Kähler submanifolds with parallel pluri-mean curvature

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1. Introduction

Some surfaces in 3-space admit isometric deformations which change the shape of the surface while preserving the intrinsic metric. Even the principal curvatures may be preserved while the principal curvature directions are rotated under the deformation; this happens precisely if the surface has constant mean curvature ("cmc"). The best known example is the deformation of the catenoid into the helicoid which transforms the meridians and the equator of the catenoid into the helicoid's ruling lines and axis, thus rotating the principal curvature directions by 45°.

In the present paper we wish to investigate submanifolds of higher dimension and codimension allowing similar deformations. The surface will be replaced by a simply connected m-dimensional complex manifold M with an immersion $f : M \to \mathbb{R}^n$ such that the induced metric on M is Kählerian, i.e., the almost complex structure J on TM is orthogonal and parallel; we call these *Kähler immersions* for short. Let α denote the second fundamental form of f and rotate it by putting $\alpha_\vartheta(x, y) = \alpha(\mathcal{R}_\vartheta x, \mathcal{R}_\vartheta y)$, where

$$\mathcal{R}_\vartheta = \cos(\vartheta)I + \sin(\vartheta)J.$$

When does there exist a family of isometric immersions $f_\vartheta : M \to \mathbb{R}^n$ with second fundamental form α_ϑ? We will see in Theorem 1 that this happens precisely if the bilinear form

$$\alpha^{(1,1)}(x, y) := \frac{1}{2}\big(\alpha(x, y) + \alpha(Jx, Jy)\big)$$

is parallel with respect to the connections on the tangent and normal bundles. In the case of a surface ($m = 1$) we have $\alpha^{(1,1)}(x, y) = \langle x, y\rangle \cdot \eta$, hence $\alpha^{(1,1)}$ is parallel if and only if the mean curvature vector $\eta = \frac{1}{2}\operatorname{trace}\alpha$ is parallel also. This motivates us to call $\alpha^{(1,1)}$ the *pluri-mean curvature* of f; in fact, for any complex curve $C \subset M$ the restriction of $\alpha^{(1,1)}$ to TC is again the metric multiplied by the mean curvature vector of the surface $f|_C$. But while surfaces with nonzero constant mean curvature can only have essential codimension 1 or 2 (cf. [13]), there are interesting substantial examples in higher dimensions and codimensions (cf. Section 7). When $\alpha^{(1,1)} = 0$, the immersion is called $(1,1)$-*geodesic* or *pluriminimal*; this case was studied earlier (cf. [5,6] and their references).

The main part of the paper is devoted to studying the relationship between a Kähler immersion $f : M \to \mathbb{R}^n$ with parallel pluri-mean curvature ("ppmc") and its Gauss map $\tau : M \to Gr$ where $\tau(p) = df_p(T_pM)$ and Gr is the Grassmannian of $2m$-dimensional subspaces of $\mathbb{R}^n$. Just as in the case of cmc surfaces (cf. [11]), ppmc submanifolds are characterized by the pluriharmonicity of their Gauss maps (Theorem 2). Pluriharmonic maps also admit an associated family of deformations, and in fact the deformed Gauss map is the Gauss map of the deformed immersion (Theorem 3).

The Gauss map τ of a Kähler immersion has a refinement τ' called the *complex Gauss map* which takes account of the complex structure: for any $p \in M$ we put $\tau'(p) = df(T'_pM)$. (Here we have extended df_p complex linearly to $T^cM = TM \otimes \mathbb{C}$ and used the J-eigenspace decomposition $T^cM = T'M + T''M$ with $J = i$ on $T'M$ and $J = -i$ on $T''M$.) The map τ' takes values in the set Z_1 of *isotropic* complex m-dimensional subspaces $E \subset \mathbb{C}^n$, i.e., the complex conjugate $\overline{E}$ is perpendicular to E with respect to the Hermitian inner product, or equivalently $\langle E, E\rangle = 0$ for the symmetric inner product $\langle x, y\rangle = \sum x_j y_j$ on $\mathbb{C}^n$. This space Z_1 can be viewed as a flag manifold fibering over Gr, and then τ' is a horizontal lift of τ. We will show that τ' is pluriharmonic if and only if τ is also and hence if and only if f is ppmc (Theorem 5). In fact we can characterize the complex Gauss maps of ppmc immersions among the pluriharmonic maps into Z_1 (Theorem 6).

Alternatively, Z_1 can also be viewed as a complex submanifold of the complex Grassmannian Gc of m-planes in $\mathbb{C}^n$. We also study the composition $j \circ \tau'$ for the inclusion $j : Z_1 \to Gc$. This map is pluriharmonic only for special ppmc immersions which we call *half isotropic* (Theorem 7). These contain two interesting subclasses, characterized also by properties of τ': the *pluri-minimal* ones with zero pluri-mean curvature (τ' is holomorphic, Theorem 4) and the *isotropic* ones where the associated family is trivial (τ and $j \circ \tau'$ are isotropic, Theorems 9 and 10). The first of these results is well known for surfaces: a surface is minimal if and only if its (complex) Gauss map is holomorphic. The second result is not interesting for surfaces in 3-space: Isotropy would mean that each tangent vector is a principal curvature direction, hence the surface must be a round sphere or a plane. But there are interesting examples in higher dimension, among them the standard embeddings of Hermitian symmetric spaces (see Section 7). We need some facts on flag manifolds which are known in principle [2] but not explicitly worked out; we shall prove these statements in Appendix A.

2. Associated families of immersions

Let M be a Kähler manifold of complex dimension m; this is a $2m$-dimensional Riemannian manifold with a parallel and orthogonal almost complex structure J on TM. Since our theory is entirely local, we do not need completeness of M, however at some points we will need simple connectivity. We consider an isometric immersion $f : M \to \mathbb{R}^n$ (a *Kähler immersion*). Let $\alpha : TM \otimes TM \to N$ be the corresponding second fundamental form defined by $\alpha(X, Y) = (\partial_X \partial_Y f)^N$ where $N = Nf = df(TM)^\perp$ denotes the normal bundle of f. Consider the parallel rotations $\mathcal{R}_\vartheta = \cos(\vartheta)I + \sin(\vartheta)J$ for any $\vartheta \in \mathbb{R}$ and let $\alpha_\vartheta : TM \otimes TM \to N$,

$$\alpha_\vartheta(x, y) = \alpha(\mathcal{R}_\vartheta x, \mathcal{R}_\vartheta y).$$

An *associated family* for f is roughly speaking a one-parameter family of isometric immersions $f_\vartheta : M \to \mathbb{R}^n$ with second fundamental form α_ϑ.[4] This is not quite correct since the second fundamental forms of the two immersions f and f_ϑ take values in different spaces, the normal bundles of f and f_ϑ. More precisely, a one-parameter family $f_\vartheta : M \to \mathbb{R}^n$ of isometric immersions will be called an *associated family* of f if their second fundamental forms α_{f_ϑ} satisfy

$$\psi_\vartheta\big(\alpha_{f_\vartheta}(x, y)\big) = \alpha_\vartheta(x, y) = \alpha(\mathcal{R}_\vartheta x, \mathcal{R}_\vartheta y) \tag{1}$$

for some parallel bundle isomorphism $\psi_\vartheta : Nf_\vartheta \to Nf$. Our first theorem below will show under which conditions such immersions exist.

We need some more notation. The complexified tangent bundle $T^cM = TM \otimes \mathbb{C}$ of a Kähler manifold M splits as $T^cM = T' \oplus T''$ where the components are the parallel eigenbundles of the almost complex structure J with $J = i$ on T' and $J = -i$ on T''. Vectors in T' are also called $(1, 0)$-vectors and those in $T'' = \overline{T'}$ are $(0, 1)$-vectors. Let $\pi'(x) = \frac{1}{2}(x - iJx)$ and $\pi''(x) = \frac{1}{2}(x + iJx)$ be the projections onto these subbundles. Extending α complex linearly to the complexified tangent and normal bundles, we put

$$\alpha^{(1,1)}(x, y) = \alpha(\pi' x, \pi'' y) + \alpha(\pi'' x, \pi' y) = \frac{1}{2}\big(\alpha(x, y) + \alpha(Jx, Jy)\big). \tag{2}$$

[4] This was called *weak associated family* in [6].

As explained in the introduction, $\alpha^{(1,1)}$ will be called the *pluri-mean curvature*, and f is called an immersion with *parallel pluri-mean curvature* (*ppmc*) if this tensor is parallel with respect to the tangent and normal connections. The following theorem which was partially obtained in [7] shows the relation to associated families.

Theorem 1. *Let $f : M \to \mathbb{R}^n$ be a Kähler immersion. Then f has an associated family if and only if it has parallel pluri-mean curvature.*

Proof. We are using the existence theorem for submanifolds (cf. [12]): Let M be a p-dimensional Riemannian manifold and N a k-dimensional euclidean vector bundle over M with a metric connection D^N. Further let $\alpha \in \mathrm{Hom}(S^2TM, N)$ where S^2TM denotes the symmetric tensor product of TM. Then there is an isometric immersion $f : M \to \mathbb{R}^{p+k}$ with normal bundle N (up to a parallel vector bundle isometry) and second fundamental form α if and only if the submanifold equations of Gauss, Codazzi and Ricci are satisfied.

Let us apply this to α_ϑ. The *Gauss equation* is

$$\big\langle R(x, y)v, w\big\rangle = \big\langle \alpha_\vartheta(x, w), \alpha_\vartheta(y, v)\big\rangle - \big\langle \alpha_\vartheta(x, v), \alpha_\vartheta(y, w)\big\rangle. \tag{G_ϑ}$$

In fact, this equation follows from (G_0), the Gauss equation of f. The easiest way to see this is to use the splitting $T^cM = T' + T''$. On T' we have $\mathcal{R}_\vartheta = \mathrm{e}^{i\vartheta}$ while $\mathcal{R}_\vartheta = \mathrm{e}^{-i\vartheta}$ on T''. We may assume that $x, y, v, w \in T' \cup T''$. In all possible cases, the right hand side of (G_ϑ) picks up a common factor $\mathrm{e}^{ik\vartheta}$ for some k. The left hand side is zero as soon x, y or v, w have the same type (both in T' or both in T''). This holds on any Kähler manifold since $R(x, y)T' \subset T'$ and $\langle T', T'\rangle = 0$, thus $\langle R(x, y)T', T'\rangle = 0$ for all $x, y \in T^c$ (where we have extended the inner product complex linearly to T^cM). For these cases (G_ϑ) follows from (G_0). In the remaining cases, two of the vectors x, y, v, w are in T' and the other two in T'', and thus (G_ϑ) is the same as (G_0).

