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Angaben zur Veröffentlichung / Publication details:

Eschenburg, Jost-Hinrich. 2002. “Higher rank curved Lie triples.” *Journal of the Mathematical Society of Japan* 54 (3): 551–64. https://doi.org/10.2969/jmsj/1191593908.

J. Math. Soc. Japan
Vol. 54, No. 3, 2002

Higher rank curved Lie triples

Dedicated to Professor Dirk Ferus

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(Received Jul. 10, 2000)
(Revised Nov. 27, 2000)

Abstract. A substantial proper submanifold M of a Riemannian symmetric space S is called a *curved Lie triple* if its tangent space at every point is invariant under the curvature tensor of S, i.e. a sub-Lie triple. E.g. any complex submanifold of complex projective space has this property. However, if the tangent Lie triple is irreducible and of higher rank, we show a certain rigidity using the holonomy theorem of Berger and Simons: M must be intrinsically locally symmetric. In fact we conjecture that M is an extrinsically symmetric isotropy orbit. We are able to prove this conjecture provided that a tangent space of M is also a tangent space of such an orbit.

0. Introduction.

There is an essential difference between submanifold theory in Euclidean and symmetric spaces. In Euclidean space or another space of constant curvature, the tangent spaces do not contain any distinguished subspaces, and the only geometric invariant of a subspace is its dimension. In fact, the main invariant for differential geometry, the curvature tensor, is zero or pointwise induced by the inner product. This is different in a symmetric space S of nonconstant curvature. On each tangent space T_pS, the curvature tensor R^S can be viewed as an algebraic structure, a triple product (i.e. a product with three factors) satisfying the curvature identities and the additional property that $R^S(v,w)$ is a derivation of R^S for any v,w; this is called a *Lie triple product*. Now there are distinguished subspaces of T_pS, namely those which are invariant under this triple product R^S; they will simply be called *Lie triples*.

So the following question is natural: What are the submanifolds whose tangent spaces are Lie triples at every point? It is well known (cf. [**H**]) that Lie triples are always tangent spaces of totally geodesic submanifolds, but we want to exclude these by requiring that the submanifold is *substantial*, i.e. not contained in any proper totally geodesic submanifold. The simplest nontrivial example is $S = \boldsymbol{C}P^n$ where the curvature tensor R^S is determined by the inner product and

2000 *Mathematics Subject Classification*. Primary 53C35; Secondary 53C24, 53C29, 53B25.
Key Words and Phrases. Symmetric spaces, restricted holonomy, submanifold equations, Gauss map, rank rigidity, extrinsic symmetry, isotropy orbits.

the complex structure only. The distinguished subspaces, the Lie triples, are just the complex and the totally real subspaces. The family of submanifolds all of whose tangent spaces are complex is very large: complex submanifolds are locally given in terms of almost arbitrary convergent power series, and likewise there are many totally real submanifolds. Another very large class which has been treated for more general symmetric spaces (cf. [**BHPP**]) contains the submanifolds where R^S is actually zero on all tangent spaces; they are called *curved flats*. Analogously, submanifolds all of whose tangent spaces are Lie triples will be called *curved Lie triples*. It turns out that all tangent spaces of a curved Lie triple are conjugate under the isometry group of S, so the type of Lie triple is fixed along the submanifold.

Lie triples always contain subspaces where R^S is zero, called *flats*. The maximal dimension of a flat is the *rank* of a Lie triple. In the present paper we will consider Lie triples which are irreducible (not a direct sum of proper subtriples) and of rank ≥ 2. We will show that the corresponding curved Lie triples are very different from the examples above: they are quite rigid. In fact, the intrinsic local geometry is completely determined (Theorem 1). In fact we can show that R^S is parallel along the submanifold which causes a restriction of the holonomy; then we can apply the holonomy theorem of Berger and Simons. For a special class of Lie triples (the tangent spaces of symmetric R-spaces or extrinsic symmetric spaces) we can even determine all corresponding curved Lie triples up to congruence: They form a one-parameter family of very singular orbits of the isotropy group of S (cf. Theorem 3 and its Corollary, Chapter 6). We conjecture that these are the only substantial higher rank curved Lie triples.

This conjecture is supported by the work of H. Naitoh (cf. [**N**] and its references) who has studied curved Lie triples where the normal spaces are also Lie triples (we call those *normal curved Lie triples*). By a case-by-case study he proved in particular that normal curved Lie triples of higher rank can be substantial only if they belong to the special class mentioned above ([**N**], p. 562). Using this result, we have determined all irreducible higher rank normal curved Lie triples. Our article was partially motivated also by the work of Carlson and Toledo who solved another special case (Hermitian normal curved Lie triples) by different methods as one step to prove rigidity of certain harmonic maps (cf. [**CT**], pp. 108, 131).

It is my pleasure to thank several people for contributing to this article by valuable hints and discussion: Martina Brück, Fran Burstall, Ernst Heintze, Bernardo Molina, Carlos Olmos and the Referee.

1. Curved Lie triples.

Let $S = G/K$ be a symmetric space of compact type with curvature tensor R^S. An immersion $f : M \to S$ will be called a *curved Lie triple* if any tangent space $\tau_p = df_p(T_pM)$ is invariant under R^S, i.e. $R^S(\tau_p, \tau_p)\tau_p \subset \tau_p$. In other words, each $\tau_p \subset T_{f(p)}S$ is a Lie triple and thus (cf. [**H**]) for any $p \in M$ there is a totally geodesic submanifold $S' = G'/K' \subset S$ tangent to f at p. We will call such an immersion f a *curved Lie triple*. Of course, totally geodesic immersions are also curved Lie triples, but we will consider only those which are *substantial*, i.e. not lying in a proper totally geodesic subspace of S. Are there such immersions, and what are they like?

If S is a sphere, this is no condition at all; any submanifold is a curved Lie triple. For $S = \boldsymbol{CP}^n$, there are two types of Lie triples, complex and totally real subspaces, and the corresponding curved Lie triples are complex or totally real (immersed) submanifolds, and there are similar examples in $\boldsymbol{HP}^n$. Examples of curved Lie triples in a more general symmetric space S are obtained by restricting the exponential map of S to an extrinsic symmetric submanifold in a tangent space of S, cf. Chapter 2.

