Directed transport in periodically rocked random sawtooth potentials

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We study directed transport of overdamped particles in a periodically rocked random sawtooth potential. Two transport regimes can be identified which are characterized by a nonzero value of the average velocity of particles and a zero value, respectively. The properties of directed transport in these regimes are investigated both analytically and numerically in terms of a random sawtooth potential and a periodically varying driving force. Precise conditions for the occurrence of transition between these two transport regimes are derived and analyzed in detail.

PACS number(s): 05.60.Cd, 05.10.Gg

I. INTRODUCTION

Directed transport in ratchet systems, i.e., devices rectifying undirected driving forces (both random and deterministic) into directed motion of the transported particles or localized structures, is much in the limelight of present activities. The reason is that the rectifying property of these systems, the so-called ratchet effect, constitutes a theoretical basis for operating Brownian motors [1–5] which, for example, can be employed to do surface smoothing [6–9] or the separation of particles [10–15]. The topic is also closely related to a variety of yet other intriguing noise-induced transport phenomena [1–4,16].

Usually the action of ratchet systems on the transported particles is described by a spatially asymmetric *periodic* potential. The assumption of strict periodicity of a ratchet potential is technically convenient, but in many systems the validity of this assumption is not guaranteed. Therefore, if a sizable spatial aperiodicity is the result of quenched disorder, it is advantageous to use a *random* ratchet potential for describing the transport properties in such systems. Within this approach, some of these properties that result from quenched disorder have already been studied [17–22].

If the noise arising from the environment can be neglected then the average drift velocity of overdamped particles in a periodic ratchet potential exhibits a threshold dependence on the amplitude of the time-periodic driving force (see also Refs. [23,24]). In this case only one transport regime with a nonzero average velocity exists whenever the amplitude of the driving force exceeds the threshold value. If the amplitude is less than threshold, the particles remain localized mainly within one period of the ratchet potential although coexisting bounded solutions may exist [10]. However, this picture can be changed drastically in ratchet systems containing quenched disorder. Indeed, if for a random ratchet potential the threshold amplitude exists and the amplitude of the driving force exceeds this threshold value then the ordinary

transport regime, i.e., transport with a *nonzero* average velocity of particles and an *arbitrary large* transport distance, emerges. But if the driving amplitude is somewhat smaller than the threshold value then a *new transport regime* characterized by a *zero* average velocity and a *finite* transport distance is expected to be realized. Moreover, since at the threshold amplitude these regimes merge, it is plausible that the transport distance approaches infinity if the driving amplitude tends to the threshold one. The aim of this paper is to study analytically and numerically these different transport regimes in ratchet systems with quenched disorder described by a random sawtooth potential.

The paper is structured as follows. In Sec. II, we introduce the overdamped equation of motion for particles in a random sawtooth potential driven by a dichotomously alternating force and formulate the main definitions and assumptions. Directed transport of particles with a nonzero average velocity is considered in Sec. III. Here we derive an explicit formula for the average velocity in the adiabatic limit and numerically study this transport regime depending on the amplitude and period of the driving force. In Sec. IV, we consider some aspects of directed transport of particles with a vanishing average velocity. Specifically, we derive the average transport distance of particles in the preferential direction and study the transition between the transport regimes with zero and nonzero average velocities. Our main findings are summarized in Sec V.

II. DEFINITIONS AND BASIC EQUATIONS

We study the directed transport of particles governed by the dimensionless overdamped equation of motion,

$$\dot{X}_t = g(X_t) + f(t). \qquad (2.1)$$

Here, X_t ($X_0=0$) denotes the particle coordinate, $f(t)$ is a periodically varying driving force of a period $2T$, and $g(x)=-dU(x)/dx= \pm g_\pm$ presents a dichotomous random force which is generated by a random sawtooth potential $U(x)$, i.e., a piecewise linear random potential such as the
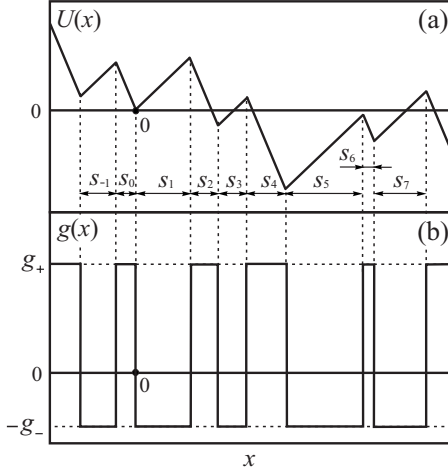
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FIG. 1. Schematic representation of (a) the random sawtooth potential $U(x)$ and (b) the corresponding dichotomous random force $g(x)$ as functions of the spatial coordinate x.

