the Rosenfeld planes and for $\mathbb{L} = \mathbb{R}$ the ones for the classical projective planes $\mathbb{KP}^2$ (together with additional factors U_1 or Sp_1). Altogether we get ten spaces in this way which all can be viewed as generalized Rosenfeld planes $(\mathbb{K} \otimes \mathbb{L})\mathbb{P}^2$ in a certain sense. Furthermore each of these ten spaces has a symmetric subspace with half the dimension and the same maximal abelian subalgebra. This is the space with the Lie triple system $(\mathbb{K} \otimes \mathbb{L})^1 \subset (\mathbb{K} \otimes \mathbb{L})^2$ and called *Rosenfeld line* $(\mathbb{K} \otimes \mathbb{L})\mathbb{P}^1$ (as a generalization of a projective line). Up to coverings, it is the real Grassmann manifold $G_l(\mathbb{R}^{k+l})$. However, only for the proper projective planes any two Rosenfeld lines intersect transversally. In the other cases (i.e. for $\mathbb{K} \otimes \mathbb{L}$ with both $\mathbb{K}, \mathbb{L} \neq \mathbb{R}$) the Rosenfeld planes and lines share a maximal torus of rank > 1, and thus the intersection of two lines may contain a (singular) subtorus (see also [N] - [NT-V]).

Next, we use the definitions given in [EH1] (which goes back to [Wi]) to construct the Lie algebras and Cartan decompositions. In the associative cases it can be seen that the given representations for the spin groups are actually the isotropy representations of symmetric spaces. Since we can show the validity of the Jacobi identity, this is also true in the exceptional cases. Using a maximal abelian Lie subalgebra, the root space decompositions can be determined and so the root systems of the corresponding symmetric spaces can be seen.

In this context we should mention the calculations from J. F. Adams and E. Witt. Adams ([Ad]) constructs the Lie algebra $\mathfrak{e}_8$ in a similar way (as the Cartan decomposition), where the Jacobi identity is proved directly with methods from representation theory, but without using concrete representations for spin groups. Furthermore the Lie group E_8 is constructed as the automorphism group of its own Lie algebra: $E_8 = \mathrm{Aut}(\mathfrak{e}_8)$. Then the other exceptional Lie groups can be found as subgroups of E_8.

Witt ([Wi]) uses a more direct and elementary approach. He defines the Lie algebra (called *Liescher Ring* in his work) over orthogonal skew symmetric matrices with the help of the properties of a Clifford algebra (called *Cliffordsches Zahlsystem*). The construction principle for this Lie algebra is essentially the same as that used in [Ad] and [EH1]. Then Witt develops a simple formula (using the trace of the representation matrices) which makes it possible to decide whether the constructed system satisfies the Jacobi identity or not.

But Adams and Witt are mainly interested in the exceptional Lie groups (resp. their Lie algebras). In contrast to them, the focus of the present work lies on the investigation of the Rosenfeld planes as symmetric spaces with the explicit construction of their isotropy representations. The key objects for all these calculations are tensor products of the division algebras. With them we can avoid long and involved computations (as used by Adams and Witt).

Here a short overview of the individual sections: Section 1 lists some elementary properties of the division algebras and their tensor products. In Sections 2 and 3 the representations for certain Clifford algebras (associated with the division algebras) and their corresponding spin groups are constructed. Section 4 determines the Lie Algebras of these groups. Section 5 introduces the Rosenfeld planes as symmetric spaces and their isotropy groups. Section 6 proves that in the classical cases the representations derived from the spin groups actually coincide with the correct isotropy representations of these spaces. Section 7 describes the Rosenfeld lines together with their maximal abelian Lie subalgebras and root space decompositions. Section 8 gives a construction of the Lie algebras for the exceptional spaces and proves the Jacobi identity with the help of the classical cases. In Section 9 the root space decompositions for all Rosenfeld planes is computed, and (together with the Appendix) the correspondence with the standard root systems is shown. Section 10 is a summary of the preceding sections.

Finally, at this point I would like to express many thanks to my supervisor Jost-Hinrich Eschenburg for his extensive support, discussions and ideas over the whole time. Also I would like to thank Peter Quast for his work as second advisor.

1. Division Algebras and Tensor Products

There are four normed division algebras $\mathbb{K}$, which are the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$. As (real) vector spaces they have dimensions $\dim \mathbb{R} = 1, \dim \mathbb{C} =$

$2, \dim \mathbb{H} = 4$ and $\dim \mathbb{O} = 8$ (where dim is always to be understood as the real dimension $\dim_{\mathbb{R}}$ if not otherwise specified). With a notation adopted from Rosenfeld (cf. [Ro]) and an order relation, which will be used for later considerations, $\mathbb{O}$ has the standard basis elements $1 < i < j < k < l < p < q < r$ with the additional relations

$$k = ij,\ p = il,\ q = jl,\ r = kl. \tag{1.1}$$

Let $m_{\mathbb{K}}$ be the maximal basis element in $\mathbb{K}$, that is $m_{\mathbb{R}} = 1, m_{\mathbb{C}} = i, m_{\mathbb{H}} = k, m_{\mathbb{O}} = r$. Then the standard basis for $\mathbb{K}$ is given by

$$B_{\mathbb{K}} = \{1, \dots, m_{\mathbb{K}}\}. \tag{1.2}$$

For two division algebras $\mathbb{K}, \mathbb{L}$ with $\mathbb{K} \supset \mathbb{L}$ the tensor product $\mathbb{K} \otimes \mathbb{L}$ (with the product $(a \otimes b)(c \otimes d) = ac \otimes bd$) is also an algebra ($\otimes$ is always to be interpreted as the tensor product $\otimes_{\mathbb{R}}$ over $\mathbb{R}$ if not otherwise specified). Generally this is not a division algebra except for the case $\mathbb{K} \otimes \mathbb{R}$ which is isomorphic to $\mathbb{K}$. An element $x \otimes 1$ will be abbreviated as x and $1 \otimes x$ as $\hat{x}$. The inner product for a division algebra (defined as $\langle x, y \rangle := \mathrm{Re}\,(\bar{x}y)$) can be extended to $\mathbb{K} \otimes \mathbb{L}$ with $x, y \in \mathbb{K}$ and $x', y' \in \mathbb{L}$:

$$\langle x\hat{x'}, y\hat{y'} \rangle := \langle x, y \rangle \cdot \langle x', y' \rangle. \tag{1.3}$$

This corresponds to the ordinary Euclidean inner product in $\mathbb{R}^n$ ($n = \dim \mathbb{K} \cdot \dim \mathbb{L}$) by identifying the canonical basis in $\mathbb{R}^n$ with the basis $B_{\mathbb{K}\otimes\mathbb{L}} = \{e\hat{f} : e \in B_{\mathbb{K}}, f \in B_{\mathbb{L}}\}$ in $\mathbb{K} \otimes \mathbb{L}$.

We can also define a *conjugation* for $\mathbb{K} \otimes \mathbb{L}$ by setting

$$\overline{x\hat{y}} := \bar{x}\hat{\bar{y}} \tag{1.4}$$

for $x \in \mathbb{K}, y \in \mathbb{L}$.

In the direct sum $(\mathbb{K} \otimes \mathbb{L})^2$ we use the standard vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.5}$$

For $u \in \mathbb{K}, v \in \mathbb{L}, k = \dim \mathbb{K}, l = \dim \mathbb{L}$, the left translation can be extended to the tensor product:

$$L_{u\hat{v}}(x\hat{y}) := ux \cdot \hat{v}\hat{y}$$

(which corresponds to a real $kl \times kl$ matrix). The right translation $R_{u\hat{v}}$ is defined accordingly. For the representations of the Clifford algebras and spin groups we will need matrices in block form containing these left and right translations. In this case we write simply $u\hat{v}$ for $L_{u\hat{v}}$ and $R(u\hat{v})$ for $R_{u\hat{v}}$ (since we will use almost only left translations, the right translations are especially distinguished). If more than one translation is used in the (nonassociative) octonionic case and $L_e L_f$ abbreviated as ef, one has to apply them in the appropriate order: $L_e(L_f x) = (L_e L_f)x$ as real matrices, but generally $L_e L_f \neq L_{ef}$, $e(fx) \neq (ef)x$ for $e, f, x \in \mathbb{O}$.

We note a few known elementary properties for division algebras which will play a key role for all later considerations ([Es4], Section 10, p. 111). Set $\mathbb{K}' := \operatorname{Im} \mathbb{K} := 1^\perp$ and let $\tilde{\mathbb{H}} \subset \mathbb{K}$ be a quaternionic subalgebra (i.e. a subalgebra isomorphic to the standard quaternions $\mathbb{H}$).

$$a \in \mathbb{K}', |a| = 1, x \in \mathbb{K} \Rightarrow a(ax) = -x. \tag{1.6}$$

$$b \in \mathbb{K}', x \in \mathbb{K} \Rightarrow b(bx) = -|b|^2 x, b^2 = -|b|^2. \tag{1.7}$$

$$a, b \in \mathbb{K}', a \perp b \Rightarrow ab \in \mathbb{K}', ab \perp a, b. \tag{1.8}$$

$$a, b \in \mathbb{K}', a \perp b \Rightarrow ab = -ba \ (\textit{anticommutativity}). \tag{1.9}$$

$$\text{Two orthonormal } a, b \in \mathbb{K}' \text{ generate } \tilde{\mathbb{H}}. \tag{1.10}$$

$$c \perp \tilde{\mathbb{H}} \Rightarrow c\tilde{\mathbb{H}} = \tilde{\mathbb{H}}c \perp \tilde{\mathbb{H}}. \tag{1.11}$$

$$a, b, c \in \mathbb{K}' \text{ orthogonal in pairs and } c \perp ab \Rightarrow (ab)c = -a(bc). \tag{1.12}$$

This *antiassociative* triple (a, b, c) is also called a *Cayley triple* if in addition a, b, c are unit vectors. Now let $x, y, z \in B_{\mathbb{K}}$ and set

$$s_x^y := \begin{cases} 1 & \text{if } x \text{ and } y \text{ commute,} \\ -1 & \text{else.} \end{cases} \tag{1.13}$$

Then

$$x(yz) = \pm(xy)z \text{ with - if it is a Cayley triple and + if not.} \tag{1.14}$$

$$xy = s_x^y yx. \tag{1.15}$$

$$x(yz) = s_x^y y(xz). \tag{1.16}$$

$$(xy)z = s_y^z (xz)y. \tag{1.17}$$

$$L_{x\hat{x}} L_{y\hat{y}} z\hat{z} = L_{y\hat{y}} L_{x\hat{x}} z\hat{z} = L_{xy \cdot \hat{x}\hat{y}} z\hat{z} = L_{yx \cdot \hat{y}\hat{x}} z\hat{z}. \tag{1.18}$$

$$R_{x\hat{x}} R_{y\hat{y}} z\hat{z} = R_{y\hat{y}} R_{x\hat{x}} z\hat{z} = R_{xy \cdot \hat{x}\hat{y}} z\hat{z} = R_{yx \cdot \hat{y}\hat{x}} z\hat{z}. \tag{1.19}$$

Proof. (1.6): $(1 + a)((1 - a)x) = (1 - a)x + a(x - ax) = x - a(ax)$ and $|1 + a||1 - a||x| = 2|x|$. Because of $|a(ax)| = |x|$, the equation $|x - a(ax)| = 2|x|$ can only be valid if the vectors x and $-a(ax)$ have the same direction, that is $a(ax) = -x$.

(1.7): Apply (1.6) to $a = b/|b|$ for $b \neq 0$.

