

ALMOST POSITIVE CURVATURE ON THE GROMOLL-MEYER SPHERE

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(Communicated by Jon G. Wolfson)

ABSTRACT. Gromoll and Meyer have represented a certain exotic 7-sphere Σ^7 as a biquotient of the Lie group $G = Sp(2)$. We show for a 2-parameter family of left invariant metrics on G that the induced metric on Σ^7 has strictly positive sectional curvature at all points outside four subvarieties of codimension ≥ 1 which we describe explicitly.

1. INTRODUCTION

Let $G = Sp(2)$ be the Lie group of unitary quaternionic 2×2 -matrices. Consider the subgroup $U \subset G \times G$,

$$(1.1) \quad U = \left\{ \left(\begin{pmatrix} q & \\ & 1 \end{pmatrix}, \begin{pmatrix} q & \\ & q \end{pmatrix} \right); q \in Sp(1) \right\},$$

which acts on G by left and right translations. D. Gromoll and W. Meyer [5] have shown that this action is free and that the orbit space $M = G/U$ is a smooth manifold which is an exotic 7-sphere (homeomorphic but not diffeomorphic to the standard 7-sphere). If G is equipped with a Riemannian metric of nonnegative sectional curvature whose isometry group contains U , then by O'Neill's formula [1] the orbit space $M = G/U$ inherits a Riemannian metric of nonnegative sectional curvature. Thus starting with the bi-invariant metric on G , Gromoll and Meyer constructed a metric of nonnegative sectional curvature on the exotic sphere M . In fact the curvature is strictly positive on some nonempty open subset of M . However, as was observed by F. Wilhelm [7], there is also an open subset with zero curvature planes in the tangent space of each of its points. But Wilhelm constructed another U -invariant metric on $Sp(2)$ (neither left nor right invariant) for which the curvature of M is strictly positive outside a subset of measure zero in M ("almost positive curvature"). In [4] the same fact was claimed for a much simpler and left invariant metric on $Sp(2)$; however, as was pointed out by the second author, the proof contains a serious mistake (see Remark 3 at the end of the present paper). The purpose of our paper is to correct this error. In fact we prove the following result, some ideas of which go back to [3] (see Theorem 4.6 for details):

Theorem 1.1. *There is a left invariant and a U -invariant metric on $G = Sp(2)$ such that the induced metric on $M = G/U$ has strictly positive curvature outside a*

finite union of subvarieties of codimension ≥ 1 . The zero curvature set $Z \subset M$ can be explicitly determined.

2. CHEEGER METRICS ON LIE GROUPS

On each Riemannian manifold, let us denote

$$(2.1) \quad \begin{aligned} \kappa(X, Y) &= \langle R(X, Y)Y, X \rangle, \\ \text{sec}(X, Y) &= \kappa(X, Y)/|X \wedge Y|^2 \end{aligned}$$

for any two tangent vectors X, Y ; the second expression is the sectional curvature of the plane σ spanned by X, Y .

Let G be a Lie group with a left invariant metric $\langle \cdot, \cdot \rangle$ of nonnegative sectional curvature. Suppose that the metric is also right invariant with respect to a compact subgroup $K \subset G$, hence the induced metric on K is bi-invariant. The Lie algebras of G and K will be denoted \mathfrak{g} and \mathfrak{k} . We may contract the metric on G in the direction of the K -cosets by viewing G as the homogeneous space $(G \times K)/\Delta K$ (where $\Delta K = \{(k, k); k \in K\}$) and choosing the metric induced from the Riemannian product metric on $G \times sK$ (Cheeger contraction; cf. [2], [1]) where sK is K with a metric scaled by $s > 0$. A vector $(X, X') \in \mathfrak{g} \times \mathfrak{k}$ is perpendicular to the ΔK -orbit (“horizontal”) iff $X + sX' \perp \mathfrak{k}$, i.e. $X' = -s^{-1}X_{\mathfrak{k}}$ where $X_{\mathfrak{k}}$ is the \mathfrak{k} -projection of X . Using the Riemannian submersion $G \times K \rightarrow G, (g, k) \mapsto gk^{-1}$, a horizontal vector $(X, -s^{-1}X_{\mathfrak{k}}) \in \mathfrak{g} \times \mathfrak{k}$ is mapped onto $X + s^{-1}X_{\mathfrak{k}} = X_{\perp} + (1 + s^{-1})X_{\mathfrak{k}} \in \mathfrak{g}$ where $X_{\perp} = X - X_{\mathfrak{k}} \in \mathfrak{k}^{\perp}$. Vice versa, the horizontal lift of $X = X_{\perp} + X_{\mathfrak{k}} \in \mathfrak{g}$ is the horizontal vector

$$(2.2) \quad \begin{aligned} \widehat{X} &= (\tilde{X}, -s^{-1}\tilde{X}_{\mathfrak{k}}), \quad \text{where} \\ \tilde{X} &= X_{\perp} + \frac{s}{s+1}X_{\mathfrak{k}}. \end{aligned}$$