one depicted in Fig. 1. This random potential $U(x)$ is characterized by (i) statistically independent random intervals of lengths s_j which are distributed with the probability densities $p_+(s)$ and $p_-(s)$ for even numbered ($j=2n$, $n=0,\pm1,\ldots$) and odd numbered ($j=2n+1$) intervals, respectively, (ii) two deterministic slopes $-g_+$ and g_- ($g_+>g_->0$), and (iii) the constraint condition $g_+s_+=g_-s_-$, with $s_\pm=\int_0^\infty ds\, s\, p_\pm(s)<\infty$ denoting the average lengths of even, s_+, and odd, s_-, intervals. The last condition implies that the average value of the dichotomous random force equals zero, i.e.,

$$\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L}dx\,g(x)=\lim_{L\to\infty}\left(g_+\frac{L_+}{L}-g_-\frac{L_-}{L}\right)=g_+s_+-g_-s_-=0,$$
(2.2)

where $2L_+$ and $2L_-$ denote the total lengths of the even and the odd intervals on the interval $(-L,L)$. In addition, we assume that at the origin of the coordinate system all sample paths of this potential change its slope from $-g_+$ (at $x=-0$) to g_- (at $x=+0$), i.e., $g(\pm0)=\mp g_\mp$. We also note that the special case of the above-defined potential when $g_+=g_-$ has been invoked to study the statistical properties of the arrival time and the arrival position of particles in a medium with quenched dichotomous disorder driven by a constant force [25,26].

This chosen form of the random potential in Fig. 1 possesses random (i.e., nonperiodic) barrier widths with corresponding (random) barrier heights and comprises the complexity of more general realizations of random landscapes while at the same time allowing for an explicit analytical treatment (see also the further remarks given before Sec. V below).

The periodic driving force $f(t)$ is assumed to be alternating, i.e., $f(t)=(-1)^{k+1}f$, where $f(>0)$ is the amplitude of $f(t)$ or the driving strength, $k=[t/T]+1$, and $[t/T]$ is the integer part of t/T. This force can therefore be explicitly written as

$$f(t)=\begin{cases}f, & (2k-2)T\le t<(2k-1)T\\ -f, & (2k-1)T\le t<2kT\end{cases}$$
(2.3)

($k=1,2,\ldots$), and since

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau dt\,f(t)=0,$$
(2.4)

its average value equals zero as well.

In accordance with Eq. (2.1), if $f\le g_-$ then the particle remains localized in the initial state for all $t>0$; the particle coordinate depends on time only if $f>g_-$. Although the average values of the forces $g(x)$ and $f(t)$ are zero, a systematic displacement of particles along the positive direction (since $g_+>g_-$) of the axis x may nevertheless exist. In other words, the potential $U(x)$ can rectify the periodic motion of particles induced by the periodic alternating force $f(t)$ into directed transport. This phenomenon is known as the ratchet effect [2–5].

III. AVERAGE TRANSPORT VELOCITY

A. Long-period limit

We next define the average transport velocity v_T of particles in the following way:

$$v_T=\lim_{t\to\infty}\frac{\langle X_t\rangle}{t},$$
(3.1)

where the angular brackets denote an averaging over the sample paths of $g(x)$. In the case of long-lasting period of the driving force, $T\to\infty$, it is possible to derive the exact formula for v_∞. The starting point is that in the definition [Eq. (3.1)] we can replace t by $2T$ and X_{2T} by X_T-Y_T, yielding

$$v_\infty=\lim_{T\to\infty}\frac{\langle X_T\rangle-\langle Y_T\rangle}{2T},$$
(3.2)