(1.8): Assume $|a| = 1$. We have the implications $b \perp a \Rightarrow ab \perp a^2 = -1$, $b \perp 1 \Rightarrow ab \perp a$ and $a \perp 1 \Rightarrow ab \perp b$.

(1.9): Assume $|a| = |b| = 1$. Then $(a + b)^2 = a^2 + b^2 + ab + ba = -2 + (ab + ba)$ and also $(a + b)^2 = -|a + b|^2 = -2$ from (1.7). It follows $ab + ba = 0$.

(1.10): The basis vectors i, j, k in the standard algebra $\mathbb{H}$ correspond to a, b, ab, since $(ab)^2 = -1$ and $(ab)a = -a(ab) = b$ from (1.8) and (1.9).

(1.11): $a, b \in \tilde{\mathbb{H}} \cap \mathbb{K}'$ and $|a| = 1 \Rightarrow ca \perp b$ since $(ca)a = -c \perp ba$.

(1.12): From (1.11) we have $bc \perp a$ and therefore $a(bc) = -(bc)a = (cb)a$. We have to show $(ab)c = -(cb)a$. It is

$$\begin{aligned} ((a+c)b)(a+c) &= (ab+cb)(a+c) \\ &= (ab)a + (cb)a + (ab)c + (cb)c \\ &= 2b + (cb)a + (ab)c, \end{aligned}$$

but also $((a+c)b)(a+c) = |a+c|^2 b = 2b$. It follows $(cb)a + (ab)c = 0$.

(1.14): This is a consequence of the antiassociativity (1.12) and the associativity in (1.10) or (1.6).

(1.15): This is a consequence of the anticommutativity (1.9).

(1.16): The case $s_x^y = 1$ is clear. Else if (x, y, z) is a Cayley triple then $x(yz) = -(xy)z = (yx)z = -y(xz)$. If it is not a Cayley triple then x, y, z are in a common associative subalgebra: $x(yz) = (xy)z = -(yx)z = -y(xz)$. (1.17) is analogous.

(1.18) and (1.19) follow from (1.14), (1.15), (1.16) and (1.17) together with the symmetry in the factors of the tensor product $\mathbb{K} \otimes \mathbb{K}$. □

2. Representations for Cl_{k+l-1}

At first we give some definitions (see [Es1] or [Ha]). The *Clifford algebra* Cl_n over $\mathbb{R}^n$ is an associative algebra with identity 1, generated by the canonical basis $e_1, \ldots, e_n$ in $\mathbb{R}^n$, and satisfying the *Clifford relations* $e_i e_j + e_j e_i = -2\delta_{ij} \cdot 1$. If Cl_n^+ (resp. Cl_n^-) denotes the part with the products constructed from an even (resp. odd) number of factors, then Cl_n splits into a direct sum of the two parts: $Cl_n = Cl_n^+ + Cl_n^-$.

The *spin group* $Spin_n$ is embedded in Cl_n^+: $Spin_n := \{v_1 \cdots v_{2r} : v_i \in \mathbb{S}^{n-1}\}$. $Spin_n$ is a twofold covering of SO_n which is universal for $n > 2$.

A *representation* for Cl_n is given by an algebra homomorphism $\rho : Cl_n \to \mathbb{R}^{p \times p}$ and from this we get a representation for the corresponding spin group.

For the division algebras $\mathbb{K}$ and $\mathbb{L}$ with $k = \dim \mathbb{K}$, $l = \dim \mathbb{L}$, we can construct a $2kl \times 2kl$ matrix representation for Cl_{k+l-1} (operating on

$(\mathbb{K} \otimes \mathbb{L})^2)$ with the following basis elements:

$$E_e = \begin{pmatrix} & -\bar{e} \\ e & \end{pmatrix} (1 \leq e \leq m_{\mathbb{K}}), E_{\hat{f}} = \begin{pmatrix} -\hat{f} & \\ & \hat{f} \end{pmatrix} (1 < f \leq m_{\mathbb{L}}). \quad (2.1)$$

With the imaginary part $\mathrm{Im}\, \mathbb{L} = 1^{\perp}$ we can write (where the subscript at the right bracket denotes that the elements generate an algebra)

$$Cl_{k+l-1} = \left\langle \begin{pmatrix} -\hat{w} & -\bar{u} \\ u & \hat{w} \end{pmatrix} : \ (u, w) \in \mathbb{K} \times \mathrm{Im}\, \mathbb{L} \right\rangle_{alg}. \quad (2.2)$$

Here we use the same notation Cl_n for the image $\rho(Cl_n)$ of the corresponding representation ρ.

The Clifford relations are valid since E_e and $E_{\hat{f}}$ anticommute. The representations for Cl_{8+4-1} and Cl_{8+8-1} are not faithful, we have the additional relations $E_1 \cdots E_r {\cdot} E_{\hat{i}} \cdots E_{\hat{k}} = I$ resp. $E_1 \cdots E_r {\cdot} E_{\hat{i}} \cdots E_{\hat{r}} = I$ in these cases.

Except for the cases $(k, l) = (2, 2), (4, 2), (4, 4)$, the representations in (2.2) correspond to the standard representations for the Clifford algebras.

The single basis element $\left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$ in Cl_1 is a complex structure (i.e. $J^2 = -1$) commuting with itself, and therefore the representation of Cl_1 is complex linear. For Cl_{2+1-1} and Cl_{2+2-1} the elements in (2.1) commute with $J = \left(\begin{smallmatrix} i & \\ & i \end{smallmatrix}\right)$ and $K = \left(\begin{smallmatrix} & \kappa \\ -\kappa & \end{smallmatrix}\right)$ (with the real 2×2 matrix $\kappa = \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)$ corresponding to the complex conjugation). J and K form a quaternionic structure (two complex structures anticommuting with each other), and therefore these representations are quaternionic linear. Cl_{4+1-1}, Cl_{4+2-1} and Cl_{4+4-1} have the quaternionic structure $J = \begin{pmatrix} R(j) & \\ & R(j) \end{pmatrix}, K = \begin{pmatrix} R(k) & \\ & R(k) \end{pmatrix}$. In this case it is essential to use *right* translations which commute with the left translations. Cl_{8+2-1} is complex linear with $J = \begin{pmatrix} \hat{i} & \\ & \hat{i} \end{pmatrix}$, and Cl_{8+4-1} is quaternionic linear with $J = \begin{pmatrix} R(\hat{j}) & \\ & R(\hat{j}) \end{pmatrix}, K = \begin{pmatrix} R(\hat{k}) & \\ & R(\hat{k}) \end{pmatrix}$.

3. Representations for $Spin_{k+l}$

The Clifford representations in Section 2 can now be used to construct representations for the spin groups. To reach this, one can proceed similarly as in [Es2], Section 11. The generating element used there is $v\omega$ ($\omega = e_1 \cdots e_n$ is the volume element and $v \in \mathbb{S}^{n-1}$). It can be used only if n is odd. But with [Ha], p. 198, it is possible to generate $Spin_n$ by ev in all cases ($e, v \in \mathbb{S}^{n-1}$ and e fixed).

Theorem 3.1. *A representation for $Spin_{k+l}$ ($k = \dim \mathbb{K}, l = \dim \mathbb{L}$) is generated by the orthogonal matrices (where the subscript at the right bracket denotes that the elements generate a group)*

$$Spin_{k+l} = \left\langle \begin{pmatrix} \hat{w} & -\bar{u} \\ u & \hat{w} \end{pmatrix} : \ (u, w) \in \mathbb{S}^{k+l-1} \subset \mathbb{K} \times \mathbb{L} \right\rangle_{group} . \tag{3.1}$$

Here we use the same notation $Spin_n$ for the image $\rho(Spin_n)$ of the corresponding representation ρ.

Proof. We set $e = e_n$ for the generator ev above and use the algebra isomorphism $j : Cl_{n-1} \to Cl_n^+$, $v \mapsto e_n v$ ($v \in \mathbb{R}^{n-1} = \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$), $1 \mapsto 1$. Since $ve_nwe_n = -ve_ne_nw = vw$ for $v, w \in \mathbb{R}^{n-1}$, j preserves the multiplication. With j, every even product (a product with an even number of factors) is mapped onto itself, and every odd product is appended by e_n. Then j maps the vectors $e_1, \dots, e_{n-1}, 1$ to $e_ne_1, \dots, e_ne_{n-1}, 1 \in Spin_n$. Every vector $v \in \mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$ can be written as $v = \alpha_1 e_1 + \cdots \alpha_{n-1}e_{n-1} + \alpha_n \cdot 1$ with $\alpha_1^2 + \cdots + \alpha_n^2 = 1$. Now for $n = k+l$ we identify the vectors $e_1, \dots, e_{n-1}$ with the basis matrices in (2.1) and apply the reverse isomorphism j^{-1}. Then v corresponds to a matrix in (2.2) added with a real multiple of the identity matrix (representing the element 1). The resulting matrix is (3.1). $\square$

Except for the cases $(k, l) = (2, 2), (4, 2), (4, 4)$ (which will be explained in Section 6), we get the standard *spin representations*. Since the representations of Cl_{11} and Cl_{15} are not faithful, only the half spin representations are obtained for $Spin_{12}$ resp. $Spin_{16}$ with the group $\mathbb{Z}_2 = \langle \omega \rangle$ as kernel.

Furthermore there are the following known isomorphisms between low dimensional spin groups and classical groups (see [Ha], Chapter 14):

$$\left.\begin{array}{rcl} Spin_2 &=& SO_2 = U_1, \\ Spin_3 &=& SU_2 = Sp_1, \\ Spin_4 &=& SU_2 \times SU_2, \\ Spin_5 &=& Sp_2, \\ Spin_6 &=& SU_4. \end{array}\right\} \tag{3.2}$$

4. The Lie Algebras of the Spin Groups

By choice of suitable curves g_t with $g_0 = 1$, basis elements for the Lie algebra for the above representations for the spin groups can be computed as follows:

The generating sphere $\mathbb{S}^{k+l-1} \subset \mathbb{K} \times \mathbb{L}$ of $Spin_{k+l}$ contains the $k+l-1$ great circles $(u_t, w_t) = (e \sin t, \cos t)$ and $(u_t, w_t) = (0, \cos t + f \sin t)$

(with $e \in B_{\mathbb{K}}, f \in \operatorname{Im} B_{\mathbb{L}}$) which intersect each other perpendicularly at the point $(0,1)$ for $t = 0$. Inserting in (3.1) and differentiating at $t = 0$ yields the following $k + l - 1$ elements of the Lie algebra $\mathfrak{spin}_{k+l} = L(Spin_{k+l})$:

$$\begin{pmatrix} & -\bar{e} \\ e & \end{pmatrix}, \begin{pmatrix} -\hat{f} & \\ & \hat{f} \end{pmatrix}. \tag{4.1}$$

These are exactly the generators of Cl_{k+l-1} in (2.1).