Next we consider the *Codazzi equation*:

$$(D_x\alpha_\vartheta)(y, z) = (D_y\alpha_\vartheta)(x, z). \tag{C_ϑ}$$

This follows from (C_0) (the Codazzi equation of f) provided that x, y have the same type. But if $x \in T'$ and $y \in T''$, we get different factors in front of the two sides of (C_ϑ). Thus (C_ϑ) follows from (C_0) precisely if $(D_{T'}\alpha)(T'', T^c)$ vanishes, but by (C_0), this is the same as $(D_{T^c}\alpha)(T', T'')$. Thus (C_ϑ) holds if and only if $\alpha^{(1,1)}$ is parallel.

It remains to consider the *Ricci equation*. For any $\xi \in N$ let A_ξ^ϑ be the symmetric endomorphism of TM defined by

$$\langle A_\xi^\vartheta x, y\rangle = \big\langle \alpha_\vartheta(x, y), \xi\big\rangle = \big\langle \alpha(\mathcal{R}_\vartheta x, \mathcal{R}_\vartheta y), \xi\big\rangle = \langle A_\xi \mathcal{R}_\vartheta x, \mathcal{R}_\vartheta y\rangle,$$

hence $A_\xi^\vartheta = \mathcal{R}_\vartheta^{-1} A_\xi \mathcal{R}_\vartheta$. Then the Ricci equation is

$$\big\langle R^N(x, y)\xi, \eta\big\rangle = \big\langle [A_\xi^\vartheta, A_\eta^\vartheta]x, y\big\rangle = \big\langle [A_\xi, A_\eta]\mathcal{R}_\vartheta x, \mathcal{R}_\vartheta y\big\rangle. \tag{R_ϑ}$$

Again this equation follows from (R_0), the Ricci equation for f, provided that $x, y \in T' \cup T''$ are of different type. But if, say, both x, y are in T', the right hand side is multiplied by $\mathrm{e}^{2i\vartheta}$. Hence (R_ϑ) follows from (R_0) if and only if $R^N(T', T') = 0$. (Note that the case $x, y \in T''$ follows by complex conjugation.) But the subsequent lemma shows that this is not a new condition; it follows also from $D\alpha^{(1,1)} = 0$. This finishes the proof of Theorem 1. □

Lemma 1. *If a Kähler immersion $f : M \to \mathbb{R}^n$ has parallel pluri-mean curvature, then $R^N(T', T') = 0$.*

Proof. Let $N^o \subset N$ denote the image of $\alpha^{(1,1)}$; since $\alpha^{(1,1)}$ is parallel, N^o is a parallel subbundle of N. Let $(N^o)^\perp \subset N$ be its orthogonal complement. For any $\xi \in (N^o)^\perp$ and $x \in T'$, $\bar{y} \in T''$ we have $\langle A_\xi x, \bar{y}\rangle = \langle \alpha(x, \bar{y}), \xi\rangle \in \langle N^o, (N^o)^\perp\rangle = 0$. Since T' and T'' are isotropic subspaces, this implies $A_\xi(T') \subset T''$, and by complex conjugation we also get $A_\xi(T'') \subset T'$.

We have to show that $\langle R^N(x, y)\xi, \eta\rangle = \langle [A_\xi, A_\eta]x, y\rangle$ vanishes for all $x, y \in T'$ and $\xi, \eta \in N$. It is sufficient to consider the following two cases:

(a) $\xi, \eta \in (N^o)^\perp$,
(b) $\xi \in N^o$ and $\eta \in N$ arbitrary.

In case (a), both A_ξ and A_η interchange T' and T''. Hence the commutator $[A_\xi, A_\eta]$ preserves T' which by isotropy of T' implies $\langle [A_\xi, A_\eta]T', T'\rangle = 0$. Case (b) will follow from the following more general fact which is well known and easy to prove by twofold covariant differentiation:

Sublemma. *Let E, F be vector bundles with connections D^E and D^F over some smooth manifold M. Let $\beta : E \to F$ be a parallel homomorphism, i.e., $\beta(D^E_X e) = D^F_X \beta(e)$ for any section e of E. Then $R^F(x, y)\beta e = \beta(R^E(x, y)e)$.*

We apply the sublemma to $\beta := \alpha^{(1,1)} : T' \otimes T'' \to N^o$. According to case (b), we may assume $\xi = \alpha(u, \bar{v})$ for some $u \in T'$ and $\bar{v} \in T''$. Since $N^o \subset N$ is parallel, we have

$$R^N(x, y)\xi = R^{N^o}(x, y)\beta(u \otimes \bar{v}) = \beta\big(R^{T'\otimes T''}(x, y)(u \otimes \bar{v})\big) = 0,$$

recalling that $R^{T'\otimes T''}(x, y)(u \otimes \bar{v}) = (R(x, y)u) \otimes \bar{v} + u \otimes R(x, y)\bar{v}$, and $R(x, y) = 0$ for $x, y \in T'$ since M is a Kähler manifold. □

3. The Gauss map

Let M be a p-dimensional smooth manifold, $f : M \to \mathbb{R}^n$ an immersion and Gr the Grassmannian of p-dimensional linear subspaces $E \subset \mathbb{R}^n$. The *Gauss map* $\tau : M \to Gr$ assigns to each $p \in M$ the subspace $\tau(p) = df_p(T_p M) \subset \mathbb{R}^n$. We view Gr as a submanifold of the vector space $S(n)$ of all symmetric real $n \times n$-matrices; this done by replacing a linear subspace E with the orthogonal projection onto E (which will be called E, too). Then the tangent space $T_E Gr$ is the subspace $S(E, E^\perp) \subset S(n)$ of all self adjoint linear maps on $\mathbb{R}^n$ sending E to $E^\perp$ and vice versa; it can be naturally identified with $\mathrm{Hom}(E, E^\perp)$. A smooth map $\phi : M \to Gr$ can be viewed as a vector bundle $\underline{\phi}$ over M whose fibre at $p \in M$ is the subspace $\phi(p) \subset \mathbb{R}^n$ (in other words, $\underline{\phi} = \phi^*\gamma$ where γ is the tautological bundle over Gr with total space $\gamma = \{(E, v) \in Gr \times \mathbb{R}^n;\ v \in E\}$). In fact $\underline{\phi}$ and $\underline{\phi}^\perp$ are subbundles of the trivial bundle $M \times \mathbb{R}^n$ and thus they inherit a natural connection which is differentiation on $\mathbb{R}^n$ followed by projection onto the fibre. We may view $\phi^* T Gr = \mathrm{Hom}(\underline{\phi}, \underline{\phi}^\perp)$, and the pull back connection on $\phi^* T Gr$ is just the natural connection on $\mathrm{Hom}(\underline{\phi}, \underline{\phi}^\perp)$. Later on we will suppress the difference between ϕ and $\underline{\phi}$ in our notation.

The differential $d\phi : TM \to \phi^* T Gr$ is computed as follows: If we differentiate ϕ with respect to a vector field X on M, the action of $\partial_X \phi$ on a section s of $\underline{\phi}$ (which is a mapping $s : M \to \mathbb{R}^n$ with

$s(p) \in \phi(p)$ for all $p \in M$) is given by

$$(\partial_X \phi) \cdot s = \partial_X(\phi \cdot s) - \phi \cdot \partial_X s = \phi^\perp \cdot \partial_X s \tag{3}$$

where ϕ and $\phi^\perp$ are considered as a projection matrices on $\mathbb{R}^n$, depending on $p \in M$. In order to apply this to the Gauss map $\phi = \tau$, we use the section $s = df(Y) = \partial_Y f$ where Y is an arbitrary vector field on M, and we obtain

$$(\partial_X \tau) \cdot df(Y) = \tau^\perp(\partial_X \partial_Y f) = \alpha(X, Y). \tag{4}$$

The following theorem due to [7] generalizes the well known result of Ruh and Vilms [11] which characterizes cmc surfaces by the harmonicity of their Gauss maps. In higher dimension, harmonicity has to be replaced by pluriharmonicity: A smooth map $\phi : M \to S$ into a symmetric space S is called *pluriharmonic* if its Levi form $Dd\phi^{(1,1)}$ (the restriction of the Hessian to $T' \otimes T''$) vanishes.[5] As always we view $d\phi$ as a section of the bundle $\mathrm{Hom}(TM, f^*TS)$ with its natural connection induced by the Levi-Civita connections on M and S. In particular, $d\tau$ is a section of $\mathrm{Hom}(TM, \mathrm{Hom}(\tau, \tau^\perp)) = \mathrm{Hom}(TM \otimes \tau, N)$. Since $f : M \to \mathbb{R}^n$ is an isometric immersion, $df : TM \to \tau$ is a parallel bundle isomorphism which will be used to identify the bundles TM and τ. Using this identification and (4) we have $d\tau = \alpha \in \mathrm{Hom}(TM \otimes TM, N)$ and $D\,d\tau = D\alpha$.

Theorem 2. *Let M be a Kähler manifold and $f : M \to \mathbb{R}^n$ an isometric immersion. Then f has parallel pluri-mean curvature if and only if its Gauss map τ is pluriharmonic.*

Proof. $D\,d\tau^{(1,1)} = 0$ if and only if for any $X \in T'$, $\overline{Y} \in T''$ and $W \in T^c$ we have $0 = (D_X d\tau)(\overline{Y}) \cdot df(W) = (D_X \alpha)(\overline{Y}, W) = (D_W \alpha)(X, \overline{Y})$, using Codazzi equation. Since T' and T'' are parallel subbundles of $T^c M$, this is equivalent to $D(\alpha^{(1,1)}) = 0$. □

The pluriharmonic map $\tau : M \to Gr$ has also an associated family: For any Kähler manifold M and any symmetric space S, a family of smooth maps $\tau_\vartheta : M \to S$ is called *associated* to $\tau = \tau_0$ if there is a parallel bundle isomorphism $\phi_\vartheta : \tau_\vartheta^* TS \to \tau^* TS$ preserving the curvature tensor R^S such that

$$\phi_\vartheta \circ d\tau_\vartheta = d\tau \circ \mathcal{R}_\vartheta. \tag{5}$$

It is known (cf. [6]) that a given smooth map $\tau : M \to S$ has a (unique) associated family if and only if it is pluriharmonic. We shall show next that the associated families of a ppmc immersion f and its pluriharmonic Gauss map τ correspond to each other.

