Compatibility of Gauß maps with metrics

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1. Introduction

Isometric hypersurface immersions of a Riemannian manifold (M, g) with dimension $m = n - 1$ into Euclidean n-space are characterized by their first and second fundamental forms, g and h. By a classical theorem going back to Bonnet, the immersion exists and is uniquely determined by g and h up to Euclidean motions provided that the pair (g, h) satisfies Gauß and Codazzi equations.

In the present paper we ask what happens if we replace (g, h) by (g, ν) where $\nu : M \to S^m$ is the Gauß map. At the first glance the new problems seems more rigid since h is obtained from the differential $d\nu : TM \to \nu^\perp$. However this observation is true only after identifying TM with the complement of the normal bundle, $\nu^\perp$. This identification is precisely the differential of the immersion which has to be constructed. In fact uniqueness might fail as it happens with minimal surfaces: All immersions in the associated family of a minimal surface have the same g and ν, but they are not congruent. A well-known example is the deformation of the catenoid to the helicoid.

The first problem is to recover the second fundamental form h from the data. In our approach, the third fundamental form $k = \langle d\nu, d\nu \rangle$ will play a major rôle since it is obtained directly from our data and the second fundamental form is its square root (using g, all 2-forms are viewed also as endomorphims). However, the square root of a self adjoint positive semi-definite matrix is not unique, and if repeated eigenvalues occur, there are even infinitely many solutions as it happens in the minimal surface case mentioned above. Moreover, in high dimensions it might be very difficult to compute.

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[1] Partially supported by DFG (SPP 1154 and GK 1523).
[2] Partially supported by CNPq-Brazil.

Fortunately, Theorems 2, 3 give other ways to recover h (using Gauß equations). The defining Eqs. (2) and (4) in Theorems 1 and 2 have been known already to Obata [23]. In the final chapter we extend these ideas to the case of higher codimension where ν takes values in the Grassmannian $G_{n-m}(\mathbb{R}^n)$. Again, uniqueness may fail in non-generic cases; examples have been provided and classified by Dajczer and Gromoll [10]. However we will restrict our attention to the generic cases. In this case, Theorems 4, 5 provide an algorithm by which we may check if the data (g, ν) are the metric and the Gauss map of some immersion.

We assume that all objects are sufficiently smooth.

2. Main results

Let (M^m, g) be a Riemannian manifold and $\nu : M^m \to S^m \subset \mathbb{R}^{m+1}$ be a smooth mapping. We are interested in the question when the given data (g, ν) are the first fundamental form and the Gauß map for an immersion $u : M^m \hookrightarrow \mathbb{R}^{m+1}$. Such data (g, ν) will be called *admissible*, and u will be called a *solution* for (g, ν). Our considerations are local, hence we may always assume that M is a simply connected open subset of $\mathbb{R}^m$.

Let $R = (R^l_{ijk})$ be the Riemann curvature tensor of the metric g, $\operatorname{Ric} = \operatorname{Tr} R = (R^l_{ijl})$ the Ricci tensor, and $s = \operatorname{Tr}\operatorname{Ric} = g^{ij}\operatorname{Ric}_{ij}$ the scalar curvature. Let $A = d\nu \in \operatorname{Hom}(TM, \nu^\perp)$ and put

$$k(v, w) = \langle Av, Aw\rangle = \langle A^*Av, w\rangle \tag{1}$$

for all $v, w \in TM$; this is a symmetric positive semi-definite bilinear form, which will be referred to as the *third fundamental form*. We can raise the indices with the help of g and consider both Ric and k as fields of operators on the tangent bundle, denoting the result by the same letter.

Theorem 1. *Let (M^m, g) and $\nu : M^m \to S^m$ be given and assume that $A = d\nu : TM^m \to \nu^\perp$ is everywhere invertible. Let k be defined by (1). Then the data (g, ν) are admissible if and only if there is $h \in S^2T^*M$ with*

$$h^2 = k \tag{2}$$

(*here we identify symmetric forms with symmetric operators with the help of g*) *such that the vector bundle homomorphism*

$$U = -(A^*)^{-1}h : TM \to \nu^\perp \tag{3}$$

is isometric and parallel with respect to the Levi-Civita connection on TM and the projection connection on $\nu^\perp$. In fact, the corresponding immersion $u : M^m \to \mathbb{R}^{m+1}$ is determined by $du = U$, and h is the second fundamental form of u, i.e. $h_{ij} = \langle u_{ij}, \nu\rangle$.