The matrices in (4.1) anticommute with each other according to the anticommutativity of the left translations ($L_e L_f + L_f L_e = 0$ for $e, f \in \operatorname{Im} B_{\mathbb{O}}$ and $e < f$). Applying the Lie brackets $[A, B] = AB - BA = 2AB$ to these matrices, the remaining $\binom{k+l-1}{2}$ basis elements of $\mathfrak{spin}_{k+l}$ are obtained (with the factor 2 omitted):

$$\left.\begin{array}{ll} \begin{pmatrix} ef & \\ & -\bar{e}f \end{pmatrix} & (1 \le e < f \le m_{\mathbb{K}}), \\ \begin{pmatrix} & e\hat{f} \\ \bar{e}\hat{f} & \end{pmatrix} & (1 \le e \le m_{\mathbb{K}}, 1 < f \le m_{\mathbb{L}}), \\ \begin{pmatrix} \hat{e}\hat{f} & \\ & \hat{e}\hat{f} \end{pmatrix} & (1 < e < f \le m_{\mathbb{L}}). \end{array}\right\} \tag{4.2}$$

If we change the sign of the matrix $\left(\begin{smallmatrix} & -I \\ I & \end{smallmatrix}\right)$ in (4.1), we can combine (4.1) and (4.2) and have only three classes of $\binom{k}{2} + kl + \binom{l}{2} = \binom{k+l}{2}$ basis matrices:

$$\left.\begin{array}{lcll} A_{ef} & = & \begin{pmatrix} ef & \\ & -\bar{e}f \end{pmatrix} & (1 \le e < f \le m_{\mathbb{K}}), \\ B_{ef} & = & \begin{pmatrix} & e\hat{f} \\ -\bar{e}\hat{\hat{f}} & \end{pmatrix} & (1 \le e \le m_{\mathbb{K}}, 1 \le f \le m_{\mathbb{L}}), \\ C_{ef} & = & \begin{pmatrix} -\hat{\bar{e}}\hat{f} & \\ & \hat{e}\hat{f} \end{pmatrix} & (1 \le e < f \le m_{\mathbb{L}}). \end{array}\right\} \tag{4.3}$$

Lemma 4.1. *The matrices* (4.3) *form an orthogonal basis of* $\mathfrak{spin}_{k+l}$ *with respect to the trace metric* $\langle X, Y \rangle = \operatorname{trace} X^t Y$.

Proof. The trace metric can be computed as

$$\operatorname{trace} X^t Y = \sum \langle X^t Y \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right), \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \rangle \tag{4.4}$$

where the sum runs over all basis vectors $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in B_{(\mathbb{K}\otimes\mathbb{L})^2}$.

Since $\mathfrak{spin}_{n+1} = \mathfrak{so}_{n+1}$ is a simple Lie algebra (except for $n = 3$), there is only one Ad-invariant metric (up to a scaling factor). Now $\mathfrak{so}_{n+1} = s + [s, s]$ with $s = \{V = \left(\begin{smallmatrix} & -v^t \\ v & \end{smallmatrix}\right) : v \in \mathbb{R}^n\}$. The matrices

E_i and $[E_i, E_j]$ for the standard basis e_i in $\mathbb{R}^n$ are orthogonal. The matrices in (4.1) correspond to the E_i and are also orthogonal with respect to the metric (4.4) and the assertion follows.

□

5. The Rosenfeld Planes

As we will see, the groups $Spin_{k+l}$ are essentially (up to a factor which will be explained presently) the isotropy groups K of the symmetric spaces G/K with rank l and Cartan decomposition $\mathfrak{g} = \mathfrak{k} + V$ where $\mathfrak{g}, \mathfrak{k}$ are the Lie algebras $\mathfrak{g} = L(G), \mathfrak{k} = L(K)$ and the representation module $V = (\mathbb{K} \otimes \mathbb{L})^2$ for K. These spaces will be defined as the *(generalized) Rosenfeld planes* $(\mathbb{K} \otimes \mathbb{L})\mathbb{P}^2$. The precise isotropy groups are $K = Spin_{k+l} \cdot S_{k+l}$ (the dot denoting the usual matrix product) where the S_{k+l} are defined as

$$S_{k+l} := \{R_{u\hat{v}} : |u| = |v| = 1, u \in \mathbb{K}, v \in \mathbb{L}\} \tag{5.1}$$

but with $u = 1$ if $k = 1$ or $k = 8$, and $v = 1$ if $l = 1$ or $l = 8$.

Depending on k and l, S_{k+l} is a representation of a product of one or two groups of type U_1 or Sp_1, and it commutes with $Spin_{k+l}$ (which can be achieved by using *right* instead of *left* multiplication). As matrix groups they have a finite intersection. That means, the (surjective) homomorphism $Spin_{k+l} \times S_{k+l} \to Spin_{k+l} \cdot S_{k+l}$ has a finite kernel (and therefore their Lie algebras are isomorphic).

Lemma 5.1. *$Spin_{k+l} \cdot S_{k+l}$ is contained in $Spin_{16}$.*

Proof. The inclusion $Spin_{k+l} \subset Spin_{16}$ is clear. It suffices to show $Sp_1 \subset Spin_{16}$ ($\hat{Sp}_1 \subset Spin_{16}$ follows similarly). We show that an element from Sp_1 can be represented by a combination of two generators from $Spin_9$:

$$\begin{pmatrix} R(h) & \\ & R(h) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} & -l \\ -l & \end{pmatrix} \begin{pmatrix} & \bar{h}l \\ lh & \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where l is the standard basis element from $B_{\mathbb{O}}$ and $x, y, h \in \mathbb{H}$ with $|h| = 1$. This means $xh = -l((lh)x)$ (resp. the equivalent equation $yh = -l((\bar{h}l)y)$ since $lh = \bar{h}l$). By linearity it is sufficient to prove this for $x, h \in B_{\mathbb{H}}$: The cases $x = 1$ or $h = 1$ are clear. For $h = x \neq 1$ we have $-l((lh)x) = -l(l(hx)) = -1 = xh$. Else (l, h, x) is a Cayley triple and $-l((lh)x) = l(l(hx)) = -hx = xh$. □

Basis elements for the Lie algebras $\mathfrak{u}_1, \hat{\mathfrak{u}}_1$ are

$$\begin{pmatrix} i & \\ & i \end{pmatrix}, \begin{pmatrix} \hat{i} & \\ & \hat{i} \end{pmatrix} \tag{5.2}$$

and for $\mathfrak{sp}_1, \hat{\mathfrak{sp}}_1$

$$\begin{pmatrix} R(e) & \\ & R(e) \end{pmatrix}, \begin{pmatrix} R(\hat{e}) & \\ & R(\hat{e}) \end{pmatrix} (e = i, j, k). \tag{5.3}$$

Lemma 5.2. *The matrices* (4.3) *with* (5.2) *or respectively* (5.3) *form an orthogonal basis of the direct sum* $\mathfrak{spin}_{k+l} + \mathfrak{s}_{k+l}$ *with respect to the trace metric* $\langle X, Y\rangle = \mathrm{trace}\ X^tY$. *The square of the norm is* $\langle X, X\rangle = 2kl$.

Proof. We have to look at the possibilities in (5.1). Clearly the matrices in (5.3) are orthogonal to each other and the same holds for matrices from a Lie algebra with and without "hat". Further, $\left(\begin{smallmatrix} i & \\ & i \end{smallmatrix}\right) \in \mathfrak{u}_1$ is orthogonal to $A_{1i} = \left(\begin{smallmatrix} i & \\ & -i \end{smallmatrix}\right)$ because of the negative sign. The same is true for $\hat{\mathfrak{u}}_1$ and C_{1i}. It remains to show the orthogonality of $\mathfrak{sp}_1$ and A_{ef} for $1 \le e < f \le k$ (the case $\hat{\mathfrak{sp}}_1$ and C_{ef} is analogous). A_{ef} consists of the matrices $\left(\begin{smallmatrix} e & \\ & e \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} e & \\ & -e \end{smallmatrix}\right)$ for $e = i, j, k$. It suffices to see that a right and a left translation R_e and L_e are orthogonal. We have $L_eR_eu = -1$ for $u = 1, e$ and $L_eR_eu = 1$ for $u \ne 1, e$ and the trace vanishes.

Since $X^t = -X$ and $X^2 = -I$ (I the identity matrix) for all matrices (5.2), (5.3) and (4.3), the expression $\langle X, X\rangle = 2kl$ follows from (4.4). $\square$

The definitions in (5.1) can be summarized in the following way: In the tensor product $\mathbb{K} \otimes \mathbb{L}$, a quaternionic factor $\mathbb{K} = \mathbb{H}$ corresponds to a factor Sp_1, a factor $\mathbb{L} = \mathbb{H}$ to $\hat{Sp}_1$, a complex factor $\mathbb{K} = \mathbb{C}$ to U_1, and $\mathbb{L} = \mathbb{C}$ to $\hat{U}_1$. The list (5.4) shows which spaces can be constructed in this way (with the corresponding root system given after each space, see Appendix). If $\mathbb{K}, \mathbb{L} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ we get certain classical spaces which will be explained in Section 6. For $\mathbb{L} = \mathbb{R}$ we have the four classical projective planes $\mathbb{KP}^2$ with rank one. The cases $\mathbb{K} = \mathbb{O}$ represent the four Rosenfeld planes with the exceptional Lie groups $G = F_4, E_6, E_7, E_8$ and Lie algebras $\mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ (where the Cayley plane $\mathbb{OP}^2$ coincides with the first Rosenfeld plane). The construction of these exceptional spaces (resp. their Cartan decompositions) will be seen in Section 8 together with the root space decomposition in Section 9.

$$\left.\begin{array}{lcll}
(\mathbb{R}\otimes\mathbb{R})\mathbb{P}^2 & = & SO_3/O_2 & (\mathfrak{b}_1),\\
(\mathbb{C}\otimes\mathbb{R})\mathbb{P}^2 & = & SU_3/S(U_2\times U_1) & (\mathfrak{bc}_1),\\
(\mathbb{H}\otimes\mathbb{R})\mathbb{P}^2 & = & Sp_3/Sp_2\times Sp_1 & (\mathfrak{bc}_1),\\
(\mathbb{O}\otimes\mathbb{R})\mathbb{P}^2 & = & F_4/Spin_9 & (\mathfrak{bc}_1),\\
\\
(\mathbb{C}\otimes\mathbb{C})\mathbb{P}^2 & = & \mathbb{CP}^2\times\mathbb{CP}^2 & (\mathfrak{bc}_1\times\mathfrak{bc}_1),\\
(\mathbb{H}\otimes\mathbb{C})\mathbb{P}^2 & = & SU_6/S(U_4\times U_2) & (\mathfrak{bc}_2),\\
(\mathbb{O}\otimes\mathbb{C})\mathbb{P}^2 & = & E_6/Spin_{10}\cdot\hat{U}_1 & (\mathfrak{bc}_2),\\
\\
(\mathbb{H}\otimes\mathbb{H})\mathbb{P}^2 & = & SO_{12}/SO_8\times SO_4 & (\mathfrak{b}_4),\\
(\mathbb{O}\otimes\mathbb{H})\mathbb{P}^2 & = & E_7/Spin_{12}\cdot\hat{Sp}_1 & (\mathfrak{f}_4),\\
\\
(\mathbb{O}\otimes\mathbb{O})\mathbb{P}^2 & = & E_8/Spin_{16} & (\mathfrak{e}_8).
\end{array}\right\} \qquad (5.4)$$

6. The Classical Spaces

In this Section we will see that in the cases $\mathbb{K},\mathbb{L}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$ the representations (3.1), (5.1) for $Spin_{k+l}\cdot S_{k+l}$ are the isotropy representations (see [EH2]) for the classical spaces in (5.4).

The next Lemma shows the reducibility for the representations (3.1) if $l>1$.

Lemma 6.1. *If $\mathbb{K},\mathbb{L}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$, the representation module $(\mathbb{K}\otimes\mathbb{L})^2$ for $Spin_{k+l}$ in* (3.1) *has an orthogonal decomposition with the l invariant subspaces of the form*

$$V=\mathbb{K}^2\cdot v,\ \ v:=\sum_{g=1}^{m}s_g g\hat{g},\ \ m:=m_{\mathbb{L}},s_g=\pm 1$$

with the setting $s_1=+1$ and an even number of negative signs s_g in the case $l=4$ (so that we have l different vectors v for a fixed $\mathbb{K}$).