Theorem 3. *Let $f : M \to \mathbb{R}^n$ be a ppmc immersion with Gauss map τ and let f_ϑ be the associated family of f. Let τ_ϑ be the Gauss map of f_ϑ. Then (τ_ϑ) is the associated family of τ.*

Proof. It suffices to show that the Gauss maps τ_ϑ of the immersions f_ϑ form an associated family. Thus we have to find a parallel bundle map $\phi_\vartheta : T_{\tau_\vartheta} Gr \to T_\tau Gr$ satisfying (5) above. Let $x, y \in T_p M$. On the

[5] For some authors (for example, [9]), maps with this property are said to be $(1,1)$-*geodesic* while a pluriharmonic map is one whose restriction to any holomorphic curve is harmonic. In the current setting, where M is Kähler, these competing definitions are equivalent.

one hand we have

$$d\tau(R_\vartheta x) : df(y) \mapsto \alpha(\mathcal{R}_\vartheta x, y),$$

on the other hand

$$d\tau_\vartheta(x) : df_\vartheta(\mathcal{R}_{-\vartheta} y) \mapsto \alpha_\vartheta(x, \mathcal{R}_{-\vartheta} y) = \psi_\vartheta\big(\alpha(\mathcal{R}_\vartheta x, y)\big).$$

Thus Eq. (5) is satisfied if for any $a \in T_{\tau_\vartheta(p)}Gr = \mathrm{Hom}(\tau_\vartheta(p), N_\vartheta(p))$ we put

$$\phi_\vartheta(a) = \psi_\vartheta \circ a \circ \mathcal{R}_{-\vartheta} \in \mathrm{Hom}\big(\tau(p), N(p)\big) = T_{\tau(p)}Gr$$

where we have identified both τ and τ_ϑ with TM using df and df_ϑ and where ψ_ϑ denotes the parallel isomorphism between the normal bundles (cf. (1) in Section 2). We see that $\phi_\vartheta(p)$ acts by conjugating a with the orthogonal $n \times n$-matrix B mapping the subspaces $\tau_\vartheta(p)$ and $N_\vartheta(p)$ onto $\tau(p)$ and $N(p)$, with

$$B|_{\tau_\vartheta(p)} = df_p \circ R_\vartheta \circ (df_\vartheta)_p^{-1}, \quad B|_{N_\vartheta(p)} = \psi_\vartheta(p).$$

Conjugation by $B \in O(n)$ is a global isometry on Gr and thus preserves the curvature tensor of Gr. Moreover, ϕ_ϑ is parallel since so are ψ_ϑ and $\mathcal{R}_\vartheta$ as well as $df : TM \to \tau$ and $df_\vartheta : TM \to \tau_\vartheta$. Thus τ_ϑ is the associated family of τ. $\square$

4. The complex Gauss map

The Gauss map τ of a Kähler manifold immersion $f : M \to \mathbb{R}^n$ records only the tangent planes without taking account of the complex structure. Therefore we introduce a refinement, the *complex Gauss map* τ'. It takes values in the set Z_1 of all m-dimensional linear subspaces $E \subset \mathbb{C}^n$ which are *isotropic*, i.e., the bilinear inner product $\langle x, y\rangle = \sum_j x_j y_j$ on $\mathbb{C}^n$ vanishes on $E \times E$. In fact we let $\tau' : M \to Z_1$,

$$\tau'(p) = df(T'_p) = \big\{df(x) - i \cdot df(Jx);\ x \in T_pM\big\} \subset \mathbb{C}^n.$$

The manifold Z_1 can be viewed in two different ways. On the one hand, it is a complex submanifold of the complex Grassmannian $Gc = G_m(\mathbb{C}^n)$ of all complex m-planes in $\mathbb{C}^n$. In fact, the complex structure on Gc is induced by the complex Lie group $GL(n, \mathbb{C})$ acting transitively on Gc, and $Z_1 \subset Gc$ is an orbit of the complex subgroup $O(n, \mathbb{C})$ inducing a complex structure on Z_1. On the other hand Z_1 can be considered also as a flag manifold fibering over the *real* Grassmannian Gr (cf. Appendix A): To any $E \in Z_1$ we may assign the orthogonal[6] decomposition ("flag") $\mathbb{C}^n = E + N + \overline{E}$ where $N = (E + \overline{E})^\perp$, and the projection $\pi : Z_1 \to Gr$ is given by $\pi(E) = E + \overline{E}$ (we view the subspaces of $\mathbb{R}^n$ as complex subspaces of $\mathbb{C}^n$ which are invariant under complex conjugation). In terms of coset spaces we have $Z_1 = O_n/(U_m \times O_k)$ where $k = n - 2m$, and $\pi : Z_1 \to Gr = O_n/(O_{2m} \times O_k)$ is the canonical projection. This is a Riemannian submersion (up to a scaling factor) for any O_n-invariant metric on Z_1 since the horizontal space (the reductive complement of $\mathfrak{so}_{2m} \oplus \mathfrak{so}_k$ in the Lie algebra $\mathfrak{so}_n$) is irreducible with

[6] The terms "orthogonal" or "perpendicular" in a complex vector space are always related to the *Hermitian* inner product $(x, y) = \langle x, \overline{y}\rangle$.

respect to the isotropy group $U_m \times O_k$ of Z_1. As a further consequence, the notions "horizontal" and "super-horizontal" agree for Z_1 (cf. Appendix A).[7]

If we take the second view point considering Z_1 as a flag manifold over Gr, we have to replace τ' by

$$\tau_1 = (\tau', N, \tau'')$$

where $\tau'' = \overline{\tau'}$ and $N = (\tau' + \tau'')^\perp$; this is the complexified normal bundle of the immersion f. Clearly, $\pi \circ \tau_1 = \tau$.

Lemma 2. *Let $f : M \to \mathbb{R}^n$ be a Kähler immersion with second fundamental form α and complex Gauss map $\tau' : M \to Z_1 \subset Gc$. Then we have for any $v \in TM$ and $x' \in T'$ (whence $df(x') \in \tau'$)*

$$d\tau'(v).df(x') = \alpha(v, x'). \tag{6}$$

Consequently $\tau_1 = (\tau', N, \tau'')$ is a (super-)horizontal lift of the real Gauss map τ.

Proof. We first view $Z_1 \subset Gc$. We may identify TM with τ and T' with τ' using df. Since $(T')^\perp = T'' + N$, we have (as for the real Grassmannian) $d\tau'(v).x' = (\partial_v X')^{(T')^\perp} = (\partial_v X')^{T''} + (\partial_v X')^N$ where X' is a $(1,0)$ vector field extending x'. But $(\partial_v X')^{T''} = (D_v X')^{T''} = 0$ because T' is parallel with respect to the Levi-Civita connection D of M. Moreover $(\partial_v X')^N = \alpha(v, x')$ which shows (6).

Now consider Z_1 as a flag manifold over Gr. Then Eq. (6) shows that $d\tau_1(v) = (d\tau'(v), dN(v), d\tau''(v))$ is a super-horizontal vector since it maps τ' into the next following space N; in other words, $d\tau_1(v).\tau'$ has no component in τ'' (cf. Eq. (A.5) in Appendix A). □

Remark. The proof shows that the horizontality of τ_1 is just another expression for the parallelity of the almost complex structure J on M.

The first occasion where the complex Gauss map turned out to be useful was the characterization of pluriminimal submanifolds by holomorphicity of τ' (cf. [10]). A similar statement for τ would not even make sense.

Theorem 4. *An Kähler immersion $f : M \to \mathbb{R}^n$ is pluriminimal (i.e., has zero pluri-mean curvature) if and only if $\tau_1 : M \to Z_1$ is holomorphic.*

Proof. The map $\tau_1 = (\tau', N, \tau'')$ is holomorphic if and only if $d\tau_1$ maps $T' = T'M$ into $T'Z$ or, more precisely (using Lemma 2), into $\mathcal{H}'_1$. In other words (cf. Appendix A), $d\tau_1(v')$ for $v' \in T'$ is a linear map sending τ' into N (which is always true by Lemma 2) and N into τ''. The latter property says that for any $w'' \in T''$ and $\xi \in N$

$$0 = -\langle d\tau_1(v').\xi, w''\rangle = \langle \xi, d\tau_1(v').w''\rangle = \langle \xi, \alpha(v', w'')\rangle$$

which means that $\alpha^{(1,1)} = 0$. □

[7] A flag manifold over a (real or complex) Grassmannian is a set of certain orthogonal decompositions $\mathbb{C}^n = E_1 + \cdots + E_r$. A vector v in the tangent space $T_E Z$ at any $E = (E_1, \ldots, E_r) \in Z$ is a linear map sending each E_i into its complement, and v is called *super-horizontal* if it maps E_i only into its nearest neighbors $E_{i-1} + E_{i+1}$. See Appendix A or [2] for a formulation of super-horizontality which is valid for any generalised flag manifold.

Theorem 5. *A Kähler immersion $f : M \to \mathbb{R}^n$ is ppmc if and only if $\tau_1 : M \to Z_1$ is a (super)horizontal pluriharmonic map.*

Proof. By Lemma 2 the complex Gauss map τ_1 of any Kähler immersion f takes values in the (super)horizontal bundle $\mathcal{H}_1$. Moreover f is ppmc if and only if its real Gauss map τ is pluriharmonic (cf. Theorem 2). But τ_1 is a horizontal lift of τ with respect to the Riemannian submersion $\pi : Z_1 \to Gr$. This implies that pluriharmonicity for τ and τ_1 are equivalent. In fact, τ is pluriharmonic if and only if for any two commuting vector fields $V' \in T'$ and $W'' \in T''$ we have $D_{W''} d\tau(V') = 0$. Since $d\tau_1(V')$ is the horizontal lift of $d\tau(V')$, this is equivalent to $D_{W''} d\tau_1(V') = 0$, see the subsequent Lemma 3 for details. □

Lemma 3. *Let Z, S be Riemannian manifolds and $\pi : Z \to S$ a Riemannian submersion. Let M be any manifold and $\tau_1 : M \to Z$ be a horizontal map, i.e., $d\tau_1(TM) \subset \mathcal{H}$ where $\mathcal{H} \subset TZ$ is the horizontal subbundle. Consider the O'Neill tensor $A : \mathcal{H} \otimes \mathcal{H} \to \mathcal{V}$ (where $\mathcal{V} = \mathcal{H}^\perp \subset TZ$ is the vertical bundle) given by*

$$A(X, Y) = [X, Y]^{\mathcal{V}} = 2(D_X Y)^{\mathcal{V}}$$

for horizontal vector fields X, Y. Then $\tau_1^* A = 0$, i.e., $(D_W d\tau_1(V))^{\mathcal{V}} = 0$ for any two vector fields V, W on M.

