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Lisa Beck, Bianca Stroffolini

Angaben zur Veröffentlichung / Publication details:

Beck, Lisa, and Bianca Stroffolini. 2013. "Regularity results for differential forms solving degenerate elliptic systems." *Calculus of Variations and Partial Differential Equations* 46 (3-4): 769–808. https://doi.org/10.1007/s00526-012-0503-6.

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Lisa Beck · Bianca Stroffolini

Abstract We present a partial Hölder regularity result for differential forms solving degenerate systems

$$d^* A(\,\cdot\,, \omega) = 0 \qquad \text{and} \qquad d\omega = 0$$

on bounded domains in the weak sense. Here certain continuity, monotonicity, growth and structure condition are imposed on the coefficients, including an asymptotic Uhlenbeck behavior close to the origin. Pursuing an approach of Duzaar and Mingione (J Math Anal Appl 352(1):301–335, 2009), we combine non-degenerate and degenerate harmonic-type approximation lemmas for the proof of the partial regularity result, giving several extensions and simplifications. In particular, we benefit from a direct proof of the approximation lemma (Diening et al. 2010) that simplifies and unifies the proof in the power growth case. Moreover, we give the dimension reduction for the set of singular points.

Mathematics Subject Classification (2000) 35J45 · 35J70

1 Introduction

In this paper we are interested in the regularity of weak solutions to possibly degenerate elliptic problems. In the easiest case considered below we study weak solutions $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ with Ω a bounded domain in $\mathbb{R}^n$, $n \geq 2$, $N \geq 1$, to nonlinear systems of the form

L. Beck (✉)
Scuola Normale Superiore di Pisa, Piazza dei Cavalieri 7, 56126 Pisa, Italy
e-mail: lisa.beck@sns.it

B. Stroffolini
Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II,
Via Cintia, 80126 Napoli, Italy
e-mail: bstroffo@unina.it

$$\operatorname{div} A(\,\cdot\,, Du) = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

where the coefficients are Hölder continuous with respect to the first variable, with some exponent $\beta \in (0, 1)$, and of class C^1 (possibly apart from the origin) with respect to the second variable with a standard p-growth condition. The main focus is set on the ellipticity condition: we allow a monotonicity or ellipticity condition which shows a degenerate (when $p > 2$) or singular (when $p < 2$) behavior in the origin and which is usually expressed by the assumption

$$\langle\, A(x, z) - A(x, \bar{z}), z - \bar{z}\,\rangle \geq \nu\,(\mu^2 + |z|^2 + |\bar{z}|^2)^{\frac{p-2}{2}}\,|z - \bar{z}|^2$$

for all $x \in \Omega$ and all $z, \bar{z} \in \mathbb{R}^N$, for some $\mu \geq 0$. The nondegenerate situation refers to the case where $\mu > 0$ (and by changing the value ν these cases can be reduced to the model case $\mu = 1$), whereas we here treat the degenerate case $\mu = 0$, meaning that we are dealing with a lack of ellipticity in the sense that no uniform bound on the ellipticity constant is available for $p \neq 2$. We highlight that the quadratic case does not impose any additional difficulties and is already covered by the standard regularity theory.

Let us first recall some of the well known facts for nondegenerate systems. In the vectorial case $N > 1$—in contrast to the scalar case $N = 1$—we cannot in general expect that a weak solution to the nonlinear elliptic system (1.1) is a classical solution (see e.g. the counterexamples in [6,24]). Instead, only a partial regularity result holds true, in the sense that we find an open subset $\Omega_0 \subset \Omega$ with $\mathcal{L}^n(\Omega \setminus \Omega_0) = 0$ such that Du is locally Hölder continuous on Ω_0 with optimal exponent β given by the exponent in the Hölder continuity assumption on the x-dependency of the coefficients. These results were obtained by Morrey [37], using an adaptation of an idea introduced by Almgren, by Giusti and Miranda [23] via the indirect blow-up technique, and then by Giaquinta, Modica and Ivert [21,27] via the direct method. Finally Duzaar and Grotowski [12] gave a new proof based on the method of $\mathcal{A}$-harmonic approximation introduced by Duzaar and Steffen [18]. For further references and in particular for related results concerning variational problems we refer to Mingione's survey article [36].

In the degenerate case $\mu = 0$ no (partial) regularity result seems to be known for such general systems. However, supposing some additional assumptions on the structure of the system, Uraltseva [42] and Uhlenbeck [41] succeeded in showing full regularity results for weak solution. In particular, for systems of what is called nowadays "Uhlenbeck" structure (cp. Proposition 3.9) Moser-type techniques may be applied, and the classical regularity results of De Giorgi, Nash and Moser can thus be extended to such systems. A prototype of these systems is the p-Laplace system with $A(z) = |z|^{p-2}z$. More precisely, she stated in the superquadratic case (for systems without explicit dependency on the space variable) that the gradient of the solution is globally Hölder continuous in the interior with an exponent depending only on the space dimension n and the ellipticity ratio ν/L. We emphasize that Uhlenbeck's proof was carried out in the more general setting of $\mathbb{R}^N$-valued closed ℓ-differential forms $\omega \in L^p(\Omega, \Lambda^\ell \mathbb{R}^n)$ solving the weak formulation to

$$d^*\rho(|\omega|)\omega = 0 \qquad \text{in}\, \Omega\,,$$

where ρ satisfies the assumptions on p. 781. Further results concerning the regularity theory under such structure assumptions can for instance be found in [40,22,1,20,25,32,33,10]. We highlight that Hamburger [25] gave an extension of Uhlenbeck's results in the setting of differential forms on Riemannian manifolds with sufficiently smooth boundary. In particular, he used an elegant duality argument to derive the subquadratic result from the superquadratic one (see also [26]). Restricting ourselves to the special case of 1-forms it is clear that the regularity result also covers weak solutions $u \in W^{1,p}(\Omega, \mathbb{R}^N)$. In what follows, we will follow the same strategy and we will give the proof of our regularity result in the setting of differential forms. Nevertheless, for convenience of the reader we describe in this introduction most of ideas behind the generalizations obtained in this paper only for classical Sobolev functions, and we postpone the notation, difficulties and modifications needed in order to deal with differential forms to Sect. 3.

Minimizers to variational integrals with possibly degenerately quasiconvex integrands were already considered by Duzaar and Mingione [14]. They observed that the non-degenerate and the degenerate theory can be combined in the following way: as long as the gradient variable keeps away from the origin, the system is also for $\mu = 0$ not singular/degenerate, and therefore a local partial regularity result holds true without an additional Uhlenbeck structure assumption. In contrast, if the origin is approached, then by requiring this crucial structure assumption one even expects local full regularity. In fact, this strategy of distinguishing the local type of ellipticity was applied successfully in [14] in case of an asymptotic behavior like the p-Laplace system close to the origin, and as a final result minimizers were proved to be locally of class $C^{1,\alpha}$ for some $\alpha > 0$ (specified in the neighborhood of points where Du does not vanish) outside a set of Lebesgue measure zero. Since we will argue in a similar way, we comment on some aspects of the regularity proof in more detail. In order to obtain an estimate for the decay of a suitable excess quantity, we employ local comparison principles based on harmonic-type approximation lemmas, see Lemmas 4.5 and 4.7 further below. These are inspired by Simon's proof of the regularity theorem of Allard and extend the method of harmonic approximation (i.e. the approximation with functions solving the Laplace equation) in a natural way to bounded elliptic operators with constant coefficients or to even more general monotone operators. Here it is worth to remark that we give a direct proof of this approximation result, motivated from [11]. In this version the approximating harmonic function preserves the boundary values of the original function (which is only approximately harmonic), and moreover, it is shown that the gradients of these two functions are close in some suitable norm, rather than just the functions. As a consequence the proof of the main partial regularity result is direct and gives a control of the regularity estimates in terms of the structure constants. The important feature of the comparison system resulting from this harmonic-type approximation is the availability of good a priori estimates for its weak solutions (more precisely, solutions to linear systems with constant coefficients are known to be smooth, and solutions to Uhlenbeck systems are known to admit at least Hölder continuous gradients). In case of systems with degeneracy in the origin the above-mentioned distinction of the two different situations is accomplished as follows: if the average of the gradient is not too small compared to the excess quantity, then we deal with the non-degenerate situation and the usual comparison with the solution to the linearized system is performed via the $\mathcal{A}$-harmonic approximation lemma (see Proposition 7.1). If in contrast the average of the gradient is very small (again compared to the excess), then we are in the degenerate situation, meaning that the solution is approximately solving an Uhlenbeck system. Therefore, it is compared to the exact solution of this Uhlenbeck system (see Proposition 7.3). These two decay estimates are then matched together in an iteration scheme as in [14], ending up with the desired partial regularity result.