Proof. First we observe the following relation for a basis element $e\in B_{\mathbb{L}}$: $e\hat{e}v=s_e v$. If $l=4$ this multiplication with $e\hat{e}$ corresponds to a product of two disjoint transpositions applied to the summands $s_g g\hat{g}$ of v (possibilities for the signs in v are: $++++,++--,+-+-,+--+$). We get the equivalent condition $\hat{e}v=s_e ev$, and therefore for a given element $w\in\mathbb{L}$ there is a $w'\in\mathbb{L}$ with $\hat{w}v=w'v$. The invariance of V follows with a generating matrix from (3.1):

$$\begin{pmatrix}\hat{\bar{w}} & -\bar{u}\\ u & \hat{w}\end{pmatrix}\begin{pmatrix}xv\\ yv\end{pmatrix}=\begin{pmatrix}x\hat{\bar{w}}v-\bar{u}yv\\ uxv+y\hat{w}v\end{pmatrix}=\begin{pmatrix}x\bar{w}'v-\bar{u}yv\\ uxv+yw'v\end{pmatrix}\in V.$$

For the orthogonality of two different spaces $U = \mathbb{K}^2 \cdot u$ and $V = \mathbb{K}^2 \cdot v$ we have to show for $x, y, x', y' \in \mathbb{K}$:

$$\left\langle \begin{pmatrix} xu \\ yu \end{pmatrix}, \begin{pmatrix} x'v \\ y'v \end{pmatrix} \right\rangle = (\langle x, x' \rangle + \langle y, y' \rangle) \sum_{g=1}^{m} s_g s'_g = 0$$

where the s_g are the signs for u and the s'_g the signs for v. The sum with the products of the signs vanishes since one half of the summands is $+1$ and the other is -1. □

Theorem 6.1. *If* $\mathbb{K}, \mathbb{L} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, *the representation for* $Spin_{k+l} \cdot S_{k+l}$ *is equivalent to the isotropy representation of the corresponding classical space in* (5.4).

Proof. With $v_1 := \sum_{g=1}^{m} g\hat{g}$ and $m := m_{\mathbb{L}}$, we have $v_1 f\hat{f} = v_1$ for $f \in B_{\mathbb{L}}$ which is equivalent to $v_1 f = v_1 \hat{\hat{f}}$. By linearity we get $v_1 q = v_1 \hat{\hat{q}}$ for any $q \in \mathbb{L}$. Set $V := \mathbb{K}^2 v_1$ (which is one of the invariant spaces in Lemma 6.1).

First the case $\mathbb{K} = \mathbb{L} = \mathbb{H}$: Define the (real) vector space isomorphism $F : V \otimes \mathbb{H} \to (\mathbb{H} \otimes \mathbb{H})^2$ by $v \otimes q \mapsto vq = v\hat{\hat{q}}$ (with kernel zero and equal dimension of both spaces). $Spin_8$ operates only on the left factor V of $V \otimes \mathbb{H}$ after Lemma 6.1. The operation of $S = Sp_1 \cdot \hat{Sp_1}$ on $(\mathbb{H} \otimes \mathbb{H})^2$ is defined as $a.vq := vq\bar{a}$ resp. $\hat{b}.vq := vq\hat{\hat{b}} = v\hat{\hat{b}}q = vbq$ for $a, b \in \mathbb{S}^3 \subset \mathbb{H}$. Then S operates (after applying F) only on the right factor $\mathbb{H}$ of $V \otimes \mathbb{H}$ as left resp. right multiplication L_b, R_a, that is as SO_4. It follows the equivariance of F and the equivalence of the isotropy representation $SO_8 \otimes SO_4$ with the representation for $Spin_8 \cdot S$.

Second the case $\mathbb{K} = \mathbb{H}, \mathbb{L} = \mathbb{C}$: Define the map $F : V \otimes_{\mathbb{C}} \mathbb{H} \to (\mathbb{H} \otimes \mathbb{C})^2$ by $v \otimes_{\mathbb{C}} q \mapsto vq$. V and $\mathbb{H}$ are complex vector spaces with the complex structure $R_{\hat{i}}$ on V and $-L_i$ on $\mathbb{H}$. Note that $R_{\hat{i}} = -R_i$ on V. Therefore $V \otimes_{\mathbb{C}} \mathbb{H}$ is a complex vector space with both complex structures. $(\mathbb{H} \otimes \mathbb{C})^2$ is a complex vector space with the complex structure $R_{\hat{i}}$. The map F is a complex vector space isomorphism resp. both structures on $V \otimes_{\mathbb{C}} \mathbb{H}$ and the structure on $(\mathbb{H} \otimes \mathbb{C})^2$ since $F(-R_i(v\hat{q})) = R_{\hat{i}} F(v\hat{q})$ and $F(-L_{\hat{i}}(v\hat{q})) = R_{\hat{i}} F(v\hat{q})$.

Since $Spin_6$ operates only on the left factor V of $V \otimes_{\mathbb{C}} \mathbb{H}$ and commutes with $R_{\hat{i}}$, it is complex linear on V and belongs to the unitary group U_4 of V, that is $Spin_6 = SU_4$. Define the operation of $S = Sp_1 \cdot \hat{U}_1$ on $(\mathbb{H} \otimes \mathbb{C})^2$ as $(a, \hat{b}).vq := vq\bar{a}\hat{\hat{b}} = vbq\bar{a}$ (since $v\hat{\hat{b}} = vb$) for $a \in \mathbb{S}^3 \subset \mathbb{H}, b \in \mathbb{S}^1 \subset \mathbb{C}$. Then S operates only on the right factor $\mathbb{H} = \mathbb{C}^2$ of $V \otimes_{\mathbb{C}} \mathbb{H}$ as U_2 since the operation $(a, b; q) \mapsto bq\bar{a}$ commutes with the complex structure $-L_i$ on $\mathbb{H}$. It follows the equivariance of F

and the equivalence of the isotropy representation $SU_4 \otimes U_2$ with the representation for $Spin_6 \cdot S$.

Third the case $\mathbb{K} = \mathbb{L} = \mathbb{C}$: Here we have $V + W = (\mathbb{C} \otimes \mathbb{C})^2$ where V and W are the orthogonal vector spaces from Lemma 6.1 (the map F in the above cases is simply the identity). $Spin_4$ operates on both spaces V and W as $SU_2 = Spin_3$. Together with U_1 and $\hat{U}_1$ we get the isotropy representation $U_2 \times U_2$ corresponding to the symmetric space $\mathbb{CP}^2 \times \mathbb{CP}^2$.

The cases $\mathbb{L} = \mathbb{R}$ correspond to the classical projective planes $\mathbb{KP}^2$. The isotropy representations are the standard ones. $\square$

Remark 6.1. *The following observation will be important in Section 8: The generators of $\mathfrak{spin}_8 = \mathfrak{so}_8$ corresponding to the matrices in* (4.3) *have squared trace norm* 8 *on* $\mathbb{R}^8$ *and hence* $32 = 8 \cdot 4$ *on* $\mathbb{R}^8 \otimes \mathbb{R}^4$. *Likewise, the generators of $\mathfrak{so}_4$ corresponding to $R(e)$ and $R(\hat{e})$ have squared norm* 4 *on* $\mathbb{R}^4$ *and* $32 = 4 \cdot 8$ *on* $\mathbb{R}^4 \otimes \mathbb{R}^8$. *Thus both sorts of generators have the same length in the isotropy representation of* $G_4(\mathbb{R}^{12})$ *and as a Lie subalgebra of $\mathfrak{spin}_{16}$, see Lemma 5.1 and 5.2. Thus the two metrics on $\mathfrak{k}$ are proportional.*

7. The Rosenfeld Lines

The upper left block of the diagonal matrices A_{ef} in (4.3) spans the Lie subalgebra $\mathfrak{spin}_k$ with one exception: If $\mathbb{K} = \mathbb{H}$ we have only three linearly independent matrices A_{ef} because of the associativity of $\mathbb{H}$ (that means $L_e L_f = L_{ef}$ in this case). To get the full Lie algebra $\mathfrak{spin}_4$ we need the three additional basis matrices from $\mathfrak{sp}_1$ in (5.3). Therefore $Spin_k$ (together with Sp_1 if $k = 4$) acts on the subspace $(\mathbb{K} \otimes 1)\mathbf{e}_1 \subset (\mathbb{K} \otimes \mathbb{L})^2$ as SO_k (SO_4 is generated by L_p and R_q with unit quaternions p, q, and SO_8 is generated by $L_u L_v$ with unit octonions u, v). Equivalently, the matrices C_{ef} in (4.3), together with $\hat{\mathfrak{sp}}_1$ in the case $l = 4$, span $\mathfrak{spin}_l$ and $Spin_l$ acts on $(1 \otimes \mathbb{L})\mathbf{e}_1$ as SO_l.

We get the isotropy representation $SO_k \otimes SO_l$ of the Grassmann manifold $G_l(\mathbb{R}^{k+l})$ (see [EH2]). As a generalization of a projective line we define the *Rosenfeld line* $(\mathbb{K} \otimes \mathbb{L})\mathbb{P}^1$ as the totally geodesic subspace of $(\mathbb{K} \otimes \mathbb{L})\mathbb{P}^2$ with the tangent space $V = (\mathbb{K} \otimes \mathbb{L})\mathbf{e}_1$, which is $G_l(\mathbb{R}^{k+l})$ up to coverings. This tangent space V is a *Lie triple system*, that is $[V, [V, V]] \subset V$ (this can be seen with the definitions of the Lie brackets (8.1) and (8.2) in Section 8).

Now we compute the Lie triple used for the root space decomposition. Essentially this is only a translation from the standard representation for $G_l(\mathbb{R}^{k+l})$ into the division algebra context. The standard Cartan decomposition of the Lie algebra is $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with $\mathfrak{g} = \mathfrak{o}_{k+l}, \mathfrak{k} = \mathfrak{o}_k + \mathfrak{o}_l$ and the tangent space $\mathfrak{p}$ consisting of matrices $\begin{pmatrix} & -X^t \\ X & \end{pmatrix}$ with the real $l \times k$ matrices X. Let X_{ab} be the basis matrix with 1 at row a and column b and 0 else. By identifying X_{ab} with a basis element $c \cdot a\hat{b} \in \mathbb{K} \otimes \mathbb{L}$, $\mathfrak{p}$ can be identified with $\mathbb{K} \otimes \mathbb{L}$. Here we are free in the choice of the constant factor c. For $c := 2$ the Lie triples in Lemmas 7.1 and 8.2 will be compatible with each other. This will become clear in Section 8.