Proof. Let V, W be local vector fields on M with $[V, W] = 0$. Locally we can write $d\tau_1(V) = \sum_i v_i (X_i \circ \tau_1)$ and $d\tau_1(W) = \sum_j w_j (X_j \circ \tau_1)$ where v_i, w_j are functions on M and $X_1, \ldots, X_n$ form a basis of horizontal vector fields on Z. Then

$$\begin{aligned} A\big(d\tau_1(V), d\tau_1(W)\big) &= \sum_{ij} v_i w_j A(X_i, X_j) \circ \tau_1 = \sum_{ij} v_i w_j (D_{X_i} X_j - D_{X_j} X_i)^{\mathcal{V}} \circ \tau_1 \\ &\overset{*}{=} \big(D_V d\tau_1(W) - D_W d\tau_1(V)\big)^{\mathcal{V}} = 0, \end{aligned}$$

due to the symmetry of the hessian $Dd\tau_1$; at $*$ we have used the identity $D_V(X_j \circ \tau_1) = D_{d\tau_1(V)} X_j = \sum_i v_i (D_{X_i} X_j) \circ \tau_1$ which is a defining property of the induced connection on vector fields along τ_1 and which implies $\sum_{ij} v_i w_j (D_{X_i} X_j) \circ \tau_1 = D_V d\tau_1(W)$. □

Now we can characterize all ppmc immersions with values in the unit sphere $S^{n-1} \subset \mathbb{R}^n$ by their complex Gauss map. In principle we are able to decide whether or not a given horizontal pluriharmonic map $\tau_1 : M \to Z_1$ is the complex Gauss map of a ppmc Kähler immersion:

Theorem 6. *Let M be a Kähler manifold. A horizontal pluriharmonic map $\tau_1 = (\tau', N, \tau'') : M \to Z_1$ is the complex Gauss map of a ppmc Kähler immersion $f : M \to S^{n-1} \subset \mathbb{R}^n$ if and only if there exists a real section f of N (a smooth map $f : M \to \mathbb{R}^n$ with $f(p) \in N_p$ for all $p \in M$) such that $df(T') = \tau'$.*

Proof. Clearly, if $f : M \to S^{n-1}$ is a Kähler immersion, the position vector f is always normal and hence a section of the normal bundle N with $df(T') = \tau'$. Further, if f is ppmc then $\tau_1 = (\tau', N, \tau'')$ is horizontal pluriharmonic by the previous theorem. Conversely, suppose that such a map $\tau_1 = (\tau', N, \tau'')$ and a real section f of N with $df(T') = \tau'$ are given. Since the values of df are perpendicular to N, hence to f, we have $\langle f, f \rangle = const \neq 0$, and we may assume that f takes values in S^{n-1}. In order to show that it is a ppmc immersion, by Theorem 2 we have to prove only that the metric induced by f on M is Kähler for the given complex structure. In general this is true (cf. [5]) if and only if

(a) $df(T')$ is isotropic and
(b) $ddf^{(1,1)}$ takes values in the normal bundle of f.

(a) is true since $df(T') = \tau'$ is isotropic by definition of Z_1, and (b) holds since τ' differentiates into N by horizontality of τ_1. More precisely, let V' and W'' be commuting $(1,0)$ and $(0,1)$ vector fields. Then $s := \partial_{V'} f$ is a section of τ', and hence $(\partial_{W''} s)^{(\tau')^\perp} = (\partial_{W''}\tau).s \in N$ (cf. (3) in Section 3). Hence $\partial_{W''}\partial_{V'} f \in \tau' + N$. Similar we obtain $\partial_{V'}\partial_{W''} f \in \tau'' + N$. Since the two expressions agree, they must be contained in the intersection of the two bundles which is N. $\square$

Returning to the first view point $Z_1 \subset Gc$ we may ask if also $\tau' : M \to Gc$ is pluriharmonic when f is ppmc. In [7] it was shown that an extra condition is needed: Let $N^o \subset N$ be the parallel subbundle spanned by the values of $\alpha^{(1,1)}$ and N^1 its orthogonal complement in N. The ppmc immersion f is called *half isotropic* if $\alpha(T', T') \subset N^1$. The reason for this notation will become clear in the next section.

Theorem 7. *Let M be a Kähler manifold and $f : M \to \mathbb{R}^n$ an isometric immersion with complex Gauss map $\tau' : M \to Gc$. Then τ' is pluriharmonic if and only if f is a half isotropic ppmc immersion.*

Proof. Recall from (6) that $d\tau' = \alpha|_{T^c \otimes T'} \in \mathrm{Hom}(T^c \otimes T', T'' + N)$. We compute $(Dd\tau')^{(1,1)}$. Let X, Z be $(1,0)$-vector fields and $\overline{Y}$ a $(0,1)$-vector field. Then

$$(D_X d\tau')(\overline{Y}).Z = \pi''\partial_X\big(\alpha(\overline{Y}, Z)\big) + \big(D^N_X \alpha\big)(\overline{Y}, Z) \tag{7}$$

where π'' is the projection onto $\tau'' \subset \mathbb{C}^n$ (which we identify with T'') and $D^N_X\alpha$ denotes the normal derivative of α. Hence τ' is pluriharmonic if and only if both terms at right hand side vanish. The first term is zero if and only if $0 = \langle \partial_X(\alpha(\overline{Y}, Z)), W\rangle = -\langle \alpha(\overline{Y}, Z), \alpha(X, W)\rangle$ for all $W \in T'$ which means that $\alpha(T', T') \in N^1 = (N^o)^\perp$. The vanishing of the second term is precisely the ppmc condition. $\square$

Remark 1. It might seem more natural to use the embedding j: $Z_1 \subset Gc$ in order to prove the above theorem; clearly, $\tau' = j \circ \tau_1 : M \to Gc$ is pluriharmonic if and only if τ_1 is pluriharmonic and $(\tau_1^*\beta)^{(1,1)} = 0$ where β denotes the second fundamental form of $Z_1 \subset Gc$. In fact, $(\tau_1^*\beta)(X, \overline{Y})$ is given by the first summand at the right hand side of (7). Proving this involves computing the normal space and the second fundamental form of the submanifold $Z_1 \subset Gc$.

Remark 2. Half isotropic ppmc immersions are studied in [7]. Such an immersion is always minimal in a sphere S^{n-1}_r if it is substantial and indecomposable as a submanifold. In fact, the mean curvature vector $\eta = \frac{1}{2m}\operatorname{trace}\alpha = \frac{1}{2m}\operatorname{trace}\alpha^{(1,1)} \in N^o$ is umbilic which can be seen as follows. First of all, η is a parallel normal vector field since $\alpha^{(1,1)}$ is parallel. Further, the symmetric bilinear form $\alpha_\eta(x, y) = \langle \alpha(x, y), \eta\rangle$ is parallel on $T' \otimes T''$ and vanishes on $T' \otimes T'$ and on $T'' \otimes T''$ since α maps these bundles into N^1 which is perpendicular to η. Thus the corresponding Weingarten map A_η is parallel. If A_η had two different eigenvalues, the corresponding eigenspace distributions would give an extrinsic splitting of the immersion. Hence $A_\eta = \kappa \cdot I$ for some constant $\kappa > 0$. Therefore $m = f + \frac{1}{\kappa}\eta$ is a constant point in

$\mathbb{R}^n$, and $f(M)$ is contained in the sphere of radius $\frac{1}{\kappa}$ centered at m. Since the mean curvature vector η is normal to this sphere, the immersion is minimal.

5. Isotropy

We have seen that a ppmc immersion $f : M \to \mathbb{R}^n$ has an associated family of isometric immersions f_ϑ with rotated second fundamental forms (cf. Eq. (1)). It may happen that this family is trivial, i.e., $f_\vartheta = f$ for all ϑ (up to Euclidean motions) which implies some symmetry for the second fundamental form α. In fact we see from (1) that $f_\vartheta = f$ for all ϑ if and only if there is a family of parallel vector bundle automorphisms $\psi_\vartheta : N \to N$ with

$$\psi_\vartheta \circ \alpha = \alpha_\vartheta \tag{8}$$

where $\alpha_\vartheta(x, y) = \alpha(\mathcal{R}_\vartheta x, \mathcal{R}_\vartheta y)$ as before. We will call such an immersion *isotropic*. By the following theorem (cf. [6]), this property can be read off from the *components of* α:

$$\alpha^{(2,0)}(x, y) = \alpha(\pi' x, \pi' y),$$
$$\alpha^{(1,1)}(x, y) = \alpha(\pi' x, \pi'' y) + \alpha(\pi'' x, \pi' y),$$
$$\alpha^{(0,2)}(x, y) = \alpha(\pi'' x, \pi'' y).$$

Theorem 8. *An isometric Kähler immersion $f : M \to \mathbb{R}^n$ is isotropic ppmc if and only if there is a parallel orthogonal decomposition of the complexified normal bundle $N^c = N' \oplus N^o \oplus N''$ such that the parallel subbundles N', N^o and N'' contain the values of $\alpha^{(2,0)}$, $\alpha^{(1,1)}$ and $\alpha^{(0,2)}$, respectively.*

Proof. If f is isotropic ppmc, then the components of α take values in the eigenbundles of ψ_ϑ corresponding to the eigenvalues $e^{2i\vartheta}$, 1 and $e^{-2i\vartheta}$. They will be called N', N^o and N''. Since ψ_ϑ is parallel, they form a parallel orthogonal decomposition of N^c. Vice versa, if such a decomposition of N^c is given, we can define a parallel bundle automorphism $\psi_\vartheta : N \to N$ by putting $\psi_\vartheta = I$ on N^o and $\psi_\vartheta = e^{\pm 2i\vartheta} I$ on N' and N'', and we obtain Eq. (8) which is equivalent to f being isotropic ppmc. □

Remark. Theorem 8 implies, in particular, that isotropic ppmc immersions are half isotropic (cf. Section 4) since $\alpha^{(2,0)}$ takes values in N' which is perpendicular to N^o. Hence, by Remark 2 in Section 4 we may assume that an isotropic ppmc immersion takes values in a sphere $S^{n-1} \subset \mathbb{R}^n$. Thus Theorem 6 applies and in principle, we can obtain these immersions from their Gauss maps.