In the present paper we will consider mainly curved Lie triples of *higher rank*. More precisely we will assume that every τ_p is a Lie triple without flat factor such that all irreducible components have rank ≥ 2. (Recall that the *rank* of a Lie triple is the dimension of its maximal flat subspaces.) The extrinsic symmetric examples mentioned above are the only substantial higher rank curved Lie triples we know and we conjecture that these are the only possible examples. The evidence for this conjecture is an intrinsic rigidity given by Theorem 1 below which relies on the following Lemma. In order to simplify notation we drop the immersion f and consider M as a submanifold of S. This is no restriction of generality since everything is local.

Lemma 1. *Let $M \subset S$ be a curved Lie triple, i.e. TM is stable under the curvature tensor R^S. Then $R^S|_{TM}$ is parallel with respect to the Levi Civita connection of the induced metric on M.*

Proof. Let D and ∇ denote the Levi-Civita connections of S and M, respectively. Consider a curve $p(t)$ on M and let a, b, c, d be ∇-parallel tangent vector fields of M along that curve. It is sufficient to show $R^S(a, b, c, d) := \langle R^S(a, b)c, d\rangle = const.$ We compute its derivative with respect to the parameter t denoting by a', b', c', d' the D-derivative of a, b, c, d. Since $DR^S = 0$, we have $R^S(a, b, c, d)' = R^S(a', b, c, d) + R^S(a, b', c, d) + R^S(a, b, c', d) + R^S(a, b, c, d')$, and all these terms vanish since a', b', c', d' are normal vector fields, but $R^S(TM, TM)TM \subset TM$. $\square$

Consequently, if $M \subset S$ is a curved Lie triple, then any parallel displacement on M preserves $R^S|_{TM}$. Hence all the Lie triples $(T_pM, R^S|_{T_pM})$ for various

$p \in M$ are isomorphic, and we may assign to M a Lie triple $\mathfrak{p}' \subset \mathfrak{p}$, fixing a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ corresponding to S.

THEOREM 1. *Let $M \subset S$ be a higher rank curved Lie triple, i.e. the corresponding Lie triple $\mathfrak{p}'$ is semisimple and all irreducible factors have rank ≥ 2. Then M with its induced metric is locally symmetric. If $\mathfrak{p}'$ itself is irreducible, the curvature tensors R^M and $R^S|_{TM}$ are proportional.*

PROOF. Fix any $p \in M$. We may assume that p is the base point of $S = G/K$. Let $\mathfrak{p} = T_pS$ and $\mathfrak{p}' = T_pM$ and let $S' = G'/K' \subset S$ be the corresponding totally geodesic submanifold with tangent space $\mathfrak{p}'$. By Lemma 1, the local holonomy group H of M at p leaves $R^S|_{\mathfrak{p}'}$ invariant. Thus $H \subset K'$, and since S' has higher rank, none of the irreducible subrepresentations of H can act transitively on the unit sphere in the corresponding subspace. Thus by the holonomy theorem of Berger and Simons (cf. [**S**]), M is locally symmetric. In fact, $(\mathfrak{p}', R^M, H)$ is a non-transitive holonomy system, hence symmetric. But the same holds for the holonomy system $(\mathfrak{p}', R^M, K')$, thus this is also symmetric which implies $H = K'$. If $R^S|_{\mathfrak{p}'}$ is indecomposable this shows that $R^S|_{\mathfrak{p}'}$ is proportional to R^M. □

2. Normal curved Lie triples.

Now we restrict our attention to a special type of Lie triples. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ corresponding to a compact symmetric space $S = G/K$. A Lie triple $\mathfrak{p}' \subset \mathfrak{p}$ will be called *normal* if $\mathfrak{p}'' = (\mathfrak{p}')^\perp \subset \mathfrak{p}$ is also a Lie triple. It is easy to see (cf. [**E**]) that in this case the reflections at the subspaces $\mathfrak{p}'$ and $\mathfrak{p}''$ are order two automorphisms of R^S. Conjugation with these automorphisms give involutions of $\mathfrak{k}$ inducing an orthogonal eigenspace decomposition $\mathfrak{k} = \mathfrak{k}' + \mathfrak{q}$ where the elements of $\mathfrak{k}'$ preserve $\mathfrak{p}'$ and $\mathfrak{p}''$ while the elements of $\mathfrak{q}$ map $\mathfrak{p}'$ into $\mathfrak{p}''$ and vice versa. Hence we obtain three symmetric pairs with the same isotropy Lie algebra $\mathfrak{k}'$, corresponding to the Cartan decomposed Lie subalgebras $\mathfrak{g}' := \mathfrak{k}' + \mathfrak{p}'$ and $\mathfrak{g}_+ := \mathfrak{k}' + \mathfrak{p}''$ and $\mathfrak{k} = \mathfrak{k}' + \mathfrak{q}$.

A special case of normal Lie triples are tangent spaces of *extrinsic symmetric spaces*. Recall that a submanifold $M \subset \boldsymbol{R}^n$ is called *extrinsic symmetric* if M is invariant under reflection at any of its affine normal spaces. By a theorem of D. Ferus (cf. [**F**], [**EH1**]), any such M is a particular orbit under the isotropy representation of another symmetric space. More precisely, $M \subset \boldsymbol{R}^n$ is extrinsic symmetric if and only if there exists a symmetric space $S = G/K$ with base point o and an isometric identification of its tangent space $\mathfrak{p} = T_oS$ with $\boldsymbol{R}^n$ such that $M = Ad(K)q$ where $q \in \mathfrak{p}$ satisfies $ad(q)^3 = -\beta^2 \cdot ad(q)$ for some $\beta > 0$.