The condition that the vector bundle homomorphism U is parallel with respect to the Levi-Civita connection on TM is equivalent to (8) below.

Since k is positive semi-definite, it has a symmetric square root h. However, as explained in Introduction, (2) is often difficult to solve. Indeed, a general solution requires finding the roots of a polynomial of the mth degree. For big m, this is impossible to do explicitly. If k has multiple eigenvalues, the following additional difficulty appears: at every point there are infinitely many solutions of (2), so even if we found one solution of (2) such that (3) is not parallel, there might exist another solution such that (3) is parallel. Hence in many cases Theorem 1 is useless unless we find a better method to compute h from the data. This is achieved by the following statements:

Theorem 2. *If the data (g, ν) are admissible, then for any solution u, the second fundamental form h and the* (*unnormalized*) *mean curvature $H = \operatorname{Tr} h = h_{ij}g^{ij}$ solve the following system*

$$hH = \operatorname{Ric} + k, \qquad H^2 = s + \operatorname{Tr} k. \tag{4}$$

Remark 1. Clearly, if $s + \operatorname{Tr} k > 0$, the equations can be solved:

$$H = \pm\sqrt{s + \operatorname{Tr} k}, \qquad h = \pm\frac{1}{\sqrt{s + \operatorname{Tr} k}}(\operatorname{Ric} + k). \tag{5}$$

Moreover, as we explain in Remark 3, the sign of H and of h is not essential for our goals.

Theorem 3. *If the data (g, ν) are admissible with $d\nu$ non-degenerate and $m = \dim M \geqslant 3$, then the second fundamental form h of any solution u solves the homogeneous linear system*

$$hk^{-1}R(\Omega) = 2\Omega h, \quad \forall \Omega \in \mathfrak{so}_g(TM), \tag{6}$$

where R is considered as curvature operator acting on $\Lambda^2 TM = \mathfrak{so}_g(TM)$. Moreover the solution h of (6) is unique up to a scalar factor.

Remark 2. Note that the missing scalar factor in Theorem 3 can be easily found using condition (2): if $\tilde{h}$ is a nonzero solution of (6) then

$$h = \pm\sqrt{\frac{\operatorname{Tr}(\tilde{h}^2)}{\operatorname{Tr} k}} \cdot \tilde{h}. \tag{7}$$

Remark 3. The sign $\pm$ in the formulas (5), (7) does not affect the existence of a solution u: If u is a solution for (g, ν) with second fundamental form h (respectively mean curvature H), then $-u$ is also a solution with second fundamental form $-h$ (respectively mean curvature $-H$).

The above theorems give us an algorithm to check admissibility of (g, ν):

1. Check if $s + \operatorname{Tr} k \geqslant 0$.[3]
2. Find h:
   (a) If $s + \operatorname{Tr} k > 0$, define h by (5).[4]
   (b) If $s + \operatorname{Tr} k = 0$ and $m \geqslant 3$, then for every $x \in M$ solve the linear system (6) in $T_x M := \nu^{\perp}$. Check whether there exists a (non-degenerate) solution $\tilde{h}$. Consider the solution h given by (7).
   (c) If $s + \operatorname{Tr} k \equiv 0$ and $m = 2$, verify the Gauß condition (Remark 6 below).
3. Check if $h^2 = k$ (this together with 2 implies that h is symmetric).
4. Finally check if $U = -(A^*)^{-1} h$ is parallel, i.e.

$$\partial_i u_j - \langle \partial_i u_j, \nu \rangle \nu = \Gamma_{ij}^k u_k \tag{8}$$

   where $u_j = U e_j$ and Γ_{ij}^k are the Christoffel symbols, the components of the Levi-Civita connection: $\nabla_i e_j = \Gamma_{ij}^k e_k$.

The data (g, ν) are admissible if and only if all checks are successful.

This answers a question raised in [12] which was discussed by the authors during and after the 10th conference on Differential Geometry and its Applications.

Remark 4. Both Gauß and Codazzi equations are hidden in the assumption that $U \in \operatorname{Hom}(TM, \nu^{\perp})$ is isometric and parallel. In fact this property is equivalent to Codazzi equations while Gauß equations follow from it, see Appendix A. The claim that the Gauß condition is a differential consequence of the Codazzi condition in the non-degenerate case is non-trivial. It shall be compared with the known fact that under some conditions the Codazzi equations are consequences of the Gauß equations [2].

