Dephasing of qubits by the Schrödinger cat

J. Dajka [a,*], M. Mierzejewski [a], J. Łuczka [a], P. Hänggi [b]

[a] Institute of Physics, University of Silesia, 40-007 Katowice, Poland
[b] Institute of Physics, University of Augsburg, D-86135 Augsburg, Germany

1. Introduction

Controlling the dynamics of open quantum systems is of crucial importance for the quantum information processing [1]. As there is no general method for analyzing the non-Markovian reduced dynamics, the exactly solvable models may provide important and unbiased results. One of the examples is the dephasing model [2–6] that describes an idealized case when the quantum system does not exchange the energy with its environment. This model has recently been studied in the context of entanglement dynamics [7–10] and the geometric phases [11]. In particular, it has been shown in Ref. [8] that the entanglement can effectively be controlled by an external *finite* bosonic quantum system prepared in so-called *non-classical* states [12].

In this paper we study the complementary case when the *infinite* bosonic system is initially prepared in the Schrödinger cat state. For a finite bosonic system such a state is defined as a superposition of two coherent states with the same amplitudes but with phases shifted by π [12]. Here we generalize this notion to the case of infinite dimensional systems composed of bath and system dynamics. We show that the reduced dynamics of the qubit depends on a specific choice of the initial Schrödinger cat state. This is in clear contrast to the situation when the initial state is purely coherent. It holds true not only for purity and coherence of a single qubit but also for entanglement of a two-qubit system.

Due to the decoherence phenomenon, the assumed initial state of an infinite bosonic bath is inaccessible in the present experiments. However, the development of experimental techniques allows one to manipulate and control systems devised from an increasing number of particles [13]. Therefore, the results presented in this paper may serve as a starting point for understanding of qubits coupled to large bosonic systems prepared in a desired quantum state. Our choice of the initial state is motivated by the fact that multiple Schrödinger cat states can accurately approximate any quantum state [14,15].

2. Model

We consider a qubit Q, which interacts with the environment R. The Hamiltonian of the total system reads [2,3]

$$H = H_Q \otimes \mathbb{I}_R + \mathbb{I}_Q \otimes H_R + H_I, \tag{1}$$

where $\mathbb{I}_Q$ and $\mathbb{I}_R$ are identity operators in corresponding Hilbert spaces of the qubit Q and the environment R, respectively. The qubit Hamiltonian H_Q is in the form

$$H_Q = \varepsilon S^z \equiv \varepsilon(|1\rangle\langle 1| - |-1\rangle\langle -1|), \tag{2}$$

where the canonical basis of the qubit is $\{|1\rangle, |-1\rangle\}$ and $\pm\varepsilon$ are the energy levels of the qubit. When Q represents a particle of spin $S = 1/2$, the energy ε is proportional to the magnitude of the external magnetic field. The environment is assumed to be a boson field described by the Hamiltonian

$$H_R = \int_0^\infty d\omega\, h(\omega) a^\dagger(\omega) a(\omega), \tag{3}$$

where the real-valued dispersion relation $h(\omega)$ specifies the environment, e.g., $h(\omega) = \omega$ describes phonon or photon environment. The operators $a^\dagger(\omega)$ and $a(\omega)$ are the creation and annihilation boson operators, respectively. The coupling of the qubit to the environment is described by the Hamiltonian

$$H_I = |1\rangle\langle 1| \otimes H_+ + |-1\rangle\langle -1| \otimes H_- \tag{4}$$

with

$$H_\pm = \pm \int_0^\infty d\omega\, G(\omega)[a(\omega) + a^\dagger(\omega)], \tag{5}$$

* Corresponding author.
E-mail address: dajka@phys.us.edu.pl (J. Dajka).

where the function $G(\omega)$ is the coupling strength. Without loosing generality, we assume that it is a real function. The Hamiltonian (1) can be rewritten in the form

$$H = |1\rangle\langle 1| \otimes H_1 + |-1\rangle\langle -1| \otimes H_{-1}, \tag{6}$$

$$H_{\pm 1} = H_R + H_{\pm} \pm \varepsilon. \tag{7}$$

Since there is no energy exchange (i.e. we use a non-demolition coupling) between the qubit and the environment, our modeling corresponds to pure dephasing. Hamiltonians like (7) have been exploited for description of the inter-conversion of electronic and vibrational energy [16], the electron-transfer reactions [17], a quantum kicked rotator [18], chaotic dynamics of a periodically driven superconducting single electron transistor [19] and the Josephson flux qubit [20], to mention but a few.