To see the normal Lie triple, extend the inner product on $\mathfrak{p}$ to an $Ad(G)$-invariant inner product on $\mathfrak{g}$. Let $\mathfrak{p}' = [\mathfrak{k}, q]$ be the tangent space and $\mathfrak{p}'' = (\mathfrak{p}')^\perp = [\mathfrak{k}, q]^\perp$ the normal space of $M \subset \mathfrak{p}$. Then $ad(q)\mathfrak{p}'' = 0$ since $\langle \mathfrak{k}, [q, \mathfrak{p}''] \rangle =$

$\langle [\mathfrak{k}, q], \mathfrak{p}'' \rangle = 0$, and on $\mathfrak{p}' = [q, \mathfrak{k}]$ we have $ad(q)^2 = -\beta^2 \cdot I$ since $ad(q)^2(ad(q)A) = ad(q)^3 A = -\beta^2 \cdot ad(q)A$. Further, $ad(q)$ vanishes on $\mathfrak{k}' = \{A \in \mathfrak{k}; [A, q] = 0\}$ which is the isotropy Lie algebra of M, and the orthogonal complement $\mathfrak{q} = (\mathfrak{k}')^\perp \subset \mathfrak{k}$ is mapped isomorphically onto the tangent space $\mathfrak{p}'$ by $ad(q)$ which is the differential of the diffeomorphism $K/K' \to Ad(K)q$. Thus the kernel of $ad(q)$ is precisely $\mathfrak{g}_+ := \mathfrak{p}'' + \mathfrak{k}'$, so we have $ad(q)^2 = -\beta^2 \cdot I$ on the orthogonal complement $\mathfrak{g}_- := \mathfrak{p}' + \mathfrak{q}$, and $ad(q)/\beta$ is a complex structure on $\mathfrak{g}_-$ interchanging $\mathfrak{p}'$ and $\mathfrak{q}$. Hence for any $s \in \boldsymbol{R}$ the $\mathfrak{g}$-automorphism $\rho_s = e^{s \cdot ad(q)/\beta}$ fixes $\mathfrak{g}_+$ and rotates $\mathfrak{g}_-$ by the angle s. In particular, ρ_π is the reflection at $\mathfrak{g}_+$ inducing on $\mathfrak{p}$ the reflection at the normal space $\mathfrak{p}''$, and of course $-\rho_\pi$ is the reflection at the tangent space $\mathfrak{p}'$. Thus $\mathfrak{p}'$ is a normal Lie triple. It has the additional property that there is an isometric $Ad(K')$-equivariant Lie triple isomorphism between $\mathfrak{p}'$ and $\mathfrak{q}$, namely $ad(q)/\beta$.

Now a *curved* Lie triple $M \subset S$ will be called *normal* if the corresponding Lie triple $\mathfrak{p}'$ is normal, i.e. all normal spaces of M are also Lie triples. Examples of substantial normal curved Lie triples arise by exponentiating an extrinsic symmetric space $M \subset \mathfrak{p} = T_o S$ into S. In fact, consider the K-equivariant map $f = \exp_o|_M : M \to S$ which is an immersion if $|q|$ is not too large. The Jacobi field equation shows that $\tau_q = df_q(T_q M) \subset T_{f(q)} S$ is the parallel translate of $T_q M \subset T_o S$ along the geodesic $\gamma(s) = \exp_o(sq)$ in S (for details see Lemma 6 at the end of this paper). But the parallel transport in a symmetric space is done by isometries of S. Therefore τ_q and $\tau_q^\perp$ are Lie triples, being conjugate to $\mathfrak{p}'$ and $\mathfrak{p}''$ respectively, and the same is true at any point $p = Ad(k)q \in M$. Hence f is a normal curved Lie triple.

In a series of papers (cf. [**N**] and its references), H. Naitoh classified the possible Lie triples which may occur as tangent spaces of substantial normal curved Lie triples; in particular, if they have higher rank they are tangent spaces of extrinsic symmetric spaces (Theorem 2.2, p. 562 in [**N**]). But it remains to determine the corresponding submanifolds.

The following improvement of Lemma 1 is basic. Consider the vector bundle $TS|_M = TM + NM$. It has two canonical connections: the Levi-Civita connection D of S and the connection ∇ which is the direct sum of the tangent and normal connections: For any tangent vector field V we put $\nabla_V X = (D_V X)^T$ and $\nabla_V \xi = (D_V \xi)^N$ where X and ξ denote tangent and normal vector fields, respectively. Then $L := D - \nabla$ is just the second fundamental form α or the Weingarten map A; in fact $L_V X = \alpha(V, X) = \alpha_{VX}$ and $L_V \xi = -A_\xi V$.

Lemma 2. *Let $M \subset S$ be a normal curved Lie triple. Then R^S is parallel along M with respect to both connections D and ∇. Consequently, if we restrict attention to some $p \in M \subset S$ considered as base point of S, then $L = D - \nabla$ takes values in $\mathfrak{k}$; more precisely, it is a linear map $L : \mathfrak{p}' \to \mathfrak{q}$.*

PROOF. We have $DR^S = 0$ since S is symmetric. In order to show $\nabla R^S = 0$, we take ∇-parallel vector fields a, b, c, d along a curve $p(t)$ in M as in the proof of Lemma 1, but now these may take values in the full bundle $TS|_M$. We may assume that either of them is a tangent or a normal vector field. Again we have to differentiate the expression $r := \langle R^S(a,b)c, d\rangle$ using the D-derivative. Recall that the D-derivative of a ∇-parallel tangent vector is normal and vice versa. If all four vector fields are tangent (or all normal), we have shown $r' = 0$ in Lemma 1; recall that also the normal spaces are Lie triples. If three of them are tangent and one is normal (or vice versa), then $r = 0$. If two are tangent and the other two normal, all terms of r' consists of either three tangent and one normal vector or vice versa, hence we have again $r' = 0$ which concludes the proof of $\nabla R^S = 0$.

Consequently, we have $L_v R^S = D_v R^S - \nabla_v R^S = 0$ for all $v \in \mathfrak{p}' = T_p M$, thus the linear map $L_v : \mathfrak{p} \to \mathfrak{p}$ is a skew adjoint derivation of R^S which means $L_v \in \mathfrak{k}$. But since L_v maps the tangent space $\mathfrak{p}'$ into the normal space $\mathfrak{p}''$ and vice versa, $L_v \in \mathfrak{q}$ and L is a linear map between $\mathfrak{p}'$ and $\mathfrak{q}$. $\square$

REMARK. Since the ∇-parallel translations preserve R^S, they are given by elements of G. Thus two tangent spaces of M are not only isomorphic Lie triples, but they are even conjugate within G.