Thus the new (left invariant) metric is

$$(2.3) \quad \begin{aligned} \langle X, Y \rangle_1 &= \langle \widehat{X}, \widehat{Y} \rangle \\ &= \langle \tilde{X}, \tilde{Y} \rangle + s \langle s^{-1}\tilde{X}_{\mathfrak{k}}, s^{-1}\tilde{Y}_{\mathfrak{k}} \rangle \\ &= \langle \tilde{X}, \tilde{Y} \rangle + s^{-1} \langle \tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{k}} \rangle \\ &= \langle \tilde{X}_{\perp}, \tilde{Y}_{\perp} \rangle + s^{-1}(s+1) \langle \tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{k}} \rangle \\ &= \langle X_{\perp}, Y_{\perp} \rangle + s(s+1)^{-1} \langle X_{\mathfrak{k}}, Y_{\mathfrak{k}} \rangle \\ &= \langle \tilde{X}, Y \rangle. \end{aligned}$$

For the curvature terms we have

$$(2.4) \quad \kappa(\widehat{X}, \widehat{Y}) = \kappa(\tilde{X}, \tilde{Y}) + s^{-3} \kappa(\tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{k}}).$$

Since all terms are nonnegative, the left hand side vanishes if and only if both summands on the right are zero. Thus a plane σ spanned by $X, Y \in \mathfrak{g}$ has zero curvature in the new metric, $\text{sec}_1(\sigma) = 0$, if and only if $\text{sec}(\tilde{\sigma}) = 0$ and $[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = 0$.¹

Example 1. Suppose that the initial metric $\langle \cdot, \cdot \rangle$ on G is bi-invariant. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the orthogonal decomposition. Consider the above metric

$$(2.5) \quad \langle X, Y \rangle_1 = \langle X_{\mathfrak{p}}, Y_{\mathfrak{p}} \rangle + \tilde{s} \langle X_{\mathfrak{k}}, Y_{\mathfrak{k}} \rangle$$

¹The “if” statement is not obvious because of the nonnegative O’Neill term. However, in all our examples starting with a bi-invariant metric on some Lie group, the vanishing of the curvature implies that the O’Neill term also vanishes; see [3], pp. 29ff, equations (1) - (4) or [8], [6].

with $\tilde{s} = \frac{s}{s+1}$. Then $\sec(\tilde{\sigma}) = 0 \iff [\tilde{X}, \tilde{Y}] = 0$, and hence $\sec_1(\sigma) = 0 \iff$

$$[\tilde{X}, \tilde{Y}] = 0, \quad [X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = 0.$$

If (G, K) is a symmetric pair, i.e. the orthogonal complement $\mathfrak{p} \subset \mathfrak{g}$ satisfies $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, then $[\tilde{X}, \tilde{Y}]_{\mathfrak{k}} = [\tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{k}}] + [\tilde{X}_{\mathfrak{p}}, \tilde{Y}_{\mathfrak{p}}]$ and $[\tilde{X}, \tilde{Y}]_{\mathfrak{p}} = [\tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{p}}] + [\tilde{X}_{\mathfrak{p}}, \tilde{Y}_{\mathfrak{k}}]$; hence $\sec_1(\sigma) = 0 \iff$

$$(2.6) \quad 0 = [X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = [X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = [X_{\mathfrak{k}}, Y_{\mathfrak{p}}] + [X_{\mathfrak{p}}, Y_{\mathfrak{k}}] = [X, Y].$$

