Isotropic ppmc immersions

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1. Introduction

Let M be a Kähler manifold (not necessarily complete) and $T = TM$ its tangent bundle. The complex structure of M defines a parallel tensor field $J : T \to T$ with $J^2 = -I$. This belongs to a one-parameter family of parallel rotations

$$R_\theta = (\cos\theta)I + (\sin\theta)J. \tag{1}$$

Now let $f : M \to \mathbb{R}^n$ be an isometric immersion with normal bundle N and second fundamental form $\alpha : S^2T \to N$ where S^2 denotes the symmetric tensor product. Using R_θ, we may define a new tensor field $\tilde{\alpha}_\theta : S^2T \to N$,

$$\tilde{\alpha}_\theta(v, w) = \alpha(R_\theta v, R_\theta w) \tag{2}$$

for any $v, w \in T$. We may ask when this happens to be also a second fundamental form, belonging to another isometric immersion $f_\theta : M \to \mathbb{R}^n$. However the normal bundles N and N_θ of f and f_θ will be different in general. Thus the

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best we can ask for is that the second fundamental form α_θ of f_θ satisfies

$$\alpha_\theta = \psi_\theta \,\tilde{\alpha}_\theta, \tag{3}$$

where $\psi_\theta : N \to N_\theta$ is a parallel vector bundle isomorphism.

Immersions f which allow such an "associated family" f_θ have been characterized in [1] in terms of *pluri-mean curvature*. The complexified tangent bundle T^c of a Kähler manifold M splits into the $(\pm i)$-eigenbundles of J, and the corresponding bundle decomposition $T^c = T' + T''$ is Levi-Civita parallel. If $f : M \to \mathbb{R}^n$ is an isometric immersion, the restrictions of α to the parallel subbundles S^2T', $T' \otimes T''$, $S^2T'' \subset S^2T$ are called $\alpha^{(2,0)}$, $\alpha^{(1,1)}$, $\alpha^{(0,2)}$, respectively. The component $\alpha^{(1,1)}$ is called *pluri-mean curvature*; it collects the mean curvature vectors of the restrictions of f to all complex one-dimensional submanifolds (complex curves) in M. The immersion f is said to have *parallel pluri-mean curvature* (*ppmc*) if $\alpha^{(1,1)}$ is parallel. It turns out that an associated family of the above type (3) exists if and only if f is a ppmc immersion.

There is a special kind of ppmc immersions: those whose associated family (3) is constant, $f_\theta = f$. Those are called *isotropic*. They are characterized by the condition that the three components of α define a parallel orthogonal decomposition of $N^c = N \otimes \mathbb{C}$ of the form

$$N^c = N' \oplus (N^o)^c \oplus N'' \tag{4}$$

such that $\alpha^{(2,0)}$, $\alpha^{(1,1)}$, $\alpha^{(0,2)}$ take values in these three subbundles, respectively (cf. Theorem 8 of [1]). In particular the real normal bundle splits orthogonally into parallel subbundles

$$N = N^o + N^1 \tag{5}$$

where N^o is the (real) image of $\alpha^{(1,1)}$ while $\alpha^{(2,0)}$ and $\alpha^{(0,2)}$ take values in the complexification of N^1.

In the present paper, we wish to show that such submanifolds must be very special: If f is not holomorphic and M does not locally split as a Riemannian product, it must be locally symmetric or a minimal surface in a sphere. The local symmetry is obtained by applying the holonomy theorem of Berger and Simons [8].

In the locally symmetric case we also get some information about the extrinsic geometry and we conjecture that f is in fact *extrinsically symmetric*, i.e. the whole second fundamental form α is parallel, not only its (1, 1)-part.