3. The submanifold equations.

Let S be an irreducible symmetric space and $M \subset S$ a normal curved Lie triple. By the Gauss equations and Lemma 2, we have for all $v, w, x, y \in \mathfrak{p}' = T_p M$:

$$\begin{aligned}\langle R^M(v,w)x, y\rangle - \langle R^S(v,w)x, y\rangle &= -\langle \alpha_{vx}\alpha_{wy}\rangle + \langle \alpha_{vy}\alpha_{wx}\rangle \\ &= -\langle L_v x, L_w y\rangle + \langle L_v y, L_w x\rangle \\ &= \langle L_w L_v x, y\rangle - \langle y, L_v L_w x\rangle \\ &= -\langle [L_v, L_w]x, y\rangle.\end{aligned}$$

We have $R^S(v,w) = -[v,w] \in \mathfrak{k}$. Further, if the Lie triple $\mathfrak{p}'$ is irreducible and of higher rank, $R^M = \lambda \cdot R^S|_{\mathfrak{p}'}$ by Theorem 1, hence

$$[L_v, L_w]x = -R^M(v,w)x + R^S(v,w)x = (\lambda - 1)\cdot [v,w]x$$

for all $v, w, x \in \mathfrak{p}'$. If moreover $\mathfrak{k}'$ acts effectively on $\mathfrak{p}'$, then we have for all $v, w \in \mathfrak{p}'$

$$(G) \qquad [L_v, L_w] = -(1 - 1/\lambda)R^M(v,w) = (\lambda - 1)\cdot [v,w].$$

Further, from the Ricci equations we obtain for all $v, w \in \mathfrak{p}'$ and $\xi, \eta \in \mathfrak{p}'' = \mathfrak{p}'^{\perp}$

$$\begin{aligned}\langle R^{\perp}(v,w)\xi,\eta\rangle - \langle R^{S}(v,w)\xi,\eta\rangle &= \langle A_{\eta}v, A_{\xi}w\rangle - \langle A_{\eta}w, A_{\xi}v\rangle \\ &= \langle L_{v}\eta, L_{w}\xi\rangle - \langle L_{v}\xi, L_{w}\eta\rangle \\ &= -\langle \eta, L_{v}L_{w}\xi\rangle + \langle L_{w}L_{v}\xi,\eta\rangle \\ &= -\langle [L_{v},L_{w}]\xi,\eta\rangle\end{aligned}$$

where $R^{\perp}$ denotes the curvature tensor of the normal bundle. Using (G) we obtain

$$(R) \qquad R^{\perp}(v,w) = R^{S}(v,w) - [L_{v},L_{w}] = -\lambda\cdot[v,w] = R^{M}(v,w)$$

where every term is considered as an element of $\mathfrak{k}'$ acting on $\mathfrak{p}''$. Finally, the Codazzi equation is the same as in Euclidean space,

$$(C) \qquad (\nabla_{u}\alpha)(v,w) = (\nabla_{v}\alpha)(u,w)$$

for all $u,v,w\in\mathfrak{p}'$; note that the curvature term $(R^{S}(u,v)w)^{\perp}$ vanishes since $\mathfrak{p}'$ is a Lie triple.

This has a surprising consequence if $\lambda>1$ in (G) (the other cases will be treated in the next sections). Let $f: M\subset S$ be a curved Lie triple such that the corresponding Lie triple $\mathfrak{p}'$ is normal, irreducible, and $\mathfrak{k}'$ acts effectively on $\mathfrak{p}'$. Let α be the second fundamental form of M and L the corresponding linear map as above. Then we have $(G),(C),(R)$. From these data we construct now a new immersion into Euclidean space:

Lemma 3. *There exists another isometric immersion $\tilde f: M\to\mathfrak{p}=\boldsymbol{R}^{n}$ such that the normal bundles of f and $\tilde f$ can be identified by a parallel isometric vector bundle isomorphism, and the second fundamental form of $\tilde f$ is $\tilde\alpha=\kappa\alpha$ with $\kappa^{2}=\lambda/(\lambda-1)$. All tangent spaces of $\tilde f$ are conjugate to $\mathfrak{p}'$ under K.*

Proof. If we put $\tilde L=\kappa\cdot L$, the Gauss equations (G) yield $R^{M}=-\lambda/(\lambda-1)[L,L]=-[\tilde L,\tilde L]$. Further from the Ricci equation (R) we also get $R^{\perp}=R^{M}=-[\tilde L,\tilde L]$. Using also (C) we see that $(M,NM,\tilde\alpha)$ satisfies Euclidean Gauss-Codazzi-Ricci equations. By the existence theorem for submanifolds, there exists an isometric immersion $\tilde f: M\to\boldsymbol{R}^{n}$ with normal bundle NM (up to a parallel isometric isomorphism) and second fundamental form $\tilde\alpha$. It remains to identify $\boldsymbol{R}^{n}$ with $\mathfrak{p}$ such that all tangent spaces are congruent to $\mathfrak{p}'$ under K. Since $T_{p}M+N_{p}M=T_{p}S$ (via the immersion f) for every $p\in M$, we have a Lie triple product R^{S}_{p} on $T_{p}M\oplus N_{p}M$ which on the other hand is identified with $\boldsymbol{R}^{n}$ using the immersion $\tilde f$. We have to show that this tensor field R^{S} along $\tilde f$ is constant. But the derivative on $\boldsymbol{R}^{n}$ splits as $\partial=\nabla+\tilde L$ since $\tilde L$ is the second fundamental form of $\tilde f$, thus $\partial_{v}R^{S}=\nabla_{v}R^{S}+\tilde L_{v}.R^{S}=0$ for any $v\in TM$ (recall that $\tilde L_{v}$ is a multiple of L_{v} and thus a derivation of R^{S}). Consequently R^{S} is constant and defines a Lie triple product structure on $\boldsymbol{R}^n$ isomorphic to $\mathfrak{p}$, and each $d\tilde{f}_p(T_pM)$ is a subtriple isomorphic to $\mathfrak{p}'$. □