Remark 5. The uniqueness of recovering the isometric immersion $u : M^m \to \mathbb{R}^{m+1}$ with fixed third quadratic form k was considered in [11]. This is similar to recovering of immersions with fixed Gauß map in the case of hypersurfaces, but not for higher codimension, see the last section.

Remark 6. The only case not covered by our theorems is $m = 2$ and $H = 0$, the case of minimal surfaces which is given by the well-known Weierstraß representation [20]; the only restriction for the metric g comes from the Gauß equation

$$K + \sqrt{\det(k)} = 0$$

and the Gauß map $\nu : M^2 \to S^2$ needs to be conformal. Any such pair (g, ν) is admissible, and to each admissible pair there exists precisely a one-parameter family of geometrically distinct isometric minimal immersions, the associated family.

3. Historical motivation

Two classical problems concern the embeddings

$$u : M^m \hookrightarrow \mathbb{R}^n. \tag{9}$$

The first is about isometric embedding, i.e. when a metric g on M can be obtained as $u^* ds^2_{\mathrm{Eucl}}$ for some u. In the PDE language this is equivalent to solvability of the system

$$\langle u_i, u_j \rangle = g_{ij}(x), \qquad 1 \leqslant i, j \leqslant m, \qquad u = \left(u^1, \dots, u^n\right) : M^m \to \mathbb{R}^n, \tag{10}$$

where $u_i := \frac{\partial}{\partial x_i} u$.

[3] In fact, a bit more is necessary: $s + \operatorname{Tr} k$ needs to allow a smooth "square root": A function H with $H^2 = s + \operatorname{Tr} k$.

[4] The sign of H is arbitrary, see Remark 3.

The Janet–Cartan theorem [9,19] guarantees this locally in the analytic category for $n=\frac{m(m+1)}{2}$, i.e. when the system (10) is determined. This was improved by Nash [22], Gromov–Rokhlin [15] and others [5,13], who relaxed analyticity to smoothness (for the price of increasing n or imposing some non-degeneracy assumptions) and so proving that embedding is always possible. Recent results include Günter [16,17], Han–Hong [18], and Andrews [4].

When $n<\frac{m(m+1)}{2}$ the system is overdetermined. Thus while rigidity (uniqueness of solutions up to Euclidean motion) is clear in many cases, no general criterion (existence) for local embedding is known (see [14] for details).

The other important problem related to embeddings (9) is to recover it from the Gauß map $\nu: M^m \to G_{n-m}(\mathbb{R}^n)$, $x \mapsto T_xM^\perp \subset \mathbb{R}^n$ (also known as Grassmann map). This problem is unsolvable for hypersurfaces ($n=m+1$) unless the Gauß map is degenerate. In general the problem can be rephrased as solvability of the following PDE system

$$\langle u_i, \nu^\alpha(x)\rangle = 0, \qquad m+1 \leqslant \alpha \leqslant n \tag{11}$$

where ν^α is an orthonormal basis of sections of $TM^\perp$ (no index for hypersurfaces).

For $2m=n=4$, the system (11) is determined while for the other $m<n-1$ overdetermined. By the results of Muto, Aminov, Borisenko [3,6,7,21] the embedding is locally recoverable upon certain non-degeneracy assumptions, up to a parallel translation and homothety. However not any m-dimensional submanifold of $G_{n-m}(\mathbb{R}^n)$ is realizable as the image of a Gauß map (except for the case 2 in 4, when no obstruction equalities exist). The conditions of realizability are not known so far (partial results can be found in [8]).

In this note we unite the systems (10) and (11) and ask when the data (g,ν) are realizable and what is the freedom. In many cases we get indeed rigidity, i.e. an embedding is recoverable up to a parallel translation (this can be obtained as a combination of the problems with the data g and the data ν above, but our conditions are wider; another approach to rigidity within the same problem was taken in [1,10,11]). However in addition to this we write the full set of constraints, thus solving the problem completely.

Notice that as $(n-m)$ grows, the amount of compatibility constraints coming from g decreases while that for ν increases, and there are always constraints for (g,ν). For the hypersurface case $n=m+1$, mainly treated here, the Gauß map alone bears no information (unless it is degenerate), making the problem for the pair (g,ν) more interesting.