Lemma 7.1. *Let $a \in B_{\mathbb{K}}, g, h, b \in B_{\mathbb{L}}$. Then*

$$[g\hat{g}, [h\hat{h}, a\hat{b}]] = \begin{cases} 4b\hat{a} & \textit{if } (g,h) \in \{(a,b),(b,a)\}, a \neq b, \\ -4a\hat{b} & \textit{if } (g,h) \in \{(a,a),(b,b)\}, a \neq b, \\ 0 & \textit{else.} \end{cases} \tag{7.1}$$

Proof. With the notations above we have

$$\left[\begin{pmatrix} & -X_{hh}^t \\ X_{hh} & \end{pmatrix}, \begin{pmatrix} & -X_{ab}^t \\ X_{ab} & \end{pmatrix} \right] = \left[\begin{pmatrix} A^t - A & \\ & B^t - B \end{pmatrix} \right]$$

where $A = X_{hh}^t X_{ab}$ is a $k \times k$ matrix. We get $A = X_{hb}$ for $h = a$ and $A = 0$ for $h \neq a$. Similarly, $B = X_{hh} X_{ab}^t$ is a $l \times l$ matrix with $B = X_{ha}$ for $h = b$ and $B = 0$ for $h \neq b$. Abbreviating $\begin{pmatrix} & -X_{ab}^t \\ X_{ab} & \end{pmatrix}$ simply with ab, we get for the Lie triple

$$[gg, [hh, ab]] = \begin{pmatrix} & -C^t \\ C & \end{pmatrix}$$

with $C = X_{gg}(A^t - A) - (B^t - B)X_{gg}$. Computing C we get (7.1) by the above identification of X_{ab} with $2a\hat{b}$. ☐

Theorem 7.1. *The elements $1, i\hat{i}, \ldots, m_{\mathbb{L}}\hat{m}_{\mathbb{L}}$ span a maximal abelian Lie subalgebra Σ in $\mathfrak{p} = \mathbb{K} \otimes \mathbb{L}$, which is a commutative and associative subalgebra in $\mathbb{K} \otimes \mathbb{L}$ (viewed as a tensor product algebra).*

Proof. The first statement follows immediately from the known fact that the maximal abelian Lie subalgebra consists of all $l \times k$ matrices X (as introduced above) with diagonal entries. The commutativity and associativity follows from the symmetry in the factors of the tensor product algebra and (1.15), (1.14). ☐

The common (i.e simultaneous) eigenspace decomposition for the endomorphisms $\mathrm{ad}(x)$ (see [Es3], p. 21) for a (compact) symmetric space G/K with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + V$ (with the Lie algebras $\mathfrak{g} = L(G), \mathfrak{k} = L(K)$ and the tangent space V) can be determined as follows. That is to solve

$$\mathrm{ad}(x)x_\alpha = [x, x_\alpha] = \alpha(x)y_\alpha,\ \mathrm{ad}(x)y_\alpha = [x, y_\alpha] = -\alpha(x)x_\alpha \tag{7.2}$$

with $x \in \Sigma \subset V$ (a maximal abelian Lie subalgebra), $x_\alpha \in V, y_\alpha \in \mathfrak{k}$ and the real roots $\alpha(x)$ (with the minus sign for the compact type in the second equation). By combining the two equations we have

$$\mathrm{ad}(x)^2 x_\alpha = [x, [x, x_\alpha]] = -\alpha(x)^2 x_\alpha. \tag{7.3}$$

Applying this for $G_l(\mathbb{R}^{k+l})$ we get the root space decomposition for $V = \mathbb{K} \otimes \mathbb{L}$:

Theorem 7.2. *Let $x = \sum_{g=1}^m \alpha_g g\hat{g}$ ($\alpha_g \in \mathbb{R}, m := m_{\mathbb{L}}$) be a general element in the maximal abelian subalgebra Σ. Then we have the following roots and common eigenspaces:*

1) The abelian Lie subalgebra Σ for $\alpha(x) = 0$,

2) $\binom{l}{2}$ one-dimensional spaces $a\hat{b} + b\hat{a}$, $a < b \in B_{\mathbb{L}}$, $\alpha(x) = 2(\alpha_a - \alpha_b)$,

3) $\binom{l}{2}$ one-dimensional spaces $a\hat{b} - b\hat{a}$, $a < b \in B_{\mathbb{L}}$, $\alpha(x) = 2(\alpha_a + \alpha_b)$,

4) l spaces with dimension $k - l$ spanned by $a\hat{b}$ with all $a \in B_{\mathbb{K}} \ominus B_{\mathbb{L}}$ and fixed $b \in B_{\mathbb{L}}$ for each space, and for $\alpha(x) = 2\alpha_b$.

Proof. The case 1) is clear since $[g\hat{g}, h\hat{h}] = 0$ for $g, h \in B_{\mathbb{L}}$. Now let $a, b \in B_{\mathbb{L}}$. Applying (7.1) yields

$$\begin{aligned}
\mathrm{ad}(x)^2 a\hat{b} &= \sum_{1 \le \mu,\nu \le m} \alpha_\mu \alpha_\nu \,\mathrm{ad}(\mu\hat{\mu})\,\mathrm{ad}(\nu\hat{\nu}) a\hat{b} \\
&= \alpha_a\alpha_b \,\mathrm{ad}(a\hat{a})\,\mathrm{ad}(b\hat{b}) a\hat{b} + \alpha_a\alpha_b \,\mathrm{ad}(b\hat{b})\,\mathrm{ad}(a\hat{a}) a\hat{b} \\
&+ \alpha_a^2 \,\mathrm{ad}(a\hat{a})\,\mathrm{ad}(a\hat{a}) a\hat{b} + \alpha_b^2 \,\mathrm{ad}(b\hat{b})\,\mathrm{ad}(b\hat{b}) a\hat{b} \\
&= 8\alpha_a\alpha_b b\hat{a} - 4(\alpha_a^2 + \alpha_b^2) a\hat{b}.
\end{aligned}$$

By interchanging a and b in this equation and then adding or subtracting these two, we get 2) and 3):

$$\mathrm{ad}(x)^2(a\hat{b} \pm b\hat{a}) = -4(\alpha_a \mp \alpha_b)^2(a\hat{b} \pm b\hat{a}).$$

If $a \in B_{\mathbb{K}} - B_{\mathbb{L}}$ and $b \in B_{\mathbb{L}}$, the above sum $\mathrm{ad}(x)^2 a\hat{b}$ has only one summand (with $(g, h) = (b, b)$ in (7.1)) and 4) follows:

$$\mathrm{ad}(x)^2 a\hat{b} = -4\alpha_b^2 a\hat{b}.$$

□

The root systems obtained in Theorem 7.2 are isomorphic to the standard root systems $\mathfrak{b}_l$ for $k \neq l$ and $\mathfrak{d}_l$ for $k = l$ (see Appendix). The difference lies in the factor 2 according to the same constant factor introduced before Lemma 7.1.

8. The Jacobi Identity

For the Lie algebra $\mathfrak{k}$ of $K = Spin_{k+l} \cdot S_{k+l}$ we define an extension for the Lie bracket on the vector space $\mathfrak{g} := \mathfrak{k} + V$ with $V = (\mathbb{K} \otimes \mathbb{L})^2$ in the following way for $A \in \mathfrak{k}$ and $v, w \in V$, where the $[v, w] \in \mathfrak{k}$ is determined by inserting all $A \in \mathfrak{k}$ in (8.2):

$$[A, v] := Av = -[v, A], \tag{8.1}$$

$$\langle A, [v, w]\rangle_{\mathfrak{k}} := \langle Av, w\rangle, \tag{8.2}$$

where $\langle\, , \rangle_{\mathfrak{k}}$ must be an invariant inner product on $\mathfrak{k}$ under the adjoint action of K and such that at the end it is the restriction of an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$ (e.g, it can be the trace metric for the representation of K on $\mathfrak{g}$).

If we have an orthogonal basis for $\mathfrak{k}$ with respect to the metric $\langle\, , \rangle_{\mathfrak{k}}$, the bracket in (8.2) can be explicitly computed as

$$[v, w] = \sum_X \alpha_X X,\ \ \alpha_X = \frac{\langle Xv, w\rangle}{\langle X, X\rangle_{\mathfrak{k}}}, \tag{8.3}$$

where the sum runs over all the basis elements $X \in \mathfrak{k}$.

This bracket (for which we have to show that is actually a Lie bracket on $\mathfrak{g}$) is invariant under K:

Lemma 8.1.

$$[gv, gw] = g[v, w]g^{-1} \ \textit{for}\ v, w \in V, g \in K. \tag{8.4}$$

Proof. The scalar product $\langle\, , \rangle$ is invariant under $g \in K$ and $\langle\, , \rangle_{\mathfrak{k}}$ is invariant under $\mathrm{Ad}_g X = gXg^{-1}, X \in \mathfrak{k}$. Then we get

$$\begin{aligned}\langle A, [gv, gw]\rangle_{\mathfrak{k}} &= \langle Agv, gw\rangle = \langle g^{-1}Agv, w\rangle = \langle g^{-1}Ag, [v, w]\rangle_{\mathfrak{k}} \\ &= \langle A, g[v, w]g^{-1}\rangle_{\mathfrak{k}}.\end{aligned}$$

□

It follows the *equivariance*

$$g[u, [v, w]] = [gu, [gv, gw]] \text{ for } u, v, w \in V, g \in K. \tag{8.5}$$

The construction with the brackets defined above works for every representation. But the essential point is to show the Jacobi identity:

$$J(x, y, z) = [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0,\ \ x, y, z \in V. \tag{8.6}$$

The exceptional cases will follow easily from subrepresentations of the classical cases. These are already known to be *s-representations* from Section 6 (see also Lemma 4.1 in [EH1]). From Remark 6.1 we make the following observation: Since $\mathfrak{spin}_{16}$ is a simple Lie algebra, there is only one Ad-invariant metric on it (up to a scaling factor), the bracket

(8.2) on $V = (\mathbb{O} \otimes \mathbb{O})^2$ depends only on the representation of K and the invariant metric on $\mathfrak{k}$, it is a true Lie triple system for $G_4(\mathbb{R}^{12})$ when restricted to $V' = (\mathbb{H} \otimes \mathbb{H})^2$.

Theorem 8.1. *$J = 0$ for all $x, y, z \in V = (\mathbb{O} \otimes \mathbb{O})^2$.*

Proof. It is sufficient to prove the Jacobi identity only for basis elements in V. Then x, y, z are of the form $a\hat{b}\mathbf{e}_1$ or $a\hat{b}\mathbf{e}_2$ with $a, b \in B_{\mathbb{O}}$. We make use of the equivariance (8.5): With three or two generators of the form (3.1) we can transform x into $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ if we set $u = a$ and $w = b$: Choose $g = \left(\begin{smallmatrix} \bar{u}\hat{\bar{w}} & \\ & u\hat{w} \end{smallmatrix}\right) = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)\left(\begin{smallmatrix} & u \\ -\bar{u} & \end{smallmatrix}\right)\left(\begin{smallmatrix} \hat{\bar{w}} & \\ & \hat{w} \end{smallmatrix}\right)$ resp. $g = \left(\begin{smallmatrix} & \bar{u}\hat{\bar{w}} \\ -u\hat{w} & \end{smallmatrix}\right) = \left(\begin{smallmatrix} & \bar{u} \\ -u & \end{smallmatrix}\right)\left(\begin{smallmatrix} \hat{w} & \\ & \hat{\bar{w}} \end{smallmatrix}\right)$. Applying such an element g to y and z, they remain basis elements (up to a sign). We assume $y = a\hat{b}\mathbf{e}_1$ (or $a\hat{b}\mathbf{e}_2$) and $z = c\hat{d}\mathbf{e}_1$ (or $c\hat{d}\mathbf{e}_2$). But the two elements a, c lie in a quaternionic subalgebra of $\mathbb{O}$ and also b, d (independently of each other) and we are in the classical case $\mathbb{H} \otimes \mathbb{H}$. Therefore the Jacobi identity must be true also in the octonionic case. $\square$

Remark 1 In the proof of Theorem 8.1 we may use the group $K = Spin_{16}$ to carry the elements y and z into the standard subspace $(\mathbb{H} \otimes \mathbb{H})^2$ (in fact we use $G_2 \times G_2 \subset Spin_7 \times Spin_7 \subset Spin_{16}$ where $G_2 = \mathrm{Aut}(\mathbb{O})$). According to (3.1), the group K contains the matrix $\left(\begin{smallmatrix} N & \\ & -N \end{smallmatrix}\right)$ where $N = L_u L_v L_{vu}$ for any two distinct orthonormal $u, v \in \mathrm{Im}\,\mathbb{O}$. These span a quaternionic subalgebra $\mathbb{H}_1$, and N is the involution on $\mathbb{O}$ with eigenvalues 1 on $\mathbb{H}_1$ and -1 on $\mathbb{H}_1^\perp$ since $\mathbb{H}_1^\perp = L_w \mathbb{H}_1$ for some $w \in \mathbb{O}$ such that L_w anticommutes with L_u, L_v, L_{vu}.