Remark. The preceding Lemma allows a new interpretation of L or rather $\tilde{L}$. Let $\hat{Q} = O(\mathfrak{p})/(O(\mathfrak{p}') \times O(\mathfrak{p}''))$ be the Grassmannian of all linear subspaces isomorphic to $\mathfrak{p}'$ in $\mathfrak{p}$. For any $E \in \hat{Q}$ the tangent space $T_E\hat{Q}$ can be viewed as the space $\mathscr{A}(E, E^\perp)$ of skew adjoint linear maps on $\mathfrak{p}$ sending E to $E^\perp$ and vice versa. Now from the immersion $\tilde{f} : M \to \mathfrak{p}$ we obtain a Gauss map $g : M \to \hat{Q}$ with $g(p) = d\tilde{f}_p(T_pM)$ for all $p \in M$, and its differential $dg_p : T_pM \to T_{g(p)}\hat{Q} = \mathscr{A}(g(p), g(p)^\perp)$ is precisely the second fundamental form $\tilde{L}$ at p. But from Lemma 3 we know that g takes values in the subset $Q = \{Ad(k)(\mathfrak{p}'); k \in K\} = K/K' \subset \hat{Q}$ which is a totally geodesic submanifold of $\hat{Q}$; to see this we just observe that Q is preserved under the reflection at the subspace $\mathfrak{q}'' = (\mathfrak{q}')^\perp$, but this reflection is the geodesic symmetry in the symmetric space $\hat{Q}$. Hence $\tilde{L}$ is the differential of the Gauss map $g : M \to Q$. Later we will show that $L : \mathfrak{p}' \to \mathfrak{q}$ is a linear isometry under certain assumptions which implies that this map g is also isometric.

4. A rigidity result for totally geodesic submanifolds.

Now we will discuss the case $\lambda = 1$ in the Gauss equation (G). This is equivalent to $R^M = R^S|_{TM}$ or in other words, M is isometric to the totally geodesic submanifold $S' \subset S$ which is tangent to M. We will show that this is impossible:

Theorem 2. *Let $S = G/K$ be a compact irreducible symmetric space with corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and let $\mathfrak{p}' \subset \mathfrak{p}$ be an irreducible higher rank normal Lie triple corresponding to a totally geodesic submanifold $S' = \exp_p(\mathfrak{p}')$. Then there is no substantial curved Lie triple $M \subset S$ tangent to S' which is intrinsically isometric to S'.*

Proof. Suppose there is such a submanifold $M \subset S$. Then $R^M = R^S|_{TM}$ and by the Gauss equations (G) we get $[L_v, L_w] = 0$ for all $v, w \in \mathfrak{p}' = T_pM$. In other words, the linear map $L : \mathfrak{p}' \to \mathfrak{q}$ takes values in a flat $\mathfrak{a} \subset \mathfrak{q}$. The Lie triple $\mathfrak{q}$ corresponds to the symmetric space $Q = K/K' = \{Ad(k)\mathfrak{p}'; k \in K\}$ which is a totally geodesic subspace of the Grassmannian $\hat{Q}$ of all linear subspaces of $\mathfrak{p}$ isomorphic to $\mathfrak{p}'$ (cf. Remark in the preceding section). The flats of the Grassmannian are well known: A flat $\hat{\mathfrak{a}} \subset T_{\mathfrak{p}'}\hat{Q}$ is determined by two orthonormal sets of vectors $v_1, \ldots, v_m \in \mathfrak{p}'$ and $\xi_1, \ldots, \xi_m \in \mathfrak{p}''$, where m is the minimum of the dimensions of $\mathfrak{p}'$ and $\mathfrak{p}''$, and $\hat{\mathfrak{a}}$ consists of all linear maps $A \in \mathscr{A}(\mathfrak{p}', \mathfrak{p}'')$ such that $A(v_i)$ is a multiple of ξ_i for each $i = 1, \ldots, m$ while $A = 0$ on $\{v_1, \ldots, v_m\}^\perp$. Since our flat $\mathfrak{a} \subset \mathfrak{q}$ can be extended to such a flat $\hat{\mathfrak{a}}$, we have $L_{v_i}v_j = \lambda_{ij}\xi_j$ for some $\lambda_{ij} \in \boldsymbol{R}$, but from the symmetry $L_{v_i}v_j = L_{v_j}v_i$ we see that $L_{v_i}v_j = 0$ for $i \neq j$. It follows that each $L_{v_i} \in \mathfrak{a} \subset \mathfrak{q} \subset \mathfrak{k}$ has rank 2 as a linear map on $\mathfrak{p}$: It just maps v_i to a multiple of ξ_i and vice versa and it is zero on $\{v_i, \xi_i\}^{\perp}$. By the subsequent Lemma 4, this is impossible. Thus all L_{v_i} are zero and M is totally geodesic, hence not substantial, a contradiction! □

Lemma 4. *Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition such that $\mathfrak{p}$ is irreducible and of higher rank. Then there is no $A \in \mathfrak{k}$ which has rank 2 as a linear endomorphism of $\mathfrak{p}$.*

Proof. Suppose that $A \in \mathfrak{k}$ is of rank 2, i.e. its kernel $V \subset \mathfrak{p}$ has codimension 2. Then the corresponding one-parameter subgroup $k(t) = \exp tA$ acts by rotations in the plane $V^{\perp}$. For any $x \in \mathfrak{p} \backslash V$, the tangent space of the K-orbit, $T_x(Kx)$, meets $V^{\perp}$ precisely in the line generated by Ax. Hence the normal space $F_x = T_x(Kx)^{\perp}$ intersects V in a hyperplane $H_x = F_x \cap V$. If x is regular, i.e. not contained in a root hyperplane, F_x is the flat through x and H_x consists of singular points. In fact, for any $y \in H_x$ consider the normal vector $\xi = y - x \in F_x$. Then the corresponding parallel normal field $\xi(t) = k(t)\xi$ along the curve $x(t) = k(t)x \in Kx$ has constant end point map $\pi_{\xi}(x(t)) = x(t) + \xi(t) = k(t)y = y$ (for the notation cf. [**T**] or [**EH2**]). Thus $H_x \subset F_x$ must be one of the root hyperplanes. Likewise $H_{x'} = F_{x'} \cap V$ is a root hyperplane in the flat $F_{x'}$ for any $x' \in Kx \backslash V$. Hence all $H_{x'}$ for $x' \in Kx$ close to x are conjugate under K which in turn implies that any orbit Ky with $y \in H_x$ is contained in the subspace $V \subset \mathfrak{p}$. But K acts irreducibly on $\mathfrak{p}$, a contradiction! □

