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Codimension of immersions with parallel pluri-mean curvature

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1. Introduction

Surfaces with constant mean curvature (cmc) in euclidean 3-space form a classical theme in differential geometry. Like minimal surfaces, they allow for a one-parameter deformation (the so called associated family) of isometric immersion preserving the principal curvatures while rotating the second fundamental form α. In higher complex dimensions $m \geqslant 2$, if $f : M \to \mathbb{R}^n$ is an isometric immersion of a Kähler manifold,[1] the rôle of the mean curvature vector $H = \operatorname{trace} \alpha$ is taken over by the so called pluri-mean curvature which is defined as follows. The complexified tangent bundle T^c of M has a parallel decomposition as $T^c = T' + T''$ where T', T'' are the $\pm i$-eigenbundles of the almost complex structure J. Its symmetric tensor product $S^2 T^c$ splits accordingly:

$$S^2 T^c = S^2 T' + S^2 T'' + T' \otimes T''. \tag{1}$$

The restrictions of α to these subspaces are called $\alpha^{20}, \alpha^{02}, \alpha^{11}$, respectively. The latter component, α^{11}, is called *pluri-mean curvature* since when restricted to any complex curve $C \subset M$, it is the mean curvature vector of $f|_C$.

When α^{11} is parallel, the immersion f is called of *parallel pluri-mean curvature* (*ppmc*), see [2] as a general reference. Such immersions also have a one-parameter family of isometric deformations f_θ which only rotate the second fundamental form α at every point by a constant angle θ. An important special case happens when $f_\theta = f$ for all θ; such ppmc immersions are called *isotropic*. Among the cmc surfaces the only example of this type is the round sphere, but there are many other examples in higher dimensions: the standard embedded Kähler symmetric (also called hermitian symmetric) spaces. In the present paper we will show that those have the least codimension among all ppmc immersions, with the only possible

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[1] A Kähler manifold is a Riemannian manifold with a parallel tensor field J with $J^2 = -I$ (almost complex structure).

exception when M has a local DeRham factor which is locally isometric to a complex Grassmannian or its noncompact dual, other than $\mathbb{CP}^m$ or $\mathbb{CH}^m$.

2. The pluri-mean curvature

Let M be a Kähler manifold with almost complex structure J. Denote T the tangent bundle of M and T^c its complexification. Since J is parallel, so are its eigenspaces T' and $T'' = \overline{T'}$ corresponding to the eigenvalues $i, -i$. Thus we have a parallel decomposition $T^c = T' + T''$. For any $P, Q \in T^c$ we denote by

$$P \circ Q = \frac{1}{2}(P \otimes Q + Q \otimes P)$$

the symmetrized tensor product. If $P = X + \bar{Y}$ and $Q = U + \bar{V}$, we have

$$P \circ Q = X \circ U + \bar{Y} \circ \bar{V} + X \circ \bar{V} + U \circ \bar{Y}$$

which shows the decomposition (1). For any $X, Y \in T'$ we consider the linear map

$$XY^* : T' \to T', \quad Z \mapsto XY^*Z = X\langle \bar{Y}, Z\rangle$$

where Y^* is the linear map $Y^* = \bar{Y}^T = \langle \bar{Y}, \ \rangle \in \mathrm{Hom}(T', \mathbb{C})$ and where $\langle\ ,\ \rangle$ denotes the $\mathbb{C}$-bilinear extension of the Kähler metric to $T^c = T \otimes \mathbb{C}$. Putting $\hat{T} = \mathrm{Hom}(T', T')$, we have a parallel bundle isomorphism

$$T' \otimes T'' \to \hat{T}, \quad X \otimes \bar{Y} \mapsto XY^* \tag{2}$$

which we will use to identify $T' \otimes T''$ with $\hat{T}$ in the sequel.

On T', the bilinear inner product vanishes, $\langle T', T'\rangle = 0$. Instead we consider the hermitian inner product

$$(X, Y) := \langle \bar{X}, Y\rangle = X^*Y.$$

For any linear map $A : T' \to T'$ its adjoint map A^* is defined as usual:

$$A^*X, Y) := (X, AY).$$

In particular we have

$$(XY^*)^* = YX^*.$$

Now let $f : M \to \mathbb{R}^n$ be an isometric immersion with normal bundle N and second fundamental form $\alpha : S^2T \to N$ which will be linearly extended to S^2T^c with values in $N^c = N \otimes \mathbb{C}$. Using the identification (2) we have

$$S^2T^c = S^2T' + S^2T'' + \hat{T}. \tag{3}$$

Let $\alpha^{11} = \alpha|_{\hat{T}}$.