2. The holonomy group

Theorem 1. *Let M be a locally irreducible Kähler manifold* (*not necessarily complete*) *and $f : M \to \mathbb{R}^n$ an isotropic ppmc immersion. Then M is locally symmetric unless f is holomorphic with values in some $\mathbb{C}^k \subset \mathbb{R}^n$ or it is an* (*isotropic*) *minimal surface in a sphere $S^{n-1} \subset \mathbb{R}^n$.*

Proof. We shall apply the holonomy theorem of Berger and Simons [8]. Recall that the holonomy group H of M consists of the parallel displacements τ_γ along all closed curves γ in M starting and ending at a point $p \in M$ fixed once and for all. Clearly H is a subgroup of the orthogonal group on the tangent space $T_p = T$ at p. Since the complex structure J is parallel, it is preserved by H and thus H is a subgroup of the unitary group on the complex vector space (T, J). Using a unitary basis we may identify T with $\mathbb{C}^m$. Local irreducibility means that there are locally no parallel subbundles of T which means that the identity component of H acts irreducibly on T. In order to show that M is locally symmetric, according to [8] we just have to prove that H does not act transitively on the unit sphere in T.

For any $\zeta \in N$ let $A_\zeta = -(\partial\zeta)_T$ be the Weingarten map. Note that A_ζ commutes (anticommutes) with J for $\zeta \in N^o$ (resp. $\zeta \in N^1$). In fact, if $\zeta \in N^o$, then $\langle A_\zeta T', T'\rangle = \langle \alpha(T', T'), \zeta\rangle \subset \langle N', N^o\rangle = 0$. Since T' is maximal isotropic,[1] this implies that $A_\zeta T' \subset T'$. Likewise, for $\zeta \in N^1$ we have $\langle A_\zeta T', T''\rangle = \langle \alpha(T', T''), \zeta\rangle \in \langle N^o, N^1\rangle = 0$, hence $A_\zeta T' \subset T''$. Thus A_ζ preserves (resp. reverses) the eigenspaces of J which proves the statement. Let us agree that ξ and η always denote vectors in N^1 and N^o, respectively.

We first consider three special cases. If $A_\eta = 0$ for all $\eta \in N^o$, then $\alpha^{(1,1)} = 0$, so f is pluriharmonic and isotropic, hence holomorphic (cf. [5]). More generally, if A_η is a multiple of the identity for all $\eta \in N^o$, then $A_\eta = 0$ for all

[1] This means that T' is maximal among all those linear subspaces of T^c where the complexified metric $\langle\,,\rangle$ vanishes.

$\eta \perp \eta_o$ where $\eta_o = \operatorname{trace} \alpha$ is the mean curvature vector of f. Hence f is a pluriharmonic immersion into a sphere in $\mathbb{R}^n$, and by [2] this must be a minimal surface (in particular $\dim M = 2$).

Further, if $A_\xi = 0$ for all $\xi \in N^1$, then $\alpha^{(2,0)} = 0 = \alpha^{(0,2)}$ since $\alpha^{(2,0)}$ and $\alpha^{(0,2)}$ take values in $N^1 \otimes \mathbb{C}$. This implies that f is extrinsic hermitian symmetric, i.e. a *standard embedding* of some hermitian symmetric space (cf. [6]).[2] In fact, in order to show $\nabla\alpha = 0$ it is enough to compute $(\nabla_Z \alpha)(X, Y)$ for vector fields X, Y, Z taking values in $T' \cup T''$. At least two of these vectors have the same type, say T'. Since $\nabla\alpha$ is symmetric (Codazzi), we may assume $X, Y \in T'$. Now $(\nabla_Z\alpha)(X, Y) = \nabla_Z(\alpha(X, Y)) - \alpha(\nabla_Z X, Y) - \alpha(X, \nabla_Z Y)$ vanishes since $X, Y, \nabla_Z X, \nabla_Z Y \in T'$. This shows that f is extrinsic symmetric. Further, from $\alpha^{(2,0)} = 0$ we get $\alpha(JX, JY) = \alpha(X, Y)$, hence by Ferus [6] we see that f is the standard embedding of a hermitian symmetric space.