4. Proof of the main results

We will work locally in M^m. Given (M,g) and $\nu: M^m \to S^m$, we want to understand whether there exists a smooth (local) map $u: M^m \to \mathbb{R}^{m+1}$ whose partial derivatives with respect to a coordinate chart satisfy

$$\langle u_i, u_j\rangle = g_{ij}, \tag{12}$$

$$\langle u_i, \nu\rangle = 0. \tag{13}$$

This is a system of algebraic equations for the partial derivatives of u_i. Once we obtained an (algebraic) solution $U=(u_i)_{i=1,\dots,m}$ of (12), (13), there exists a smooth mapping $u=(u^1,\dots,u^n): M \to \mathbb{R}^n$ if and only if the integrability conditions

$$u_{ij} = u_{ji} \tag{14}$$

are fulfilled, where the second index means partial derivative: $u_{ij} := \frac{\partial}{\partial x_j} u_i$. Eq. (14) splits into a tangent and a normal part. The tangent part ($\nu^\perp$-part) can be interpreted as follows. Eqs. (12), (13) mean that U is a bundle isometry between TM and $\nu^\perp$. Now the tangent part of (14) says that the canonical connection (via projection) on $\nu^\perp \subset M \times \mathbb{R}^n$ is torsion free when $\nu^\perp$ and TM are identified using U. Since the connection is also metric preserving, it is Levi-Civita:

$$(u_{ij})^T = \Gamma_{ij}^k u_k \tag{15}$$

where $(\ )^T$ denotes the tangent component ($\nu^\perp$-component). In other words, the tangent part of (14) under the assumptions (12), (13) says precisely that $U: TM \to \nu^\perp$ is parallel (affine, connection preserving).

The normal part (ν-part) of (14), in view of (13), is equivalent to

$$h_{ij} = h_{ji}$$

where

$$h_{ij} = \langle u_{ij}, \nu\rangle = -\langle u_i, \nu_j\rangle, \quad h = -A^* U. \tag{16}$$

Once we have got h, we obtain $U=-(A^*)^{-1}h: TM \to \nu^\perp$ from (16) and check orthogonality (12) and parallelity (15).

Next we show that $h^2=k$ is necessary. If an immersion $u: M \to \mathbb{R}^n$ with Gauß map ν is given, then $h=-A^*U$ where $A=d\nu$ and $U=du$ because $h_{ij}=\langle u_{ij},\nu\rangle = -\langle u_i, \nu_j\rangle$. Since h is self adjoint and U orthogonal, we have

$$h^2 = hh^* = A^*UU^*A = A^*A = k.$$

Now let us show that our assumptions are sufficient. Assuming h symmetric with $h^2=k$ and choosing $U=-(A^*)^{-1}h: TM \to \nu^\perp$, we obtain

$$UU^* = \left(A^*\right)^{-1} h^2 A^{-1} = \left(A^*\right)^{-1} k A^{-1} = \left(A^*\right)^{-1} A^* A A^{-1} = I,$$

thus U is an isometry. Moreover $h=-A^*U$ is symmetric, i.e.

$$\langle u_i, \nu_j\rangle = \langle u_j, \nu_i\rangle.$$

Since U takes values in $\nu^\perp$, we have $\langle u_i, \nu\rangle = 0$ and hence

$$\langle u_{ij}, \nu\rangle = \langle u_{ji}, \nu\rangle.$$

This is the normal part of (14). The tangent part is obtained from the parallelity assumption (15), since the Christoffel symbols Γ_{ij}^k are symmetric in (ij). Thus the integrability condition (14) is proved and hence we obtain a map $u: M \to \mathbb{R}^n$ with $du = U$. This finishes the proof of Theorem 1.

Theorem 2 is an obvious consequence of the Gauss equations

$$R_{ijkl} = h_{il}h_{jk} - h_{ik}h_{jl}. \tag{17}$$

In fact, taking the trace over jk, i.e. multiplying by g^{jk} and summing we obtain

$$\mathrm{Ric} = h \cdot H - k. \tag{18}$$

Taking again the trace on both sides,

$$s = H^2 - \mathrm{Tr}\, k. \tag{19}$$

This shows (4). Theorem 2 is proved.

