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A characterization of submanifolds of extrinsic symmetric spaces

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1. Introduction

The present article is motivated by two classical problems in submanifold geometry, reduction of codimension and umbilicity. By Erbacher [1], the codimension of an immersion $f : M \to \mathbb{R}^n$ can be reduced iff there is a parallel bundle $E \subset M \times \mathbb{R}^n$ containing TM. On the other hand, an immersion f takes values in the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ iff there is a normal vector field ξ with

$$\nabla^{\perp}\xi = 0, \qquad \langle \alpha(v, w), \xi \rangle = \langle v, w \rangle \tag{1}$$

where $\nabla^{\perp}$ is the connection in the normal bundle and α is the second fundamental form of f, cf. [5]. What happens if we replace the sphere by any extrinsic symmetric space $S \subset \mathbb{R}^n$? These are natural generalizations of the sphere in many respects. Clearly we need a bundle $E \subset M \times \mathbb{R}^n$ which contains TM and carries the structure of TS, and some condition on the $E^{\perp}$-part of α replacing (1) will be needed.

Notation. We do not distinguish between a vector bundle E and its space of sections ΓE. By $a \in E$ we mean that a is a (locally defined) section of E, and $v \in TM$ says that v is a (local) tangent vector field of M.

2. The result

Let $S \subset \mathbb{R}^n$ be a full extrinsically symmetric space [4], i.e. a compact submanifold whose second fundamental form $\beta^S : TS \otimes TS \to NS$ is parallel and onto. Consider a submanifold M of S and put $E = TS|_M$. The restriction of β^S is a parallel bundle map $\beta : E \otimes E \to E^{\perp}$ which is related to the "second fundamental form" $\beta^E(v, a)$ of the bundle E as follows:

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$$\beta^E(v,a) := (\partial_v a)^{E^\perp} = \beta(v,a) \tag{2}$$

for all $v \in TM$ and $a \in E$. It is our aim to show that these properties are already sufficient for a submanifold M of $\mathbb{R}^n$ to lie actually in S:

Theorem 1. *Let M be a submanifold of some Euclidean space $\mathbb{R}^n$ which also contains a full extrinsic symmetric space S. Suppose that there is a subbundle $E \subset M \times \mathbb{R}^n$ with $TM \subset E$ and a parallel bundle homomorphism $\beta : E \otimes E \to E^\perp$ satisfying (2) such that β is congruent to the second fundamental form β^S at some point. Then M is contained in S, up to a rigid motion of $\mathbb{R}^n$.*

Remark 1. In the case where S is the sphere, $S = \mathbb{S}^{n-1}$, the assumptions of our theorem are those of the umbilicity theorem (1). In fact, in this case $E = \xi^\perp$ for some normal vector field ξ on M and $\beta(a,b) = \langle a,b\rangle\xi$ is always parallel. Condition (2) says $-\langle \partial_v\xi, a\rangle = \langle \partial_v a, \xi\rangle = \langle v, a\rangle$ for all $v \in TM$, $a \in E$. In particular $\langle \partial_v \xi, a\rangle = 0$ for every $a \perp TM$ which shows that ξ is a parallel normal vector field with $\langle \alpha(v,w), \xi\rangle = -\langle \partial_v \xi, w\rangle = \langle v, w\rangle$ for all $v, w \in TM$. Vice versa, if $M \subset \mathbb{R}^n$ carries a parallel normal vector field ξ with $-\langle \partial_v \xi, w\rangle = \langle \alpha(v,w), \xi\rangle = \langle v, w\rangle$, we have $\partial_v \xi = (\partial_v \xi)^{TM} = -v$ and hence $(\partial_v a)^{E^\perp} = \langle \partial_v a, \xi\rangle\xi = -\langle \partial_v \xi, a\rangle\xi\rangle = \langle v, a\rangle\xi = \beta(v,a)$, as in the assumptions of our theorem.

Remark 2. In [2] we determined under some genericity assumptions, which maps ν from a Riemannian manifold M^m to the Grassmannian $G_m(\mathbb{R}^n)$ can be the Gauss maps of an isometric immersion $M \hookrightarrow \mathbb{R}^n$. A related question is if we can see from ν when M actually lies in a sphere or in an extrinsic symmetric space within $\mathbb{R}^n$. Our theorem gives an answer: it is necessary and sufficient that the bundle $\tau = \nu^\perp$ can be extended to a bundle $E \supset \tau$ with the properties as in the assumptions of our theorem.