Thus from now on we may assume that there are normal vectors $\eta \in N^o$ and $\xi \in N^1$ such that $A_\xi \neq 0$ and A_η has at least two different eigenvalues. Since N^o, N^1 are parallel subbundles of N, the Weingarten maps A_η and A_ξ commute by the Ricci equation, and hence they have a compatible eigenspace decomposition. Let $F \subset T$ be an eigenspace of A_ξ corresponding to some nonzero eigenvalue λ. Further let

$$T = E_1 \oplus \cdots \oplus E_r \tag{6}$$

be the eigenspace decomposition with respect to A_η. Due to the compatibility we obtain a decomposition

$$F = F \cap E_1 \oplus \cdots \oplus F \cap E_r \tag{7}$$

with $r \geqslant 2$. We will show next that (7) still holds when F is replaced by the space hF for any $h \in H$.

In fact, let $h \in H$ correspond to the parallel displacement along a curve γ on M starting and ending at p. Let $\tilde{h}$ be the parallel displacement in N^o along the same curve γ. Since $\alpha^{(1,1)}$ is parallel and $\alpha^{(2,0)}, \alpha^{(0,2)}$ take values in $(N^1)^c$, the linear map $A = (\eta \mapsto A_\eta) : N^o \to \operatorname{End}(TM)$ is also parallel. In fact, let η be a parallel normal field and v, w parallel tangent fields along some curve c in M. Then $\langle A_\xi v, w\rangle = \langle \alpha_{vw}, \xi\rangle = \langle \alpha^{(1,1)}_{vw}, \xi\rangle$ is constant, hence $A_\xi v$ is parallel along c. Thus A intertwines the parallel displacements of N^o and $\operatorname{End}(T)$. Therefore $A_{\tilde{h}\eta} = hA_\eta h^{-1}$, and the eigenspace decomposition corresponding to $A_{\tilde{h}\eta}$ is $T = hE_1 \oplus \cdots \oplus hE_r$. Replacing η by $\tilde{h}\eta$ in (7), we get a decomposition $F = F \cap hE_1 + \cdots + F \cap hE_r$. Hence putting $\tilde{F} = h^{-1}F$, we obtain

$$\tilde{F} = \tilde{F} \cap E_1 \oplus \cdots \oplus \tilde{F} \cap E_r. \tag{8}$$

We call a subspace $\tilde{F} \subset T$ *split* if (8) holds. We just have shown that all hF, $h \in H$, are split.

Since the complex structure J anticommutes with A_ξ, the nonzero eigenvalues of A_ξ come in pairs $\pm\lambda$ and the corresponding eigenspaces F_λ and $F_{-\lambda}$ are interchanged by J. Hence $\hat{F} = F_\lambda + F_{-\lambda}$ is a complex subspace which is also split, and the same holds for $h\hat{F}$ for any $h \in H$. Now we have to consider two cases: $\hat{F} \neq T$ and $\hat{F} = T$.

Case 1. $\hat{F} \neq T$. Then it is an element of some complex Grassmannian $P = G_k(T)$ where $k = \dim_{\mathbb{C}} \hat{F}$. The H-orbit of $\hat{F}$ is contained in a connected component of the set of split spaces. This is a proper totally geodesic submanifold $Q \subset P$, more precisely the Riemannian product of r Grassmannians $G_{k_j}(E_j)$ with $k_j = \dim_{\mathbb{C}}(\hat{F} \cap E_j)$. Let S be the smallest totally geodesic submanifold of Q containing the H-orbit $H\hat{F} = \{h\hat{F};\ h \in H\}$. Clearly, S is invariant under H. Let $G = U(m)$ be the unitary group on $T = \mathbb{C}^m$ which acts as the transvection group on the Grassmannian P, and let $G_S = \{g \in G;\ gS = S\}$ be the subgroup leaving S invariant. The induced action of G_S on S (which need not be effective) contains the full transvection group of S; this must be a subgroup of $U(E_1) \times \cdots \times U(E_r)$ since S is totally geodesic in Q which is a product of r Grassmannians. Thus the action of the holonomy group H on S induces a Lie group homomorphism $\phi : H \to U(E_1) \times \cdots \times U(E_r)$. This is trivial only if $S = H\hat{F} = \{\hat{F}\}$ which is impossible since H acts irreducibly on T.