3. Exact reduced dynamics

The model we study is exactly solvable [2,4,6], i.e., the Schrödinger equation for the wave function $|\Psi(t)\rangle$ of the total system can be solved exactly. Here we follow the method presented in Ref. [9]. First, one needs to specify an initial state $|\Psi(0)\rangle$. Let us assume that at the initial time $t = 0$, the wave function has the form

$$|\Psi(0)\rangle = (b_1|1\rangle + b_{-1}|-1\rangle) \otimes |R\rangle, \tag{8}$$

where b_1 and b_{-1} determine the qubit initial state and $|R\rangle$ is the initial state of the environment. Then

$$|\Psi(t)\rangle = b_1|1\rangle \otimes |\psi_1(t)\rangle + b_{-1}|-1\rangle \otimes |\psi_{-1}(t)\rangle, \tag{9}$$

where $|\psi_i(t)\rangle = \exp[-H_i t]|R\rangle$ $(i = \pm 1)$ can be rewritten in the form [9]

$$|\psi_1(t)\rangle = e^{-i\Lambda_1(t)} D(g_t^+ - g^+) e^{-iH_R t}|R\rangle,$$
$$|\psi_{-1}(t)\rangle = e^{-i\Lambda_{-1}(t)} D(g^- - g_t^-) e^{-iH_R t}|R\rangle. \tag{10}$$

The phases $\Lambda_1(t)$ and $\Lambda_{-1}(t)$ are given by

$$\Lambda_{1/-1}(t) = \pm\varepsilon t - \int_0^\infty d\omega\, g^2(\omega)\{h(\omega)t - \sin[h(\omega)t]\}, \tag{11}$$

where the abbreviation $g(\omega) = G(\omega)/h(\omega)$ has been introduced. For any function f, the notation f_t stands for

$$f_t(\omega) = e^{-ih(\omega)t} f(\omega). \tag{12}$$

For an arbitrary square-integrable function f, the displacement operator $D(f)$ is defined as [21]

$$D(f) = \exp\left\{\int_0^\infty d\omega [f(\omega)a^\dagger(\omega) - f^*(\omega)a(\omega)]\right\}. \tag{13}$$

The reduced qubit dynamics can be obtained for any factorizable initial state of the form

$$\varrho(0) = \sum_{i,j=1,-1} p_{ij}|i\rangle\langle j| \otimes |R\rangle\langle R|, \tag{14}$$

where $\varrho(0)$ is the initial statistical operator of the total system and p_{ij} are non–negative parameters. The reduced statistical operator $\rho(t)$ for the qubit alone can be obtained by tracing the environment degrees of freedom, namely,

$$\begin{aligned}\rho(t) &= \mathrm{Tr}_R[\varrho(t)] \\ &= \sum_{i,j=1,-1} p_{ij}|i\rangle\langle j| \otimes \mathrm{Tr}_R(e^{-iH_i t}|R\rangle\langle R|e^{iH_j t}) \\ &= \sum_{i,j=1,-1} p_{ij}c_{ji}(t)|i\rangle\langle j|,\end{aligned} \tag{15}$$

where Tr_R denotes the partial tracing over the environment variables, H_i for $i = \pm 1$ is given by Eq. (7) and $c_{ji}(t) = \langle\psi_j(t)|\psi_i(t)\rangle$ is a scalar product between the functions $|\psi_j(t)\rangle$ and $|\psi_i(t)\rangle$ in the environmental Hilbert space. The initial state of the qubit $|\theta,\phi\rangle$ is commonly parametrized by two angles on the Bloch sphere: the polar angle θ and azimuthal angle ϕ. Then

$$|\theta,\phi\rangle = \cos(\theta/2)|1\rangle + e^{i\phi}\sin(\theta/2)|-1\rangle. \tag{16}$$

In this parametrization $b_1 = \cos(\theta/2)$ and $b_{-1} = e^{i\phi}\sin(\theta/2)$ (see Eq. (8)) and the initial density matrix $\rho(0)$ of the reduced qubit dynamics reads

$$\rho(0) = \begin{pmatrix} \cos^2(\theta/2) & (1/2)\sin\theta e^{-i\phi} \\ (1/2)\sin\theta e^{i\phi} & \sin^2(\theta/2) \end{pmatrix}. \tag{17}$$

From Eq. (15) we obtain the density matrix $\rho(t)$ in the form

$$\rho(t) = \begin{pmatrix} \cos^2(\theta/2) & (1/2)A(t)\sin\theta e^{-i\phi} \\ (1/2)A^*(t)\sin\theta e^{i\phi} & \sin^2(\theta/2) \end{pmatrix}. \tag{18}$$

All information about influence of the environment on the qubit is incorporated in the dephasing function $A(t) = c_{-1,1}(t)$.