5. The second fundamental form.

Lemma 5. *Let G be a compact Lie group and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ its Lie algebra with a Cartan decomposition. Let $\mathfrak{p}' \subset \mathfrak{p}$ and $\mathfrak{q} \subset \mathfrak{k}$ be Lie triples of equal dimension with $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{k}' = [\mathfrak{p}', \mathfrak{p}']$. Assume $\mathfrak{p}'$ to be irreducible. Let $L : \mathfrak{p}' \to \mathfrak{q}$ be linear with*

$$[Lv, Lw] = \rho \cdot [v, w] \tag{1}$$

for all $v, w \in \mathfrak{p}'$ and some constant $\rho \neq 0$. Then $\rho > 0$ and $L' = L/\sqrt{\rho}$ is a K'-equivariant isometric Lie triple isomorphsm.

Proof. L must be injective by (1), hence it is a linear isomorphism by the dimension assumption. We may assume $\rho = \varepsilon := \pm 1$ by passing to $L/\sqrt{|\rho|}$. Choosing an $Ad(G)$-invariant inner product on $\mathfrak{g}$ we have $\langle A, [x, y]\rangle = \langle Ax, y\rangle$ for all $A \in \mathfrak{k}$ and $x, y \in \mathfrak{p}$. Thus for any $A \in \mathfrak{k}'$ and all $v, w \in \mathfrak{p}'$,

$$\langle Av, w\rangle = \langle A, [v, w]\rangle = \varepsilon\langle A, [Lv, Lw]\rangle = \varepsilon\langle ALv, Lw\rangle$$

where A is considered as a linear map on $\mathfrak{p}'$ and on $\mathfrak{q}$ (we write Av and ALv in place of $[A,v]$ and $[A,Lv]$). Hence

$$L^T AL = \varepsilon A \tag{2}$$

for all $A \in \mathfrak{k}'$. In particular we have $Av = 0 \Leftrightarrow ALv = 0$ which means that the isotropy Lie algebras of v and Lv agree:

$$\mathfrak{k}'_v = \mathfrak{k}'_{Lv} \tag{3}$$

for any $v \in \mathfrak{p}'$. Consequently, the principal orbits of K' on $\mathfrak{p}'$ and $\mathfrak{q}$ have the same dimension. Thus also the maximal flat subspaces in $\mathfrak{p}'$ and $\mathfrak{q}$ have the same dimension, being sections of the K'-actions. But by (1), L maps a flat F' of $\mathfrak{p}'$ into a flat F of $\mathfrak{q}$, thus inducing a linear isomorphism between the two flats. From (3) we see that L also maps the singular hyperplanes (root hyperplanes) of F' onto the singular hyperplanes of F. This implies that $L|_{F'}$ is a multiple of an isometry (see Sublemma below). But since any two flats in $\mathfrak{p}'$ are connected by a finite chain of flats each of which intersects its successor in a nonzero subspace, this multiple must be the same on each flat, and the whole map L is a multiple of an isometry, i.e. $LL^T = \mu \cdot I$ for some $\mu > 0$.

Now for any $A, B \in \mathfrak{k}$ we apply (2) to A, B and $[A,B]$ in place of A and obtain

$$[A,B] = [L^T AL, L^T BL] = \mu \cdot L^T [A,B] L = \varepsilon\mu \cdot [A,B]$$

which shows $\mu = 1$ and $\varepsilon = 1$. Thus $L^T = L^{-1}$ and L commutes with any $A \in \mathfrak{k}'$ by (2). Together with (1) this shows that L is an isometric isomorphism of Lie triples. $\square$

Sublemma. *Let F', F be two k-dimensional Euclidean vector spaces with root systems $R' \subset F'$ and $R \subset F$ such that R' is indecomposable. Let $L : F' \to F$ be a linear isomorphism such that L and L^{-1} map root hyperplanes onto root hyperplanes. Then L is an isometry up to scaling.*

Proof. We put $\hat{R}' = \boldsymbol{R} \cdot R' = \{\lambda \cdot r; \lambda \in \boldsymbol{R}, r \in R'\}$ and similar $\hat{R} = \boldsymbol{R} \cdot R$. Since L maps the root hyperplanes $r^\perp \subset F'$ (for all $r \in R'$) bijectively onto the root hyperplanes in F, its inverse transposed map $L' := (L^T)^{-1} : F' \to F$ maps $\hat{R}'$ onto $\hat{R}$. Consequently, for any 2-dimensional linear subspace ("plane") $E' \subset F'$ meeting R' we have $L'(\hat{R}' \cap E') = \hat{R} \cap E$ with $E = L'(E')$. The unit vectors in $\hat{R}' \cap E'$ are the vertices of a regular $2n$-gon with $n \in \{1,2,3,4,6\}$, and the same is true for the unit vectors of $\hat{R} \cap E$ (with the same value of n). We then will call E', E planes *of type n*. For $n \geq 3$ it follows easily that $L'|_{E'}$ is a multiple of an isometry. In fact, if $n = 3$ or $n = 6$, the set of unit vectors of $\hat{R}' \cap E$ always contains a triple $a, b, a+b$, and after a suitable identification of E' and E we may assume that $L'(a) = t_1 a$, $L'(b) = t_2 b$ and $L'(a+b) = t_3(a+b)$ for some $t_1, t_2, t_3 \in \boldsymbol{R}$. But by linearity we have also $L'(a+b) = t_1 a + t_2 b$ which shows $t_1 = t_2 = t_3$. If $n = 4$, we have a triple $a, b, (a+b)/\sqrt{2}$ instead and a similar argument holds. Since the root system R' does not split into two perpendicular subsets, any two roots can be joined by a (finite) chain of roots r_i with r_i, r_{i+1} linearly independent and $\langle r_i, r_{i+1} \rangle \neq 0$. Then the planes $E_i = \mathrm{Span}\{r_i, r_{i+1}\}$ have type ≥ 3 and $E_i \cap E_{i+1} \neq 0$. Now L' is a multiple of an isometry on each E_i, but due to the nontrivial intersection, this multiple must be the same for all i. Hence $L' = (L^T)^{-1}$ is an isometry up to scaling. Thus the same is true for L. □