Case 2. $\hat{F} = T$. Then A_ξ has just two eigenspaces F and JF, and $T = F \oplus JF$. Thus F belongs to the set of maximal totally real subspaces of $T = \mathbb{C}^m$. These form another symmetric space $P' = U(m)/O(m)$. In fact, F lies in the totally geodesic subspace $Q' \subset P'$ consisting of the split spaces (8); we have $Q' = Q_1 \times \cdots \times Q_r$ where

[2] If M is a hermitian symmetric space, the complex structure J_p in every tangent space T_p is a derivation of the curvature tensor and hence can be considered as an element of the Lie algebra $\mathfrak{g}$ of the isometry group of M. The *standard embedding* is the map $f : M \to \mathfrak{g}$, $p \mapsto J_p \in \mathfrak{g}$.

$Q_i = U(m_i)/O(m_i)$ with $m_i = \dim_{\mathbb{C}} E_i$. Since the full H-orbit of F is contained in Q', there is again a totally geodesic subspace $S' \subset Q'$ which is preserved by H and contains F, and as above we obtain a nontrivial Lie group homomorphism $\phi : H \to U(E_1) \times \cdots \times U(E_r)$.

Now if $H \subset U(m)$ acts transitively on the unit sphere, its identity component H_o is one of the three subgroups $U(m)$, $SU(m)$, $Sp(m/2)$. But $U(m)$ acts transitively on both P and P' and hence it cannot preserve a proper totally geodesic subspace. The other two groups are simple. Since the homomorphism ϕ is nontrivial, one of its components $\phi_i : H_o \to U(E_i)$ must be nontrivial and hence injective. But there are no representations of $SU(m)$ or $Sp(m/2)$ with degree $< m$. Hence $\dim E_i = m$ and thus E_i is the whole space T in contradiction to our assumption that A_η has at least two different eigenvalues. Thus H does not act transitively on the sphere and M is locally symmetric. □

Remark. The only known isotropic ppmc immersions (besides holomorphic maps and isotropic minimal surfaces in spheres) are the so called *extrinsic symmetric* ones, those with $\nabla\alpha = 0$. They split into two subclasses: the standard embeddings of hermitian symmetric spaces where $\alpha^{(2,0)} = 0$ and the Grassmannian $G_2(\mathbb{R}^{m+2})$ of 2-planes in $\mathbb{R}^{m+2}$, doubly covered by the complex quadric Q^m (the space of *oriented* 2-planes) and embedded as symmetric rank 2 projection matrices into the euclidean space of all symmetric endomorphisms with trace 2 on $\mathbb{R}^{m+2}$. This is an example for the second case $\hat{F} = T$ of the previous proof, and it is the only known case (besides surfaces) where both N^o and N^1 are nontrivial. We conjecture that there are no other examples. This would require to prove $\nabla\alpha = 0$ for locally symmetric isotropic ppmc immersions. In the following we give some evidence for this conjecture (see also [4]).

3. Extrinsic geometry

Theorem 2. *Let M be a locally symmetric Kähler manifold and $f : M \to \mathbb{R}^n$ an isotropic ppmc immersion. Then at every point $p \in M$, the values of $\nabla\alpha$ are perpendicular to the first normal space spanned by the values of α.*

Proof. Let $X, Y, Z, W, V \in T'$. From the Gauss equation we get

$$\langle R_{X\bar{Y}} Z, \bar{W}\rangle = \langle \alpha_{X\bar{W}}, \alpha_{\bar{Y}Z}\rangle - \langle \alpha_{XZ}, \alpha_{\bar{Y}\bar{W}}\rangle. \tag{9}$$