By Theorem 3, isotropy of a ppmc immersion $f : M \to \mathbb{R}^n$ implies the isotropy of its Gauss map $\tau : M \to Gr$. The converse statement however cannot be true: If $f : M \to \mathbb{R}^n$ is *pluriminimal*, i.e., a pluriharmonic isometric immersion, its associated family f_ϑ satisfies

$$df_\vartheta = df \circ \mathcal{R}_\vartheta$$

up to a rigid motion of $\mathbb{R}^n$ (cf. [6]), hence we also conclude $\tau_\vartheta = \tau$ (another argument for the isotropy of τ will be given below). But we will see in the next theorem that these are essentially the only two cases where the Gauss map is isotropic.

We need some preparations. For any complex vector bundle $E \subset M \times \mathbb{C}^n$, let us define a linear map $d : T^c \to \mathrm{Hom}(E, E^\perp)$ (the *differential* or *shape operator* of E) by assigning to each vector $v \in T^c$ and any section s of E the $E^{\perp}$-component of $\partial_v s$. According to the splitting $T^c = T' + T''$, the differential splits as $d = d' + d''$.

Lemma 4. *For any isotropic ppmc immersion $f : M \to \mathbb{R}^n$ we get the following chain of differentials*:

$$d' : N'' \to \tau'' \to N^o \to \tau' \to N' \to 0,$$
$$d'' : N' \to \tau' \to N^o \to \tau'' \to N'' \to 0.$$

Proof. Since N', N^o, N'' are parallel subbundles of N^c, being eigenbundles of the parallel bundle automorphism $\psi_\vartheta : N \to N$, the differential of any of them takes values in τ^c. Similarly, τ' and τ'' are mapped into N^c, being parallel subbundles of τ^c. Hence $d'\tau'' = \alpha(T', T'') = N^o$. Further, $\langle d'N'', \tau''\rangle = \langle N'', d'\tau''\rangle = \langle N'', N^o\rangle = 0$ and consequently $d'N'' \subset \tau''$ since $\tau'' \subset \tau^c$ is maximal isotropic. Next, $\langle d'N^o, \tau'\rangle = \langle N^o, d'\tau'\rangle = \langle N^o, N'\rangle = 0$, thus $d'N^o \subset \tau'$. Further, $d'\tau' = N'$. Finally, $\langle d'N', \tau'\rangle = \langle N', N'\rangle = 0$ since N' is isotropic (being perpendicular to $N'' = \overline{N'}$), and $\langle d'N', \tau''\rangle = \langle N', N^o\rangle = 0$, thus we get $d'N' = 0$. This proves the first chain of differentials. The second one follows by complex conjugation. □

Lemma 5. *Let $M = M_1 \times M_2$ be a Riemannian product of Kähler manifolds and $f : M \to \mathbb{R}^n$ an isometric immersion. Let $x_1 \in TM_1$ and $x_2 \in TM_2$. Then $|\alpha^{(1,1)}(x_1, x_2)| = |\alpha^{(2,0)}(x_1, x_2)|$. In particular $\alpha(x_1, x_2) = 0$ if and only if $\alpha^{(2,0)}(x_1, x_2) = 0$. If this holds for all such x_1, x_2, the splitting is extrinsic, i.e., we have an orthogonal decomposition $\mathbb{R}^n = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$ such that $f = f_1 \oplus f_2$ for isometric immersions $f_i : M_i \to \mathbb{R}^{n_i}$.*

Proof. Since all mixed curvature tensor components of the Riemannian product M are zero, we obtain from the Gauss equation that, for any $y_1 \in T^cM_1$ and $y_2 \in T^cM_2$,

$$0 = \big\langle R(y_1, \bar{y}_1)y_2, \bar{y}_2\big\rangle = \big\langle \alpha(y_1, \bar{y}_2), \alpha(\bar{y}_1, y_2)\big\rangle - \big\langle \alpha(y_1, y_2), \alpha(\bar{y}_1, \bar{y}_2)\big\rangle = \big|\alpha(y_1, \bar{y}_2)\big|^2 - \big|\alpha(y_1, y_2)\big|^2.$$

Thus $|\alpha(y_1, y_2)| = |\alpha(y_1, \bar{y}_2)|$ and in particular, putting $y_1 = \pi' x_1$ and $y_2 = \pi' x_2$, we get

$$\big|\alpha(\pi' x_1, \pi' x_2)\big| = \big|\alpha(\pi' x_1, \pi'' x_2)\big|.$$

The extrinsic splitting is obvious if $\alpha(TM_1, TM_2) = 0$. □

Lemma 6. *Let $H \subset O(2m)$ be a group acting on $V = \mathbb{R}^{2m}$ and let $J, \tilde{J} \in O(2m)$ be two H-invariant complex structures on V. Then there is an H-invariant decomposition $V = \sum_j V_j$ such that on each V_j we have either $\tilde{J} = \pm J$ or there is an H-invariant quaternionic structure on V_j.*

Proof. Using the complex structure J, we consider $\mathbb{R}^{2m}$ as a complex vector space, and we decompose $\tilde{J}$ into its complex linear and antilinear components (called L and A). Hence $\tilde{J} = L + A$ with $L = \frac{1}{2}(\tilde{J} - J\tilde{J}J)$ and $A = \frac{1}{2}(\tilde{J} + J\tilde{J}J)$. From $\tilde{J}^2 = -I$ we get

$$-I = L^2 + A^2 + LA + AL,$$

and since $L^2 + A^2$ is linear while $LA + AL$ is antilinear, this implies $L^2 + A^2 = -I$ and $LA + AL = 0$. Since both L and A are antisymmetric, L^2 and $A^2 = -L^2 - I$ are symmetric and decompose $V = \mathbb{R}^{2m}$ into common eigenspaces $W_1, \ldots, W_r$ with non-positive real eigenvalues. Let $W = W_j$ be any of these eigenspaces and $-c^2$, $-s^2$ with $c^2+s^2=1$ the corresponding eigenvalues of L^2 and A^2. If $s=0$, we have $A=0$ and $\tilde{J}J=J\tilde{J}$ on W. Thus there is an H-invariant splitting $W=W_+ + W_-$ with $\tilde{J}=J$ on W_+ and $\tilde{J}=-J$ on W_-. If $s\neq 0$, we may put $J_2=\frac{1}{s}A$ and obtain $(J_2)^2=\frac{1}{s^2}A^2=-I$. This is an antisymmetric complex structure, hence orthogonal (since $(J_2)^T=-J_2$ and $(J_2)^2=-I$ imply $(J_2)^TJ_2=I$), and J_2 anti-commutes with J. Thus $J_1:=J$ together with J_2 and $J_3:=J_1J_2$ form an H-invariant quaternionic structure on W. $\square$

Corollary. *Let M be a locally irreducible Riemannian manifold with two linear independent parallel almost complex structures. Then M is locally hyper-Kähler, i.e., locally there exist three anti-commuting parallel almost complex structures on M.*

Proof. We apply Lemma 6 for $V=T_pM$ where H is the local holonomy group of M at the point p. By assumption this acts irreducibly, so the H-invariant decomposition $V=\sum V_j$ must be trivial. Since the two almost complex structures are linearly independent, we get a quaternionic structure (J_1, J_2, J_3) on T_pM which is invariant under the local holonomy group and thus allows a parallel extension on a neighborhood of p. $\square$

Theorem 9. *Let M be a Kähler manifold such that no local factor of M is hyper-Kähler, and let $f:M\to\mathbb{R}^n$ be an isometric immersion with Gauss map $\tau:M\to Gr$. Then τ is isotropic pluriharmonic if and only if f is either pluriminimal or isotropic ppmc.*

Proof. The map $\tau:M\to Gr$ is isotropic pluriharmonic if and only if there is a holomorphic super-horizontal lift $\hat{\tau}:M\to Z$ into some flag manifold Z fibering over Gr (cf. [6]). We classify these flag manifolds in the appendix and obtain $Z=Z_r$ for some $r\in\mathbb{N}$, where Z_r is the set of all $(2r+1)$-tuples of complex subspaces $E_{-r},\dots,E_r$ with given dimensions forming an orthogonal decomposition $\mathbb{C}^n=\sum_{j=-r}^{r}E_j$ such that $E_{-j}=\overline{E_j}$ for all j. Thus the lift $\hat{\tau}$ is a "moving" orthogonal decomposition $(E_{-r},\dots,E_r)$ of subbundles $E_j\subset M\times\mathbb{C}^n$ with $E_{-j}=\overline{E_j}$, and the fact that $\hat{\tau}$ is holomorphic super-horizontal means that $d'E_j=E_{j+1}$. Since $\hat{\tau}$ is a lift of τ, we have either $\tau^c=E_{\text{even}}$ or $\tau^c=E_{\text{odd}}$ where $E_{\text{even}}=\sum_{j+r\text{ even}}E_j$ and $E_{\text{odd}}=\sum_{j+r\text{ odd}}E_j$.

Now $f:M\to\mathbb{R}^n$ is pluriminimal if and only if τ' is holomorphic which means $d''\tau'=0$. Consequently $d''N\subset\tau'$ (since $\langle d''N,\tau'\rangle=\langle N,d''\tau'\rangle=0$) and $d''\tau''\subset N$ (since $\tau''\subset\tau^c$ is parallel), hence

$$d'':\tau''\to N\to\tau'\to 0,\qquad d':\tau'\to N\to\tau''\to 0.$$

Thus $\hat{\tau}=(\tau',N,\tau'')$ is a (super-)horizontal holomorphic lift into the corresponding flag manifold Z_1 (and in particular, τ is isotropic pluriharmonic).

If $f:M\to\mathbb{R}^n$ is isotropic ppmc, then $\hat{\tau}=(N'',\tau'',N^o,\tau',N')$ is a super-horizontal holomorphic lift into the corresponding flag manifold Z_2 (cf. Lemma 4).