Lemma 2.1. *For all $X, Y \in T'$, the expression $\xi = \alpha(X^*Y + Y^*X)$ is real while $\eta = \alpha(X^*Y - Y^*X)$ is imaginary.*

Proof. Recall $\alpha(X^*Y) = \alpha(X, \bar{Y})$. But $\overline{\alpha(X, \bar{Y})} = \alpha(\bar{X}, Y) = \alpha(Y, \bar{X})$. Hence $\bar{\xi} = \xi$ and $\bar{\eta} = -\eta$. $\square$

We will split $\hat{T} = \mathrm{Hom}(T', T')$ further as $\hat{T} = \hat{T}_+ + \hat{T}_-$ where $\hat{T}_+$ and $\hat{T}_-$, respectively, denote the subspaces of hermitian and antihermitian linear maps. This splitting is parallel, too. If $\alpha^{11} : \hat{T} \to N^c$ is parallel, the same holds for its restriction $\alpha^{11}_+ = \alpha^{11}|_{\hat{T}_+} : \hat{T}_+ \to N$. Thus the image N^o of α^{11}_+ is a parallel subbundle of N.

3. Holonomy

Let M be Kähler and $f : M \to \mathbb{R}^n$ ppmc. Fix some $p \in M$. Let Hol^T be the set of all parallel transports τ_γ along closed curves ("loops") γ starting and ending at p. Since the metric and the almost complex structure J are preserved under parallel transport, Hol^T is a subset of the unitary group $U(T_p)$ on the tangent space at p. Since concatenation of loops results in composition of the corresponding parallel transports, Hol^T is a group, a subgroup of $U_m = U(T_p)$, called *holonomy group*. It acts also on $\hat{T} = T' \otimes T''$ in the obvious way, $h(X \otimes \bar{Y}) = hX \otimes h\bar{Y}$, as the holonomy group of the vector bundle $\hat{T}$ preserving the subbundles $\hat{T}_\pm$. Further we will consider the holonomy group Hol^{N^o} of $N^o \subset N$.

Proposition 3.1. *If $f : M \to \mathbb{R}^n$ is ppmc, there is a surjective group homomorphism $\phi : \mathrm{Hol}^T \to \mathrm{Hol}^{N^o}$ such that $\alpha^{11}_+ : \hat{T}_+ \to N^o$ is equivariant with respect to ϕ,*

$$\alpha^{11}_+ \circ h = \phi(h) \circ \alpha^{11}_+ \tag{4}$$

for all $h \in \mathrm{Hol}^T$.

Proof. Let $h \in \mathrm{Hol}^T$. Then there is a loop γ at p such that $h = \tau_\gamma$. Since α_+^{11} preserves parallel transport, we have

$$\alpha_+^{11} \circ \tau_\gamma = \tau_\gamma^o \circ \alpha_+^{11}$$

where τ_γ^o denotes parallel transport in the vector bundle N^o. Thus there is an element $h^o \in \mathrm{Hol}^{N^o}$ with $h^o = \tau_\gamma^o$. Now we obtain the equality

$$\alpha_+^{11} \circ h = h^o \circ \alpha_+^{11} \tag{5}$$

from which we see that $h^o \in \mathrm{Hol}^{N^o}$ is determined by h, i.e. independent of the choice of γ (mind that α_+^{11} is surjective onto N^o). We put $\phi(h) = h^o$. The homomorphism property follows easily from concatenation of loops and the equivariance of α_+^{11} from (5). $\square$

Now suppose that M is locally irreducible as a Riemannian manifold, i.e. it does not split locally as a nontrivial Riemannian product. By the holonomy theorem of Berger and Simons [9,10] we know that $\mathrm{Hol}^T \supset SU_m$ unless M is locally Kähler symmetric. Thus our last Proposition 3.1 implies:

Theorem 3.2. *Let M be a locally irreducible Kähler manifold of complex dimension m and $f : M \to \mathbb{R}^n$ a ppmc immersion. Then f has real codimension $\geqslant m^2$ unless f is pluriminimal, i.e. $\alpha^{11} = 0$, or M is locally Kähler symmetric.*