Taking covariant differentiation on both sides we get from $\nabla_V R = 0$:

$$0 = -\big\langle (\nabla_V \alpha)_{XZ}, \alpha_{\bar{Y}\bar{W}}\big\rangle, \tag{10}$$

recall that by Codazzi equations, $(\nabla_A \alpha)_{BC} = 0$ for all $A, B, C \in T' \cup T''$ unless A, B, C have the same type. Thus the values of $(\nabla\alpha)^{(3,0)}$ are perpendicular to the values of $\alpha^{(2,0)}$ (with respect to the hermitian inner product $(A, B) = \langle A, \bar{B}\rangle$). On the other hand let us recall that the three components of α take values in mutual orthogonal parallel subbundles of N^c, hence the values of $(\nabla\alpha)^{(3,0)}$ are also perpendicular to those of $\alpha^{(1,1)}$ and $\alpha^{(0,2)}$. □

Theorem 3. *Let M be a locally irreducible Kähler manifold and $f : M \to \mathbb{R}^n$ an isotropic ppmc immersion of codimension $\leqslant 6$. Then $f(M)$ is extrinsically symmetric or a minimal surface in a sphere or f is holomorphic.*

Proof. If f is a minimal immersion, then it is pluriminimal, i.e. $\alpha^{(1,1)} = 0$ (cf. [7]) and moreover isotropic and hence holomorphic (cf. [5]). Thus we may assume $\eta_o := \operatorname{trace}\alpha \neq 0$. Since $\eta_o = \sum \alpha(E_i, \bar{E}_i)$ for some unitary basis $E_1, \ldots, E_m$ of T', we see that $\eta_o = \operatorname{trace}\alpha^{(1,1)} \in N^o$ is parallel. But for any parallel section η of N^o, the corresponding Weingarten map A_η is parallel too since $\langle A_\eta v, w\rangle = \langle \alpha_{vw}, \eta\rangle = \langle \alpha_{vw}^{(1,1)}, \eta\rangle$ for any two tangent vectors v, w (the other components of α_{vw} are perpendicular to η). The eigendistributions of A_η would give a product decomposition of M. Thus by irreducibility, $A_\eta = \lambda \cdot I$ for some $\lambda \in \mathbb{R}$. For $\eta = \eta_o$, this constant is nonzero by assumption which shows that $f(M)$ lies in a sphere S^{n-1} of radius $r = 1/|\lambda|$. If $\dim N^o = 1$, then f is pluriminimal or $(1,1)$-geodesic in a sphere and hence a minimal surface, cf. [3]. If $\dim N^o = 2$, the same conclusion holds: There is (up to multiples) just one other parallel section $\eta \perp \eta_o$ in N^o, thus $A_\eta = \lambda \cdot I$. But this time we have $\lambda = 0$ since $\operatorname{trace} A_\eta = \langle \operatorname{trace}\alpha, \eta\rangle = \langle \eta_o, \eta\rangle = 0$. Thus f is again pluriminimal in a sphere and thus a minimal surface.

Hence we may assume $\dim N^o \geqslant 3$. By Theorem 1 we know that M is locally symmetric. We consider the decomposition (4) of the complexified normal bundle N^c. By Theorem 2, the subbundle N' contains two mutually orthogonal subbundles N'_1 and N'_2 containing the values of $\alpha^{(2,0)}$ and $(\nabla\alpha)^{(3,0)}$, respectively. If both tensors are nonzero, the dimension of N' is at least 2, and since the same holds for $N'' = \overline{N'}$, the dimension of N^c (the codimension of f) must be at least $3 + 2 + 2 = 7$. Otherwise either $\alpha^{(2,0)} = 0$ and f is a standard embedding of an hermitian symmetric space, cf. [6], or $(\nabla\alpha)^{(3,0)} = 0$ and hence $\nabla\alpha = 0$ (by the vanishing of $\nabla(\alpha^{(1,1)})$ and Codazzi) and f is extrinsic symmetric. $\square$

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