Conversely, let $f:M\to\mathbb{R}^n$ be any Kähler immersion such that the Gauss map $\tau:M\to Gr$ is isotropic pluriharmonic and let $\hat{\tau}=(E_{-r},\dots,E_r)$ be the holomorphic super-horizontal lift of τ. Then $\tau^c=E_{-r'}+E_{-r'+2}+\cdots+E_{r'}$ where $r'\in\{r-1,r\}$, and since $d'E_j\subset E_{j+1}$ and $d''E_j\subset E_{j-1}$, the subbundles E_j of τ^c are parallel. Let $\tau_j^c=E_j+E_{-j}$. Then $\tau_j^c=\tau_j\otimes\mathbb{C}$ for some parallel real subbundle $\tau_j\subset\tau$, and $E_{\pm j}=(I\mp iJ_j)\tau_j$ for a parallel complex structure J_j on τ_j, if $j\neq 0$. By the corollary of Lemma 6 and the present assumption we may assume $J_j=\pm J$ where J is the complex structure of TM, transplanted by df onto τ. (Maybe we yet have to split τ_j into holonomy irreducible subbundles.) Thus $E_j = \pi'(\tau_j)$ or $E_j = \pi''(\tau_j)$. If $E_i = \tau_i'$ and $E_j = \tau_j'$ for some $i \neq \pm j$, using the symmetry of α we have

$$\alpha(E_i, E_j) = d(E_i).E_j \subset E_{j-1} \cap E_{i-1} = 0$$

and likewise, if $E_i = \tau_i'$ and $E_j = \tau_j''$, we have

$$\alpha(E_i, E_{-j}) = d(E_i).E_{-j} \subset E_{-j-1} \cap E_{i-1} = 0.$$

In both cases we get $\alpha^{(2,0)}(\tau_i, \tau_j) = 0$ which by Lemma 5 is equivalent to $\alpha(\tau_i, \tau_j) = 0$ (recall that the parallel subbundles τ_j define a local Riemannian product structure on M). So we see that the splitting is also extrinsic and we may assume $r' = 1$.

The remaining possibilities for our moving flag are the following four cases: (τ', N, τ''), (τ'', N, τ'), $(N'', \tau', N^o, \tau'', N')$, and $(N'', \tau'', N^o, \tau', N')$ (the bundles N' and N'' are interchangeable). In the first case we have $d''\tau' = 0$, so τ' is holomorphic and hence f is pluriminimal by Theorem 4. The second case is equivalent to $d'\tau' = 0$ which means $\alpha^{(2,0)} = 0$. This implies $(D\alpha)^{(2,1)} = 0$ and hence $D\alpha = 0$ by the Codazzi equation, and in particular f is a ppmc immersion. In fact these are the standard embeddings of compact Hermitian symmetric spaces (cf. Section 7). In the third case we get $d'\tau'' \subset N'$ and $d''\tau' \subset N''$. Hence $\alpha(T', T'') \in N' \cap N'' = 0$ and thus $\alpha^{(1,1)} = 0$. So we are back to the first case. Finally in the last case, $\alpha^{(2,0)}$, $\alpha^{(1,1)}$ and $\alpha^{(0,2)}$ take values in the parallel subbundles N', N^o and N'' which shows isotropy by Theorem 8. $\square$

6. Isotropy and complex Gauss map

Using the complex Gauss map with values in the complex Grassmannian, we can characterize isotropy avoiding the unpleasant extra condition of Theorem 9:

Theorem 10. *A Kähler immersion $f : M \to \mathbb{R}^n$ is isotropic ppmc if and only if its complex Gauss map $\tau' : M \to Gc$ is isotropic pluriharmonic, but not holomorphic.*

Proof. Assume first that $f : M \to \mathbb{R}^n$ is isotropic ppmc. Then we have an orthogonal decomposition ("moving flag") $\mathbb{C}^n = N' \oplus \tau' \oplus Q$ with $Q := N^o + \tau'' + N''$ (where $\mathbb{C}^n$ denotes the trivial vector bundle $M \times \mathbb{C}^n$), and by Lemma 4 we have the differentials $d' : Q \to \tau' \to N' \to 0$ and $d'' : N' \to \tau' \to Q \to 0$. Thus the map (Q, τ', N') into the corresponding flag manifold over Gc with the projection $(Q, \tau', N') \mapsto \tau'$ is horizontal and holomorphic and thus τ' is isotropic pluriharmonic.

Conversely, let us assume that τ' is isotropic pluriharmonic, i.e., there is a one parameter group $\phi_\vartheta \in \mathrm{Aut}(\tau'^*(TGc))$ with $\phi_\vartheta \circ d\tau' = d\tau' \circ \mathcal{R}_\vartheta$. By [6] we have a horizontal holomorphic lift $\hat{\tau}'$ of τ' into some flag manifold Z over Gc, i.e. (cf. Appendix A) there are decompositions $\tau' = \tau_1' \oplus \cdots \oplus \tau_r'$ and $(\tau')^\perp = P_1 \oplus \cdots \oplus P_{r+1}$ (where P_1 and P_{r+1} might be zero) such that $d' : P_i \to \tau_i \to P_{i+1}$ and $d'' : P_{i+1} \to \tau_i \to P_i$ for $i = 1, \ldots, r$. By the following argument we may assume $r = 1$ and thus $\hat{\tau}'$ is a "moving decomposition" of the type

$$\mathbb{C}^n = P_1 \oplus \tau' \oplus P_2.$$

In fact, the parallel decomposition $\tau' = \tau_1' \oplus \cdots \oplus \tau_r'$ induces a corresponding real parallel decomposition $TM = T_1 \oplus \cdots \oplus T_r$ and hence the manifold M can be (locally) decomposed as a Riemannian product of Kähler manifolds. This splitting is even extrinsic: For any $x_i' \in T_i'$ and $x_j' \in T_j'$ we have (using df to identify T' and τ')

$$\alpha(x_i', x_j') = d\tau'(x_i').x_j' \subset d'\tau_j \subset P_{j+1},$$
$$\alpha(x_j', x_i') = d\tau'(x_j').x_i' \subset d'\tau_i \subset P_{i+1},$$

hence $\alpha(x_i', x_j') = 0$ and thus $\alpha^{(2,0)}(T_i, T_j) = 0$. But by Lemma 5 this implies $\alpha(T_i, T_j) = 0$. Hence we may assume $r = 1$.

Now we claim for any $(1, 0)$ vector fields X, Y, Z (while still identifying TM with τ)

$$(D_{\overline{Z}} d\tau')(\overline{X}).Y = \big(D^N_{\overline{Z}}\alpha\big)(\overline{X}, Y) + \big(\partial_{\overline{Z}}\big(\alpha(\overline{X}, Y)\big)\big)^{T''}. \tag{9}$$

In fact, recall from (6) (Lemma 2) that

$$d\tau' : TM \to \tau'^*(TGc) = \mathrm{Hom}(\tau', \tau'^{\perp}), \qquad d\tau'(V).Y = (\partial_V Y)^{T''+N} = \alpha(V, Y)$$

for any $V \in T^c$ and $Y \in T'$. Then

$$(D_{\overline{Z}} d\tau')(\overline{X}).Y = \big(\partial_{\overline{Z}}\big(d\tau'(\overline{X}).Y\big)\big)^{T''+N} - d\tau'(D_{\overline{Z}}X).Y - d\tau'(X).D_{\overline{Z}}Y. \tag{10}$$

Now we may replace $d\tau'$ by α. Consider the right hand side of (10) ("rhs (10)"). The first term splits into its components with respect to T'' and N. Its N-component together with the 2nd and 3rd terms gives $(D^N_{\overline{Z}}\alpha)(\overline{X}, Y)$ (which is the first term of rhs (9)) while the remaining term $\partial_{\overline{Z}}(\alpha(\overline{X}, Y)^{T''}$ is the second summand of rhs (9). Thus Eq. (9) is proved.

On the other hand we have

$$(D_{\overline{Z}} d\tau')(\overline{X}) = D_{\overline{Z}}\big(d\tau'(\overline{X})\big) - d\tau'(D_{\overline{Z}}\overline{X}). \tag{11}$$

If τ' is isotropic pluriharmonic, then both terms at rhs (11) are eigenvectors of ϕ_θ with respect to the eigenvalue $\mathrm{e}^{-i\theta}$: the second one because $D_{\overline{Z}}\overline{X} \in T''$ and $\phi_\theta \circ d\tau' = d\tau' \circ \mathcal{R}_\theta$, and the first one because the eigenbundle of ϕ_θ is parallel. Thus these vectors lift to $(0, 1)$ super-horizontal tangent vectors of Z (cf. [6]) which map $P_2 \to T' \to P_1$.

It follows that $(D_{\overline{Z}} d\tau')(\overline{X})$ maps T' into P_1, and since the first term of rhs (9) vanishes by the ppmc property, we conclude from (9) that

$$\big(\partial_{\overline{Z}}\alpha(\overline{X}, Y)\big)^{T''} \in P_1.$$

Thus putting $T_0'' = T'' \cap P_1$ and letting T_1'' be the orthogonal complement of T_0'' in T'', we have $(\partial_{\overline{Z}}\alpha(\overline{X}, Y))_{T_1''} = 0$, and therefore we obtain for all $W \in T'$ with $\overline{W} \in T_1''$:

$$\big\langle \alpha(\overline{X}, Y), \alpha(\overline{Z}, W)\big\rangle = \big\langle \partial_{\overline{Z}}\alpha(\overline{X}, Y), W\big\rangle = 0.$$

In other words, $\alpha(\overline{Z}, W) = 0$ for all $Z \in T'$ which says that W and hence all of T_1'' lies in the subbundle

$$\ker\alpha^{(1,1)} := \big\{W \in T';\ \alpha(\overline{Z}, W) = 0\ \forall Z \in T'\big\}.$$

By parallelity of $\alpha^{(1,1)}$, this is a parallel subbundle of T' which can be split off, using Lemma 5 (yielding a pluriminimal factor). Thus we may assume that $\ker\alpha^{(1,1)} = 0$ and hence $T_1'' = 0$, i.e., $T'' \subset P_1$.

Just as in (11) we have that $(D_Z d\tau').X$ is in the $\mathrm{e}^{i\theta}$-eigenspace of ϕ_ϑ whose elements map T' into P_2, and as in (9) we have

$$(D_Z d\tau')X).Y = \big(D^N_Z\alpha\big)(X, Y) + \big(\partial_Z(\alpha(X, Y)\big)^{T''} \in P_2. \tag{12}$$

But the second term of rhs (12) is in $T'' \subset P_1$ while the first one is in $N \perp T''$. Hence the sum can be perpendicular to T'' (recall that $P_2 \perp P_1 \supset T''$) only if its T''-component (the second term of rhs (12)) vanishes. Taking the inner product of this term with any $W \in T'$ we obtain

$$\big\langle \alpha(X, Y), \alpha(Z, W)\big\rangle = 0$$

for arbitrary $X, Y, Z, W \in T'$. Thus $\langle N', N'\rangle = 0$ or in other words $N' \perp \overline{N'} = N''$. Since f is already half isotropic (cf. Theorem 7), we also have $N' \perp N^o$. Now the proof is finished by Theorem 8. □

7. Examples

Clearly, if $f_i : M_i \to \mathbb{R}^{n_i}$ are any two ppmc Kähler immersions ($i = 1, 2$), then so is $f = f_1 \times f_2 : M_1 \times M_2 \to \mathbb{R}^{n_1+n_2}$. Therefore it is enough to study ppmc immersions $f : M \to \mathbb{R}^n$ which are *irreducible*, i.e., they do not split as above, and *substantial*, i.e., their image is not contained in any proper affine subspace of $\mathbb{R}^n$. Three classes of such immersions are known:

(1) surfaces with nonzero parallel mean curvature vector,
(2) pluriminimal submanifolds,
(3) extrinsic symmetric Kähler immersions.