In this paper we have two primary goals. On the one hand, we give a generalization of the existing results concerning possibly degenerate problems. We pursue an approach proposed by Duzaar and Mingione [17] in order to extend the known results dealing with a possible degeneracy at the origin like the p-Laplace system to more general ones that may behave at the origin like *any* arbitrary system of Uhlenbeck structure; a similar generalization was also suggested by Schmidt [39] who obtained the corresponding partial regularity result for degenerate variational functionals under (p, q)-growth conditions. This first aim is essentially achieved by the use of an extension of the p-harmonic approximation lemma from [15] (and similar to the one in [11]), namely the a-harmonic approximation Lemma 4.5. Moreover, the context is the one of differential forms (but as already pointed out before the corresponding result for weak solutions $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ follows immediately by restricting ourselves to 1-forms). The intention of this further generalization requires some variants of the Lemmas which are usually applied in the regularity proofs. However, the arguments needed in case of differential forms are available from the literature (e.g. a Poincaré-type inequality can be obtained by means of the Hodge decomposition [30]).

On the other hand we present a unified and simplified proof of the partial regularity result for the sub- and the superquadratic case. This is accomplished by the use of a slightly different excess functional (which is up to a constant equivalent to the ones in [17] and which is in a similar form already used in [39]) and some elementary observations. More precisely, in order to derive an excess-decay estimate for elliptic problems of p-growth with $p > 2$, the existing proofs usually make use of a minimizing polynomial in the following sense: given a map $u \in L^2(B_\rho, \mathbb{R}^N)$ let us denote by P_ρ the unique affine function which minimizes $P \mapsto \int_{B_\rho} |u - P|^2\, dx$ among all affine functions $P \colon \mathbb{R}^n \to \mathbb{R}^N$. With an explicit formula for P_ρ in terms of u at hand it is then possible to control differences of P_ρ on balls of different radii, which in turn is used (via its minimization property) to gain suitable decay estimates also in case of powers different from 2. In contrast, we here avoid the use of this minimizing affine function and we instead take advantage a technical lemma involving an immediate *quasi-minimizing* property which is adjusted to our specific excess-functional and which gives the right excess-decay estimate in only one step. Furthermore, the combination of the degenerate and the non-degenerate decay estimates is performed in a simplified way. Nevertheless, all the arguments are essentially known, but we believe that some of them have not been used in the same way before.

Once the partial regularity result is achieved, it is natural to ask whether the Hausdorff dimension of the singular set can still be improved. We first note that for degenerate Uhlenbeck systems the (interior) singular set is indeed empty—due to the special structure of the coefficients. Turning our attention to the non-degenerate situation without any structure assumptions, much less is known. Indeed, in the course of proving regularity of the gradient Du for classical solutions $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, the set of regular points is characterized, which in turn yields as a first and immediate consequence of a measure density result that the singular set is of Lebesgue measure zero. An estimate of the Hausdorff dimension was first investigated in the case of differentiable systems by Campanato in the 80's. The proof relied on the possibility of differentiating the system and obtaining existence of second-order derivatives of the solution. The first one who built a bridge between Hölder continuity of the coefficients and size of the singular set was Mingione [35,34]: he showed that the singular set $\Omega \setminus \Omega_0$ is not only negligible with respect to the Lebesgue measure, but that its Hausdorff dimension is actually not greater than $n - 2\beta$ (with β the degree of Hölder continuity of the coefficients). For related results on dimension reduction of the singular set in the context of convex variational integrals we refer to [31]. By means of the machinery of fractional Sobolev spaces and the differentiability of the system in a fractional sense developed in the previous papers, this upper bound on the Hausdorff dimension of the singular set is shown to be still valid for the solutions under consideration in this paper.

In conclusion, the main regularity result of our paper in the special case of classical weak solution can be stated as follows:

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $p \in (1, \infty)$, and consider a weak solution $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ to the system (1.1) under assumptions corresponding to (H1)–(H1) given* *in Sect. 2. Then there exists an open subset* $\Omega_0 \subset \Omega$ *such that*

$$u \in C^{1,\sigma}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^N) \quad \textit{and} \quad \dim_{\mathcal{H}}(\Omega \setminus \Omega_0) \leq n - 2\beta,$$

where σ *is an exponent depending only on* n, N, p, L, ν *and* β.

As noted above, this result will be established in the general framework of differential forms, see the corresponding results in Theorems 2.1 and 2.2. Lastly, we mention that the theory provided in this paper could also serve to deal with other regularity issues for differential forms, such as partial regularity (and dimension reduction for the singular set) in the context of parabolic systems as in [16], or Calderón-Zygmund-type estimates with nonstandard growth as in [2].

A possible application of our results might be a nonlinear weighted projection where the weight is a Hölder continuous function of the point or a bundle map coming from a metric tensor which is Hölder continuous. In addition, the partial regularity result might apply to Yang-Mills connections with weighted Hölder continuous norms.

2 Structure conditions and main results

In what follows we shall denote by $\Lambda^\ell := (\Lambda^\ell(\mathbb{R}^n))^N$ the vector bundle of all exterior differential ℓ-forms over an open subset in $\mathbb{R}^n$ taking values in $\mathbb{R}^N$, suppressing in the notation the space dimensions n, N. Furthermore, we shall use d and d^* for the usual exterior derivative and codifferential (see also Sect. 3 further below). We start with Ω a bounded domain in $\mathbb{R}^n$ and we suppose that $\omega \in L^p(\Omega, \Lambda^\ell)$, with $1 < p < \infty$, is a weak solution to the elliptic system

$$d^* A(\,\cdot\,, \omega) = 0 \quad \text{and} \quad d\omega = 0 \quad \text{in } \Omega\,, \tag{2.1}$$

for a vector field $A \colon \Omega \times \Lambda^\ell \to \Lambda^\ell$ satisfying some structure conditions: the mapping $\omega \mapsto A(x, \omega)$ is of class $C^0(\Lambda^\ell, \Lambda^\ell) \cap C^1(\Lambda^\ell \setminus \{0\}, \Lambda^\ell)$, and for fixed numbers $0 < \nu \leq L$, all $x, \bar{x} \in \Omega$ and all $\omega, \bar{\omega} \in \Lambda^\ell$ the following assumptions concerning growth, ellipticity and continuity hold true:

(H1) A is Lipschitz continuous with respect to ω with

$$|A(x,\omega) - A(x,\bar{\omega})| \leq L\,(|\omega|^2 + |\bar{\omega}|^2)^{\frac{p-2}{2}}\,|\omega - \bar{\omega}|\,,$$

(H2) $D_\omega A$ is Hölder continuous with some exponent $\alpha \in (0, |p-2|)$ such that

$$|D_\omega A(x,\omega) - D_\omega A(x,\bar{\omega})| \leq L\,(|\omega|^2 + |\bar{\omega}|^2)^{\frac{p-2-\alpha}{2}}\,|\omega - \bar{\omega}|^\alpha$$

holds for $p > 2$, whereas in the subquadratic case $p \in (1, 2)$ there holds for all $\omega, \bar{\omega} \neq 0$

$$|D_\omega A(x,\omega) - D_\omega A(x,\bar{\omega})| \leq L\,|\omega|^{p-2}|\bar{\omega}|^{p-2}(|\omega|^2 + |\bar{\omega}|^2)^{\frac{2-p-\alpha}{2}}\,|\omega - \bar{\omega}|^\alpha,$$

(H3) A is degenerately monotone:

$$\langle\, A(x,\omega) - A(x,\bar{\omega}), \omega - \bar{\omega}\,\rangle \geq \nu\,(|\omega|^2 + |\bar{\omega}|^2)^{\frac{p-2}{2}}\,|\omega - \bar{\omega}|^2,$$

(H4) A is Hölder continuous with respect to the first variable with exponent $\beta \in (0, 1)$:

$$|A(x,\omega) - A(\bar{x},\omega)| \leq L\,|\omega|^{p-1}\,|x - \bar{x}|^\beta,$$

(H5) A is of Uhlenbeck structure at 0, i.e. there exists a non-decreasing function $\widetilde{\mu} \colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $\tilde{\omega} \in \Lambda^\ell$ with $|\tilde{\omega}| \leq \widetilde{\mu}(t)$ there holds

$$|A(x,\tilde{\omega}) - \rho_x(|\tilde{\omega}|)\,\tilde{\omega}| \leq t\,|\tilde{\omega}|^{p-1}$$

uniformly for all $x \in \Omega$, where ρ_x is a family of functions satisfying (G1)–(G3) introduced on p. 781 further below.