Example 2. Let $G \supset K \supset H$ be a chain of subgroups and suppose that both (G, K) and (K, H) are symmetric pairs. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\mathfrak{k} = \mathfrak{h} + \mathfrak{q}$ be the corresponding decompositions. Start with the metric $\langle \cdot, \cdot \rangle_1$ defined by Example 1, depending on a parameter $s > 0$, and define the metric $\langle \cdot, \cdot \rangle_2$ by Cheeger contraction along H (depending on a new parameter $t > 0$) as in (2.3), where K is replaced by H and $\langle \cdot, \cdot \rangle_1$ takes the role of $\langle \cdot, \cdot \rangle$:

$$(2.7) \quad \begin{aligned} \langle X, Y \rangle_2 &= \langle X_{\mathfrak{p}}, Y_{\mathfrak{p}} \rangle_1 + \langle X_{\mathfrak{q}}, Y_{\mathfrak{q}} \rangle_1 + \tilde{t} \langle X_{\mathfrak{h}}, Y_{\mathfrak{h}} \rangle_1 \\ &= \langle X_{\mathfrak{p}}, Y_{\mathfrak{p}} \rangle + \tilde{s} \langle X_{\mathfrak{q}}, Y_{\mathfrak{q}} \rangle + \tilde{s} \tilde{t} \langle X_{\mathfrak{h}}, Y_{\mathfrak{h}} \rangle \end{aligned}$$

with $\tilde{t} = \frac{t}{t+1}$. Then $\sec_2(\sigma) = 0 \iff \sec_1(\tilde{\sigma}) = 0$ and $[\tilde{X}_{\mathfrak{h}}, \tilde{Y}_{\mathfrak{h}}] = 0 \iff$

$$(2.8) \quad 0 = [\tilde{X}, \tilde{Y}] = [\tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{k}}] = [X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = [X_{\mathfrak{q}}, Y_{\mathfrak{q}}] = [X_{\mathfrak{h}}, Y_{\mathfrak{h}}],$$

where $\tilde{X} = X_{\mathfrak{p}} + X_{\mathfrak{q}} + \frac{t}{t+1} X_{\mathfrak{h}}$ and $\tilde{Y} = Y_{\mathfrak{p}} + Y_{\mathfrak{q}} + \frac{t}{t+1} Y_{\mathfrak{h}}$, as in (2.2).

3. ZERO CURVATURE PLANES ON $Sp(2)$

Let us consider the chain $G \supset K \supset H$ for $G = Sp(2)$, $K = Sp(1) \times Sp(1)$ and $H = \Delta Sp(1) = \{ \begin{pmatrix} q & \\ & q \end{pmatrix}; q \in Sp(1) \}$. The pairs (G, K) and (K, H) are symmetric, corresponding to the rank-one symmetric spaces S^4 and S^3 . We start with the bi-invariant trace metric $\langle X, Y \rangle = \text{Re trace } X^* Y = \text{Re } \sum \overline{x_{ij}} y_{ij}$ on $\mathfrak{g} = \mathfrak{sp}(2)$, apply Cheeger contraction in the K -direction, and Cheeger-contract again in the H -direction, defining metrics $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ as in Example 2.

Since $G/K = S^4$ as well as $K/H = S^3$ and $H = S^3$ have positive curvature, the vanishing of the last three brackets in (2.8) means the linear dependence of the factors. In particular we may assume $Y_{\mathfrak{p}} = 0$, i.e. $\tilde{Y} = \tilde{Y}_{\mathfrak{k}} = \begin{pmatrix} y_1 & \\ & y_2 \end{pmatrix}$.

Case 1. $X_{\mathfrak{p}} = 0$, i.e. $\tilde{X} = \tilde{X}_{\mathfrak{k}} = \begin{pmatrix} x_1 & \\ & x_2 \end{pmatrix}$.

From $[\tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{k}}] = 0$ we obtain that the imaginary quaternions x_1, y_1 as well as x_2, y_2 are linearly dependent. Moreover, from $[X_{\mathfrak{q}}, Y_{\mathfrak{q}}] = [X_{\mathfrak{h}}, Y_{\mathfrak{h}}] = 0$ we also see that $x_1 \pm x_2$ and $y_1 \pm y_2$ are linearly dependent. Putting $y = y_1, y_2$, we may assume

$$(3.1) \quad \tilde{Y} = \begin{pmatrix} y & \\ & 0 \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} 0 & \\ & y \end{pmatrix}.$$

Case 2. $X_{\mathfrak{p}} \neq 0$, i.e. $X = \begin{pmatrix} x_1 & -\bar{x} \\ x & x_2 \end{pmatrix}$ with $x \neq 0$.