In order to prove Theorem 3, we transform Eqs. (17) into its curvature operator form

$$R(\Omega) = 2 \cdot h\Omega h, \quad \text{i.e., } R^{ij}{}_{kl}\Omega^{kl} = 2 \cdot h^i_l \Omega^{kl} h^j_k$$

(which must be fulfilled for every $\Omega \in \Lambda^2 TM = \mathfrak{so}_g(TM)$). Multiplying by $h^{-2} = k^{-1}$ from the left, we get

$$k^{-1}R(\Omega) = 2 \cdot h^{-1}\Omega h \tag{20}$$

which is equivalent to

$$hk^{-1}R(\Omega) = 2 \cdot \Omega h. \tag{21}$$

This is the linear Eq. (6) for h which we wanted to prove. It remains to show uniqueness of the solution provided $m \geqslant 3$. This will be done in the following

Lemma. *Assume $m \geqslant 3$. If both $h, \tilde{h}$ solve (21) for all $\Omega \in \mathfrak{so}(\mathbb{R}^m)$, then h and $\tilde{h}$ are proportional.*

Proof. Both h and $\tilde{h}$ satisfy (20) and hence

$$h^{-1}\Omega h = \tilde{h}^{-1}\Omega\tilde{h}$$

for all $\Omega \in \mathfrak{so}(\mathbb{R}^m)$. Thus $g := \tilde{h}h^{-1}$ commutes with all $\Omega \in \mathfrak{so}(\mathbb{R}^m)$. If $m \geqslant 3$, the centralizer of $\mathfrak{so}(\mathbb{R}^m)$ contains only the scalar matrices, hence $g = \lambda I$ and $\tilde{h} = \lambda h$ for some $\lambda \in \mathbb{R}$. □

5. The general case

When we study immersions $M^m \to \mathbb{R}^n$ with arbitrary $n > m$, it is not easy to get a closed formula for h as in the main theorems. But as we will see, in the generic case h can still be effectively computed from the data.

The given data are again a Riemannian manifold (M^m, g) and a smooth map $\nu: M \to G_{n-m}(\mathbb{R}^n)$ into the Grassmannian of $(n-m)$-planes in $\mathbb{R}^n$, and we ask if ν is the Gauß map of some isometric immersion $u: M \to \mathbb{R}^n$. Choose an orthonormal basis $(\nu^\alpha)_{\alpha=1,\dots,n-m}$ of ν. Let

$$A^\alpha = \left(d\nu^\alpha\right)^T$$

denote the corresponding Weingarten operators, where $(\ )^T$ again denotes the tangent component ($\nu^\perp$-component), and let

$$k = \sum k^{\alpha\alpha}, \qquad k^{\alpha\beta} = \left(A^\alpha\right)^* A^\beta \tag{22}$$

the corresponding third fundamental on M induced by ν from the standard symmetric metric on $G_{n-m}(\mathbb{R}^n)$. The second fundamental form h which we search for, is ν-valued and has also several components $h^\alpha = \langle h, \nu^\alpha\rangle$: Given an isometric immersion $u: M \to \mathbb{R}^n$ and $U = du = (u_1, \dots, u_m)$, we have

$$h^\alpha_{ij} = \left\langle u_{ij}, \nu^\alpha\right\rangle = -\left\langle u_i, \nu^\alpha_j\right\rangle, \qquad h^\alpha = -\left(A^\alpha\right)^* U.$$

Consequently

$$h^\alpha h^\beta = h^\alpha (h^\beta)^* = (A^\alpha)^* U U^* A^\beta = k^{\alpha\beta}.$$

There exist deformations of isometric immersions with fixed k, see [24], which are different from ours for codimensions exceeding 1. However in this case k bears significantly less information than the Gauß map ν. With the latter we can restore the immersion up to a translation in a generic case.

Theorem 4. *Let (M^m, g) and $\nu : M \to G_{n-m}(\mathbb{R}^n)$ be given and assume that at least one of the $A^\alpha = (d\nu^\alpha)^T : TM \to \nu^\perp$ is everywhere invertible. Let $k^{\alpha\beta}$ be as in (22). Then the data (g, ν) are admissible if and only if there exist $h^\alpha \in S^2 T^* M$ with*

$$h^\alpha h^\beta = k^{\alpha\beta} \tag{23}$$

and a vector bundle homomorphism $U : TM \to \nu^\perp$ with

$$h^\alpha = -(A^\alpha)^* U \tag{24}$$

for all α, such that U is parallel with respect to the Levi-Civita connection on TM and the projection connection on $\nu^\perp$. In fact, the corresponding immersion $u : M^m \to \mathbb{R}^n$ is determined by $du = U$.