where X_T is the displacement of the particle during the first half period of $f(t)$, and $Y_T=X_T-X_{2T}$ is the displacement during the second half period. As pointed out above, at $f\le g_-$ the particle stays in the initial state for all t, and so $v_\infty=0$. A simple analysis of Eq. (2.1) shows that if $f>g_-$ then $X_T>0$ and $\lim_{T\to\infty}\langle X_T\rangle/T\in[f-g_-,f+g_+]$, i.e., this limit is always nonvanishing. In contrast, the sign of Y_T depends on f and $g(X_T)$, and the limit $\lim_{T\to\infty}\langle Y_T\rangle/T$ can either be zero or nonzero. Specifically, if $g_-<f\le g_+$ then $Y_T>0$ at $g(X_T)=-g_-$ and $Y_T\le0$ at $g(X_T)=g_+$. Since in this case the particle moves during the second half-period only in one odd interval [if $g(X_T)=-g_-$] or in one even interval [if $g(X_T)=g_+$], we obtain $\lim_{T\to\infty}\langle Y_T\rangle/T=0$. On the contrary, if $f>g_+$ then $Y_T>0$ and the particle passes during the second half period an infinite number of intervals s_j when $T\to\infty$. In this case $\lim_{T\to\infty}\langle Y_T\rangle/T\in[f-g_+,f+g_-]$, i.e., the limit is nonzero. One therefore expects that the dependence of v_∞ on f differs in these two cases.

Let us first derive the average velocity for $g_-<f\le g_+$. In this case $\lim_{T\to\infty}\langle Y_T\rangle/T=0$, and Eq. (3.2) becomes

$$v_\infty = \lim_{T\to\infty} \frac{\langle X_T \rangle}{2T}. \qquad (3.3)$$

Representing X_T in the form $X_T = X_T^+ + X_T^-$, where X_T^+ and X_T^- are the total lengths of the even and odd intervals s_j on the interval $(0, X_T)$, respectively, and using the relation $g_+\langle X_T^+\rangle = g_-\langle X_T^-\rangle$, which follows from the condition $g_+ s_+ = g_- s_-$, we find

$$\langle X_T \rangle = \langle X_T^- \rangle \left(1 + \frac{g_-}{g_+}\right). \qquad (3.4)$$

On the other hand, because the particle passes the even and odd intervals with the velocities $f + g_+$ and $f - g_-$, respectively, we have

$$T = \frac{\langle X_T^+ \rangle}{f + g_+} + \frac{\langle X_T^- \rangle}{f - g_-} = \langle X_T^- \rangle \left(1 + \frac{g_-}{g_+}\right)\frac{f + g_+ - g_-}{(f+g_+)(f-g_-)}. \qquad (3.5)$$

Finally, substituting Eqs. (3.4) and (3.5) into Eq. (3.3), the resulting average velocity assumes the form

$$v_\infty = \frac{(f+g_+)(f-g_-)}{2(f+g_+-g_-)}. \qquad (3.6)$$

If $f > g_+$ then the average velocity of particles is defined by Eq. (3.2). Introducing by analogy with the previous case the total lengths Y_T^+ and Y_T^- of the even and odd intervals s_j on the interval $(0, Y_T)$, for the second half period of $f(t)$ we obtain

$$\langle Y_T \rangle = \langle Y_T^- \rangle \left(1 + \frac{g_-}{g_+}\right) \qquad (3.7)$$

and, likewise,

$$T = \frac{\langle Y_T^+ \rangle}{f - g_+} + \frac{\langle Y_T^- \rangle}{f + g_-} = \langle Y_T^- \rangle \left(1 + \frac{g_-}{g_+}\right)\frac{f - g_+ + g_-}{(f-g_+)(f+g_-)}. \qquad (3.8)$$

Calculating with the help of these formulas the limit $\lim_{T\to\infty}\langle Y_T \rangle / T$, Eq. (3.2) gives the result

$$v_\infty = \frac{(f+g_+)(f-g_-)}{2(f+g_+-g_-)} - \frac{(f-g_+)(f+g_-)}{2(f-g_++g_-)}, \qquad (3.9)$$

which can be simplified to read

$$v_\infty = \frac{g_+ g_-(g_+ - g_-)}{f^2 - (g_+ - g_-)^2}. \qquad (3.10)$$

Thus, in the adiabatic limit the average velocity of particles in a random sawtooth potential $U(x)$ driven by a periodically alternating force $f(t)$ is given by Eq. (3.6) if $g_- < f \leq g_+$ and Eq. (3.10) if $f > g_+$. It is important to note that these results do not depend on the concrete distributions of the intervals s_j. In fact, in evaluating the average velocity we used only the condition that the probability densities $p_\pm(s)$ possess finite first moments $s_\pm$.