Then $0 = [\tilde{X}, \tilde{Y}]_{\mathfrak{p}} = [X_{\mathfrak{p}}, \tilde{Y}] \iff y_2 = xyx^{-1}$ for $y := y_1$, and $0 = [\tilde{X}, \tilde{Y}]_{\mathfrak{k}} = [\tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{k}}] \iff x_1 = \alpha y_1, x_2 = \beta y_2$ for real numbers α, β ; hence

$$(3.2) \quad \tilde{Y} = \begin{pmatrix} y & \\ & xyx^{-1} \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} \alpha y & -\bar{x} \\ x & -\alpha xyx^{-1} \end{pmatrix},$$

where $x, y \in \mathbb{H}$, y is imaginary and $\alpha \in \mathbb{R}$. We have $\beta = -\alpha$ since we require $\langle \tilde{X}, \tilde{Y} \rangle = 0$.

Case 2a. $\alpha = 0$; hence

$$(3.3) \quad \tilde{Y} = \begin{pmatrix} y & \\ & xyx^{-1} \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} & -\bar{x} \\ x & \end{pmatrix}.$$

Case 2b. $\alpha \neq 0$; hence (without loss of generality) $\alpha = 1$.

Then $[X_{\mathfrak{h}}, Y_{\mathfrak{h}}] = 0$ iff $y + xyx^{-1}$ and $y - xyx^{-1}$ are proportional, which means $xyx^{-1} = \beta y$. Comparing the norms on both sides we get

$$(3.4) \quad xyx^{-1} = \pm y$$

and

$$(3.5) \quad \tilde{Y} = Y_{\pm} = \begin{pmatrix} y & \\ & \pm y \end{pmatrix}, \quad \tilde{X} = X_{\pm} = \begin{pmatrix} y & -\bar{x} \\ x & \mp y \end{pmatrix}.$$

Lemma 3.1. *The zero curvature planes in $\mathfrak{g} = T_e G$ for $G = Sp(2)$ and the metric $\langle \cdot, \cdot \rangle_2$ are spanned by $X, Y \in \mathfrak{g}$ with \tilde{X}, \tilde{Y} given by either (3.1) or (3.3) or (3.5).*

4. THE GROMOLL-MEYER SPHERE

The metric $\langle \cdot, \cdot \rangle_2$ on $G = Sp(2)$ is invariant under the action of U (cf. (1.1)) and hence it induces a metric on the orbit space $M = G/U$. Consider any

$$(4.1) \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

Since g is unitary, the rows and columns are unit vectors, in particular

$$(4.2) \quad |a|^2 + |b|^2 = 1.$$

The vertical space at g of the submersion $\pi : G \rightarrow G/U$ is $T_g(U.g) = gV_g$ with $V_g = \{v_g; v \in \text{Im } \mathbb{H}\}$ where

$$(4.3) \quad v_g = g^{-1} \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix} g - \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} \bar{a}va - v & \bar{a}vb \\ \bar{b}va & \bar{b}vb - v \end{pmatrix}.$$

Thus according to (2.3), a vector $gX \in T_g G$ is horizontal for π iff

$$(4.4) \quad 0 = \langle X, v_g \rangle_2 = \langle \tilde{X}, v_g \rangle_1$$

for all $v \in \text{Im } \mathbb{H}$. Note that $\langle \tilde{X}, v_g \rangle_1$ is just a multiple of $\langle \tilde{X}, v_g \rangle$ if one of the components of $\tilde{X} = \tilde{X}_{\mathfrak{p}} + \tilde{X}_{\mathfrak{t}}$ are zero. Now we discuss which of the zero curvature planes in $G = Sp(2)$ (see Lemma 3.1) can be horizontal at any $g \in G$. By a slight abuse of language, a plane $\tilde{\sigma}$ spanned by $\tilde{X}, \tilde{Y} \in \mathfrak{g}$ will be called *horizontal at g* if

$$(4.5) \quad \langle \tilde{X}, v_g \rangle_1 = \langle \tilde{Y}, v_g \rangle_1 = 0$$

for all $v \in \text{Im } \mathbb{H}$.

Case 1.

Lemma 4.1. *A plane of type (3.1) is nowhere horizontal.*

Proof. $\langle \tilde{Y}, v_g \rangle = \langle y, \bar{a}va - v \rangle = \langle ay\bar{a} - y, v \rangle$ vanishes for all $v \in \text{Im } \mathbb{H}$ iff $y = ay\bar{a}$, and likewise $\langle \tilde{X}, v_g \rangle$ vanishes for all v iff $y = by\bar{b}$. But this implies $|a| = |b| = 1$ in contradiction to (4.2). \square

Case 2a.