6. Irreducible curved Lie triples of extrinsic symmetric type.

THEOREM 3. *Let S be a compact irreducible symmetric space and $M \subset S$ be a substantial curved Lie triple such that the corresponding Lie triple $\mathfrak{p}'$ is normal, irreducible and of higher rank. Suppose further that $\mathfrak{k}'$ acts effectively on $\mathfrak{p}'$ and that $\mathfrak{p}'$ and $\mathfrak{q}$ have equal dimension. Then $M = \exp_o(\tilde{M})$ where $\tilde{M} \subset \mathfrak{p} = T_o S$ is an extrinsic symmetric space.*

PROOF. By the Gauss equations (G), the second fundamental form L of $M \subset S$ satisfies $[L_v, L_w] = (\lambda - 1)[v, w]$, and from Theorem 2 we have $\lambda \neq 1$. Now Lemma 5 shows that $\lambda > 1$ and L is a constant multiple of a linear isometry. The same holds for $\tilde{L} = \kappa L$ which is the differential of the Gauss map $g : M \to Q \subset \hat{Q}$ associated to the immersion $\tilde{f} : M \to \mathfrak{p}$ (cf. Lemma 3 and the subsequent remark). Thus g is a local isometry up to scaling. In particular, $\tilde{L} = dg$ is parallel, sending parallel vector field on M onto parallel vector fields on Q. So the second fundamental form $\tilde{\alpha} : TM \otimes TM \to NM$ is parallel and hence $\tilde{f} : M \to \mathfrak{p}$ is extrinsic symmetric (cf. [**F**], [**EH1**]). Thus we may assume that $\tilde{f}(M) =: \tilde{M} = Ad(K)q$ for some $q \in \mathfrak{p}$ centralized by K' and with $ad(q)^3 = -\beta^2 \cdot ad(q)$ for some $\beta > 0$.

We want to show that $M \subset S$ is congruent to one of the immersions $f_s : \tilde{M} \to S$ given by $f_s(x) = \exp_o(sx)$ with $s > 0$. By the subsequent Lemma 6 and the following remark, the metrics of all $M_s = f_s(\tilde{M})$ are proportional to that of $\tilde{M}$ with the factor $\mu_s = \sin(s\beta)/\beta$, and $f_s(\tilde{M})$ is totally geodesic where μ_s takes its maximum, i.e. for $s_o = \pi/2\beta$. For $s = s_o$ we have $R^{M_s} = R^S|_{TM}$. Making s smaller we shrink the metric enlarging the curvature. Since $\lambda > 1$, we find some $s < \pi/2\beta$ with $R^{M_s} = \lambda \cdot R^S|_{TM}$; in fact we have to choose s such that $\sin(s\beta) = \mu_s/\mu_{s_o} = 1/\sqrt{\lambda}$. On the other hand, in Lemma 6 we will compute also the ratio between the second fundamental forms of $\tilde{M}$ and M which is $\alpha_s/\tilde{\alpha} = \cos(s\beta) = \sqrt{1 - \sin^2(s\beta)} = \sqrt{1 - 1/\lambda} = 1/\kappa$ with $\kappa^2 = \lambda/(\lambda - 1)$ as in Lemma 3. Thus $M \subset S$ and $M_s \subset S$ have the same metric and second fundamental form (up to parallel isometries of the normal bundles), thus they are the same up to an isometry of S, by the congruence theorem for submanifolds (cf. [**ET**]). □

If $\mathfrak{p}' \subset \mathfrak{p}$ is of *extrinsic symmetric type*, i.e. the tangent space of an extrinsic symmetric space in $\mathfrak{p}$, then $\mathfrak{p}'$ is normal and $\mathfrak{k}'$ acts effectively on $\mathfrak{p}'$. Thus we obtain from our theorem:

COROLLARY. *Let S be a compact irreducible symmetric space and $M \subset S$ a substantial curved Lie triple such that the corresponding Lie triple $\mathfrak{p}'$ is irreducible of rank ≥ 2 and of extrinsic symmetric type. Then $M = \exp_o(\tilde{M})$ where $\tilde{M} \subset \mathfrak{p} = T_oS$ is the (up to congruence and scaling unique) extrinsic symmetric space corresponding to $\mathfrak{p}'$.*

LEMMA 6. *Let $S = G/K$ be a compact symmetric space and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding Cartan decomposition. Suppose that $\mathfrak{p} = T_oS$ contains an extrinsic symmetric space $\tilde{M} = Ad(K)q$ for some $q \in \mathfrak{p}$ with $ad(q)^3 = -\beta^2 \cdot ad(q)$. For $0 < s < \pi/\beta$ let $f_s : \tilde{M} \to S$, $f_s(x) = \exp_o(sx)$ and put $M_s = f_s(\tilde{M})$. Then f_s is a K-equivariant homothetic immersion with metric scaling factor $\sin(s\beta)/\beta$. Further, for any $x \in \tilde{M}$ the normal space $N_x\tilde{M} \subset T_oS$ and $N_{f_s(x)}M_s \subset T_{f_s(x)}S$ are mapped onto each other by parallel translation along the geodesic $\gamma_x(s) = \exp_o(sx)$ which defines a parallel isometric vector bundle isomorphism between $N\tilde{M}$ and NM_s. Using this identification we have $\alpha_s = \cos(s\beta)\tilde{\alpha}$ for the second fundamental forms $\tilde{\alpha}$ of $\tilde{M}$ and α_s of M_s.*

PROOF. We have seen that $ad(q)\beta$ acts trivially on $\mathfrak{p}'' + \mathfrak{k}'$, and on $\mathfrak{p}' + \mathfrak{q}$ it is a complex structure interchanging $\mathfrak{p}'$ and $\mathfrak{q}$. For any $v \in T_q\tilde{M} = \mathfrak{p}'$ we have $(df_s)_q v = J_v(s)$ where J_v is the Jacobi field along the geodesic $\gamma_q(s) = \exp_o(sq)$ with $J_v(0) = 0$ and $J_v'(0) = v$. Up to parallel translation along γ_q, the Jacobi field J_v satisfies the ODE $J_v'' + ad(q)^2 J_v = 0$. Since $ad(q)^2 = -\beta^2 \cdot I$ on $\mathfrak{p}'$, we obtain $J_v(s) = (\sin(\beta s)/\beta) \cdot v$. Hence f_s is an isometry up to scaling by the factor $\sin(s\beta)/\beta$, and the tangent spaces $T_q\tilde{M}$ and $T_{f_s(q)}M_s$ are parallel along γ_q.