The proof is almost the same as before and will be omitted. But as before we need an effective method to compute h^α from the given data. This is given by the next theorem:

Theorem 5. *Assume that the data (g, ν) are admissible with $|H| = \sqrt{s + \operatorname{Tr} k} \neq 0$ and $\operatorname{Ric} + k$ invertible. Then*

$$h^\beta = \sum_\alpha H^\alpha (\operatorname{Ric} + k)^{-1} k^{\alpha\beta} \tag{25}$$

where $H^\alpha = \operatorname{Tr} h^\alpha$ are the components of the mean curvature vector $H = \operatorname{Tr} h$ which is a fixed vector with length $\sqrt{s + \operatorname{Tr} k}$ for the matrix $\rho = (\rho_{\alpha\beta})$ on ν defined by

$$\rho_{\alpha\beta} = \operatorname{Tr}\big((\operatorname{Ric} + k)^{-1}\big) k^{\alpha\beta}. \tag{26}$$

Proof. Suppose that an isometric immersion $u : M \to \mathbb{R}^n$ with Gauß map ν is given. The Gauß equations are

$$R_{ijkl} = \sum_\alpha h^\alpha_{il} h^\alpha_{jk} - h^\alpha_{ik} h^\alpha_{jl}.$$

Taking the trace over jk yields

$$\operatorname{Ric} = \sum_\alpha \big(h^\alpha H^\alpha - (h^\alpha)^2\big) = \sum_\alpha h^\alpha H^\alpha - k$$

and hence

$$\sum_\alpha H^\alpha h^\alpha = \operatorname{Ric} + k. \tag{27}$$

Tracing again we obtain the length of the mean curvature vector,

$$|H| = \sqrt{s + \operatorname{Tr} k}. \tag{28}$$

From (27) we can compute the h^α since the products $h^\alpha h^\beta = k^{\alpha\beta}$ are known:

$$(\operatorname{Ric} + k) h^\beta = \sum_\alpha H^\alpha k^{\alpha\beta}$$

and (25) follows. In order to compute H^α we take the trace of (25):

$$H^\beta = \sum_\alpha H^\alpha \operatorname{Tr}\big((\operatorname{Ric} + k)^{-1} k^{\alpha\beta}\big) = \sum_\alpha H^\alpha \rho_{\alpha\beta}$$

with $\rho_{\alpha\beta}$ as in (26). Thus H is a fixed vector of the matrix $\rho = (\rho_{\alpha\beta})$. In the generic case this fixed space is only one-dimensional (it is at least one-dimensional since it contains $H \neq 0$). Using (28) we see that H is uniquely determined up to sign.[5] □

[5] The sign of H cannot be fixed. Indeed, as in Remark 3, if $u : M \to \mathbb{R}^n$ is an immerion with mean curvature vector H, then $-u$ has the same Gauß map and mean curvature vector $-H$.

Once again we have got an algorithm by which we may check if the data (g, ν) are admissible, belonging to some isometric immersion u. From the data we form the matrix ρ and check if it has a fixed vector. In the generic case, the fixed space is at most one-dimensional. We choose a fixed vector H using (28). Then we define the quadratic form h^β by (25) and check if it satisfies (23). Setting $U = -(A^{\alpha *})^{-1} h^\alpha$ for one α we check if (24) holds true for the other indices α and if U is parallel. The data (g, ν) are admissible if and only if all the tests are successful.

In non-degenerate cases, similar to the hypersurface case, the Gauß equation follows from Codazzi and Ricci equations. Non-uniqueness of solution for this system means isometric deformation with fixed Gauß image, and was completely understood by Dajczer and Gromoll in [10].

Acknowledgement

This work began when the authors met at the 10th International Conference on Differential Geometry and its Applications in Olomouc. We wish to express our thanks for hospitality.

Appendix A

We want to show that under the assumptions (12), (13) and $(14)^\perp$, Codazzi equations are equivalent to parallelity of U, $(14)^T$, and they imply Gauß equations.