We first note that—due to the growth condition (H1), the monotonicity in (H1) and the Uhlenbeck type behavior at 0 in (H1)—the coefficients $A(x,\omega)$ exhibit a polynomial growth with respect to the variable ω, namely for all $x \in \Omega$, $\omega \in \Lambda^\ell$ there holds

$$\nu\,|\omega|^{p-1} \leq |A(x,\omega)| \leq L\,|\omega|^{p-1}. \tag{2.2}$$

Secondly, in view of the differentiability of $\omega \mapsto A(x,z)$, we remark that (H1) and (H1) imply a growth and (degenerate) ellipticity condition for $D_\omega A(x,\omega)$, more precisely, we have

$$|D_\omega A(x,\omega)| \leq L\,|\omega|^{p-2}, \tag{2.3}$$

$$\langle\, D_\omega A(x,\omega)\,\xi, \xi\,\rangle \geq \nu\,|\omega|^{p-2}\,|\xi|^2 \tag{2.4}$$

for all $\xi \in \Lambda^\ell$, every $x \in \Omega$, and all $\omega \in \Lambda^\ell \setminus \{0\}$ (for $p > 2$ these inequalities are also valid for $\omega = 0$).

Example A simple example or model case for the systems under consideration in this paper are the following x-depending versions of the p-Laplace system:

$$A(x,\omega) := \beta(x)\,|\omega|^{p-2}\,\omega$$

for all $\omega \in \Lambda^\ell$ and with $\beta(\cdot)$ a continuous function in Ω taking values in $[\nu, L]$ with Hölder exponent β. In order to verify that this system satisfies all assumptions (H1)–(H1) we first observe that (H1) follows from Lemma 3.6 further below. The second condition is easy to check for $p > 2$, whereas for $1 < p < 2$ it is derived by distinguishing the cases where $|\omega - \bar{\omega}|^2 \geq |\omega||\bar{\omega}|$ holds or were the opposite inequality is satisfied. Next, (H1) is exactly formula [15, (10)], and the last two conditions are trivially satisfied (noting that $A(x,\cdot)$ as a definition for the family $\rho_x(\cdot)$ is admissible).

For a form $\omega \in L^p(B_r(x_0), \Lambda^\ell)$ we now introduce the excess

$$\Phi(\omega; x_0, r, \omega_0) := \fint_{B_r(x_0)} |V_{|\omega_0|}(\omega - \omega_0)|^2 \qquad \text{for every } \omega_0 \in \Lambda^\ell,$$

where $V_\mu(\xi) := (\mu^2 + |\xi|^2)^{(p-2)/4}\xi$. In the sequel this excess shall frequently be used for the choice $\omega_0 = (\omega)_{x_0,\rho}$, where $(\omega)_{x_0,r} = \fint_{B_r(x_0)} \omega$ is an abbreviation for the meanvalue of ω on the ball $B_r(x_0)$. As mentioned in [39] this excess is equivalent to

$$\fint_{B_r(x_0)} |V_0(\omega) - V_0(\omega_0)|^2 \tag{2.5}$$

up to a constant depending only on n, N, p and ℓ, and also to the one used in [14] (see Remark 3.8 below). With this notation at hand we can now state our main regularity result for weak solutions to (2.1) on a bounded domain in $\mathbb{R}^n$:

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $p \in (1,\infty)$ and consider a weak solution $\omega \in L^p(\Omega, \Lambda^\ell)$ to the homogeneous system (2.1) under the assumptions (H1)–(H1). Then there exists $\sigma = \sigma(n, N, p, \ell, L, \nu, \beta)$ and an open subset $\Omega_0(\omega) \subset \Omega$ such that*

$$\omega \in C^{0,\sigma}_{\mathrm{loc}}(\Omega_0(\omega), \Lambda^\ell) \qquad \text{and} \qquad |\Omega \setminus \Omega_0(\omega)| = 0$$

with the following characterization of the set of regular points:

$$\Omega_0(\omega) := \left\{ x_0 \in \Omega : \; : \liminf_{r\searrow 0} \Phi(\omega; x_0, r, (\omega)_{x_0,r}) = 0 \quad \textit{and} \quad \limsup_{r\searrow 0} \left|(\omega)_{x_0,r}\right| < \infty \right\}.$$

Moreover, if $x_0 \in \Omega_0(\omega)$ *and*

$$\limsup_{r\searrow 0} \frac{\left|(\omega)_{x_0,r}\right|^p}{\Phi(\omega; x_0, r, (\omega)_{x_0,r})} = \infty\,, \tag{2.6}$$

then ω *is locally Hölder continuous with exponent* $\min\{\beta, 2\beta/p\}$. *Furthermore, if* $\omega(x_0) \neq 0$, *then* $\omega \in C^{0,\beta}(B_s(x_0), \Lambda^\ell)$ *for some* $s > 0$.

Remark More precisely, in regular points $x_0 \in \Omega_0(\omega)$ where (2.6) is not satisfied, the local Hölder continuity is determined by the exponent from the Hölder continuity of the coefficients with respect to the first variable and the asymptotic degenerate system in the origin in a neighborhood of x_0. In this case the exponent σ is given by $\min\{\gamma, \beta\}$ in the subquadratic case and by $\min\{2\gamma/p, 2\beta/p\}$ in the superquadratic case. Here $\gamma \in (0, 1)$ is the number from the a priori estimate for weak solutions to systems of Uhlenbeck-type given in Proposition 3.9 below (we note that γ does not depend on the point x_0 since the parameters n, N, ℓ, p, L and ν remain fixed for all functions ρ_x).

As a second result we give the dimension reduction for the singular set, which states a relation between the degree of regularity of the coefficients and the size of the Hausdorff dimension of the singular set:

Theorem 2.2 *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain,* $p \in (1, \infty)$ *and consider a weak solution* $\omega \in L^p(\Omega, \Lambda^\ell)$ *to the system* (2.1) *under the assumptions* (H1)–(H1). *Then we have*

$$\dim_{\mathcal{H}} (\Omega \setminus \Omega_0(\omega)) \leq n - 2\beta.$$

At the end of this paper we shall deal briefly with a related regularity questions: in Sect. 9 we study the inhomogeneous situation, sketching the modifications required in the proofs in order to obtain the same results concerning partial regularity and dimension reduction as above in the homogeneous situation.

3 Notation and preliminaries

We now explain some general terminology that shall be used in the whole paper, introduce the general setting, give some technical results, and recall a by-now classical regularity result.