Since by assumption $g_+ > g_-$, the transport of particles occurs with the average velocity v_∞ in the positive direction of the axis x. In accordance with Eqs. (3.6) and (3.10), v_∞ is
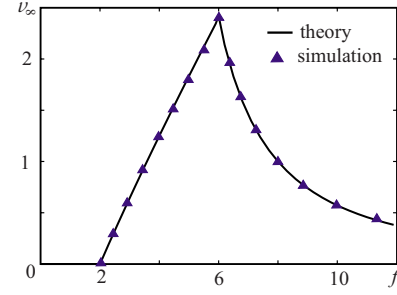

FIG. 2. (Color online) Average transport velocity v_∞ of particles as a function of the driving strength f in the adiabatic limit. The solid lines represent the theoretical results obtained from Eqs. (3.6) and (3.10), and the triangular symbols (blue) indicate results derived from the numerical simulations of Eq. (2.1). The presented results correspond to the dichotomous random force $g(x)$ with $g_+ = 6$ and $g_- = 2$.

a nonmonotonic function of f which assumes the maximum value

$$v_{\infty\,\mathrm{max}} = \frac{g_+(g_+ - g_-)}{2g_+ - g_-} \qquad (3.11)$$

for $f = g_+$, i.e., at the point where the character of v_∞ as a function of f changes qualitatively. If, on the other hand, $g_+ < g_-$ then the transport of particles occurs in the negative direction of the axis x. In this case, the average velocity (and the corresponding conditions for f) is determined by Eqs. (3.6) and (3.10) in which g_+ and g_- must be replaced by g_- and g_+, respectively. At $g_+ = g_-$ the average velocity v_∞ equals zero for all f.

The predicted dependence of v_∞ on f and the numerical results obtained by simulation of Eq. (2.1) are depicted in Fig. 2 for the case with $g_+ = 6$ and $g_- = 2$. For each f the numerical average velocity,

$$v_{\mathrm{sim}} = \frac{1}{N}\sum_{i=1}^{N} \frac{X_{2T}^{(i)}}{2T}, \qquad (3.12)$$

shown in this figure by the triangular symbols, was calculated for $N = 2 \times 10^2$ runs of the motion Eq. (2.1). Before each run, a new realization of the dichotomous force $g(x)$ was generated in accordance with the uniform probability densities

$$p_\pm(s) = \begin{cases} (2d_\pm)^{-1}, & 0 \leq s \leq 2d_\pm \\ 0, & s > 2d_\pm. \end{cases} \qquad (3.13)$$

Since in this case $s_\pm = d_\pm$, we chose $d_+ = 0.5$ and $d_- = 1.5$ in order to satisfy the condition $g_+ s_+ = g_- s_-$. Finally, to ensure that the particle passes a large number of the intervals s_j, we chose $T = 10^2 d_-/(f-g_-)$ if $f \in (g_-, g_+]$, and $T = 10^2 \max d_\pm/(f-g_\pm)$ if $f > g_+$. As seen from Fig. 2, our numerical results obtained in such a way are in excellent agreement with theory. It should be noted, however, that we used the uniform distributions for the intervals s_j only for illustrative purposes: the average velocity v_T of particles does not depend on $p_\pm(s)$ in the limit $T \to \infty$.
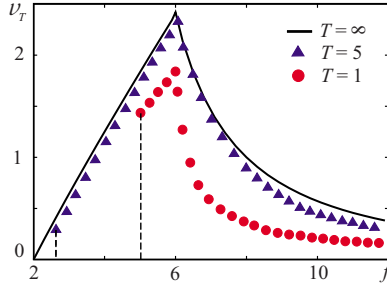
FIG. 3. (Color online) Average transport velocity v_T as a function of the driving strength f for different values of the half-period T. The triangular (blue) and circular (red) symbols represent the numerical results obtained via $N = 10^3$ runs of Eq. (2.1) and by using the numerical average velocity [Eq. (3.12)] in which $2T$ is replaced by $40T$. The theoretical dependence of v_T on f for $T = \infty$ (solid lines) reproduces the average velocity v_∞ from Fig. 2 and is shown for comparison only. The parameters characterizing the dichotomous random force $g(x)$ whose intervals s_j are distributed with uniform probability densities [Eq. (3.12)] are chosen to be $g_+ = 6$, $g_- = 2$, $d_+ = 0.5$, and $d_- = 1.5$. According to Eq. (3.14), $f_{th} = 2.6$ for $T = 5$ and $f_{th} = 5$ for $T = 1$.