Lemma 4.2. *If a plane of type (3.3) is horizontal at g , then either $a = 0$ or $b = 0$ or*

$$(4.6) \quad \det(I - \text{Ad}(a^{-1}) - \text{Ad}(b^{-1})) = 0.$$

Proof. The matrix \tilde{X} is horizontal at g if and only if

$$(4.7) \quad 0 = \langle \tilde{X}, v_g \rangle = 2\langle x, \bar{b}va \rangle = 2\langle bx\bar{a}, v \rangle$$

for all $v \in \text{Im } \mathbb{H}$. This is equivalent to $bx\bar{a} \in \mathbb{R}$. Hence, either $a = 0$ or $b = 0$ or $bx = ra$ for some nonzero $r \in \mathbb{R}$. In the latter case we have, in particular

$$(4.8) \quad \text{Ad}(bx) = \text{Ad}(a),$$

$$(4.9) \quad \text{Ad}(x) = \text{Ad}(b^{-1}) \text{Ad}(a),$$

provided that $b \neq 0$. On the other hand, the matrix \tilde{Y} is horizontal at g if and only if

$$(4.10) \quad 0 = \langle \tilde{Y}, v_g \rangle = \langle |a|^2 \text{Ad}(a)y - y + |b|^2 \text{Ad}(bx)y - \text{Ad}(x)y, v \rangle$$

for all $v \in \text{Im } \mathbb{H}$. Since $y \in \text{Im } \mathbb{H}$, this means

$$(4.11) \quad \begin{aligned} 0 &= |a|^2 \text{Ad}(a)y + |b|^2 \text{Ad}(bx)y - y - \text{Ad}(x)y \\ &\stackrel{4.8}{=} \text{Ad}(a)y - y - \text{Ad}(x)y \\ &\stackrel{4.9}{=} \text{Ad}(a)y - y - \text{Ad}(b^{-1}) \text{Ad}(a)y, \end{aligned}$$

where we have also used $|a|^2 + |b|^2 = 1$ (4.2). If $a \neq 0$, we obtain from the last equality

$$\text{Ad}(a)y \in \ker(I - \text{Ad}(a^{-1}) - \text{Ad}(b^{-1}))$$

and in particular

$$(4.6) \quad \det(I - \text{Ad}(a^{-1}) - \text{Ad}(b^{-1})) = 0.$$

□

Lemma 4.3. *There exists a plane of type (3.3) which is horizontal at g if and only if either (4.6) holds or*

$$(4.12) \quad a = 0, \quad |\text{Im } b| \geq \frac{1}{2} \quad \text{or} \quad b = 0, \quad |\text{Im } a| \geq \frac{1}{2}.$$

Proof. Suppose first that $a, b \neq 0$. If (4.6) is satisfied, there is a nonzero $w \in \ker(I - \text{Ad}(a^{-1}) - \text{Ad}(b^{-1}))$. Then defining $y = \text{Ad}(a^{-1})w$ and $x = b^{-1}a$, we obtain a horizontal plane of type (3.3) at g . The converse conclusion was done before.

Now suppose $b = 0$. Then $|a| = 1$ and equation (4.11) becomes

$$(4.13) \quad \text{Ad}(a)y - y = \text{Ad}(x)y.$$

Geometrically, this equality means that $\text{Ad}(a)$ rotates y by the angle $\frac{\pi}{3}$ (the three vectors $\text{Ad}(a)y$, y , and $\text{Ad}(x)y$ form the sides of an equilateral triangle). Hence (4.13) has a solution (x, y) if and only if the rotation angle of the rotation $\text{Ad}(a)$ is $\geq \frac{\pi}{3}$. This in turn is equivalent to $\angle(a, 1) \geq \frac{\pi}{6}$, i.e. $|\text{Im } a| \geq \frac{1}{2}$. Inserting the solution (x, y) into (3.3) defines a horizontal plane of type (3.3). The case $a = 0$ is similar. □

Case 2b.