Exterior forms and pullback. We here recall some basic notations and technical details associated with differential forms. For a more extensive discussion we refer to [30, Sect. 2–5]. Let E denote a real vector space of dimension n endowed with an inner product $\langle\, , \,\rangle$ and let $\{e_1, \dots, e_n\}$ be an orthonormal basis. We first note that the inner product on E induces in a natural way an inner product on the dual space E' via

$$\langle\, \xi_1, \zeta_1 \,\rangle := \sum_{k=1}^{n} \xi_1(e_k)\, \zeta_1(e_k)\,,$$

and then also on general ℓ-forms for all $\ell = 0, 1, \dots, n$, according to the relation

$$\langle\, \xi, \zeta \,\rangle := \det\langle\, \xi_i, \zeta_j \,\rangle\,,$$

where $\xi, \zeta \in \Lambda^\ell E$ with $\xi = \xi_1 \wedge \ldots \wedge \xi_\ell, \zeta = \zeta_1 \wedge \ldots \wedge \zeta_\ell$ and $\xi_i, \zeta_i \in E'$ for all $i \in \{1, \ldots, \ell\}$. We further denote the dual basis by $\{e^1, \ldots, e^n\}$ (in the sense $e^i(e_j) = \delta_{ij}$) and we refer to $\mathbf{e} = e^1 \wedge \ldots \wedge e^n$ as an orientation on E. We next introduce the Hodge star operator

$$*\colon \Lambda^\ell E \to \Lambda^{n-\ell} E \qquad (\text{for } \ell = 0, 1, \ldots, n)$$

which is uniquely determined by

$$*1 = \mathbf{e} \qquad \text{and} \qquad \xi \wedge *\zeta = \langle \xi, \zeta \rangle \, \mathbf{e}.$$

We observe that $*$ is an isometry and $**$ is the multiplication by $(-1)^{\ell(n-\ell)}$ on $\Lambda^\ell E$. To define the orthogonal and the normal part of a differential form $\omega \in \Lambda^\ell E$, let V be a subspace of E and let us denote by $\pi : E \to V$ the orthogonal projection. Then the tangential part of ω with respect to V is given by

$$\omega_T(X_1, \ldots, X_\ell) := \omega(\pi X_1, \ldots, \pi X_\ell)$$

for all $X_1, \ldots, X_\ell \in E$. Hence, we have $\omega_T \in \Lambda^\ell E$ with the property that

$$\omega_T(X_1, \ldots, X_\ell) = \omega(X_1, \ldots, X_\ell)$$

whenever $X_1, \ldots, X_\ell \in V$. Furthermore, the normal part of ω is defined as $\omega_N := \omega - \omega_T$. This definition of tangential and normal part is to be understood pointwise if applied to sections on $\bar{\Omega}$.

Function spaces of differential forms. For what follows, Ω shall always be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. We denote by $L^p(\Omega, \Lambda^\ell)$, $p \in [1, \infty]$, the Lebesgue space of all measurable ℓ-forms ω on Ω for which

$$\|\omega\|_{L^p(\Omega,\Lambda^\ell)} := \begin{cases} \left(\int_\Omega |\omega|^p\right)^{1/p} & \text{if} 1 \leq p < \infty, \\ \operatorname{ess\,sup}_\Omega |\omega| & \text{if } p = \infty \end{cases}$$

is finite (with norm $|\omega| := \langle \omega, \omega \rangle^{1/2}$ inherited from the scalar product). We here consider as measure the one induced by the volume form $*1$ and we will always omit the notation of the measure under the integral sign. If $p, q \in [1, \infty]$ are Hölder conjugate exponents, then the scalar product of $\omega \in L^p(\Omega, \Lambda^\ell)$ and $\varphi \in L^q(\Omega, \Lambda^\ell)$ is finite and given by

$$\langle \omega, \varphi \rangle_\Omega := \int_\Omega \langle \omega, \varphi \rangle := \int_\Omega \omega \wedge *\varphi = \int_\Omega \varphi \wedge *\omega.$$

We will further employ the standard definitions for the Sobolev spaces $W^{k,p}(\Omega, \Lambda^\ell)$ and the Hölder space $C^{k,\alpha}(\Omega, \Lambda^\ell)$, $k \in \mathbb{N}$, $\alpha \in [0, 1]$, and we refer for more details on general Sobolev spaces to the literature since we will mostly deal only with partly Sobolev classes introduced below). If we restrict ourselves to functions with vanishing tangential or normal part, we will indicate this restriction by the notion $W_T^{k,p}(\Omega, \Lambda^\ell)$ and $W_N^{k,p}(\Omega, \Lambda^\ell)$, respectively. At this stage, we mention that the Meyers and Serrin approximation by smooth functions is also possible, in the sense that for $1 \leq p < \infty$ the space $C^\infty(\bar{\Omega}, \Lambda^\ell)$ is dense in $W^{1,p}(\Omega, \Lambda^\ell)$, $C_0^\infty(\bar{\Omega}, \Lambda^\ell)$ is dense in $W_0^{1,p}(\Omega, \Lambda^\ell)$, and the same holds for the Sobolev spaces with vanishing tangential and normal part; see [30, Corollary 3.2, 3.3, 3.4].

We now recall the notion of (weak) exterior derivatives. As usually, we denote by d the exterior derivative

$$d\colon C^\infty(\Omega, \Lambda^\ell) \to C^\infty(\Omega, \Lambda^{\ell+1}),$$

for which we have a product formula of the form

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\ell} \omega_1 \wedge d\omega_2$$

(with ℓ the degree of the form ω_1), whereas the Hodge co-differential d^*, the formal adjoint of d, is given by

$$d^* = (-1)^{n\ell+1} * d*: C^{\infty}(\Omega, \Lambda^{\ell+1}) \to C^{\infty}(\Omega, \Lambda^{\ell})$$

for $\ell = 1, \ldots, n$ (we note that also the opposite sign in the definition of d^* is found in the literature). Taking into account the product formula given above, it is easy to see that the following integration by parts formula—a duality relation between the two differential operators—holds true for all $\omega \in C^{\infty}(\bar{\Omega}, \Lambda^{\ell})$ and $\varphi \in C^{\infty}(\bar{\Omega}, \Lambda^{\ell+1})$:

$$\int_{\Omega} \langle \omega, d^*\varphi \rangle - \int_{\Omega} \langle d\omega, \varphi \rangle = \int_{\partial\Omega} \omega_T \wedge *\varphi_N .$$

Similar to the notion of weak derivative, we can introduce weak or generalized exterior (co-) derivatives by requiring only the validity of the integration by parts formula for arbitrary test functions with compact support. More precisely, we have the following

Definition ([30], Definition 3.1) A differential form $\omega \in L^1(\Omega, \Lambda^{\ell})$ has a generalized exterior derivative if there exists an integrable $(\ell + 1)$-form, denoted by $d\omega$, which satisfies

$$\int_{\Omega} \langle \omega, d^*\varphi \rangle = \int_{\Omega} \langle d\omega, \varphi \rangle$$

for every test form $\varphi \in C_0^{\infty}(\Omega, \Lambda^{\ell+1})$. If this identity holds for every $\varphi \in C^{\infty}(\Omega, \Lambda^{\ell+1})$, then $\omega_T = 0$ and we say that ω has vanishing tangential part. The notion of generalized exterior coderivative and vanishing normal part are defined analogously. Lastly, we denote by

$$\ker d : = \left\{\omega \in L^1(\Omega, \Lambda^{\ell}) \text{ with } d\omega = 0\right\}$$
$$\ker d^* : = \left\{\omega \in L^1(\Omega, \Lambda^{\ell}) \text{ with } d^*\omega = 0\right\}$$

the set of closed and coclosed ℓ-forms, respectively.

One feature of the system under consideration is that we demand only the partial differentiation of the system with respect to d. So there are spaces of differential forms coming naturally by differential equations where the differentiation occurs only via the operators d or d^*, these are the so-called partly Sobolev classes of first order for which we require only that both the form and its generalized exterior derivative or coderivative are L^p integrable:

$$W^{d,p}(\Omega, \Lambda^{\ell}) : = \left\{\omega \in L^p(\Omega, \Lambda^{\ell}) : d\omega \in L^p(\Omega, \Lambda^{\ell+1})\right\},$$
$$W^{d^*,p}(\Omega, \Lambda^{\ell}) : = \left\{\omega \in L^p(\Omega, \Lambda^{\ell}) : d^*\omega \in L^p(\Omega, \Lambda^{\ell-1})\right\},$$

and both spaces are Banach spaces when equipped with the norm $\|\omega\|_{W^{d,p}} := \|\omega\|_{L^p} + \|d\omega\|_{L^p}$ resp. $\|\omega\|_{W^{d^*,p}} := \|\omega\|_{L^p} + \|d^*\omega\|_{L^p}$. It is worth mentioning that the Sobolev space $W^{1,p}(\Omega, \Lambda^{\ell})$ coincides with the intersection of the two partly Sobolev spaces $W^{d,p}(\Omega, \Lambda^{\ell})$ and $W^{d^*,p}(\Omega, \Lambda^{\ell})$. Like before, we can also introduce the subspaces which are suitable to consider boundary value problems, namely the spaces $W_T^{d,p}(\Omega, \Lambda^{\ell})$ and $W_N^{d^*,p}(\Omega, \Lambda^{\ell})$ of differential forms with vanishing tangential resp. normal part on the boundary $\partial\Omega$. Furthermore, an according result concerning the approximation by smooth functions analogous to above holds true for the partly Sobolev spaces $W^{d,p}_{(T)}(\Omega, \Lambda^\ell)$ and $W^{d^*,p}_{(N)}(\Omega, \Lambda^\ell)$ (whereas if we consider $\bar{\Omega}$ the density holds only true if the tangential resp. normal part vanishes). An important inequality that relates Sobolev spaces and partly Sobolev spaces is the following extension of Gaffney's inequality:

Theorem 3.1 ([30], Theorem 4.8) *For every $p \in (1, \infty)$ there exists a constant c depending only on p and Ω such that*

$$\|\omega\|_{W^{1,p}} \le c \left(\|\omega\|_{L^p} + \|d\omega\|_{L^p} + \|d^*\omega\|_{L^p}\right)$$

holds true for all $\omega \in W^{1,p}_T(\Omega, \Lambda^\ell) \cup W^{1,p}_N(\Omega, \Lambda^\ell)$.

Before stating (Sobolev-)Poincaré-type inequalities in the setting of differential forms, we introduce the spaces of exact, coexact and harmonic (meaning both exact and coexact) forms, integrable to the power p:

$$\begin{aligned} dW^{1,p}_{(T)}(\Omega, \Lambda^{\ell-2}) &:= \left\{d\omega \colon \omega \in W^{1,p}_{(T)}(\Omega, \Lambda^{\ell-2})\right\}, \\ d^*W^{1,p}_{(N)}(\Omega, \Lambda^{\ell}) &:= \left\{d^*\omega \colon \omega \in W^{1,p}_{(N)}(\Omega, \Lambda^{\ell})\right\}, \\ H^p_{(N,T)}(\Omega, \Lambda^{\ell-1}) &:= \left\{h \in W^{1,p}_{(N,T)}(\Omega, \Lambda^{\ell-1}) \colon dh = 0 = d^*h\right\}. \end{aligned}$$

These spaces allow us to state a result on the decomposition of differential forms in L^p into exact, coexact and harmonic components, which is known as Hodge decomposition (for the related theory on $\mathbb{R}^n$ see also [29]). In order to get a direct sum decomposition, we have several choices to partially fix the boundary values of the components, see [30, Theorem 5.4]:

$$\begin{aligned} L^p(\Omega, \Lambda^{\ell-1}) &= dW^{1,p}_T(\Omega, \Lambda^{\ell-2}) \oplus d^*W^{1,p}_N(\Omega, \Lambda^{\ell}) \oplus H^p(\Omega, \Lambda^{\ell-1}), \\ L^p(\Omega, \Lambda^{\ell-1}) &= dW^{1,p}_T(\Omega, \Lambda^{\ell-2}) \oplus d^*W^{1,p}(\Omega, \Lambda^{\ell}) \oplus H^p_T(\Omega, \Lambda^{\ell-1}), \\ L^p(\Omega, \Lambda^{\ell-1}) &= dW^{1,p}(\Omega, \Lambda^{\ell-2}) \oplus d^*W^{1,p}_N(\Omega, \Lambda^{\ell}) \oplus H^p_N(\Omega, \Lambda^{\ell-1}). \end{aligned}$$

We wish to take advantage of the first one, and therefore, we shall also work with the orthogonal projections associated to this decomposition:

- Exact projection $E_T(\cdot\,; \Omega) \colon L^p(\Omega, \Lambda^{\ell-1}) \to dW^{1,p}_T(\Omega, \Lambda^{\ell-2})$,
- Coexact projection $E^*_N(\cdot\,; \Omega) \colon L^p(\Omega, \Lambda^{\ell-1}) \to d^*W^{1,p}_N(\Omega, \Lambda^{\ell})$,
- Harmonic projection $H(\cdot\,; \Omega) \colon L^p(\Omega, \Lambda^{\ell-1}) \to H^p(\Omega, \Lambda^{\ell-1})$.

Hence, the identity operator $\mathrm{Id} \colon L^p(\Omega, \Lambda^{\ell-1}) \to L^p(\Omega, \Lambda^{\ell-1})$ may in particular be rewritten as

$$\mathrm{Id} = E_T(\cdot\,; \Omega) + E^*_N(\cdot\,; \Omega) + H(\cdot\,; \Omega). \tag{3.1}$$

For 1-forms in the Sobolev space $W^{d,p}$, a Poincaré-type inequality can be proved by subtracting the meanvalue. For forms of higher order instead, such an inequality can be obtained by replacing the mean values with a suitable, more complicated averaging operator applied to the form under consideration (which by construction is also a closed form), see the Poincaré and the Sobolev-Poincaré-inequalities [28, Corollary 4.1, 4.2]. A different approach based on the Hodge decomposition was given in [30, Theorem 6.4], and the results read as follows:

Lemma 3.2 (Poincaré inequality) *Let $p \in (1, \infty)$ and consider $\omega \in W^{d,p}(B_r, \Lambda^\ell)$ (resp. $\omega \in W^{d^*,p}(B_r, \Lambda^\ell)$). Then $\omega_0 = E_T(\omega; B_r) + H(\omega; B_r)$ (resp. $\omega_0 = E_N^*(\omega; B_r) + H(\omega; B_r)$) is a closed (coclosed) form, and $\omega - \omega_0$ belongs to $W^{1,p}(B_r, \Lambda^\ell)$ with the estimate*

$$\|\omega - \omega_0\|_{W^{1,p}(B_r,\Lambda^\ell)} \le c(n, p)\, \|d\omega\|_{L^p(B_r,\Lambda^\ell)} \qquad \left(c(n, p)\, \|d^*\omega\|_{L^p(B_r,\Lambda^\ell)}\right).$$

Remark 3.3 There exists a formulation for differential forms $\omega \in W^{d,p}(B_r, \Lambda^\ell)$ with vanishing tangential part ω_T, stating that if $\omega_T = 0$ then there exists a closed ℓ-form ω_0 satisfying $(\omega_0)_T = 0$ such that $\omega - \omega_0 \in W_T^{1,p}(B_r, \Lambda^\ell)$ with the corresponding estimate (see [30, Theorem 6.4]).

Lemma 3.4 (Sobolev-Poincaré inequality) *Let $p \in (1, n)$ and consider $\omega \in W^{d,p}(B_r, \Lambda^\ell)$ (resp. $\omega \in W^{d^*,p}(B_r, \Lambda^\ell)$). Then there exists $\omega_0 \in L^p(B_r, \Lambda^\ell) \cap \ker d$ (resp. $\omega_0 \in L^p(B_r, \Lambda^\ell) \cap \ker d^*$) such that $\omega - \omega_0$ belongs to $L^{np/(n-p)}(B_r, \Lambda^\ell)$ with the estimate*

$$\|\omega - \omega_0\|_{L^{np/(n-p)}(B_r,\Lambda^\ell)} \le c(n, p)\, \|d\omega\|_{L^p(B_r,\Lambda^\ell)} \qquad \left(c(n, p)\, \|d^*\omega\|_{L^p(B_r,\Lambda^\ell)}\right).$$

Some technical lemmas. In what follows, the functions $V_\mu, V : \Lambda^\ell \to \Lambda^\ell, \ell \in \{0, 1, \ldots, n\}$, will be useful. For $\xi \in \Lambda^\ell$, $\mu \ge 0$ and $p > 1$ they are defined by

$$V_\mu(\xi) = \left(\mu^2 + |\xi|^2\right)^{\frac{p-2}{4}} \xi \qquad \text{and} \qquad V(\xi) = V_0(\xi) = |\xi|^{\frac{p-2}{2}} \xi,$$

which are locally bi-Lipschitz bijections on Λ^ℓ. Some algebraic properties of the function V_μ, V and a Young-type inequality are given in the following