In the following we assume that initially the environment is in the pure Schrödinger cat state, which is defined by the relation

$$|R\rangle = \frac{1}{\sqrt{N}}[|\alpha\rangle + e^{i\Phi}|-\alpha\rangle], \tag{19}$$

where $|\alpha\rangle = D(\alpha)|\Omega\rangle$ is the coherent state determined by the function $\alpha = \alpha(\omega)$ and $|\Omega\rangle$ is the vacuum state of the bosonic bath. The normalization constant

$$N = 2 + 2\cos(\Phi)\exp\left[-2\int_0^\infty d\omega|\alpha(\omega)|^2\right]. \tag{20}$$

The phase Φ allows to manipulate the initial state of the environment. In this case, the dephasing function becomes

$$\begin{aligned}A(t) = N^{-1}[&\langle\alpha_{-1}(t)|\alpha_1(t)\rangle + \langle\alpha_{-1}(t)|-\alpha_1(t)\rangle e^{i\Phi} \\ &+ \langle-\alpha_{-1}(t)|\alpha_1(t)\rangle e^{-i\Phi} + \langle-\alpha_{-1}(t)|-\alpha_1(t)\rangle]\end{aligned} \tag{21}$$

with $|\alpha_{\pm 1}(t)\rangle = \exp(-iH_{\pm 1}t)|\alpha\rangle$. For the sake of brevity we calculate the explicit form of the dephasing function $A(t)$ for the case of given coherent states $|\alpha\rangle$ determined by real functions $\alpha(\omega)$ only. As a first main result we find

$$A(t) = N^{-1}A_0(t)e^{-2i\varepsilon t}\{A_+(t)e^{-i\Phi} + A_-(t)e^{i\Phi} + 2\cos[4\Lambda_\alpha(t)]\}, \tag{22}$$

where

$$\Lambda_\alpha(t) = \int_0^\infty d\omega\, \alpha(\omega)g(\omega)\sin(h(\omega)t), \tag{23}$$

$$A_0(t) = \exp\left\{-4\int_0^\infty d\omega\, g^2(\omega)[1 - \cos(h(\omega)t)]\right\}, \tag{24}$$

$$A_\pm(t) = \exp\left\{-2\int_0^\infty d\omega\, \alpha^2(\omega) \mp 4\int_0^\infty d\omega\, \alpha(\omega)g(\omega)[1 - \cos(h(\omega)t)]\right\}. \tag{25}$$

As we show next, the dephasing function $A(t)$ determines certain quantifiers describing various aspects of quantum information.

4. Purity and coherence

We start with basic quantifiers describing the information loss of the qubit. The first one is the *purity* defined by

$$\mathscr{P}(t) = \mathrm{Tr}(\rho^2(t)) = \tfrac{1}{2}(|A(t)|^2 - 1)\sin^2\theta + 1. \tag{26}$$

Its interpretation is clear: the environment results in a decrease of the purity. It is equal to 1 for pure states and 1/2 for maximally mixed states. To quantify coherence, we introduce the coherence factor $\mathscr{C}(t)$ which is determined by the evolution of the non-diagonal elements of the qubit reduced density matrix,

$$|\rho_{12}(t)| = \mathscr{C}(t)|\rho_{12}(0)|. \tag{27}$$

Comparison of Eqs. (17) and (18) yields

$$\mathscr{C}(t) = |A(t)|. \tag{28}$$

The coherence factor is maximal in the absence of the qubit–bath interaction, i.e., $\mathscr{C}(t) = 1$, and vanishes for the case of complete decoherence, $\mathscr{C}(t) = 0$.