Remark 2 Given two quaternionic subalgebras $\mathbb{H}_2$ and $\mathbb{H}$ spanned by elements of $B_{\mathbb{O}}$, we can use such N to map $\mathbb{H}_2$ on $\mathbb{H}$ as follows. The two subalgebras correspond to two lines in the projective plane over the field $\mathbb{F}_2 = \{0, 1\}$ (*Fano plane*), whose "points" are the seven basis elements of $\mathrm{Im}\,\mathbb{O}$, thus $\mathbb{H}$ and $\mathbb{H}_2$ have precisely one basis element in common, say i. Then $\mathbb{H} = \mathrm{span}\,\{1, i, j, ij\}$ and $\mathbb{H}_2 = \mathrm{span}\,\{1, i, p, ip\}$ for some $p \in B_{\mathbb{O}} \cap \mathrm{Im}\,\mathbb{O}$. We claim that $i, j+p, ij+ip$ span another quaternionic subalgebra $\mathbb{H}_1$. In fact, $(j+p)(ij+ip) = jij + p(ij) + j(ip) + pip = 2i$, note that (p, i, j) is a Cayley triple and hence $p(ij) = -(pi)j = j(pi) = -j(ip)$. Using the map N for this quaternionic subalgebra $\mathbb{H}_1$ we obtain $N(j+p) = j+p$ while $N(j-p) = -(j-p)$ since $j-p \perp \mathbb{H}_1$. Subtracting the two equations we see $N(p) = j$, and similarly we obtain $N(ip) = ij$. Moreover, N fixes $i \in \mathbb{H}_1$. Thus $N(\mathbb{H}_2) = \mathbb{H}$.

Remark 3 Another proof for the Jacobi identity would be using Witt's equation (Theorem 17 in [Wi]):

$$\operatorname{trace} \sum_{i,j} (D_i D_j)^2 = \frac{1}{2} f^2 r. \tag{8.7}$$

Here we have to sum over all pairs of the $\binom{k+l}{2}$ basis matrices D_i in (4.3), f denotes the dimension of the corresponding representation module, and r the dimension (called *rank* by Witt) of the Lie algebra. Then Witt's Theorem says that the Jacobi identity is valid for a given representation if and only if the above equation is valid. For the matrices in (4.3) we have the relations $D_i D_j = \pm D_j D_i$, $D_i^2 = -I$ and trace $I = f$ (with the identity matrix I). To verify the formula, we have to count the number of matrices with which a given matrix D_i commutes or anticommutes. It can be shown that D_i commutes with $1 + \binom{k+l-2}{2}$ and anticommutes with the remaining $\binom{k+l}{2} - 1 - \binom{k+l-2}{2} = 2(k+l) - 4$ matrices. Then with $f = 2kl$ and $r = \binom{k+l}{2}$ equation (8.7) can be reduced to

$$k^2 + l^2 - 9(k+l) + 16 = 0. \tag{8.8}$$

The formula does not work with additional factors S_{k+l}, only for pure spin groups (presumably it can be adapted to this case). Inserting $(k,l) = (8,1)$ for $Spin_9$ or $(k,l) = (8,8)$ for $Spin_{16}$ we see that (8.8) is valid in these cases.

The next lemma determines the Lie triple where the first two elements are in $(\mathbb{K} \otimes \mathbb{L})\mathbf{e}_1$ and the third in $(\mathbb{K} \otimes \mathbb{L})\mathbf{e}_2$. Together with Lemma 7.1 this will be used for the root space decomposition in Section 9. As in Lemma 7.1 we are also free in choosing a constant factor (which must not be the same for all spaces, but only fixed within a given space). Here we simply omit the denominator $\langle X, X \rangle_{\mathfrak{k}}$ in (8.3).

Lemma 8.2. *Let $u, v, w, z \in B_{\mathbb{K}\otimes\mathbb{L}}$ and R_u the right translation. Then*

$$[u\mathbf{e}_1, [v\mathbf{e}_1, w\mathbf{e}_2]] = -R_u R_{\bar{v}} w \mathbf{e}_2. \tag{8.9}$$

Proof. We apply (8.3), where we have to consider only the antidiagonal matrices of type $X = B_z = \left(\begin{smallmatrix} & z \\ -\bar{z} & \end{smallmatrix} \right)$ in (4.3) with the inner product

$$\langle X \cdot v\mathbf{e}_1, w\mathbf{e}_2 \rangle = -\langle \bar{z} v, w \rangle.$$

Since there is only one such X where this inner product does not vanish, namely for $\bar{z} = \pm w\bar{v}$, we get (by choosing the inner product such that the denominator $\langle X, X \rangle_{\mathfrak{k}}$ is 1 in (8.3))

$$[v\mathbf{e}_1, w\mathbf{e}_2] = -\langle \bar{z} v, w \rangle X \tag{8.10}$$

and

$$[u\mathbf{e}_1, [v\mathbf{e}_1, w\mathbf{e}_2]] = \langle \bar{z}v, w\rangle X \cdot u\mathbf{e}_1 = -\langle \bar{z}v, w\rangle \bar{z}u\mathbf{e}_2 = -R_u R_{\bar{v}} w\mathbf{e}_2.$$

$\square$

For $x \in \Sigma, w \in B_{\mathbb{K}\otimes\mathbb{L}}$ (with the algebra Σ from Theorem 7.1) we get

Corollary 8.1.

$$\mathrm{ad}(x\mathbf{e}_1)^2 w\mathbf{e}_2 = -(R(x)^2 w)\mathbf{e}_2. \tag{8.11}$$

Now we can explain the factor 4 connecting the Lie triple in Lemma 7.1 with that in 8.2, resulting from the special choice of the constant factors in the Lie triples.

First we compute $[\mathbf{e}_1, \mathbf{e}_2]$ from (8.2) and (8.3). We have to look for all basis matrices $A \in \mathfrak{spin}_{16}$ from (4.3) where $\langle A, [\mathbf{e}_1, \mathbf{e}_2]\rangle = \langle A\mathbf{e}_1, \mathbf{e}_2\rangle \neq 0$. There is only one such matrix, namely $A = B_{11} = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$ and we get $[\mathbf{e}_1, \mathbf{e}_2] = -B_{11}$ and therefore $[\mathbf{e}_1, [\mathbf{e}_1, \mathbf{e}_2]] = -\mathbf{e}_2$.

Next we compute $[\mathbf{e}_1, i\mathbf{e}_1]$. In (4.3) there are the four matrices $A = A_{1i}, A_{jk}, A_{lp}, A_{qr} \in \mathfrak{spin}_{16}$ where $\langle A\mathbf{e}_1, i\mathbf{e}_1\rangle \neq 0$, and $[\mathbf{e}_1, i\mathbf{e}_1]$ is the sum of these matrices (with certain signs ± 1). In this case we get $[\mathbf{e}_1, [\mathbf{e}_1, i\mathbf{e}_1]] = -4i\mathbf{e}_1$.

Thus the operator $-\mathrm{ad}(\mathbf{e}_1)^2$ has eigenvalues 1 and 4 which are the extreme values of the curvature in $\mathbb{CP}^2$.

In the next theorem we see that the Rosenfeld plane $(\mathbb{K}\otimes\mathbb{L})\mathbb{P}^2$ and the Rosenfeld line $(\mathbb{K}\otimes\mathbb{L})\mathbb{P}^1$ have the same maximal abelian subalgebra.

Theorem 8.2. *The vectors $\mathbf{e}_1, \ldots, m\hat{m}\mathbf{e}_1$ with $m := m_{\mathbb{L}}$ span a maximal abelian Lie subalgebra $\Sigma \subset V = (\mathbb{K}\otimes\mathbb{L})^2$.*

Proof. That these vectors span an abelian Lie subalgebra in V follows already from Theorem 7.1. For the same reason this algebra can not be extended within $(\mathbb{K}\otimes\mathbb{L})\mathbf{e}_1$. That this is also true for the other component $(\mathbb{K}\otimes\mathbb{L})\mathbf{e}_2$ can be seen from (8.10): This bracket never vanishes, even if we use a general element instead of only a basis element $a\hat{b}\mathbf{e}_2$. $\square$

9. The Root Space Decomposition

Theorem 9.1. *For the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + V$ with $V = (\mathbb{K}\otimes\mathbb{L})^2$, the equation* (7.3) *has the following roots and common eigenspaces:*

The component $(\mathbb{K}\otimes\mathbb{L})\mathbf{e}_1 \subset V$ is decomposed into eigenspaces stated in Theorem 7.2 corresponding to the Grassmann manifold $G_l(\mathbb{R}^{k+l})$.

For the decomposition of the component $(\mathbb{K}\otimes\mathbb{L})\mathbf{e}_2 \subset V$ we use the notations: Σ is the maximal abelian subalgebra from Theorem 8.2, $x =$

$\sum_{g=1}^{m} \alpha_g g\hat{g}\mathbf{e}_1 \in \Sigma$ *with* $m := m_{\mathbb{L}}, g \in B_{\mathbb{L}}, \alpha_g \in \mathbb{R}$, *and the linear form* $\alpha(x) = \sum_{g=1}^{m} s_g \alpha_g$ *with the signs* $s_g = \pm 1$. *For* $e, f \in B_{\mathbb{K}}$ s_f^e *is defined as in* (1.13) *(that is* $s_f^e = 1$ *if* e, f *commute, and* $s_f^e = -1$ *if* e, f *anticommute) and* R_e *denotes the right translation. A vector* $y\mathbf{e}_2 \in V$ *is abbreviated as* y *(since we are working only in this component from now on). Then we distinguish the following four cases:*

$\mathbb{K}\otimes\mathbb{R}$: *Set* $s_1 := 1$. *There is one root and the* k*-dimensional eigenspace spanned by the vectors* $R_e 1$ *for* $e \in B_{\mathbb{K}}$.

$\mathbb{K}\otimes\mathbb{C}$: *Set* $s_1 := 1$ *and* s_i *arbitrary. There are two roots where each of them has the* k*-dimensional eigenspace spanned by the vectors* $R_e v_e$ *with* $v_e := 1 + s_i s_i^e \, i\hat{i}$ *for* $e \in B_{\mathbb{K}}$.

$\mathbb{H}\otimes\mathbb{H}$: *Set* $s_1 := 1, s_k := s_i s_j$ *and* s_i, s_j *arbitrary. There are four roots (all with an even number of negative signs) where each of them has the four-dimensional eigenspace spanned by the vectors* $R_e v_e$ *with* $v_e := (1 + s_i s_i^e \, i\hat{i})(1 + s_j s_j^e \, j\hat{j})$ *for* $e \in B_{\mathbb{H}}$.

$\mathbb{O}\otimes\mathbb{H}$: *Set* $s_1 := 1, s_k := -s_i s_j$ *and* s_i, s_j *arbitrary. There are four roots (all with an odd number of negative signs) where each of them has the four-dimensional eigenspace spanned by the vectors* $R_e v_e$ *with* $v_e := (1 - s_i s_i^e \, i\hat{i})(1 - s_j s_j^e \, j\hat{j})$ *for* $e \in B_{\mathbb{H}^\perp}$. *In addition we have the roots and eigenspaces from* $\mathbb{H}\otimes\mathbb{H}$.