Lemma 3.5 (cf. [14], Lemma 1; [5], Lemma 2.1) *Let $p \in (1, \infty)$, $\mu \ge 0$, and consider the function $V_\mu : \mathbb{R}^k \to \mathbb{R}^k$ defined above. Then for all $\xi, \eta \in \mathbb{R}^k$ and $t > 0$ there hold:*

(i) $2^{\frac{p-2}{4}} \min\{\mu^{\frac{p-2}{2}} |\xi|, |\xi|^{\frac{p}{2}}\} \le |V_\mu(\xi)| \le \min\{\mu^{\frac{p-2}{2}} |\xi|, |\xi|^{\frac{p}{2}}\}$ *for* $p \in (1, 2)$,
$\max\{\mu^{\frac{p-2}{2}} |\xi|, |\xi|^{\frac{p}{2}}\} \le |V_\mu(\xi)| \le 2^{\frac{p-2}{4}} \max\{\mu^{\frac{p-2}{2}} |\xi|, |\xi|^{\frac{p}{2}}\}$ *for* $p \ge 2$,
(ii) $|V_\mu(t\xi)| \le \max\{t, t^{\frac{p}{2}}\} |V_\mu(\xi)|$,
(iii) $|V_\mu(\xi + \eta)| \le c(p) \left(|V_\mu(\xi)| + |V_\mu(\eta)|\right)$,
(iv) $c^{-1}(p) \left(\mu^2 + |\eta|^2 + |\xi|^2\right)^{\frac{p-2}{4}} |\eta - \xi| \le |V_\mu(\eta) - V_\mu(\xi)|$
$\le c(k, p) \left(\mu^2 + |\eta|^2 + |\xi|^2\right)^{\frac{p-2}{4}} |\eta - \xi|$,
(v) $|V_\mu(\xi) - V_\mu(\eta)| \le c(k, p)\, |V_\mu(\xi - \eta)|$ *for* $p \in (1, 2)$,
$|V_\mu(\xi) - V_\mu(\eta)| \le c(k, p, M)\, |V_\mu(\xi - \eta)|$, *provided* $|\xi| \le M\mu$, *for* $p \ge 2$,
(vi) $|V_\mu(\xi - \eta)| \le c(p, M)\, |V_\mu(\xi) - V_\mu(\eta)|$, *provided* $|\xi| \le M\mu$, *for* $p \in (1, 2)$,
$|V_\mu(\xi - \eta)| \le c(p)\, |V_\mu(\xi) - V_\mu(\eta)|$ *for* $p \ge 2$,
(vii) $(\mu^2 + |\eta|^2)^{\frac{p-2}{2}} |\eta|\, |\xi| \le \varepsilon\, |V_\mu(\eta)|^2 + \max\{\varepsilon^{-1}, \varepsilon^{1-p}\}\, |V_\mu(\xi)|^2$ *for every* $\varepsilon \in (0, 1)$.

Lemma 3.6 (cf. [1], Lemma 2.1) *Let ξ, η be vectors in $\mathbb{R}^k$, $\mu \in [0, 1]$ and $q > -1$. Then there exist constants $c_1, c_2 \ge 1$, which depend only on q but which are independent of μ, such that*

$$c_1^{-1}\, (\mu + |\xi| + |\eta|)^q \le \int_0^1 (\mu + |\xi + t\eta|)^q \, dt \le c_2\, (\mu + |\xi| + |\eta|)^q .$$

In [38, Lemma 6.2] it was proved that the mapping $\xi \mapsto \int_\Omega |V_\mu(\omega - \xi)|^2$ is quasi-minimized by the mean value of ω on Ω, i.e. this mapping is minimized by $(\omega)_\Omega$ up to a multiplicative constant which depends only on the parameter p. In the sequel, a slight modification of this statement (adjusted to the excess used in this paper) involving different indices of the V_μ-function will be useful:

Lemma 3.7 *Let $\omega \in L^p(B, \Lambda^\ell)$, $p \in (1, \infty)$, with $B \subset \mathbb{R}^n$ a ball. Then, for all $\chi \in \Lambda^\ell$ we have*

$$\int_B \left| V_{|(\omega)_B|}(\omega - (\omega)_B) \right|^2 \leq c(p) \int_B \left| V_{|\chi|}(\omega - \chi) \right|^2$$

for a constant c depending only on p.

Proof We first note that the function $W_\mu(\xi) := (\mu + |\xi|)^{(p-2)/2} \xi$ is equivalent to $V_\mu(\xi)$ up to a constant depending only on p (and independent of $\mu \geq 0$). It is thus sufficient to prove the statement with V replaced by W. Furthermore, the mapping $\mu \mapsto W_\mu(\xi)$ is differentiable with derivative $d/d\mu\, W_\mu(\xi) = (\mu + |\xi|)^{(p-4)/2} \xi\, (p-2)/2$ for every $\xi \neq 0$. Therefore, due to the technical Lemma 3.6 (applying Young's inequality and keeping in mind the definition of the W_μ-function in the superquadratic case) we observe

$$\begin{aligned}
&\left| W_{|(\omega)_B|}(\omega - (\omega)_B) - W_{|\chi|}(\omega - (\omega)_B) \right| \\
&\quad \leq c(p) \int_0^1 (|\chi + t((\omega)_B - \chi)| + |\omega - (\omega)_B|)^{\frac{p-4}{2}}\, dt\, |\omega - (\omega)_B|\, |(\omega)_B - \chi| \\
&\quad \leq c(p) \int_0^1 (|\chi + t((\omega)_B - \chi)| + |\omega - (\omega)_B|)^{\frac{p-2}{2}}\, dt\, |(\omega)_B - \chi| \\
&\quad \leq c(p)\, (|\chi| + |(\omega)_B - \chi| + |\omega - (\omega)_B|)^{\frac{p-2}{2}}\, |(\omega)_B - \chi| \\
&\quad \leq c(p) \left| W_{|\chi|}(\omega - (\omega)_B) \right| + c(p) \left| W_{|\chi|}((\omega)_B - \chi) \right|
\end{aligned}$$

(note that the left-hand side vanishes trivially in points where $\omega = (\omega)_B$). Since $|W_\mu(\xi)|^2$ is convex with respect to ξ, we then conclude the desired inequality via Jensen's inequality

$$\begin{aligned}
&\fint_B \left| W_{|(\omega)_B|}(\omega - (\omega)_B) \right|^2 \\
&\quad \leq 2 \fint_B \left(\left| W_{|\chi|}(\omega - (\omega)_B) \right|^2 + \left| W_{|(\omega)_B|}(\omega - (\omega)_B) - W_{|\chi|}(\omega - (\omega)_B) \right|^2 \right) \\
&\quad \leq c(p) \fint_B \left| W_{|\chi|}(\omega - (\omega)_B) \right|^2 + c(p) \left| W_{|\chi|}((\omega)_B - \chi) \right|^2 \\
&\quad \leq c(p) \fint_B \left| W_{|\chi|}(\omega - \chi) \right|^2 + c(p) \left| W_{|\chi|}\left(\fint_B (\omega - \chi) \right) \right|^2 \leq c(p) \fint_B \left| W_{|\chi|}(\omega - \chi) \right|^2.
\end{aligned}$$