Proof. If the mean curvature vector H vanishes, f is pluriminimal, see [3]. Thus H is a nonzero parallel section of N^o which is fixed under Hol^{N^o}. Consider the decomposition of $\hat{T}_+$ into irreducible SU_m-modules:

$$\hat{T}_+ \cong \hat{T}_- \cong \mathfrak{u}_m \cong \mathbb{R} \oplus \mathfrak{su}_m, \tag{6}$$

where SU_m acts trivially on the $\mathbb{R}$-factor and by the adjoint representation on $\mathfrak{su}_m$. If M is not locally Kähler symmetric, Hol^T contains SU_m and therefore the decomposition (6) is irreducible also with respect to Hol^T. Now consider the kernel of α_+^{11} which is also Hol^T-module, due to the equivariance (5). If $\mathfrak{su}_m \subset \ker \alpha_+^{11}$, then $N^o = \mathbb{R}H$ which is possible only for $m = 1$, cf. [6].[2] Thus we may assume $\mathfrak{su}_m \not\subset \ker \alpha_+^{11}$, and by irreducibility of $\mathfrak{su}_m$ we have in fact $\mathfrak{su}_m \cap \ker \alpha_+^{11} = 0$. Thus the Hol^T-homomorphism α_+^{11} is injective on $\mathfrak{su}_m$, and therefore also N^o contains a holonomy submodule W isomorphic to $\mathfrak{su}_m$.[3] Since W must be perpendicular to the fixed space of Hol^{N^o} containing H, the dimension of N^o is at least one more than the dimension of $\mathfrak{su}_m$ which is $m^2 - 1$. Thus $\operatorname{codim}(f) \geqslant \dim N^o \geqslant m^2$. $\square$

Remark. There is a distinguished element in $\hat{T}_+$, the identity I. This is mapped by α_+^{11} onto the mean curvature vector H. In fact, we have $I = \sum_j e_j e_j^*$ for any unitary basis (e_j) of T', and therefore

$$\alpha(I) = \sum_j \alpha\left(e_j e_j^*\right) = \sum_j \alpha(e_j, \overline{e_j}) = H. \tag{7}$$

4. Kähler symmetric spaces

Now we consider the remaining case where M is locally irreducibly Kähler symmetric (hermitian symmetric). Thus M is locally isometric to an irreducible Kähler symmetric space $\hat{M}$ acted on holomorphically by its transvection group G. More precisely, at any point $p \in \hat{M}$, the isotropy group $G_p = K$ acts complex linearly in $T_p\hat{M}$ with respect to its complex structure J_p, and K contains the complex unitary scalars $\mathbb{S}^1$ as a central subgroup, generated by J_p. In particular, $J_p \in \mathfrak{g}_p = \mathfrak{k} \subset \mathfrak{g}$ and hence the mapping $p \mapsto J_p$ defines an embedding of $\hat{M}$ into its transvection Lie algebra $\mathfrak{g}$, the so called *standard embedding*. This is ppmc with $\alpha^{20} = 0$, cf. [7], and since the Cartan decomposition $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ agrees with the decomposition $\mathfrak{g} = T_p\hat{M} + N_p\hat{M}$ into tangent and normal spaces of $\hat{M}$, the codimension of this embedding is the dimension of $\mathfrak{k}$.

Now let us consider an arbitrary ppmc immersion $f : M \to \mathbb{R}^n$ where M can be viewed (at least locally) as an open subset of $\hat{M}$. Again we have $\hat{T}_+ \cong \hat{T}_- = \mathfrak{u}_m$, but this time the holonomy group Hol^T at p is smaller: it is equal to the isotropy group K (cf. [4,8]). Let $\mathfrak{q}$ be the complement of $\mathfrak{k}$ in $\mathfrak{u}_m$, i.e.

$$\hat{T}_+ \cong \mathfrak{u}_m = \mathfrak{k} + \mathfrak{q} = \mathbb{R} + \mathfrak{k}' + \mathfrak{q}. \tag{8}$$

[2] Here is the argument: Since M is Kähler, we have $R(X, Y) = 0$ for all $X, Y \in T'$. On the other hand, the Gauss equations yield $\langle R(X, Y)\bar{Y}, \bar{X}\rangle = \langle \alpha(X, \bar{X}), \alpha(Y, \bar{Y})\rangle - \langle \alpha(X, \bar{Y}), \alpha(Y, \bar{X})\rangle$. If α^{11} is parallel and $N^o = \mathbb{R}H$, then $\alpha(X, \bar{Y}) = \beta(X, \bar{Y})H$ for some parallel bilinear form β. By irreducibility, β is the metric, $\alpha(X, \bar{Y}) = \langle X, \bar{Y}\rangle H$. Thus

$$0 = \left\langle R(X, Y)\bar{Y}, \bar{X}\right\rangle = \left(|X|^2|Y|^2 - (X, Y)^2\right)|H|^2$$

which implies either X, Y linear dependent ($m = 1$) or $H = 0$ (f superminimal).