$\mathbb{O}\otimes\mathbb{O}$: *Set* $s_1 := 1, s_{ef} := s_e s_f, s_{-e} := s_e$ *and* s_i, s_j, s_l *arbitrary. There are* 64 *roots (all with an even number of negative signs) where each of them has the one-dimensional eigenspace spanned by the vector* $R_e v_e$ *with* $v_e := (1 + s_i s_i^e \, i\hat{i})(1 + s_j s_j^e \, j\hat{j})(1 + s_l s_l^e \, l\hat{l})$ *for* $e \in B_{\mathbb{O}}$.

Proof. $\mathbb{K}\otimes\mathbb{R}$: With $x = \left(\begin{smallmatrix}\alpha_1 \\ 0\end{smallmatrix}\right)$ this is clear from the eigenspace equation (7.3) and Corollary 8.1: $\operatorname{ad}(x)^2 e = -\alpha_1^2 e$. The operator $\operatorname{ad}(x)^2$ leaves all vectors e invariant up to the sign.

$\mathbb{K}\otimes\mathbb{C}$: With $a\hat{b} \in \mathbb{K}\otimes\mathbb{C}$ and $x = \left(\begin{smallmatrix}\alpha_1 + \alpha_i \, i\hat{i} \\ 0\end{smallmatrix}\right)$ we get

$$\operatorname{ad}(x)^2 a\hat{b} = -(\alpha_1 + \alpha_i R_{i\hat{i}})^2 a\hat{b} = -(\alpha_1^2 + 2\alpha_1\alpha_i R_{i\hat{i}} + \alpha_i^2) a\hat{b}. \tag{9.1}$$

The operator $R_{i\hat{i}}$ leaves v_e invariant up to the sign: $R_{i\hat{i}} v_e = i\hat{i} + s_i s_i^e = s_i s_i^e v_e$. Also $R_{g\hat{g}}$ commutes with R_e up to the sign: For $e \in B_{\mathbb{K}}, a, g \in B_{\mathbb{L}}$ we have $R_{g\hat{g}} R_e a\hat{a} = s_g^e R_e R_{g\hat{g}} a\hat{a}$. We see that our eigenvectors must be of the form $R_e v_e$ to be invariant: $R_{i\hat{i}} R_e v_e = s_i^e R_e R_{i\hat{i}} v_e = s_i R_e v_e$. Inserting $R_e v_e$ in (9.1), the proof is finished for $\mathbb{K}\otimes\mathbb{C}$:

$$\operatorname{ad}(x)^2 R_e v_e = -(\alpha_1 + s_i\alpha_i)^2 R_e v_e = -(\alpha_1^2 + 2 s_i \alpha_1 \alpha_i + \alpha_i^2) R_e v_e.$$

$\mathbb{H}\otimes\mathbb{H}$: Here we follow the same pattern. First we note that $v_e = xy$ is a product of two elements $x = 1 + s_i s_i^e\, i\hat{i}$ and $y = 1 + s_j s_j^e\, j\hat{j}$ in the commutative and associative subalgebra Σ (Theorem 7.1). Therefore we can apply $R_{g\hat{g}}$ in any order: $R_{g\hat{g}}(xy) = (R_{g\hat{g}}x)y = R_{g\hat{g}}(yx) = (R_{g\hat{g}}y)x$. Then $R_{g\hat{g}}v_e = s_g s_g^e v_e$ and $R_{g\hat{g}}R_e v_e = s_g R_e v_e$ for $g = i, j$ as in the case $\mathbb{K}\otimes\mathbb{C}$. For $g = k$ we get with (1.19)

$$R_{k\hat{k}}v_e = R_{i\hat{i}}R_{j\hat{j}}v_e = s_i s_j s_i^e s_j^e v_e = s_{ij}s_{ij}^e v_e = s_k s_k^e v_e \tag{9.2}$$

and $R_{g\hat{g}}R_e v_e = s_g R_e v_e$ is also valid for $g = k$. For two arbitrary operators $R_{g\hat{g}}$ and $R_{h\hat{h}}$ for $g, h \in B_{\mathbb{H}}$ we have

$$R_{g\hat{g}}R_{h\hat{h}}R_e v_e = R_{h\hat{h}}R_{g\hat{g}}R_e v_e = s_g s_h R_e v_e.$$

Now multiplying this equation with $\alpha_g\alpha_h$, we see that the left side corresponds to a summand at the left side of the eigenspace equation (7.3) and the right side to a summand at the right side of (7.3) (with the factor $\alpha(x)^2 = (\alpha_1 + s_i\alpha_i + s_j\alpha_j + s_k\alpha_k)^2)$.

$\mathbb{O}\otimes\mathbb{H}$: There is no difference to $\mathbb{H}\otimes\mathbb{H}$ up to a slight deviation, but the result is the same: Since $s_{ij} = s_k = -s_i s_j$ and $s_i^e = s_j^e = -1$ for all $e \in B_{\mathbb{H}^\perp}$ we have in (9.2): $s_i s_j s_i^e s_j^e = -s_{ij}(-s_{ij}^e) = s_{ij}s_{ij}^e = s_k s_k^e$.

$\mathbb{O}\otimes\mathbb{O}$: The method is exactly as in $\mathbb{H}\otimes\mathbb{H}$. Analogous to $k = ij$, every element $e \in B_{\mathbb{O}}$, except 1 and r, can be written as a product of two elements in $\{i, j, l\}$ (see (1.1)). Only for r we need all three: $r = (ij)l$ and we have to apply three right translations in this case. Again the order and setting of parentheses is not relevant because of the commutativity and associativity in Σ:

$$R_{r\hat{r}}v_e = R_{i\hat{i}}R_{j\hat{j}}R_{l\hat{l}}v_e = s_i s_j s_l s_i^e s_j^e s_l^e v_e = s_{ijl}s_{ijl}^e v_e = s_r s_r^e v_e.$$

$\square$

Writing $v_e(\mathbb{K})$ instead of v_e for the eigenvectors, denoting their corresponding division algebra, we see a recursive structure:

$$\begin{aligned} v_e(\mathbb{R}) &= 1, \\ v_e(\mathbb{C}) &= 1 \pm i\hat{i}, \\ v_e(\mathbb{H}) &= (1 \pm i\hat{i})(1 \pm j\hat{j}), \\ v_e(\mathbb{O}) &= (1 \pm i\hat{i})(1 \pm j\hat{j})(1 \pm l\hat{l}). \end{aligned}$$

This can be written as

$$v_e(\mathbb{K}) = v_e(\mathbb{K}') \pm v_e(\mathbb{K}')\; n_{\mathbb{K}}\hat{n}_{\mathbb{K}}, \tag{9.3}$$

where $\mathbb{K}'$ denotes the division algebra with half the dimension of $\mathbb{K}$, and $n_{\mathbb{K}}$ denotes the "smallest" element in $B_{\mathbb{K}'^\perp} \subset B_{\mathbb{K}}$. Here the orthogonality of the eigenvectors can be seen inductively.

Now we can explain the connection between the roots obtained from Theorems 7.2 and 9.1 and the standard root systems (see Appendix) for the symmetric spaces in (5.4).

$\mathbb{K}\otimes\mathbb{R}$: From Theorem 7.2 we have the roots $\pm 2e_1$ and from 9.1 the roots $\pm e_1$. Together they form the nonreduced root system $\mathfrak{bc}_1$ (except for $\mathbb{K}=\mathbb{R}$ with the root system $\mathfrak{b}_1$ corresponding to the real projective plane $\mathbb{RP}^2$).

$\mathbb{K}\otimes\mathbb{C}$: First the case $\mathbb{K}=\mathbb{C}$: Multiplying $\mathfrak{bc}_1$ with $\sqrt{2}$ and rotating it with 45 degree clockwise and also anticlockwise, we get a root system isomorphic to the direct sum $\mathfrak{bc}_1+\mathfrak{bc}_1$. In complex notation this corresponds to the left translations L_{1-i} and L_{1+i} (with basis $e_1=1, e_2=i$). This isomorphic root system consists of the roots from Theorem 7.2 together with the roots in 9.1. Therefore the resulting symmetric space is $\mathbb{CP}^2\times\mathbb{CP}^2$.

For the other two cases $\mathbb{K}=\mathbb{H},\mathbb{O}$ we apply the left translation L_{1+i} to $\mathfrak{bc}_2$ and get the roots in 7.2 and 9.1.

$\mathbb{K}\otimes\mathbb{H}$: For $\mathbb{K}=\mathbb{H}$ we use quaternionic notation (with basis $e_1 = 1, e_2=i, e_3=j, e_4=k$) and apply the left translation $L_{1+i+j+k}$ to $\mathfrak{b}_4$ and get the roots from 7.2 and 9.1. For $\mathbb{K}=\mathbb{O}$ there is only a factor 2 between the standard root system $\mathfrak{f}_4$ and the roots from 7.2 and 9.1. Therefore these two root systems are also isomorphic.

$\mathbb{O}\otimes\mathbb{O}$: Here we have also a factor 2 between $\mathfrak{e}_8$ and the roots in 7.2 and 9.1.

10. Conclusion

Every Clifford algebra Cl_{n-1} contains $\mathbb{R}^n=\mathbb{R}\cdot 1\oplus\mathbb{R}^{n-1}$. The unit sphere $\mathbb{S}^{n-1}\subset\mathbb{R}^n$ consists of invertible elements since $(t+v)(t-v) = t^2-v^2 = t^2+|v|^2$ for any $(t,v)\in\mathbb{R}\times\mathbb{R}^{n-1}$. The group generated by $\mathbb{S}^{n-1}$ is the spin group $Spin_n\subset Cl_{n-1}$. A representation of Cl_{n-1} restricts to a representation of $Spin_n$. In our context, two particular cases for n are important, $n=k$ and $n=k+l$ for $k,l\in\{1,2,4,8\}$:

(a) $\underline{n=k=\dim\mathbb{K}}$ where $\mathbb{K}$ is one of the normed division algebras $\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$. Then $\mathbb{R}^k=\mathbb{K}$ acts by left (or right) multiplication on $\mathbb{K}$ itself. The action of $\mathbb{S}^{k-1}\subset Spin_k$ commutes with the right translations $R(S_\mathbb{K})$ with $S_\mathbb{C}=\mathbb{S}^1\subset\mathbb{C}$, $S_\mathbb{H}=\mathbb{S}^3\subset\mathbb{H}$ and $\mathbb{S}_\mathbb{K}=\{1\}$ for $\mathbb{K}=\mathbb{R}$ and $\mathbb{K}=\mathbb{O}$. Then the group
$$Spin'_k := Spin_k\cdot S_\mathbb{K} \tag{10.1}$$
acts on $\mathbb{K}$ as SO_k.

(b) $\underline{n = k + l}$ where k, l are the dimensions of $\mathbb{K}, \mathbb{L} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. Then $\mathbb{R}^{k+l} = \mathbb{K} \oplus \mathbb{L}$ acts on $(\mathbb{K} \otimes \mathbb{L})^2$ by

$$(u, w) \mapsto \begin{pmatrix} \hat{\bar{w}} & -\bar{u} \\ u & \hat{w} \end{pmatrix}. \tag{10.2}$$

This action by matrices whose entries are left translations commutes with right multiplications by $S_{\mathbb{K}} \otimes S_{\mathbb{L}}$, and we put

$$Spin'_{k+l} := Spin_{k+l} \cdot R(S_{\mathbb{K}} \otimes S_{\mathbb{L}}). \tag{10.3}$$

The subgroup of $Spin'_{k+l}$ preserving $\mathbb{K} \otimes \mathbb{L} \subset (\mathbb{K} \otimes \mathbb{L})^2$ acts on $\mathbb{K} \otimes \mathbb{L}$ as the tensor product representation of $Spin'_k \times Spin'_l$ where $Spin'_k$, $Spin'_l$ act on $\mathbb{K}$, $\mathbb{L}$ as in (a). This is the isotropy representation of the Grassmannian $G_l(\mathbb{R}^{k+l})$.