□

Remark 3.8 As emphasized above, the excess $\Phi(\omega; x_0, r, (\omega)_{x_0,r})$ used throughout this paper is equivalent to the one from [14]. For the superquadratic case their excess defined as

$$\fint_{B_r(x_0)} |(\omega)_{x_0,r}|^{p-2} |\omega - (\omega)_{x_0,r}|^2 + |\omega - (\omega)_{x_0,r}|^p$$

is obviously equivalent up to a constant depending only on p. With the previous two lemmas at hand, the equivalence to $\fint_{B_r(x_0)} |V(\omega) - (V(\omega))_{x_0,r}|^2$ in the subquadratic case can be seen as follows: Since V_μ is surjective, we find $(\omega)^V_{x_0,r}$ such that $V((\omega)^V_{x_0,r}) = (V(\omega))_{x_0,r}$, and we then obtain

$$
\begin{aligned}
\fint_{B_r(x_0)} \left|V_{|(\omega)_{x_0,r}|}\left(\omega - (\omega)_{x_0,r}\right)\right|^2 &\leq c(p) \fint_{B_r(x_0)} \left(|(\omega)^V_{x_0,r}|^2 + |\omega|^2\right)^{\frac{p-2}{2}} |\omega - (\omega)^V_{x_0,r}|^2 \\
&\leq c(p) \fint_{B_r(x_0)} \left|V(\omega) - (V(\omega))_{x_0,r}\right|^2 \\
&\leq c(p) \fint_{B_r(x_0)} \left|V(\omega) - V((\omega)_{x_0,r})\right|^2 \\
&\leq c(n,N,p,\ell) \fint_{B_r(x_0)} \left(|(\omega)_{x_0,r}|^2 + |\omega|^2\right)^{\frac{p-2}{2}} |\omega - (\omega)_{x_0,r}|^2 \\
&\leq c(n,N,p,\ell) \fint_{B_r(x_0)} \left|V_{|(\omega)_{x_0,r}|}\left(\omega - (\omega)_{x_0,r}\right)\right|^2.
\end{aligned}
$$

A regularity result for degenerate Uhlenbeck systems. We next state a comparison estimate for special nonlinear degenerate systems which exhibit a particular structure that allows to prove everywhere regularity of the solution. More precisely, we consider vector fields of the form

$$a(\bar{\omega}) = \rho(|\bar{\omega}|)\,\bar{\omega}$$

for every $\bar{\omega} \in \Lambda^\ell$. For the function $\rho\colon [0,\infty) \to [0,\infty)$ we shall assume the following continuity, ellipticity and growth conditions:

(G1) The function $t \mapsto \rho(t)$ is of class $C^0([0,\infty]) \cap C^1((0,\infty])$,

(G2) There hold the inequalities

$$\nu\, t^{p-2} \leq \rho(t) \leq L\, t^{p-2}$$

and

$$\nu\, t^{p-2} \leq \rho(t) + \rho'(t)\, t \leq L\, t^{p-2}\,,$$

(G3) There exists a Hölder exponent $\beta_\rho \in (0, \min\{1, |p-2|\})$ such that

$$|\rho'(s)\, s - \rho'(t)\, t| \leq L\,(|s|^2 + |t|^2)^{\frac{p-2-\beta_\rho}{2}} |s-t|^{\beta_\rho}.$$

for all $s, t \in (0,\infty)$, and some $p \geq 2$, $0 < \nu \leq L$. The model case of a vector field satisfying these conditions is the p-Laplace system, i.e. the vector field give by $a(\bar{\omega}) = |\bar{\omega}|^{p-2}\bar{\omega}$ for all $\bar{\omega} \in \Lambda^\ell$. For systems satisfying the above structure assumptions the following regularity result can be retrieved from [41, Theorem 3.2], [33] and [25, Theorem 4.1]:

Proposition 3.9 *Let $p \in (1,\infty)$. There exists a constant $c \geq 1$ and an exponent $\gamma \in (0,1)$ depending only on n, N, ℓ, p, L and ν such that the following statement holds true: whenever $h \in L^p(B_R(x_0), \Lambda^\ell)$ is a weak solution of the system*

$$d^*(\rho(|h|)\,h) = 0 \quad \text{and} \quad dh = 0 \quad \text{in } B_R(x_0)\,,$$

where $\rho(\cdot)$ fullfills the assumptions (G1)–(G1)*, then for every* $0 < r < R$ *there hold*

$$\sup_{B_{R/2}(x_0)} |h|^p \leq c \fint_{B_R(x_0)} |h|^p \quad \text{and} \quad \Phi(h; x_0, r, (h)_{x_0,r}) \leq c \left(\frac{r}{R}\right)^{2\gamma} \Phi(h; x_0, R, (h)_{x_0,R}).$$

4 Harmonic approximation lemmas

In this section we shall state two harmonic-type approximation lemmas which are adapted to the degenerate and the non-degenerate situation and which will allow us to compare the solution to the original system to the solution of an easier systems (for which good a priori estimates are available). To this aim we first need a result on Lipschitz-truncation, which from its original formulation can be restated as follows:

Proposition 4.1 (Lipschitz truncation, cf. [19], Prop. 4.1) *Let $B \subset \mathbb{R}^n$ be a ball. There exists a constant c depending only on n, N, ℓ and B such that whenever $\chi_k \rightharpoonup 0$ weakly in $W^{1,p}_T(B, \Lambda^\ell)$, then for every $\lambda > 0$ there exists a sequence $\{\chi_k^\lambda\}_{k\in\mathbb{N}}$ of maps $\chi_k^\lambda \in W^{1,\infty}_T(B, \Lambda^\ell)$ such that*

$$\|\chi_k^\lambda\|_{W^{1,\infty}} \leq c\,\lambda.$$

Moreover, up to a set of Lebesgue measure zero we have

$$\{z \in B\colon \chi_k^\lambda(z) \neq \chi_k(z)\} \subset \{z \in B\colon M(D\chi_k)(z) > \lambda\}\,,$$

where M denotes the maximal operator restricted to B, i. e.

$$M(D\chi_k)(z) = \sup_{r>0, B_r(z)\subset B} \fint_{B_r(z)} |D\chi_k|.$$

Due to the direct approach for the proof of Lemma 4.5 we in fact need it only in a simpler version, namely for single functions instead of weakly converging sequences. However, there are much more involved Lipschitz truncation lemmas available in the literature, such as on general domains, versions involving sequences of truncations and variable exponent in [9, Theorem 2.5, Theorem 4.4]), or versions truncating at two different levels (one for the function itself, the second one as above for its gradient). In this paper we shall use a consequence of the previous truncation Lemma 4.1 from [11] for a version concerning the existence of a good truncation level in the setting of Sobolev-Orlicz spaces $W^{1,\phi}_T(B, \Lambda^\ell)$. The assumptions on ϕ are the following:

Assumptions 4.2 *Let $\phi\colon [0,\infty) \to [0,\infty)$ be a convex function, hence in particular of class C^1 in $(0,\infty)$, such that $\phi(0) = 0$, $\phi'(t) \searrow 0$ as $t \searrow 0$ and $\phi'(t) \nearrow +\infty$ as $t \to \infty$. Furthermore, denoting by $\phi^*(t) := \sup_{s\geq 0}\{st - \phi(s)\}$ its conjugate function, we require that both ϕ and ϕ^* verify the Δ_2-condition, meaning that $\phi(2t) \leq c\phi(t)$ and $\phi^*(2t) \leq c\phi^*(t)$ for all $t \geq 0$ and universal constants c (the smallest such constant is usually denoted by $\Delta_2(\phi)$ and $\Delta_2(\phi^*)$, respectively).*

Some useful properties for functions satisfying Assumptions 4.2 can be immediately deduced. We first observe $(\phi^*)^* = \phi$ and $\phi(t) \leq t\phi(t)$ for all $t \geq 0$. We further recall the well known Young-type inequalities

$$t\,s \leq \delta\,\phi(s) + c(\delta, \Delta_2(\phi^*))\,\phi^*(t),$$
$$\phi'(s)\,t \leq \delta\,\phi(s) + c(\delta, \Delta_2(\phi))\,\phi(t)$$

for all $\delta, s, t \geq 0$. Under the previous assumption we then have:

Corollary 4.3 **([11])** *Let ϕ be a function verifying Assumptions 4.2. For every $\varepsilon > 0$ there exists $c > 0$ depending only on n, N, ℓ and $\Delta_2(\{\phi, \phi^*\})$ such that the following statement holds: If $B \subset \mathbb{R}^n$ is a ball and $\chi \in W_0^{1,\phi}(B, \Lambda^\ell)$, then for every $m_0 \in \mathbb{N}$ and $\gamma > 0$ there exists $\lambda \in [\gamma, 2^{m_0}\gamma]$ such that the Lipschitz truncation $\chi^\lambda \in W_T^{1,\infty}(B, \Lambda^\ell)$ of Theorem 4.1 satisfies*

$$\|D\chi^\lambda\|_\infty \leq c\,\lambda\,,$$
$$\fint_B \phi\left(|D\chi^\lambda|\,\mathbb{1}_{\{\chi^\lambda \neq \chi\}}\right)\,dx \leq c \fint_B \phi\left(\lambda\,\mathbb{1}_{\{\chi^\lambda \neq \chi\}}\right)\,dx \leq \frac{c}{m_0} \fint_B \phi(|D\chi|)\,dx.$$