[3] The injectivity of $\alpha_+^{11}|_{\mathfrak{su}_m}$ together with (5) shows $\mathfrak{su}_m \cong W$ as SU_m-modules. In particular, the derivative of the Lie group homomorphism ϕ maps the Lie algebra of $SU_m \subset \mathrm{Hol}^T$ injectively into the Lie algebra of Hol^{N^o}.

Table 1

	G/K	$\dim \mathfrak{u}_m = m^2$	$\dim \mathfrak{k}$	$\dim \mathfrak{q}$
1	$SU_{p+q}/S(U_p \times U_q)$	p^2q^2	p^2+q^2-1	$(p^2-1)(q^2-1)$
2	SO_{2p}/U_p	$\frac{(p-1)^2}{4}p^2$	p^2	$(\frac{(p-1)^2}{4}-1)p^2$
3	$SO_{p+2}/(SO_p \times SO_2)$	p^2	$\frac{p(p-1)}{2}+1$	$\frac{p(p+1)}{2}-1$
4	Sp_p/U_p	$\frac{(p+1)^2}{4}p^2$	p^2	$(\frac{(p+1)^2}{4}-1)p^2$
5	$E_6/(Spin_{10} \cdot U_1)$	256	46	210
6	$E_7/(E_6 \cdot U_1)$	729	79	650

It has been shown by Wang and Ziller [11] that the K-module $\mathfrak{q}$ is irreducible, and in most cases $\mathfrak{k}'$ is irreducible too, thus (8) is an irreducible decomposition of $\hat{T}$ with respect to K. As before, α_+^{11} maps the identity map I spanning the $\mathbb{R}$-factor of $\hat{T}_+$ onto $H \in N^o$. Thus the K-module $\ker \alpha_+^{11}$ must contain $\mathfrak{k}'$ or $\mathfrak{q}$. But unless $\hat{M} = \mathbb{CP}^m$ or $\mathbb{CH}^m$ (where the holonomy group is U_m, a case which was treated in the last section), the dimension of $\mathfrak{q}$ is much bigger than the one of $\mathfrak{k}'$, see Table 1.

Theorem 4.1. *Let M be irreducible and locally isometric to the Kähler symmetric space $\hat{M} = G/K$ where G is the transvection group of $\hat{M}$ and $K = K' \cdot \mathbb{S}^1$ the isotropy group, and suppose that the Lie algebra $\mathfrak{k}'$ of K' is simple. Then every ppmc immersion $f : M \to \mathbb{R}^n$ has codimension at least as big as* $\dim \mathfrak{k}$ *(cf. Table 1). If f is isotropic ppmc, then the codimension is at least* $4 + \dim \mathfrak{k}$ *unless f is the standard embedding.*

Remarks.

(1) If f is isotropic ppmc, M is locally symmetric [5].
(2) Any irreducible Kähler symmetric space $\hat{M} = G/(K' \cdot \mathbb{S}^1)$ has simple $\mathfrak{k}'$ unless it is a complex Grassmannian of higher rank or its noncompact dual. In this case we will see from the proof that $\dim \mathfrak{k}$ in the theorem has to be replaced by the minimum of $p^2 = \dim \mathfrak{u}_p$ and $q^2 = \dim \mathfrak{u}_q$.

Proof. If $\ker \alpha_+^{11}$ contains $\mathfrak{k}' + \mathfrak{q}$, the space N^o is one-dimensional (generated by H) which is impossible, see above. Thus it contains at most one of these two factors, and the other one is mapped isomorphically into N^o. Since $\dim \mathfrak{q} > \dim \mathfrak{k}$, we have $\mathrm{codim}(f) \geqslant \dim N^o \geqslant \dim \mathfrak{k}' + 1 = \dim \mathfrak{k}$.