Class (1) has been investigated by Yau [13]; these examples occur only in $\mathbb{R}^3$ or S^3 unless they are minimal surfaces in S^{n-1}. Class (2) contains many examples in all dimensions, cf. [3] and the recent paper [1]. We will now briefly describe class (3).

Recall that an isometric (irreducible, substantial) immersion $f : M \to \mathbb{R}^n$ is called *extrinsic symmetric* if the full second fundamental form $\alpha \in \mathrm{Hom}(TM \otimes TM, N)$ is parallel. These immersions have been classified by Ferus ([8], also cf. [4]). It is not difficult to see that α is parallel if and only if f is invariant under reflection at each of its normal spaces. In particular all point reflections or geodesic symmetries on M extend to (extrinsic) isometries, hence M is globally symmetric. Moreover, M is isotropy irreducible, i.e., the full extrinsic isotropy group of M acts irreducibly on the tangent space (cf. [4]). The corresponding Gauss map $\tau : M \to Gr$ is a totally geodesic isometric immersion of the symmetric space M into the real Grassmannian Gr. In fact, since τ is equivariant and M is isotropy irreducible, it is isometric (up to a scaling factor). Moreover, the image of τ is invariant under the corresponding point reflections of Gr and thus totally geodesic; note that the point reflection of the Grassmannian at some $\tau(p) \in Gr$ is just the reflection at the normal space $\tau(p)^\perp = N_p$.

Hence, if $f : M \to \mathbb{R}^n$ is an extrinsic symmetric immersion which is also Kähler (with almost complex structure J), then f is clearly ppmc since the parallelity of $\alpha^{(1,1)}$ is a weaker condition. Moreover, if f is also substantial and irreducible, it is isotropic. To see this recall that a symmetric space M with a Kähler metric is in fact *Hermitian symmetric*, i.e., the rotations $\mathcal{R}_\vartheta(p) = \cos(\vartheta)I + \sin(\vartheta)J$ on T_pM for any $p \in M$ extend to isometries ρ_ϑ on M fixing p. But these isometries are generated by point reflections which extend to orthogonal linear maps on $\mathbb{R}^n$, hence ρ_ϑ also extends to some $A_\vartheta \in O(n)$ with $f \circ \rho_\vartheta = A_\vartheta \circ f$. We put $\psi_\vartheta(p) = A_{2\vartheta}|_{N_p}$. Since A_ϑ (being an extrinsic isometry) commutes with α, we obtain

$$\psi_\vartheta\big(\alpha(v, w)\big) = \alpha(\mathcal{R}_\vartheta v, \mathcal{R}_\vartheta w) \tag{13}$$

for all $v, w \in T_pM$. In particular this equation implies that $p \mapsto \psi_\vartheta(p)$ is parallel (as an endomorphism of the normal bundle N), since so are $\mathcal{R}_\vartheta$ and α and since $N = \alpha(TM \otimes TM)$. Thus f is isotropic ppmc.

Since $\psi_\pi = I$ by (13), the eigenvalues of $\psi_{\pi/2}$ can only be ± 1. Accordingly, class (3) has two subclasses: If 1 is the only eigenvalue, i.e., $\psi_{\pi/2} = I$, then we get from (13)

$$\alpha(Jv, Jw) = \alpha(v, w)$$

for all v, w, hence $\alpha^{(2,0)} = 0$. These immersions have been characterized already by Ferus [8]: They are the so called *standard embeddings* of an Hermitian symmetric space $M = G/K$ into the Lie algebra $\mathfrak{g}$ of G via the map $p \mapsto J_p$ (recall that the complex structure J_p on T_pM is a skew-symmetric derivation of the curvature tensor of M at p, hence it extends to an infinitesimal isometry, i.e., to an element of $\mathfrak{g}$).

In the remaining examples, the eigenvalue -1 occurs for $\psi_{\pi/2}$. Inspection shows that these are precisely the extrinsic symmetric $2:1$ immersions of $Gr_2^+ = G_2^+(\mathbb{R}^N)$, the Grassmannian of *oriented* 2-planes in $\mathbb{R}^N$, factorizing over the ordinary real Grassmannian Gr_2. In fact, Gr_2^+ is an Hermitian symmetric space (which can be identified with the complex quadric $\{[z] \in \mathbb{C}P^{N-1};\ \langle z, z\rangle = 0\}$ via the map $E = \operatorname{Span}\{x, y\} \mapsto [x + iy]$, where (x, y) is any oriented orthonormal basis of the oriented plane $E \subset \mathbb{R}^N$). We put $f = \tilde{f} \circ \pi$ where $\pi : Gr_2^+ \to Gr_2$ is the canonical projection and $\tilde{f} : Gr_2 \to S(N)$ the usual (extrinsic symmetric) embedding of the Grassmannian into the space of symmetric real $N \times N$-matrices by assigning to each plane $E \in Gr_2$ the orthogonal projection of $\mathbb{R}^N$ onto E. In this case, the (-1)-eigenspace is 2-dimensional. The easiest example is the Veronese immersion

$$S^2 \to \mathbb{R}P^2 \to S^4 \subset \mathbb{R}^5 \cong \left\{X \in S(3);\ \operatorname{trace} X = 1\right\}.$$

It is an open problem how to construct further classes of examples. Using our Theorem 6, we hope that a better understanding of horizontal pluriharmonic maps into Z_1 will lead to new ppmc immersions.

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Appendix A. Canonical embeddings of flag manifolds

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{g}^c = \mathfrak{g} \otimes \mathbb{C}$ be the complexification of $\mathfrak{g}$. We consider adjoint orbits ("flag manifolds") $Z = Ad(G)\xi$ for $\xi \in \mathfrak{g}$. An orbit can always be represented as a coset space G/H where H is the stabilizer subgroup; in the present case $H = C(\xi) = \{g \in G;\ Ad(g)\xi = \xi\}$ is the centralizer of ξ. More precisely, Z is the image of the equivariant embedding $j_\xi : G/H \to \mathfrak{g}$, $j_\xi(gH) = Ad(g)\xi$. Of course, if we fix H, many $\xi \in \mathfrak{g}$ may have H as centralizer and give different embeddings j_ξ of the same coset space G/H, but there are distinguished such ξ: We call $\xi \in \mathfrak{g}$ a *canonical element* and j_ξ a *canonical embedding* of G/H for $H = C(\xi)$ if

C1 The eigenvalues of $\frac{1}{i} ad(\xi)$ are integers (where $i = \sqrt{-1}$), C2 $\mathfrak{g}_1 + \mathfrak{g}_{-1}$ generates $\mathfrak{g}^c$, where $\mathfrak{g}_k \subset \mathfrak{g}^c$ denotes the k-eigenspace of $\frac{1}{i}ad(\xi)$.[8]

The Jacobi identity implies $[\mathfrak{g}_j, \mathfrak{g}_k] \subset \mathfrak{g}_{j+k}$. Since $\mathfrak{g}_1 + \mathfrak{g}_{-1}$ is a generating subspace and $\mathfrak{g}_{-j} = \overline{\mathfrak{g}_j}$, the eigenvalues of $\frac{1}{i}ad(\xi)$ form a set $\{-r, \dots, r\}$ for some positive integer r (called the *height* of the flag manifold) where $\mathfrak{g}_0 = \mathfrak{h}^c$ is the complexified Lie algebra of H, and we have a direct decomposition $\mathfrak{g}^c = \sum_{j=-r}^{r} \mathfrak{g}_j$.

The flag manifold $Z = G/H$ fibres over a symmetric space $S = G/K$ defined by the corresponding (complexified) Cartan decomposition as follows:

$$\mathfrak{k}^c = \sum_{j \text{ even}} \mathfrak{g}_j, \quad \mathfrak{p}^c = \sum_{j \text{ odd}} \mathfrak{g}_j. \tag{A.1}$$

In fact, the Cartan relations $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ are obvious from $[\mathfrak{g}_j, \mathfrak{g}_k] \subset \mathfrak{g}_{j+k}$, and clearly $\mathfrak{h}^c = \mathfrak{g}_0 \subset \mathfrak{k}^c$. Thus Z defines a unique symmetric space S which is *inner*, i.e., its symmetry is an inner automorphism (namely $Ad(e^{\pi\xi})$). But conversely there are *several* flag manifolds which fibre over S as described. As an example we shall determine all canonical elements and corresponding flag manifolds over complex and real Grassmannians, using only elementary linear algebra.