B. Nonadiabatic driving

In contrast to the previous case, if T is finite then the average velocity v_T depends on the explicit form of the probability densities $p_\pm(s)$. It is important to note that in this case the condition $f > g_-$ does not guarantee that $v_T > 0$ (at $g_+ > g_-$). Specifically, if $\int_a^\infty ds\, p_-(s) > 0$ for arbitrary large (but finite) a, i.e., if $p_-(s)$ is the probability density with unbounded support, then $\langle X_\infty \rangle < \infty$ and so $v_T = 0$ for all finite f and T. However, even if $p_-(s) = 0$ for $s > b$, i.e., if $p_-(s)$ has bounded support, the average velocity equals zero as well if $T < T_{th} = b/(f - g_-)$, where T_{th} is the threshold half period. From a physical point of view, the condition $v_T = 0$, i.e., the absence of directed transport of particles toward infinity, arises from the fact that there is a nonzero probability of those (odd) intervals s_j that cannot be overcome by particles during a positive pulse of $f(t)$. The existence of directed transport of particles with a zero average velocity and a finite transport distance will be considered in more detail in the next section.

In accordance with the above discussion, directed transport of particles with a nonzero average velocity exists only if both conditions, $f > g_-$ and $T > T_{th}$, hold true. Since the latter condition is more restrictive than the former, it is the latter that determines the criterion of directed transport of particles with nonzero average velocity v_T. In particular, if $p_-(s)$ is the uniform probability density [see Eq. (3.13)], then this criterion can be written in the form

$$f > f_{th} = g_- + \frac{2d_-}{T}, \qquad (3.14)$$

where f_{th} is the threshold amplitude of the driving force $f(t)$. The dependencies of v_T on f obtained by the numerical simulation of the motion Eq. (2.1) are shown in Fig. 3. Typical solutions of this equation for $f \in (f_{th}, g_+)$ and $f > g_+$ are illustrated in Fig. 4.
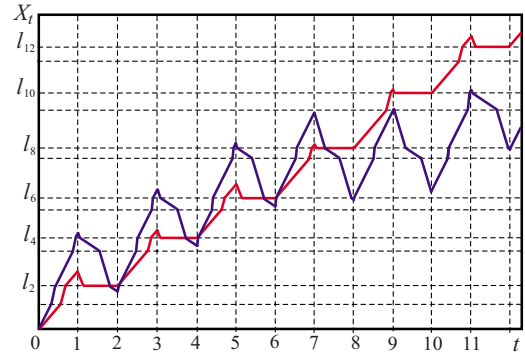

FIG. 4. (Color online) Illustrative realizations of the particle coordinate X_t in the case of nonzero average velocity v_T. The parameters of the dichotomous random force $g(x)$ are chosen as in Fig. 3, $T = 1$, and $f_{th} = 5$. The transport regime with $v_T > 0$ occurs at $f > f_{th}$, and X_t displays different behavior for $f \in (f_{th}, g_+)$ and $f > g_+$. The line with horizontal segments (red) represents X_t for $f = 5.5$ (in this case $f < g_+ = 6$ and $v_{T=1} = 1.62$), and the other line (blue) represents X_t for $f = 8$ ($f > g_+$, $v_{T=1} = 1.00$). For convenience, the distances $l_n = \sum_{j=1}^n s_j$, which correspond to a given sample path of $g(x)$, are shown for even n only.

IV. DIRECTED TRANSPORT WITH ZERO AVERAGE VELOCITY

In the case of zero average velocity the particles cannot be transported to an arbitrary large distance along the axis x in a common way. Instead, for each realization of $g(x)$ the particles are transported to any position in whose vicinity they oscillate (see Fig. 5). For random $g(x)$ these positions are random as well, and since $g_+ > g_-$, they are preferably distributed at $x > 0$. Our next objective is to find the average distance $\langle l \rangle$ from the origin of the coordinate system to these positions in the positive direction of the axis x.

Let us assume that the interval s_{2n+1} with some $n (\geq 0)$ is the *first* odd interval satisfying the condition $s_{2n+1} > \Delta$, where
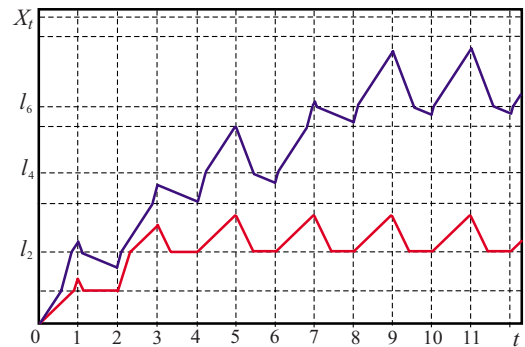