5. Entanglement decay

In order to study the influence of the dephasing on the quantum non-locality, we extend the previous model and include a second, completely independent, qubit q. The Hamiltonian of such a composite system thus reads

$$H = [H_Q + H_R + H_I] \otimes \mathbb{I}_q + H_q, \tag{29}$$

$$H_q = \mathbb{I}_Q \otimes \mathbb{I}_R \otimes \varepsilon S_q^z. \tag{30}$$

We assume that the correlations between both the qubits are encoded in their initial entanglement. For simplicity, we take the depolarized Bell states as the initial state, i.e.,

$$\rho(0) = (1-p)\rho_i + \frac{p}{4}\mathbb{I}_Q \otimes \mathbb{I}_q, \quad i = 1, \ldots, 4 \tag{31}$$

with

$$\rho_{1/2} = \tfrac{1}{2}[|-1,1\rangle \pm |1,-1\rangle][\langle -1,1| \pm \langle 1,-1|], \tag{32}$$

$$\rho_{3/4} = \tfrac{1}{2}[|-1,-1\rangle \pm |1,1\rangle][\langle -1,-1| \pm \langle 1,1|]. \tag{33}$$

The depolarization accounts for an imperfect preparation of the initial state.

In a general case, the state of an open system is mixed. To quantify its entanglement, several useful measures have been proposed [22]. One of the operational measures is the negativity, defined by $N(\rho) = \max(0, -\sum_i \lambda_i)$, where λ_i are the negative eigenvalues of the partially transposed density matrix of two qubits [23]. For the model under consideration the negativity can straightforwardly be evaluated for an arbitrary evolution time t. One obtains

$$N(\rho(t)) = \max\left(0, \frac{1-p}{2}|A(t)| - \frac{p}{4}\right). \tag{34}$$

The negativity is positive for an entangled mixed state, whereas it vanishes for unentangled states. Moreover, it presents an entanglement monotone and can be used to quantify the degree of entanglement.

6. Discussion

The main quantifiers like purity (26), coherence factor (28) or negativity (34) depend directly on the dephasing function $A(t)$. Therefore, we start with discussing its properties in further detail. The dephasing function depends on the qubit–environment coupling via the functions $g(\omega) = G(\omega)/h(\omega)$ and $\alpha(\omega)$. The latter one defines the initial Schrödinger cat state. For convenience, we can assume that both functions are real. We also introduce the new function $J(\omega) \equiv \omega^2 g^2(\omega)$. Then, the comparison of the function $A_0(t)$ (see Eq. (24)) with the standard expression for the decoherence function (see e.g. Ref. [6, Eq. (4.51)]), allows one to identify $J(\omega)$ as the spectral density. In the literature, there are several examples of $J(\omega)$ in use. A frequently used one is the generalized Drude form defined by [2]

$$J(\omega) = \lambda\, \omega^{1+\mu} \exp(-\omega/\omega_c), \tag{35}$$

where $\mu > -1$ and ω_c is the cut-off frequency. The case $\mu \in (-1, 0)$ corresponds to a sub-ohmic, $\mu = 0$ to the conventional ohmic and $\mu \in (0, \infty)$ to a super-ohmic environment.

One can observe that the long-time limit is given by

$$A_0 = \lim_{t\to\infty} A_0(t) = \exp\left\{-4\int_0^\infty d\omega J(\omega)/\omega^2\right\}. \tag{36}$$

The integral in this expression is infinite for a sub-ohmic and an ohmic environment. Then $A_0 = 0$ and the dephasing function diminishes to zero, $\lim_{t\to\infty} A(t) = 0$. Consequently, purity (26), coherence factor (28) and negativity (34) asymptotically take on the following asymptotic long-time values:

$$\mathscr{P} = 1 - \tfrac{1}{2}\sin^2\theta, \quad \mathscr{C} = 0, \quad N = 0. \tag{37}$$

One can see that for the sub-ohmic and ohmic environments all the quantifiers are independent of any particular choice of $|\alpha\rangle$. It means that in the long-time regime the qubit properties do not depend any longer on the initial Schrödinger cat state. The super-ohmic case is more intriguing because $A_0 > 0$. As it follows from the expression for the normalization constant N (see Eq. (20)), the function $\alpha(\omega)$ is square-integrable. Starting from the Cauchy–Schwarz inequality one also can find that integrals (23) and (25) exist, are finite and their values depend on the function $\alpha(\omega)$. In consequence, the dephasing function depends on both α and Φ, i.e. on the initial state of the environment. Therefore, all characteristics (26), (28) and (34), do depend on the initial environment state, provided the environment is *super-ohmic*.

It is instructive to compare Eq. (22) with the dephasing function obtained for an initial, purely coherent state $|R\rangle = |\alpha\rangle$ of the environment. In this case the dephasing function $A(t)$ becomes

$$A(t) = \exp(-2i\varepsilon t)\exp[-4i\Lambda_\alpha(t)]A_0(t). \tag{38}$$

In clear contrast to the initial Schrödinger cat state, $|A(t)|$ now depends only on $A_0(t)$. This fact implies that purity (26), coherence factor (28) and negativity (34) are independent of the initial state of the environment also for a super-ohmic bath.