3. The proof

The main idea of the proof is to apply the general existence and uniqueness theorem for submanifolds of symmetric spaces, cf. [3]: If S is a symmetric space and $f : M \to S$ an immersion, then the differential of f is a bundle homomorphism $F = df : TM \to E := f^*TS$ with

$$(\nabla_v F)w = (\nabla_w F)v, \tag{3}$$

$$R^E(v,w)a = R^S(Fv, Fw)a \tag{4}$$

for all $v, w \in TM$ and $a \in E$, where R^S is the curvature tensor of S and R^E the curvature tensor of the vector bundle E. Vice versa, if data (E, ∇, R^S, F) are given, where E is a vector bundle over M with a metric connection ∇ and $R^S : E \otimes E \otimes E \to E$ a ∇-parallel isomorphic copy of the curvature tensor of S and $F : TM \to E$ an injective vector bundle homomorphism with (3), (4), then F is the differential of an immersion $\tilde f : M \to S$ or more precisely, there exist an immersion $\tilde f : M \to S$ and a parallel bundle isometry $\Phi : \tilde f^*TS \to E$ carrying R^S on $\tilde f^*TS$ into R^S on E such that

$$d\tilde f = \Phi \circ F. \tag{5}$$

In our case, S is an extrinsic symmetric space in $\mathbb{R}^n$, and the parallel curvature tensor R^S enjoys a parallel "square root" taking care of the normal geometry as well: the second fundamental form $\beta : E \otimes E \to E^\perp$ which is related to R^S by the (quadratic) Gauss equation

$$\langle R^S(c,d)a, b\rangle = \langle \beta(c,b), \beta(d,a)\rangle - \langle \beta(c,a), \beta(d,b)\rangle. \tag{6}$$

Now suppose that we have got an immersion $f : M \to \mathbb{R}^n$ with the properties assumed in the theorem. We apply the existence and uniqueness theorem to $F = df : M \to E$. Eq. (3) is obvious from $F = df$. In order to prove (4), one needs a version of the Gauss equation for subbundles:

Lemma 2. *Let M be a manifold and $E \subset M \times \mathbb{R}^n$ a vector bundle, equipped with the projection connection $\nabla_v a = (\partial_v a)^E$ where $a \in E$ and $v \in TM$. Let $\beta^E : TM \otimes E \to E^\perp$ be its "second fundamental form"*

$$\beta^E(v,a) = (\partial_v a)^{E^\perp} \tag{7}$$

with $v \in TM$ and $a \in E$. This is related to the curvature tensor R^E of (E, ∇) as follows:

$$\langle R^E(v,w)a, b\rangle = \langle \beta^E(v,b), \beta^E(w,a)\rangle - \langle \beta^E(v,a), \beta^E(w,b)\rangle. \tag{8}$$

Proof. The lemma is proved like Gauss equations for submanifolds: From the decomposition $\partial_w a = \nabla_w a + \beta^E(w,a)$ we obtain

$$\langle \partial_v \partial_w a, b\rangle = \langle \nabla_v \nabla_w a, b\rangle - \langle \beta^E(w,a), \beta^E(v,b)\rangle$$

and hence

$$0=\left\langle R^{\partial}(v,w)a,b\right\rangle=\left\langle R^{E}(v,w)a,b\right\rangle-\left\langle \beta^{E}(w,a),\beta^{E}(v,b)\right\rangle+\left\langle \beta^{E}(v,a),\beta^{E}(w,b)\right\rangle$$

which finishes the proof. □

Now putting $c=v$ and $d=w$ in the Gauss equation (6) and using $\beta=\beta^E$ for these entries (cf. (2)), we obtain (4) by comparing (6) and (8). Using the general existence theorem (5) we obtain an immersion $\tilde f: M\to S\subset\mathbb{R}^n$ with $df=\Phi\circ d\tilde f$ for some parallel and isometric bundle isomorphism $\Phi:\tilde f^*TS\to E$; in particular, the two immersions f and $\tilde f$ are (intrinsically) isometric. It remains to show that they just differ by an isometry of the ambient space $\mathbb{R}^n$.

In order to prove the congruence of $\tilde f$ and f we extend Φ to an isometric bundle isomorphism $\hat\Phi$ of

$$V=M\times\mathbb{R}^n=E+E^\perp=\tilde E+\tilde E^\perp$$

where $\tilde E:=\tilde f^*TS$. This is possible by the geometry of extrinsic symmetric spaces:

Lemma 3. *There is an extension of $\Phi:\tilde E\to E$ to a bundle isometry $\hat\Phi: V\to V$ such that*

$$\hat\Phi\big(\tilde\beta(a,b)\big)=\beta(\Phi a,\Phi b) \tag{9}$$

for all $a,b\in\tilde E$, where $\tilde\beta:\tilde E\otimes\tilde E\to\tilde E^\perp$ is the second fundamental form of S along $\tilde f$.