FIG. 5. (Color online) Illustrative realizations of the particle coordinate X_t in the case of zero average velocity v_T. The parameters of the dichotomous random force $g(x)$ are the same as in Fig. 3, $T = 0.5$, and $f_{th} = 8$. In this case, the transport regime with $v_T = 0$ occurs only if $f \in (g_-, f_{th})$, and X_t shows different behavior for $f < g_+$ and $f > g_+$. The lower line (red) represents X_t for $f = 5.5$ ($f < g_+ = 6$ and $\langle l \rangle = 1,93$), and the upper line (blue) for $f = 7.5$ ($f > g_+$ and $\langle l \rangle = 20,63$).

$$\Delta = (f - g_-)T \tag{4.1}$$

is the minimal displacement of particles during a positive pulse of the driving force $f(t)$. In other words, s_{2n+1} is the first interval which is not crossed by particles in the positive direction of the axis x. The distance from the coordinate origin to this interval is given by

$$l_{2n} = \sum_{j=1}^{2n} s_j \tag{4.2}$$

if $n \geq 1$, and $l_0 = 0$ if $n = 0$. We also introduce the probability

$$w = 1 - \int_\Delta^\infty ds\, p_-(s) = \int_0^\Delta ds\, p_-(s) \tag{4.3}$$

that the length of the odd interval is smaller than Δ. In accordance with these definitions, the probability density $P(l)$ that $l = l_{2n}$ (with $n \geq 0$) can be written in the form

$$P(l) = (1 - w) \sum_{n=1}^\infty \int_0^\Delta \cdots \int_0^\Delta \left(\prod_{j=1}^n ds_{2j-1} p_-(s_{2j-1}) \right)$$
$$\times \int_0^\infty \cdots \int_0^\infty \left(\prod_{k=1}^n ds_{2k} p_+(s_{2k}) \right) \delta(l - l_{2n}) + (1 - w)\delta(l), \tag{4.4}$$

where $\delta(\cdot)$ is the Dirac δ function. Using the geometric series formula $\sum_{n=0}^\infty w^n = (1-w)^{-1}$, it is not difficult to verify that $P(l)$ is properly normalized, i.e.,

$$\int_0^\infty dl\, P(l) = (1 - w) \sum_{n=1}^\infty w^n + 1 - w = (1 - w) \sum_{n=0}^\infty w^n = 1. \tag{4.5}$$

The average transport distance, i.e., the mean value of l_{2n}, is defined in the usual way,

$$\langle l \rangle = \int_0^\infty dl\, l\, P(l). \tag{4.6}$$

In order to calculate this integral, we first note that, according to Eq. (4.2),

$$\int_0^\infty dl\, l\, \delta(l - l_{2n}) = \sum_{m=1}^n s_{2m-1} + \sum_{m=1}^n s_{2m}. \tag{4.7}$$

Then, substituting the probability density [Eq. (4.4)] into Eq. (4.6) and taking into account the formulas

$$\int_0^\Delta \cdots \int_0^\Delta \left(\prod_{j=1}^n ds_{2j-1} p_-(s_{2j-1}) \right) \sum_{m=1}^n s_{2m-1} = n\tilde{s}_- w^{n-1}, \tag{4.8}$$

where $\tilde{s}_- = \int_0^\Delta ds\, s\, p_-(s)$, and

$$\int_0^\infty \cdots \int_0^\infty \left(\prod_{k=1}^n ds_{2k} p_+(s_{2k}) \right) \sum_{m=1}^n s_{2m} = ns_+, \tag{4.9}$$

expression (4.6) can be reduced to read

$$\langle l \rangle = (1 - w)(\tilde{s}_- + s_+ w) \sum_{n=1}^\infty n w^{n-1}. \tag{4.10}$$

Finally, using the formula $\sum_{n=1}^\infty n w^{n-1} = (1-w)^{-2}$, we obtain for the average transport distance the following remarkably simple result:

$$\langle l \rangle = \frac{\tilde{s}_- + s_+ w}{1 - w}. \tag{4.11}$$

It is important to note that Eq. (4.11) represents the average distance to the first impassable interval in the *positive* direction of the axis x, i.e., the average value of the maximum displacement of particles in the preferred direction. If $f \in (g_-, g_+)$ then $X_t \geq 0$ for all sample paths of $g(x)$ and thus the average displacement of particles, $\lim_{\tau\to\infty}(1/\tau)\int_0^\tau dt\, X_t$, relates closely to $\langle l \rangle$. But when $f > g_+$ then there exists a set of sample paths, whose total probability is nonzero, on which the particles are transported in the negative direction of the axis x. As a consequence, in this case the average displacement of particles is, in general, smaller than $\langle l \rangle$.