Lemma 4.4. *If a plane of type (3.5) is horizontal at g , then*

$$(4.14) \quad |a| = |b| = 1/\sqrt{2}$$

and $w := \operatorname{Im} a^{-1}b$ satisfies

$$(4.15) \quad \langle w - 2a^{-1}wa, w \rangle = 0.$$

Proof.

$$(4.16) \quad \langle v_g, Y_+ \rangle = \langle \bar{a}va + \bar{b}vb - 2v, y \rangle = \langle v, ay\bar{a} + by\bar{b} - 2y \rangle,$$

$$(4.17) \quad \langle v_g, Y_- \rangle = \langle \bar{a}va - \bar{b}vb, y \rangle = \langle v, ay\bar{a} - by\bar{b} \rangle.$$

Thus $\langle \tilde{Y}, V_g \rangle = 0$ iff one of the following equations holds:

$$\begin{aligned} ay\bar{a} + by\bar{b} &= 2y, \\ ay\bar{a} - by\bar{b} &= 0. \end{aligned}$$

The first of these equations is impossible by the triangle inequality together with (4.2):

$$|ay\bar{a} + by\bar{b}| \leq |ay\bar{a}| + |by\bar{b}| \leq (|a|^2 + |b|^2)|y| = |y| < |2y|.$$

Thus we are left with the second equation,

$$(4.18) \quad ay\bar{a} = by\bar{b},$$

which implies $|a| = |b|$.

Note that we have also shown that Y_+ cannot be horizontal. Thus we need only consider $\tilde{X} = X_-$ and $\tilde{Y} = Y_-$ in (3.5), and

$$(4.19) \quad xyx^{-1} = -y,$$

which means that x is imaginary and nonzero with $x \perp y$.

Now let \tilde{X}, \tilde{Y} be as above spanning $\tilde{\sigma}$. By (3.5) we have

$$(4.20) \quad \tilde{Y} = \begin{pmatrix} y & \\ & -y \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} y & x \\ x & y \end{pmatrix}$$

with $y \perp x \in \operatorname{Im} \mathbb{H}$. Thus according to (2.5) we get for all $v \in \operatorname{Im} \mathbb{H}$

$$(4.21) \quad \begin{aligned} 0 = \langle \tilde{X}, v_g \rangle_1 &= 2\langle x, \bar{b}va \rangle + \tilde{s}\langle y, \bar{a}va + \bar{b}vb - 2v \rangle \\ &= 2\langle bx\bar{a}, v \rangle + \tilde{s}\langle ay\bar{a} + by\bar{b} - 2y, v \rangle \\ &= \langle bxa^{-1} + \tilde{s}(aya^{-1} - 2y), v \rangle, \end{aligned}$$

where we have used $2\bar{a} = a^{-1}$ and $ay\bar{a} = by\bar{b} = \frac{1}{2}aya^{-1}$ from (4.14) and (4.18). Putting $p = a^{-1}b/\tilde{s}$, we obtain

$$(4.22) \quad \operatorname{Im} apxa^{-1} = 2y - aya^{-1}.$$

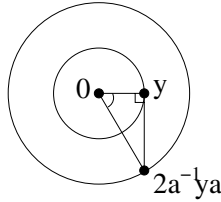
From $aya^{-1} = byb^{-1}$ we see $yp = py$; thus $p \in \mathbb{C}_y := \mathbb{R} + \mathbb{R}y$ and the left multiplication with p preserves \mathbb{C}_y and \mathbb{C}_y^\perp . By (4.19) we have $x \in \mathbb{C}_y^\perp$ and therefore $px \in \mathbb{C}_y^\perp$. Conjugating (4.22) by a^{-1} we obtain

$$(4.23) \quad 2a^{-1}ya - y = \operatorname{Im}(px) \perp y,$$

$$(4.24) \quad \langle 2a^{-1}ya - y, y \rangle = 0.$$

Since $w = \operatorname{Im} \tilde{s}p \in \mathbb{C}_y$ is a multiple of y , we may replace y by w in equation (4.24) and obtain (4.15). \square

Remark 1.