Next consider a normal vector $\tilde{\xi} \in N_q\tilde{M}$ and let $\xi \in N_{f_s(q)}M_s$ its image under parallel transport along γ_q. This parallel transport is done by the transvection $g = \exp(sq)$, hence $\xi = g\tilde{\xi}$. For any $A \in \mathfrak{q}$ let $\tilde{\xi}(t) = \exp(tA)\tilde{\xi}$. This is a normal vector field along the curve $q(t) = \exp(tA)q$ in $\tilde{M}$ which is ∇-parallel since $(d/dt)\tilde{\xi}(t)|_{t=0} = A\tilde{\xi} \in \mathfrak{p}'$ has zero normal component. Now consider the normal vector field $\xi(t) = \exp(tA)\xi$ along the curve $f_s(q(t))$ in M_s. In fact $\xi(t)$ can also be obtained by parallel translating $\tilde{\xi}(t)$ along the geodesic $\gamma_{q(t)}$. We want to compute its derivative with respect to the Levi-Civita connection D in S. This is associated to the horizontal distribution on the principal bundle $G \to S$ obtained from the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Using left translation by g^{-1} back to the base point o we have

$$g^{-1}\frac{D}{dt}\xi(t)|_{t=0} = \frac{D}{dt}(g^{-1}\exp(tA)g\tilde{\xi})_{t=0} = (Ad(g^{-1})A)_{\mathfrak{k}}\tilde{\xi}$$

where the subscript $(\ )_{\mathfrak{k}}$ denotes the $\mathfrak{k}$-component in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. But on the other hand,

$$Ad(g^{-1})A = Ad(\exp(-sq))A = e^{-s\beta\cdot ad(q)/\beta}A = \cos(s\beta)A - \frac{\sin(s\beta)}{\beta}[q, A],$$

hence $(Ad(g^{-1})A)_{\mathfrak{k}} = \cos(s\beta)A$. It follows that $g^{-1}(D/dt)\xi(t)|_{t=0} = \cos(s\beta)A\tilde{\xi}$. This is in $\mathfrak{p}'$ again, hence $(D/dt)\xi(t)$ has zero normal component, and its tangent component (the Weingarten map) is proportional to that of $\tilde{\xi}(t)$ with the constant factor $\cos(s\beta)$. This shows that parallel normal vector fields on $\tilde{M}$ become parallel normal vector fields along M_s, using the identification of $N\tilde{M}$ and NM_s by parallel transport along radial geodesics, and for the second fundamental forms we have $\alpha_s = \cos(s\beta)\alpha$. □

Remark. In particular it follows that $\alpha_s = 0$ for $s\beta = \pi/2$, hence $M_s \subset S$ is totally geodesic for $s = \pi/2\beta$. In fact, f_{2s} maps $\tilde{M}$ to an "antipodal" point $\bar{o}$ of o which is also fixed by the group K, and M_s is an "equator".

7. Concluding remarks.

H. Naitoh ([**N**], p. 562) has given a classification of the possible tangent spaces of normal curved Lie triples. The ones with higher rank which admit substantial curved Lie triples are of extrinsic symmetric type. Using this classification, we get from Theorem 3 and its Corollary:

Theorem 4. *Let S be a compact irreducible symmetric space. Then the substantial irreducible higher rank curved Lie triples $M \subset S$ are precisely of the form $M = \exp_o(\tilde{M})$ where $\tilde{M} \subset T_oS$ is an extrinsic symmetric space.*

Unfortunately, this theorem is not quite the end of the story since in some irreducible symmetric spaces there are extrinsic symmetic spaces which are locally reducible. E.g. for $\mathfrak{p} = \boldsymbol{R}^n \otimes \boldsymbol{R}^n$, corresponding to the Grassmannian of n-planes in $\boldsymbol{R}^{2n}$, the submanifold $\tilde{M} = \{v \otimes w \in \mathfrak{p}; |v| = |w| = 1\} \cong (S^{n-1} \times S^{n-1})/(\pm I)$ is such an example.

The long classification of Naitoh was not used up to Theorem 4. In fact, in the case where $\mathfrak{p}'$ is irreducible of higher rank with $\dim \mathfrak{p}' = \dim \mathfrak{q}$, our Theorem 3 implies Naitoh's result. It would be most desirable to show directly that there are no substantial higher rank normal curved Lie triples if $\dim \mathfrak{p}' < \dim \mathfrak{q}$ (the other case $\dim \mathfrak{p}' > \dim \mathfrak{q}$ is excluded by the injectivity of $L : \mathfrak{p}' \to \mathfrak{q}$ following from (1)). If $\mathfrak{p}$ is of higher rank but not normal, we conjecture that there are no substantial curved Lie triples of this type at all.

If S is a symmetric space of *noncompact* type, then many of the arguments are still valid. However, in the noncompact case Theorem 3 cannot hold as it stands since there are noncompact curved Lie triples analogous to horospheres in real hyperbolic space. They are obtained as the limit for $r \to \infty$ of submanifolds $M_r = \exp_{\gamma(r)}(\tilde{M}_r)$ where γ is a unit speed geodesic in S and $\tilde{M}_r \subset T_{\gamma(r)}S$ is an extrinsic symmetric space of radius r such that all M_r have the same tangent space at the point $p = \gamma(0)$. Moreover, there are other examples which are analogous to the remaining umbilic hypersurfaces in hyperbolic space. These examples have been investigated recently by D. Osipova ([**O**]).

REMARK ADDED IN PROOF. Recently it has been shown by a different method that the assumption "irreducible and of higher rank" in Theorem 4 and in the Corollary of Theorem 3 can be dropped provided that S is not constantly curved, cf. [**BENT**].

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