According to Eq. (4.11), the average distance $\langle l \rangle$ is finite if $w \neq 1$. If the probability density $p_-(s)$ has unbounded support then this condition holds for *all* half-period T's of the driving force $f(t)$. Otherwise, i.e., in the case of bounded support, $\langle l \rangle$ may be finite or infinite depending on the value of T. In order to illustrate the distinctive features of directed transport in these two cases, we next calculate $\langle l \rangle$ for the exponential and uniform probability densities $p_\pm(s)$, which represent the probability densities with unbounded and bounded support, respectively.

A. Exponentially distributed intervals

For the exponential probability densities

$$p_\pm(s) = \lambda_\pm e^{-\lambda_\pm s}, \tag{4.12}$$

where $\lambda_\pm$ are the rate parameters, we have $s_\pm = \lambda_\pm^{-1}$, $w = 1 - e^{-\lambda_- \Delta}$, and

$$\tilde{s}_- = \frac{1}{\lambda_-}(1 - e^{-\lambda_- \Delta} - \lambda_- \Delta e^{-\lambda_- \Delta}). \tag{4.13}$$

Therefore, in this case formula (4.11) becomes

$$\langle l \rangle = \frac{\lambda_- + \lambda_+}{\lambda_- \lambda_+}(e^{\lambda_- \Delta} - 1) - \Delta. \tag{4.14}$$

For given g_+ and g_-, the parameters $\lambda_\pm$ and $g_\pm$ are not independent because, in accordance with Eq. (2.2), the condition $g_+ \lambda_- = g_- \lambda_+$ must hold. Eliminating with the help of this relation the parameter λ_+, Eq. (4.14) yields

$$\langle l \rangle = \frac{1}{\lambda_-}\left(1 + \frac{g_-}{g_+}\right)(e^{\lambda_- \Delta} - 1) - \Delta. \tag{4.15}$$

According to this result, the average distance $\langle l \rangle$ is finite, and so $v_T = 0$ for all finite $f > g_-$ and T. In other words, in the case of exponential distributions of the interval s_j the directed transport of particles always occurs with zero average velocity. This feature of directed transport arises from the
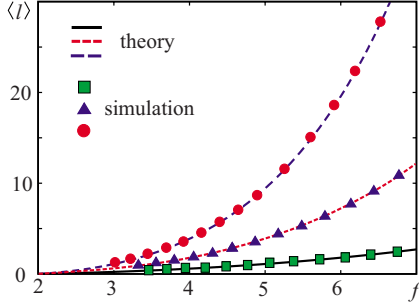
FIG. 6. (Color online) Average transport distance $\langle l \rangle$ as a function of the driving strength f for exponentially distributed intervals s_j. The theoretical curves are derived from Eq. (4.15) with $g_+ = 6$, $g_- = 2$, $\lambda_- = 1/3$, and $T = 0.5$ (solid line), $T = 1$ (short-dashed line, red), and $T = 1.5$ (long-dashed line, blue). The symbols depict the numerical results obtained by (i) generating a sample path of $g(x)$ in accordance with exponential distributions [Eq. (4.12)], (ii) finding the distance [Eq. (4.2)] to the first interval s_{2n+1} whose length exceeds Δ, and (iii) averaging this distance over 10^3 realizations of $g(x)$.

fact that the probability density $p_-(s)$ has unbounded support. As it follows from Eq. (4.15), the average distance grows linearly with Δ, $\langle l \rangle = (g_- / g_+)\Delta$, if $\lambda_- \Delta \ll 1$, and exponentially, $\langle l \rangle = \lambda_-^{-1}(1 + g_- / g_+)e^{\lambda_- \Delta}$, if $\lambda_- \Delta \gg 1$. Our analytical results are in full agreement with the numerical simulations (see Fig. 6).

B. Uniformly distributed intervals

If the intervals s_j are distributed with uniform probability densities [Eq. (3.13)] then $s_\pm = d_\pm$,

$$w = \begin{cases} \Delta(2d_-)^{-1}, & 0 < \Delta < 2d_-, \\ 1, & \Delta \geq 2d_-, \end{cases} \tag{4.16}$$

and

$$\tilde{s}_- = \begin{cases} \Delta^2(4d_-)^{-1}, & 0 < \Delta < 2d_- \\ d_-, & \Delta \geq 2d_-. \end{cases} \tag{4.17}$$