Geometrically, (4.24) means that the angle between y and $a^{-1}ya$ is $\pi/3 = 60^\circ$. Thus the rotation angle of $\text{Ad}(a^{-1})$ (and of $\text{Ad}(b^{-1})$; see (4.18)) must be $\geq \pi/3$; hence $\angle(1, a) \geq \pi/6$, or in other words,

$$(4.25) \quad \frac{|\text{Im } a|}{|a|} \geq \frac{1}{2}.$$

Lemma 4.5. *Suppose that $a, b \in \mathbb{H}$ satisfy (4.14), (4.15) and (4.25). Then there exists a horizontal plane of type (3.5) at $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.*

Proof. First suppose that $\tilde{p} = a^{-1}b = \tilde{s}p$ is real which in view of (4.14) means $a = \pm b$. By (4.25), the rotation angle of $\text{Ad}(a^{-1})$ is $\geq \pi/3$; hence there exists a nonzero $y \in \text{Im } \mathbb{H}$ which is rotated precisely by the angle $\pi/3$ and thus satisfies (4.24). Put $x = 2a^{-1}ya - y \perp y$ and define \tilde{X}, \tilde{Y} as in (4.20). This matrix pair is of type (3.5), and it is perpendicular to V_g by (4.17) and (4.21).

Now suppose that $w = \text{Im } \tilde{p} \neq 0$; in this case (4.15) implies (4.25). Then we choose $y = w$ and $x = \text{Im } (p^{-1}(2a^{-1}wa - w))$; compare (4.23). Since $w - 2a^{-1}wa \in \mathbb{C}_y^\perp$ (it is imaginary and perpendicular to $w = y$), we also have $p^{-1}(w - 2a^{-1}wa) \in \mathbb{C}_y^\perp$; hence $x \perp y$ and thus $xyx^{-1} = -y$. Defining matrices \tilde{X}, \tilde{Y} using (4.20), these are of type (3.5) and perpendicular to V_g by (4.17) and (4.21). \square

Remark 2. Clearly, the relations (4.6), (4.12), (4.14), (4.15) and (4.25) must be invariant under the action of U . In fact, if $u = \left(\begin{pmatrix} q & \\ & 1 \end{pmatrix}, \begin{pmatrix} q & \\ & q \end{pmatrix} \right)$, we have $u.g = \tilde{g} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$ with $\tilde{a} = qa^{-1}$ and $\tilde{b} = qbq^{-1}$.

Now we have proved the following.

Theorem 4.6. *Let $G = \text{Sp}(2)$ with the left invariant metric (2.7) and $U \subset G \times G$ defined by (1.1). The orbit space $M = G/U$ inherits a Riemannian metric such that the canonical projection $\pi : G \rightarrow M$ is a Riemannian submersion. Let*

$$Z = \{p \in M; \exists \sigma \subset T_p M : \sec(\sigma) = 0\}.$$

Then $Z = Z_1 \cup Z_2 \cup Z_3 \cup Z_4$ where

$$\begin{aligned} \pi^{-1}Z_1 &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b \neq 0, \det(I - \text{Ad}(a^{-1}) - \text{Ad}(b^{-1})) = 0 \right\}, \\ \pi^{-1}Z_2 &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; |a| = |b|, w := \text{Im } a^{-1}b \perp w - 2a^{-1}wa, |\text{Im } a| \geq |a|/2 \right\}, \\ \pi^{-1}Z_3 &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; b = c = 0, |\text{Im } a| \geq 1/2 \right\}, \\ \pi^{-1}Z_4 &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a = d = 0, |\text{Im } b| \geq 1/2 \right\}, \end{aligned}$$

where all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are supposed to belong to $\text{Sp}(2)$. \square

Remark 3. The mistake in [4] is in the third line of the proof of the theorem, page 1166. The computation of $\langle v_g, X \rangle$ holds only for $X \in \mathfrak{k}$, but X may have a nonzero \mathfrak{p} -component as well. Thus the matrix X in (4), p. 1166, is too special and must be replaced with the more general $X = \begin{pmatrix} ry & -\bar{x} \\ x & -rxyx^{-1} \end{pmatrix}$ for arbitrary $r \in \mathbb{R}$, and

instead of (5) $\text{Im}(bx\bar{a}) = 0$ we obtain (5') $\text{Im}(bx\bar{a}) = r(y - ay\bar{a})$, while equation (6) ($ay\bar{a} - y + bxyx^{-1}\bar{b} - xyx^{-1} = 0$) remains unchanged. We have 15 variables, $(a, b) \in S^7$, $x \in \mathbb{H}$, $y \in \text{Im}(\mathbb{H})$, $r \in \mathbb{R}$, with 2 arbitrary real constants (the lengths of x and y), and 6 constraint equations (5') and (6) which reduce the number of free variables to 7. Thus the solution set is likely to project onto a subset with positive measure in the (a, b) -space S^7 ; this would imply that the metric considered in [4] fails to have almost positive curvature.

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