For any skew adjoint linear action of a Lie algebra $\mathfrak{k}$ on a euclidean vector space V and for a suitable $\mathrm{ad}(\mathfrak{k})$-invariant metric on $\mathfrak{k}$, the trilinear map

$$\mathfrak{k} \times V \times V \to \mathbb{R}, \quad (A, v, w) \mapsto \langle Av, w \rangle = -\langle Aw, v \rangle \tag{10.4}$$

defines a linear map

$$\Lambda^2 V \to \mathfrak{k}, \quad v \wedge w \mapsto [v, w] \text{ with } \langle A, [v, w] \rangle = \langle Av, w \rangle \tag{10.5}$$

for all $A \in \mathfrak{k}$.

This map is equivariant with respect to the linear action of the group $K = \exp \mathfrak{k}$ on V and (by the adjoint representation) on $\mathfrak{k}$. Further, the construction is compatible with subrepresentations of Lie subalgebras. It defines a Lie triple on V if and only if the Jacobi identity is satisfied,

$$[[u, v], w] + [[v, w], u] + [[w, u], v] = 0. \tag{10.6}$$

Clearly, (10.6) holds if $(\mathfrak{k}, V)$ is an "s-representation", which is the isotropy representation of a symmetric space. It holds also if a subrepresentation $(\mathfrak{k}', V')$ is an s-representation provided that there is a basis of V such that any three basis elements can be mapped into V' by some $k \in K$. We apply this for $V = (\mathbb{O} \otimes \mathbb{O})^2$ with its standard basis

$$B = \{(p \otimes q)\mathbf{e}_i : i \in \{1, 2\},\ p, q \in B_{\mathbb{O}}\} \tag{10.7}$$

where $\mathbf{e}_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\mathbf{e}_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ and $B_{\mathbb{O}} = \{1, i, j, k, l, il, jl, kl\}$, and we let $V' = (\mathbb{H} \otimes \mathbb{H})^2$. Clearly, using antidiagonal and diagonal matrices of type (10.2) and their products, we can map any $(p \otimes q)\mathbf{e}_i$ to $\mathbf{e}_1$ while keeping B invariant. Thus we may assume $u = \mathbf{e}_1$, $v = (p \otimes q)\mathbf{e}_i$, $w = (r \otimes s)\mathbf{e}_j$ with $i, j \in \{1, 2\}$. But the two octonions p, r lie in some quaternionic subalgebra, and the same is true for q, s. Thus u, v, w lie in a copy of $(\mathbb{H} \otimes \mathbb{H})^2 \subset (\mathbb{O} \otimes \mathbb{O})^2$ (which can be mapped to the standard subspace $V' = (\mathbb{H} \otimes \mathbb{H})^2$ by some element of $\mathrm{Aut}(\mathbb{O}) \times \mathrm{Aut}(\mathbb{O}) \subset Spin_{16}$). On the other hand, $(\mathbb{H} \otimes \mathbb{H})^2$ is the Lie triple system of the Grassmannian $G_4(\mathbb{R}^{12})$ since $Spin_8' \times Spin_4'$ acts on it as the tensor product representation of $SO_8 \times SO_4$ on $\mathbb{R}^8 \otimes \mathbb{R}^4 \cong (\mathbb{H} \otimes \mathbb{H})^2$. To see this (cf. Section 6), we use the common eigenspace decomposition for the commuting linear maps $R(q \otimes q)$ on $(\mathbb{H} \otimes \mathbb{H})^2$ for $q \in \{1, i, j, k\}$. The eigenspaces are preserved by $Spin_8 \times Spin_4$ and interchanged by $S_{\mathbb{H}} \otimes S_{\mathbb{H}}$ (recall from (10.3) that $Spin_4' = Spin_4 \cdot S_{\mathbb{H}}$).

These arguments suffice to show that $(\mathbb{K} \otimes \mathbb{L})^2$ is a Lie triple system, hence it defines a symmetric space X. In order to identify the type of X we need to compute the Lie triple $[u, [v, w]]$ for u, v in a maximal abelian subspace $\Sigma \subset V = (\mathbb{K} \otimes \mathbb{L})^2$ and $w \in V$, which yields the root system of X. In fact, Σ can be chosen as a maximal abelian subspace in $(\mathbb{K} \otimes \mathbb{L})^1 \subset (\mathbb{K} \otimes \mathbb{L})^2$, since $[\alpha \mathbf{e}_1, \beta \mathbf{e}_2]$ is nonzero for all $\alpha \in B_{\mathbb{K}\otimes\mathbb{L}}$ and nonzero $\beta \in \mathbb{K} \otimes \mathbb{L}$, see (10.8) below. Because $\mathbb{K} \otimes \mathbb{L}$ corresponds to the Grassmannian $G_l(\mathbb{R}^{k+l})$, we know its maximal abelian subspace; it is spanned by the $p \otimes p$ where p runs through the basis $B_{\mathbb{L}} \subset B_{\mathbb{O}}$ of $\mathbb{L}$ (we may assume $\mathbb{L} \subset \mathbb{K}$). Since the Lie triple product is known on the subspace $V' = (\mathbb{K} \otimes \mathbb{L})^1$, it suffices to compute $[\beta \mathbf{e}_1, \omega \mathbf{e}_2]$ for any $\beta, \omega \in B_{\mathbb{K}\otimes\mathbb{L}}$, using (10.5)

$$[\beta \mathbf{e}_1, \omega \mathbf{e}_2] = \begin{pmatrix} & -\beta\bar{\omega} \\ \omega\bar{\beta} & \end{pmatrix} \tag{10.8}$$

where the conjugation on $\mathbb{K} \otimes \mathbb{L}$ is defined by $\overline{p \otimes q} := \bar{p} \otimes \bar{q}$. Hence

$$[\alpha \mathbf{e}_1, [\beta \mathbf{e}_1, \omega \mathbf{e}_2]] = -(\omega\bar{\beta})\alpha \mathbf{e}_2 = -R(\alpha)R(\bar{\beta})\omega \mathbf{e}_2. \tag{10.9}$$

Thus we only have to determine the common eigenvalues of $R(\alpha)R(\bar{\beta})$ on $\mathbb{K} \otimes \mathbb{L}$ for $\alpha, \beta \in \Sigma$ (cf. Section 9). Since $\Sigma \subset \mathbb{K} \otimes \mathbb{L}$ is commutative and fixed under the conjugation, at the end we only need the common eigenvalues of $R(\alpha)$, $\alpha \in \Sigma$. These are easy to compute since R_α just permutes $\pm B_{\mathbb{K}\otimes\mathbb{L}}$. The result is the table in the appendix.

We have seen in this work how the very peculiar properties of the normed division algebras generate the ten generalized Rosenfeld planes. In particular we have used the representation (10.2) of Cl_{k+l-1} and the fact that any two octonions lie in a common quaternionic subalgebra.

Appendix

In the table below we list the roots (with multiplicities) obtained in Theorems 7.2 and 9.1 and their standard root systems (see also [He], Table V, p. 518, and Table VI, pp. 532-534).

The eigenvalues of $\mathrm{ad}(x)^2$ with $x = \sum_{g=1}^{m_{\mathbb{L}}} \alpha_g g\hat{g}\mathbf{e}_1 \in \Sigma$ are of the type $-\alpha(x)^2$ where $\alpha : \Sigma \to \mathbb{R}$ is a linear form (root). By m we denote the multiplicity of α, by R the root system, and by $r = \dim \Sigma$ the rank. e, f are indices with $1 \leq e < f \leq m_{\mathbb{L}}$. We look separately at the root systems for each $\mathbb{K} \otimes \mathbb{L}$,
(a) first for the corresponding Rosenfeld line,
(b) then the extension for the Rosenfeld plane.

No	$\mathbb{K}\otimes\mathbb{L}$	r	(a) α	(a) m	(a) R	(b) α	(b) m	(b) R
1	$\mathbb{R}\otimes\mathbb{R}$	1	none	0	$\emptyset$	α_1	1	$\mathfrak{b}_1$
2	$\mathbb{C}\otimes\mathbb{R}$	1	$2\alpha_1$	1	$\mathfrak{b}_1$	α_1	2	$\mathfrak{bc}_1$
3	$\mathbb{H}\otimes\mathbb{R}$	1	$2\alpha_1$	3	$\mathfrak{b}_1$	α_1	4	$\mathfrak{bc}_1$
4	$\mathbb{O}\otimes\mathbb{R}$	1	$2\alpha_1$	7	$\mathfrak{b}_1$	α_1	8	$\mathfrak{bc}_1$
5	$\mathbb{C}\otimes\mathbb{C}$	2	$2(\alpha_1 \pm \alpha_i)$	1	$\mathfrak{d}_2$	$\alpha_1 \pm \alpha_i$	2	$\mathfrak{bc}_1 + \mathfrak{bc}_1$
6	$\mathbb{H}\otimes\mathbb{C}$	2	$2(\alpha_1 \pm \alpha_i)$	1				
			$2\alpha_e$	2	$\mathfrak{b}_2$	$\alpha_1 \pm \alpha_i$	4	$\mathfrak{bc}_2$
7	$\mathbb{O}\otimes\mathbb{C}$	2	$2(\alpha_1 \pm \alpha_i)$	1				
			$2\alpha_e$	6	$\mathfrak{b}_2$	$\alpha_1 \pm \alpha_i$	8	$\mathfrak{bc}_2$
8	$\mathbb{H}\otimes\mathbb{H}$	4	$2(\alpha_e \pm \alpha_f)$	1	$\mathfrak{d}_4$	$\sum s_e\alpha_e,\ \prod s_e = 1$	4	$\mathfrak{b}_4$
9	$\mathbb{O}\otimes\mathbb{H}$	4	$2(\alpha_e \pm \alpha_f)$	1				
			$2\alpha_e$	4	$\mathfrak{b}_4$	$\sum s_e\alpha_e$	4	$\mathfrak{f}_4$
10	$\mathbb{O}\otimes\mathbb{O}$	8	$2(\alpha_e \pm \alpha_f)$	1	$\mathfrak{d}_8$	$\sum s_e\alpha_e,\ \prod s_e = 1$	1	$\mathfrak{e}_8$

The following standard root systems were used (see [He], pp. 462-464, 472-475):

$\mathfrak{b}_l$ $(l \geq 1) :$ $R = \{\pm e_i\ (1 \leq i \leq l),\ \pm e_i \pm e_j\ (1 \leq i < j \leq l)\}$,

$\mathfrak{d}_l$ $(l \geq 2) :$ $R = \{\pm e_i \pm e_j\ (1 \leq i < j \leq l)\}$,

$\mathfrak{f}_4 :$ $R = \{\pm e_i,\ \pm e_i \pm e_j\ (i < j),\ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}$, the cardinality of R is $2 \cdot 4 + \binom{4}{2} \cdot 4 + 2^4 = 48$,

$\mathfrak{e}_8 :$ $R = \{\pm e_i \pm e_j\ (i < j),\ \frac{1}{2}\sum_{i=1}^{8}(-1)^{\nu(i)} e_i\ (\sum_{i=1}^{8} \nu(i) \text{ even})\}$, the cardinality of R is $\binom{8}{2} \cdot 4 + 2^7 = 240$,

$\mathfrak{bc}_l$ $(l \geq 1) :$ $R = \{\pm e_i \pm e_j\ (1 \leq i < j \leq l),\ \pm e_i\ (1 \leq i \leq l),\ \pm 2e_i\ (1 \leq i \leq l)\}$.

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