Proof. Since every normal vector $\xi\in\tilde E^\perp$ is of the form $\tilde\beta(a,b)$ for some $a,b\in\tilde E$, it suffices to show that $\hat\Phi$ is well defined by (9), i.e. for all $a,b,a',b'\in\tilde E$

$$\tilde\beta(a,b)=\tilde\beta\big(a',b'\big)\quad\Rightarrow\quad\beta(\Phi a,\Phi b)=\beta\big(\Phi a',\Phi b'\big). \tag{10}$$

By assumption of our theorem, β and $\tilde\beta$ are congruent at some point $p\in M$. Hence we may assume $E_p=\tilde E_p$ and $\Phi_p=I$, and (10) is obvious at p. At any other point $q\in M$ we consider a curve γ joining p to q and the parallel displacements of the various bundles along this curve: $\tau,\tilde\tau$ along $E,\tilde E$ and $\tau^\perp,\tilde\tau^\perp$ along $E^\perp,\tilde E^\perp$. Since $\Phi,\tilde\beta,\beta$ are parallel, its values at p and q are joined by parallel displacements: We have $\Phi_q\tilde\tau=\tau\Phi_p$ and $\tau^*\beta=\tau^\perp\beta$, $\tilde\tau^*\tilde\beta=\tilde\tau^\perp\tilde\beta$, hence

$$\tilde\beta_q(\tilde\tau a,\tilde\tau b)=\tilde\tau^\perp\tilde\beta_p(a,b),\qquad \beta_q(\Phi_q\tilde\tau a,\Phi_q\tilde\tau b)=\tau^\perp\beta_p(\Phi_p a,\Phi_p b)$$

for all $a,b\in\tilde E_p$. This proves (10) at q. □

Lemma 4. *$\hat\Phi: V\to V$ is parallel with respect to ∂.*

Proof. There are several connections on V, the trivial connection ∂ and the two projection connections $\nabla,\tilde\nabla$ for the decompositions $V=E+E^\perp$ and $V=\tilde E+\tilde E^\perp$; e.g. we have $\nabla(a+\xi)=(\partial a)^E+(\partial\xi)^{E^\perp}$ for $a\in E$ and $\xi\in E^\perp$. We know already that $\hat\Phi$ is parallel with respect to the connections $\tilde\nabla$ on the domain and ∇ on the range. For the E-component this is due to the parallelity of Φ, see (5), and for the $E^\perp$-part it follows from (9) and the parallelity of $\tilde\beta$, β and Φ.

To prove ∂-parallelity of $\hat\Phi$ we have to see that $\hat\Phi$ carries $\tilde A=\partial-\tilde\nabla$ onto $A=\partial-\nabla$. From $\partial_v a=\nabla_v a+\beta(v,a)$ and $\partial_v\xi=\nabla_v\xi-A_\xi v$ we find

$$A(v,a)=\beta(v,a),\qquad A(v,\xi)=-A_\xi v \tag{11}$$

where $A_\xi=-(\partial\xi)^E$ is the "Weingarten map" of E, and similar for $\tilde A$. Using (9) we have

$$\hat\Phi\tilde A(v,a)=\hat\Phi\tilde\beta(v,a)=\beta(v,\Phi a)=A(v,\Phi a) \tag{12}$$

where we consider TM both as a subbundle of E (using df) and of $\tilde E$ (using $d\tilde f$). Moreover, for $a\in\tilde E$ and $v\in TM$ and $\xi\in\tilde E^\perp$ we have

$$\left\langle A(v,\hat\Phi\xi),\Phi a\right\rangle=\left\langle\beta(v,\Phi a),\hat\Phi\xi\right\rangle=\left\langle\hat\Phi\beta(v,a),\hat\Phi\xi\right\rangle=\left\langle\beta(v,a),\xi\right\rangle=\left\langle\tilde A(v,\xi),a\right\rangle$$

from which we conclude $\Phi^{-1}A(v,\hat\Phi\xi)=\tilde A(v,\xi)$ and hence

$$\Phi\tilde A(v,\xi)=A(v,\hat\Phi\xi). \tag{13}$$

Thus $\hat\Phi\tilde A(v,x)=A(v,\hat\Phi x)$ for any $x\in V$, and therefore

$$\partial_v\hat\Phi x=\nabla_v\hat\Phi x+A(v,\hat\Phi x)=\hat\Phi\big(\tilde\nabla_v x+\tilde A(v,x)\big)=\hat\Phi\partial_v x.\qquad\square$$

Now $\hat\Phi: V\to V$ is a constant orthogonal map relating df and $d\tilde f$, see (5), and the proof of the theorem is finished.

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