It remains to prove the claim on *isotropic* ppmc immersions $f : M \to \mathbb{R}^n$. Then the image of α^{20} is isotropic and perpendicular to N^o, cf. [2]. If $\alpha^{20} = 0$ it follows easily that $\nabla\alpha = 0$,[4] and moreover that f is the standard embedding [7]. If $\alpha^{20} \neq 0$, the dimension of N is at least two more than the one of N^o and hence $\geqslant 2 + \dim \mathfrak{k}$. Further, the image of $\nabla\alpha$ is perpendicular to that of α, cf. [5]. Thus we obtain at least another two normal dimensions unless $\nabla\alpha = 0$. But the latter extrinsic symmetric case is well known by [7]; besides the standard embeddings there is only the case of the Grassmannian of oriented 2-planes

$$G_2^+(\mathbb{R}^n) = SO_n/(SO_{n-2} \times SO_2)$$

with its embedding into the space of symmetric $n \times n$-matrices with trace 0 where the codimension is $n - 1 + \dim \mathfrak{k}$. However, $G_2^+(\mathbb{R}^n)$ is the 2-sphere for $n = 3$ and reducible for $n = 4$ since $G_2^+(\mathbb{R}^4) = \mathbb{S}^2 \times \mathbb{S}^2$. Thus $n \geqslant 5$ and the codimension is $\geqslant 4 + \dim \mathfrak{k}$ in this case too. $\square$

Table 1 contains the dimensions of the representations (cf. [1], pp. 311–317). We have $\mathfrak{q} = 0$ iff $G/K = \mathbb{CP}^m$ or its dual. This happens in No. 1 for $p = 1$ or $q = 1$, in Nos. 2 for $p = 3$, and in Nos. 3 and 4 for $p = 1$. In No. 2, cases $p = 1$ and $p = 2$, the action of G on G/K is strongly ineffective, i.e. its kernel has positive dimension. In No. 3, for $p = 2$, the space $G_2^+(\mathbb{R}^4) = \mathbb{S}^2 \times \mathbb{S}^2$ is reducible, for $p = 3$ we have $G_2^+(\mathbb{R}^5) = SO_5/(SO_3 \times SO_2) = Sp_2/U_2$ (cf. No. 4) and for $p = 4$, the space $G_2^+(\mathbb{R}^6) = SO_6/(SO_4 \times SO_2) = SU_4/S(U_2 \times U_2) = G_2(\mathbb{C}^4)$ is a complex Grassmannian (No. 1).

5. Reducible Kähler manifolds

Theorem 5.1. *Let M be an open subset of a Riemannian product $M_1 \times M_2$ where M_i $(i = 1, 2)$ are m_i-dimensional Kähler manifolds without a flat local DeRham factor. Let $f : M \to \mathbb{R}^n$ be a ppmc immersion which is not a product of immersions. Then f has codimension at least $2m_1m_2$.*

Proof. Since our claim is local, we may assume $M = M_1 \times M_2$. Let $\pi_i : M \to M_i$ be the canonical projection. Put $T_i = \pi_i^*(TM_i)$ and define $T_i', \hat{T}_i, \hat{T}_{i+}$ accordingly. We have the parallel decomposition

$$\hat{T}_+ = \hat{T}_{1+} + \hat{T}_{2+} + \mathrm{Hom}(T_1', T_2'). \tag{9}$$

[4] $\nabla\alpha$ is symmetric by Codazzi equations, and all of its four components of types 30, 21, 12, 03 are derivatives of α^{20} or α^{02} which are zero.

The holonomy group H_i of M_i acts irreducibly on T'_i. Thus $\mathrm{Hom}(T'_1, T'_2)$ is irreducible under the action of $H_1 \times H_2$. If $\mathrm{Hom}(T'_1, T'_2) \subset \ker \alpha_+^{11}$, using the interior product structure and the Gauss equations we have for any $X \in T'_1$, $Y \in T'_2$:

$$\langle \alpha(X, \bar{Y}), \alpha(\bar{X}, Y) \rangle = \langle \alpha(X, Y), \alpha(\bar{X}, \bar{Y}) \rangle. \tag{10}$$

Thus from $\alpha(T'_1, T''_2) = 0$ we obtain $\alpha(T'_1, T'_2) = 0$ and hence $\alpha(T_1, T_2) = 0$. This implies the splitting of the immersion f which was excluded. Hence by irreducibility we may assume $\mathrm{Hom}(T'_1, T'_2) \cap \ker \alpha_+^{11} = 0$. Therefore we find a copy of $\mathrm{Hom}(T'_1, T'_2)$ inside N^0. Since $\mathrm{Hom}(T'_1, T'_2)$ has complex dimension $m_1 m_2$, the codimension of f (the dimension of $N \supset N^o$) is bounded from below by $2m_1 m_2$. $\square$

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