Theorem 6. *Let (M, g) and $\nu : M \to S^m$ be given and $d\nu$ nondegenerate. Let $U = (u_1, \ldots, u_m) : TM \to \nu^\perp$ be a vector bundle isometry such that $b_{ij} := \langle u_i, \nu_j \rangle = -h_{ij}$ is symmetric (normal integrability condition). Then*

$$\nabla U = 0 \quad \Leftrightarrow \quad (\nabla_i b)_{jk} = (\nabla_j b)_{ik}, \tag{29}$$

$$\nabla U = 0 \quad \Rightarrow \quad R_{ijkl} = b_{il} b_{jk} - b_{ik} b_{jl}. \tag{30}$$

Proof. "$\Rightarrow$" of (29): From $b = U^* d\nu = \langle U, d\nu \rangle$ we obtain

$$\nabla b = \langle \nabla U, d\nu \rangle + \langle U, \nabla d\nu \rangle. \tag{31}$$

Since $\nabla U = 0$, we are left with $\nabla b = \langle U, \nabla d\nu \rangle$ or more precisely,

$$(\nabla_i b)_{jk} = \big\langle u_j, (\nabla_i d\nu)_k \big\rangle.$$

Since the right-hand side (the Hessian of the map ν) is symmetric in ik, we have proved our claim.

"$\Leftarrow$" of (29): We still have (31), more precisely

$$(\nabla_i b)_{jk} = \big\langle (\nabla_i U)_j, \nu_k \big\rangle + \big\langle u_j, (\nabla_i d\nu)_k \big\rangle.$$

From the symmetry of ∇b and $\nabla d\nu$ in ik we see

$$\big\langle (\nabla_i U)_j, \nu_k \big\rangle = \big\langle (\nabla_k U)_j, \nu_i \big\rangle. \tag{32}$$

On the other hand, by covariant differentiation of the isometry property $\langle U, U \rangle = U^* U = g$ we obtain $\langle \nabla U, U \rangle + \langle U, \nabla U \rangle = 0$, more precisely

$$\big\langle (\nabla_i U)_j, u_k \big\rangle + \big\langle u_j, (\nabla_i U)_k \big\rangle = 0. \tag{33}$$

Since $U^* d\nu = b$ is self adjoint with respect to g, we can choose local coordinates in such a way that both tensors g and b are diagonal at the considered point and hence $u_k = \lambda_k \nu_k$ for each k (where we have used the non-degeneracy of b). Substituting this into (33) and putting $\theta_{ijk} = \langle (\nabla_i U)_j, \nu_k \rangle$, we get

$$\lambda_k \theta_{ijk} + \lambda_j \theta_{jik} = 0$$

(no summation). Cycling (ijk) and using (32), $\theta_{ijk} = \theta_{kji}$, we get 3 equations in the 3 unknowns $\theta_{ijk}, \theta_{jki}, \theta_{kij}$. The determinant of this linear system equals $2\lambda_i \lambda_j \lambda_k \neq 0$, and therefore $\langle (\nabla_i U)_j, \nu_k \rangle = \theta_{ijk} = 0$. Since the vectors ν_k form a basis of $\nu^\perp$, we obtain $\nabla U = 0$.

Proof of (30): Since $U : TM \to \nu^\perp$ is isometric and parallel, it carries the Riemannian curvature tensor on TM into the curvature tensor of the projection connection ∇ on $\nu^\perp$ which is computed as usual:

$$\nabla_j u_k = (u_{kj})^T = u_{kj} + b_{kj} \nu$$

$$\nabla_i \nabla_j u_k = \big(\partial_i (\nabla_j u_k)\big)^T = (u_{kji})^T + b_{kj} \nu_i,$$

$$\langle \nabla_i \nabla_j u_k, u_l \rangle = \big\langle \partial_i (\nabla_j u_k), u_l \big\rangle = \langle u_{kji}, u_l \rangle + b_{kj} b_{il},$$

$$R_{ijkl} = \big\langle (\nabla_i \nabla_j - \nabla_j \nabla_i) u_k, u_l \big\rangle = b_{kj} b_{il} - b_{ki} b_{jl},$$

using the symmetry of $u_{kji} = \partial_i \partial_j u_k$ in ij. The last equality is the Gauß equation (30) which finishes the proof. □

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