First let $G = U_n$ the unitary group. Then $\mathfrak{g} = \mathfrak{u}_n$ is the space of skew-Hermitian matrices. Any $\xi \in \mathfrak{g}$ determines an orthogonal eigenspace decomposition of $\mathbb{C}^n$, and the eigenvalues are imaginary. Thus there is an orthogonal decomposition $\mathbb{C}^n = \sum_{j=1}^m E_j$ such that $\xi = i \cdot \sum_{j=1}^m \lambda_j E_j$ for real numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_m$, where for any subspace $E \subset \mathbb{C}^n$ we use the same symbol E to denotes the orthogonal projection matrix onto E. If $E, F \subset \mathbb{C}^n$ are subspaces with $E \perp F$, we embed $\mathrm{Hom}(E, F)$ into $\mathrm{End}(\mathbb{C}^n) = \mathfrak{g}^c$ by putting $L|_{E^\perp} = 0$ for any $L \in \mathrm{Hom}(E, F)$. Then we have for any $L_{EF} \in \mathrm{Hom}(E, F)$:

$$[E, L_{EF}] = -L_{EF}, \qquad [F, L_{EF}] = L_{EF}. \tag{A.2}$$

Thus for all $L_{jk} \in H_{jk} := \mathrm{Hom}(E_j, E_k)$ we obtain

$$ad(\xi)L_{jk} = i \cdot (\lambda_k - \lambda_j) \cdot L_{jk}. \tag{A.3}$$

Hence, if ξ is canonical, then $\lambda_k - \lambda_j$ are integers for all j, k, by property C1. Next we claim $\lambda_{j+1} - \lambda_j = 1$ for all j. This is due to property C2 saying that $\mathfrak{g}_1 + \mathfrak{g}_{-1}$ generates $\mathfrak{g}^c$. In fact, if $\lambda_{k+1} - \lambda_k \geqslant 2$ for some k, we may decompose $\mathbb{C}^n = E \oplus F$ with $E = \sum_{j=1}^k E_j$ and $F = \sum_{l=k+1}^m E_l$. Then $\lambda_l - \lambda_j \geqslant 2$ for all $j \in \{1, \dots, k\}$ and $l \in \{k+1, \dots, m\}$, and hence $H_{jl} = \mathrm{Hom}(E_j, E_l)$ and $H_{lj} = \mathrm{Hom}(E_l, E_j)$ belong to some $\mathfrak{g}_k$ with $|k| \geqslant 2$. In other words, $\mathfrak{g}_1 + \mathfrak{g}_{-1}$ is contained in $\mathrm{Hom}(E, E) \oplus \mathrm{Hom}(F, F)$ which is a proper Lie subalgebra of $\mathfrak{u}_n^c$. This contradicts property C2. Thus we have seen (the converse statement is obvious):

Proposition A.1. *An element $\xi \in \mathfrak{g} = \mathfrak{u}_n$ is canonical if and only if $\xi = i(\lambda_0 \cdot I + \sum_{j=1}^m j \cdot E_j)$ for some orthogonal decomposition $\mathbb{C}^n = \sum_{j=1}^m E_j$ and any $\lambda_0 \in \mathbb{R}$. Then $\mathfrak{g}_k = \sum_j H_{j,j+k}$.*

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[8] A canonical element ξ is not uniquely determined by H. But there is only one such ξ (up to adding an element in the center of $\mathfrak{g}$) in any Weyl chamber C of $\mathfrak{g}$ which is adjacent to the subtorus T' centralized by H (where "adjacent" means that $\overline{C} \cap \mathfrak{t}'$ contains an open subset of $\mathfrak{t}' = L(T')$). In fact $\xi = \sum_{j \in J} \alpha_j^*$, where $\{\alpha_1, \dots, \alpha_l\}$ are the simple roots of $\mathfrak{g}$ corresponding to C and $\alpha_1^*, \dots, \alpha_l^*$ the dual root vectors (i.e., $\alpha_j(\alpha_k^*) = \delta_{jk}$) and where $J = \{j \in \{1, \dots, l\};\ \mathfrak{g}_{\alpha_j} \cap \mathfrak{h} = 0\}$ (cf. [2, p. 42]). Using this extra structure we can represent G/H as the complex coset space G^c/P for the parabolic subgroup $P = \{g \in G^c;\ Ad(g)\xi \in \xi + \sum_{k>0} \mathfrak{g}_k\}$, and our definition of "canonical element" agrees with that of [2, p. 41].

The corresponding flag manifold is a "classical" flag manifold Z consisting of all orthogonal decompositions of $\mathbb{C}^n$ with the same dimensions as $E_1, \dots, E_r$, and Z is embedded as the adjoint orbit $Ad(U_n)\xi$. What is the corresponding symmetric space S over which Z fibres? Let us put $E_{odd} = \sum_{j \text{ odd}} E_j$ and $E_{ev} = \sum_{j \text{ even}} E_j$. Then we have

$$\mathfrak{k}^c = \mathrm{End}(E_{ev}) \oplus \mathrm{End}(E_{odd}), \qquad \mathfrak{p}^c = \mathrm{Hom}(E_{ev}, E_{odd}) \oplus \mathrm{Hom}(E_{odd}, E_{ev}). \tag{A.4}$$

This is the complexified Cartan decomposition of a symmetric space, namely the Grassmannian of all subspaces in $\mathbb{C}^n$ with the same dimension as E_{ev} (or as E_{odd}).

Now let $G = SO_n$ be the orthogonal group which we consider as a subgroup of U_n. Let $\xi \in \mathfrak{so}_n \subset \mathfrak{u}_n$. As before, we have $\xi = i \cdot \sum_{j=1}^m \lambda_j E_j$ for some orthogonal decomposition $\mathbb{C}^n = \sum_j E_j$ where $\lambda_1 < \dots < \lambda_m$ are real. But now ξ is a real matrix, i.e., we also have $\xi = \bar{\xi} = -i \cdot \sum_j \lambda_j \overline{E_j}$. Since the projections E_j are linearly independent and nonnegative, there is a permutation σ of $\{1, \dots, m\}$ such that $\overline{E_j} = E_{\sigma j}$ and $\lambda_{\sigma j} = -\lambda_j$. Thus

$$\widehat{H}_{jk} := \mathrm{Hom}(E_j, E_k) + \mathrm{Hom}(E_{\sigma k}, E_{\sigma j})$$

is the eigenspace of $ad(\xi)$ corresponding to the eigenvalue $\lambda_k - \lambda_j$, according to (A.3). Now $\mathfrak{so}_n^c = \{A \in \mathbb{C}^{n \times n};\ A^T = -A\}$ is generated as a vector space by $M_{jk} := L_{jk} - (L_{jk})^T$ for all $L_{jk} \in \mathrm{Hom}(E_j, E_k)$ and all $j, k \in \{1, \dots, m\}$. We claim that $M_{jk} \in \widehat{H}_{jk}$.

In fact, it is sufficient to show that $(L_{jk})^T \in \mathrm{Hom}(E_{\sigma k}, E_{\sigma j})$. Put $y = (L_{jk})^T x$ for some $x \in \mathbb{C}^n$. Let us denote the symmetric inner product on $\mathbb{C}^n$ by $\langle v, w \rangle = \sum v_j w_j$. Then for all $w \in \mathbb{C}^n$ we have $\langle y, w \rangle = \langle x, L_{jk} w \rangle$, and the latter is nonzero only if $w \in E_j$ and $x \in \overline{E_k}$. Moreover $\langle y, w \rangle \neq 0$ implies $y \in \overline{E_j}$. Thus $(L_{jk})^T$ maps $\overline{E_k} = E_{\sigma k}$ into $\overline{E_j} = E_{\sigma j}$ and vanishes on the orthogonal complement of $E_{\sigma k}$; this proves the claim.

Hence $ad(\xi)$ takes the same eigenvalues $\lambda_j - \lambda_k$ on $\mathfrak{so}_n^c$ as on $\mathfrak{u}_n^c$. Thus by C1, these differences are integers and by C2 we even have $\lambda_{j+1} - \lambda_j = 1$ as before; otherwise $\mathfrak{so}_n^c$ had to be contained in a subalgebra $\mathrm{Hom}(E, E) + \mathrm{Hom}(F, F) \subset \mathfrak{u}_n^c$ for some nontrivial decomposition $\mathbb{C}^n = E \oplus F$, but the inclusion $SO_n \subset U_n$ is an irreducible representation. Thus we conclude that the set of eigenvalues λ_j of $\frac{1}{i}\xi$ is of the form $\{-r, -r+1, \dots, r-1, r\}$ for some positive integer or half integer r. Relabelling E_j we obtain:

Proposition A.2. *An element $\xi \in \mathfrak{g} = \mathfrak{so}_n$ is canonical if and only if $\xi = \sum_{j=-r}^r j \cdot E_j$ for some orthogonal decomposition $\mathbb{C}^n = \sum_{j=-r}^r E_j$ such that $E_{-j} = \overline{E_j}$ for all $j \in \{-r, \dots, r\}$, for some $r \in \frac{1}{2}\mathbb{N}$. Then $\mathfrak{g}_k = \sum_j \widetilde{H}_{j,j+k}$ where $\widetilde{H}_{j,l} = \{A \in \widehat{H}_{j,l};\ A^T = -A\}$.*

The corresponding symmetric space S is a subset of the complex Grassmannian obtained from E_{ev}, namely $S = \{A(E_{ev});\ A \in SO_n\}$, where $E_{ev} := \sum_{j+r \text{ even}} E_j$. We have to distinguish two cases:

(a) $r \in \mathbb{N}$: Then the eigenvalues of $\frac{1}{i} ad(\xi)$ are the integers $j \in \{-r, \dots, r\}$. If $j + r$ is even, then so is $-j + r$. Hence E_{ev} is invariant under conjugation and thus the complexification of a subspace of $\mathbb{R}^n$. Hence S is the real Grassmannian containing all subspaces of $\mathbb{R}^n$ with the same dimension as E_{ev}.

(b) $r \notin \mathbb{N}$: Then all eigenvalues $j \in \{-r, \dots, r\}$ are proper half integers. If $j + r$ is even, $-j + r$ is odd, and hence $\overline{E_{ev}} = \sum_{j+r \text{ odd}} E_j = (E_{ev})^\perp$. Thus the dimension n is even and E_{ev} is a maximal isotropic subspace. Therefore S is the space of all maximal isotropic subspaces of $\mathbb{C}^n$, or as a coset space, $S = SO_n/U_{n/2}$.

Corollary. *The flag manifolds over real Grassmannians are precisely the manifolds of all orthogonal decompositions $\mathbb{C}^n = \sum_{j=-r}^{r} E_j$ for some $r \in \mathbb{N}$, where $E_{-j} = \overline{E_j}$ and the dimensions of $E_0, \ldots, E_r$ are fixed arbitrarily.*

The complexified tangent space of a general canonically embedded flag manifold $Z = Ad(G)\xi$ at the point ξ is $T^c = ad(\mathfrak{g}^c)\xi = ad(\xi)(\sum_j \mathfrak{g}_j) = \sum_{j\neq 0} \mathfrak{g}_j$. Moreover, Z is also a complex manifold (a coset space of the complex group G^c), and the space of $(1,0)$ tangent vectors is $T' = \sum_{j>0} \mathfrak{g}_j$. Further, the complexified horizontal subspace for the fibration $\pi : Z \to S$ is $\mathcal{H} = \sum_{k \text{ odd}} \mathfrak{g}_k$ while the $(1,0)$ super-horizontal space is just $\mathcal{H}'_1 = \mathfrak{g}_1 \subset \mathcal{H}$.

In particular, for a flag manifold Z over a real Grassmannian we obtain using the previous notation:

$$T^c = \sum_{j\neq k} \widetilde{H}_{jk}, \qquad T' = \sum_{j<k} \widetilde{H}_{jk}, \qquad \mathcal{H}'_1 = \sum_{j} \widetilde{H}_{j,j+1}. \tag{A.5}$$

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