7. Conclusions

Dephasing characteristics of qubits coupled to a bosonic environment and prepared in a Schrödinger cat state has been investigated. The properties of the reduced dynamics, as reflected in the purity and coherence factor, have been shown to exhibit an explicit phase-dependence Φ as a parameter of the Schrödinger cat state. Qualitatively the same behavior has been obtained for the entanglement feature, being quantified by the negativity. The main conclusion is the following: if the initial state of the environment is the coherent (or vacuum) state then the informational quantifiers do not depend on the initial state. However, if the initial state is a linear combination of two coherent states as the Schrödinger cat state then such quantifiers as the purity, coherence factor or the negativity do depend on the initial state—via the function α and the phase Φ—of the environment, at least in the short-to-intermediate time regime. For the super-ohmic environment this result holds true also in the long-time limit. Moreover, the Φ-dependence allows one to selectively control the dephasing characteristics and the entanglement characteristics.

Acknowledgments

The work supported by Polish Ministry of Science and Higher Education under the Grant no. 202 131 32/3786. We also gratefully acknowledge (P.H.) the financial support by the German Excellence Initiative via the "Nanosystems Initiative Munich (NIM)" and the DFG through the collaborative research centre SFB 631.

References

[1] R. Alicki, Controlled open quantum systems irreversible quantum dynamics, Lecture Notes in Physics, vol. 622, Springer, Berlin, 2003.
[2] J. Łuczka, Phys. A 167 (1990) 919.
[3] H. Spohn, Comm. Math. Phys. 123 (1987) 277;
R. Alicki, Open Syst. Inform. Dyn. 11 (2004) 53.
[4] G.W. Unruh, Phys. Rev. A 51 (1995) 992.
[5] G.M. Palma, K.A. Suominen, A.K. Ekert, Proc. Roy. Soc. London Ser. A 452 (1996) 567.
[6] H.-P. Breuer, F. Petruccione, The Theory of Open Quantum Systems, Oxford University Press, Oxfrord, 2002.
[7] J. Dajka, M. Mierzejewski, J. Łuczka, J. Phys. A Math. Theor. 40 (2007) F879.
[8] J. Dajka, M. Mierzejwski, J. Łuczka, Phys. Rev. A 77 (2008) 042316.
[9] J. Dajka, J. Łuczka, Phys. Rev. A 77 (2008) 062303.
[10] R. Doll, M. Wubs, P. Hänggi, S. Kohler, Europhys. Lett. 76 (2006) 547;
R. Doll, M. Wubs, P. Hänggi, S. Kohler, Phys. Rev. B 76 (2007) 045317.
[11] J. Dajka, M. Mierzejewski, J. Łuczka, J. Phys. A Math. Theor. 41 (2008) F012001;
J. Dajka, J. Luczka, J. Phys. A Math. Theor. 41 (2008) F442001.
[12] C.C. Gerry, P.L. Knight, Introductory Quantum Optics, Cambridge University Press, Cambridge, 2006.
[13] S. Haroche, M. Brune, J.-M. Raimond, European Phys. J. Special Topics 159 (2008) 19.
[14] P. Domokos, I. Ianszky, P. Adam, T. Larsen, Quantum Opt. 6 (1994) 187.
[15] J. Janszky, A. Petak, C. Sibilia, C. Bertolotti, P. Adam, Quantum Semiclass. Opt. 7 (1995) 145.
[16] J.J. Markham, Rev. Modern Phys. 31 (1956) 956;
S.H. Lin, J. Chem. Phys. 44 (1966) 3759.
[17] J. Tang, Chem. Phys. 188 (1994) 143.
[18] S.A. Gardiner, J.I. Cirac, P. Zoller, Phys. Rev. Lett. 97 (1997) 4790.
[19] S. Montangero, A. Romito, G. Benenti, R. Fazio, Europhys. Lett. 71 (2005) 893.
[20] E.N. Pozzo, D. Dominguez, Phys. Rev. Lett. 98 (2007) 057006.
[21] O. Brattelli, D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics, Springer, Berlin, 1997.
[22] F. Verstraete, K. Audenaert, J. Dehaene, B. De Moor, J. Phys. A Math. Gen. 34 (2001) 10327;
G. Vidal, R.F. Werner, Phys. Rev. A 65 (2002) 032314.
[23] A. Peres, Phys. Rev. Lett. 77 (1996) 1413.