According to these results, Eq. (4.11) for $\Delta \geq 2d_-$, i.e., $f \geq f_{th}$, yields $\langle l \rangle = \infty$. In contrast, if $0 < \Delta < 2d_-$, i.e., $f \in (g_-, f_{th})$, then Eq. (4.11) reduces to

$$\langle l \rangle = \frac{\Delta(2d_+ + \Delta)}{2(2d_- - \Delta)}. \tag{4.18}$$

Since $g_+ d_+ = g_- d_-$, the last formula can be rewritten in the form

$$\langle l \rangle = d_- \left(1 + \frac{g_-}{g_+}\right)\frac{f - g_-}{f_{th} - f} - \frac{T}{2}(f - g_-). \tag{4.19}$$

Thus, depending on f, two regimes of directed transport exist. The first occurs at $f \in (g_-, f_{th})$ and is characterized by a zero average velocity v_T and a finite transport distance [Eq. (4.19)]. The second, with a nonzero v_T and an infinite $\langle l \rangle$, takes place at $f > f_{th}$. At the threshold amplitude $f = f_{th}$ the transition between these regimes occurs. Like in the previous case, the dependencies of the average transport distance $\langle l \rangle$ on f, which follow from Eq. (4.19), are fully corroborated by our numerical simulations (see Fig. 7).
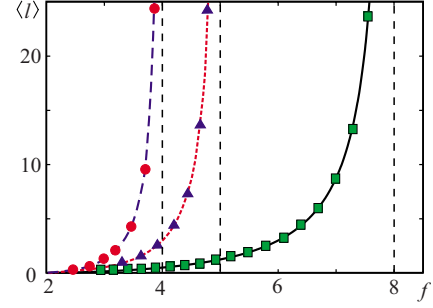

FIG. 7. (Color online) Average transport distance $\langle l \rangle$ as a function of the driving strength f for uniformly distributed intervals s_j. The theoretical curves are obtained from Eq. (4.19) for $g_+ = 6$, $g_- = 2$, and $d_- = 1.5$. The solid line corresponds to the half-period $T = 0.5$, the short-dashed line (red) to $T = 1$ and the long-dashed line (blue) to $T = 1.5$. In these cases $f_{th} = 8$, 5 and 4, respectively. The symbols depict the numerical results that are obtained in the same way as in Fig. 6.

We note that the random sawtooth potentials account for the influence of quenched disorder in nonperiodic ratchet systems and at the same time allow for a full analytical description of the ratchet effect. In the case of other random ratchet potentials a rigorous theoretical analysis of directed transport becomes extremely cumbersome without providing prominent additional insight. Put differently, the above analysis evidences that qualitatively the same results hold for a wider class of random ratchet potentials that produce the random forces $g(x)$ varying in the interval $(-g_-, g_+)$ and assuming a zero mean value. Specifically, if the distances between the nearest global maxima of $g(x)$ are distributed with unbounded support then only one transport regime of particles with $v_T = 0$ can be realized. The reason for this is the same as in the case of a dichotomous random force: for any finite half-period T of the driving force $f(t)$ there is always a nonzero probability for distances that cannot be overcome by particles during a positive pulse of $f(t)$. Accordingly, if the support is bounded then two transport regimes with $v_T > 0$ (when T is sufficiently large) and $v_T = 0$ (when T is sufficiently short) exist.

V. CONCLUSIONS

We have studied the directed transport of particles in absence of noise which are driven by a periodically alternating force in a viscous medium with quenched disorder. The influence of quenched disorder is modeled by a random sawtooth potential that generates a dichotomous random force with zero mean. We could show that, depending on the characteristics of the dichotomous and driving forces, two regimes of directed transport occur, namely, with a nonzero average velocity and with a vanishing average velocity.

The main result which we have obtained for the former regime is an explicit formula for the average transport velocity in the long-period limit of the driving force. An important feature of this limiting formula is that it does not depend on the probability densities of the intervals characterizing the dichotomous random force. We have shown numerically that

for finite periods of the driving force the average transport velocity is always less than the limiting one if all other parameters are kept the same.

In order to characterize the transport regime with a zero average velocity, we have calculated analytically the average value of the maximum displacement of particles in the preferred transport direction. This quantity is finite and so the average velocity of particles is zero if the probability density of the odd intervals characterizing the dichotomous force has unbounded support. Otherwise, i.e., if this probability density has bounded support, the average velocity can be either zero or nonzero, depending on the characteristics of the dichotomous and driving forces. We have applied the uniform probability densities for the quantitative study of the transport properties in these regimes and for describing the transition between them. All our theoretical predictions are nicely confirmed by our numerical simulations.

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