Using this tool and assuming an additional assumption on second order derivatives of the form $\phi'(t) \sim t\phi''(t)$, an extension of the p-harmonic approximation lemma to general convex function in the framework of Sobolev-Orlicz spaces, the ϕ-harmonic approximation lemma, was proved in [11]. For sake of completeness, we recall it here:

Lemma 4.4 **(ϕ-harmonic approximation lemma, [11])** *Let ϕ satisfy Assumptions 4.2. For every $\varepsilon > 0$ and $\theta \in (0,1)$ there exists $\delta > 0$ which only depends on ε, θ, and the characteristics of ϕ such that the following holds. If $u \in W^{1,\phi}(B, \mathbb{R}^N)$ is almost ϕ-harmonic on a ball $B \subset \mathbb{R}^n$ in the sense that*

$$\left|\fint_B \phi'|Du|)\frac{Du}{|Du|}D\varphi\,dx\right| \leq \delta\Big(\fint_B \phi(|Du|)\,dx + \phi\left(\|D\varphi\|_\infty\right)\Big) \tag{4.1}$$

for all $\varphi \in C_0^\infty(B, \mathbb{R}^N)$, then the unique ϕ-harmonic map $h \in W^{1,\phi}(B, \mathbb{R}^N)$ with $h = u$ on ∂B satisfies

$$\left(\fint_B |V_\phi(Du) - V_\phi(Dh)|^{2\theta}\,dx\right)^{\frac{1}{\theta}} \leq \varepsilon \fint_B \phi(|Du|)\,dx.$$

where $V_\phi(Q) = \sqrt{\frac{\phi'(|Q|)}{|Q|}}\,Q$.

Note that this definition of *almost ϕ-harmonic* slightly differs from the original definition of *almost p-harmonic* from [15]. As it is easily seen, it is weaker; so any *almost p-harmonic* function in the sense of [15] is *almost ϕ-harmonic* for $\phi(t) = \frac{1}{p}t^p$ in the sense of (4.1). The reason for choosing this version of *almost harmonic* is, that (4.1) has very good scaling properties.

The ϕ-harmonic approximation Lemma improves the result of Duzaar and Mingione [15] in three different directions. First, it is proved by a direct approach (and not via contradiction), and therefore allows to keep good track of the dependencies of the constants involved in the approximation. Second, the boundary values of the original function are preserved (i. e. $u = h$ on ∂B). Third, h and u are close with respect to the gradients rather than just the functions.

Restricting ourselves to the case of power growth in order to keep the setting as simple as possible, we now derive by similar techniques a suitable version in the context of differential forms which will apply not only to the p-Laplace system, but also to more general monotone operators. In what follows we consider vector fields $a\colon \Omega \times \Lambda^\ell \to \Lambda^\ell$ which are measurable with respect to the first variable, continuous in the second, and which satisfy growth and monotonicity conditions of the form

$$\begin{cases} \big|a(x,\omega)\big| \le L\,\big(\mu^2+|\omega|^2\big)^{\frac{p-1}{2}}\,, \\ a(x,\omega)\cdot\omega \ge \nu\,\big(\mu^2+|\omega|^2\big)^{\frac{p-2}{2}}\,|\omega|^2\,, \\ (a(x,\omega)-a(x,\bar\omega))\cdot(\omega-\bar\omega) \ge \nu\,\big(\mu^2+|\bar\omega|^2+|\omega|^2\big)^{\frac{p-2}{2}}\,|\omega-\bar\omega|^2 \end{cases} \tag{4.2}$$

for all $x\in\Omega$, $\omega,\bar\omega\in\Lambda^\ell$, $p>1$, $\mu\in[0,1]$ and $0<\nu\le L$. Furthermore, we assume that $a(\cdot,\cdot)$ is uniformly continuous on bounded subsets, i.e. that

$$|a(x,\omega)-a(x,\bar\omega)| \le K(|\omega|+|\bar\omega|)\,\vartheta(|\omega-\bar\omega|) \tag{4.3}$$

whenever $x\in\Omega$, $\omega,\bar\omega\in\Lambda^\ell$, where $K\colon[0,\infty)\to[0,\infty)$ is a locally bounded, nondecreasing function and $\vartheta\colon[0,\infty)\to[0,1]$ is a nondecreasing function with $\lim_{t\searrow 0}\vartheta(t)=0$. We note that these assumptions are in particular satisfied with $\mu=0$ for vector fields $a(\omega):=\rho(|\omega|)\,\omega$ where ρ fullfills conditions (G1) and (G1) from the previous section. Following the notation of [7], we define for a convex function $\phi\in C^1((0,\infty))$ and $\mu\ge 0$ the *shifted function* ϕ_μ by

$$\phi_\mu(t) := \int_0^t \phi'_\mu(s)\,ds \qquad \text{with} \qquad \phi'_\mu(t) := \frac{\phi'(\mu+t)}{\mu+t}\,t$$

for $t>0$. If ϕ satisfies the Assumptions 4.2, then also the shifted function satisfies the Assumptions 4.2 uniformly for every $\mu\ge 0$, see also [7, Appendix]. In the case of powers $\phi(t):=t^p$, the excess function $V_\mu(t)$ introduced in Sect. 3 is equivalent to the shifted function $(\phi_\mu(t))^{1/2}$ (up to a constant depending only on p) and relates to the operator $a(\cdot,\cdot)$ satisfying the assumption (4.2) above via the inequalities:

$$\begin{cases} |a(x,\omega)| \le c(p,L)\,\phi'_\mu(|\omega|)\,, \\ a(x,\omega)\cdot\omega \ge \nu\,|V_\mu(\omega)|^2 \ge c^{-1}(p)\,\nu\,\phi_\mu(|\omega|)\,, \\ (a(x,\omega)-a(x,\bar\omega))\cdot(\omega-\bar\omega) \ge c^{-1}(p)\,\nu\,|V_\mu(\omega)-V_\mu(\bar\omega)|^2. \end{cases} \tag{4.4}$$

We may now introduce the notion of an a-harmonic form: a form $\omega\in L^p(\Omega,\Lambda^\ell)\cap\ker d$ is called a-harmonic in a domain Ω (with a slight abuse of notation because in case of 1-forms the function and not its derivative is called a-harmonic) if $a(\cdot,\cdot)$ fulfills the growth assumption $(4.2)_1$ and if

$$\int_\Omega \langle\, a(x,\omega), d\varphi\,\rangle = 0 \qquad \text{for every } \varphi\in C_T^\infty(\Omega,\Lambda^{\ell-1}).$$

Lemma 4.5 **(a-harmonic approximation; cf. [17], Lemma 3.2)** *Let $p\in(1,\infty)$ and $\phi(t)=t^p$ for all $t\ge 0$. For every $\varepsilon>0$ and every $\theta\in(0,1)$ there exists $\delta>0$ which depends only on n,N,p,ℓ,ν,L,θ and ε such that the following statement holds true: Let $B\subset\mathbb{R}^n$ be a ball. Whenever $a(\cdot,\cdot)\colon B\times\Lambda^\ell\to\Lambda^\ell$ is a vector field satisfying (4.2) and (4.3) and whenever $\chi\in W^{d,p}(B,\Lambda^{\ell-1})$ is a differential form such that $d\chi$ is approximately a-harmonic in the sense that*

$$\Big|\fint_B \langle\, a(x,d\chi), d\varphi\,\rangle\Big| \le \delta\Big(\fint_B \phi_\mu(|d\chi|)+\phi_\mu(\|d\varphi\|_\infty)\Big) \tag{4.5}$$

holds for all $\varphi \in C^1_T(B, \Lambda^{\ell-1})$, *then there exists a form* $h \in W^{d,p}(B, \Lambda^{\ell-1})$, $(h)_T = (\chi)_T$ *on* ∂B, *such that* dh *is a-harmonic and such that it satisfies*

$$\fint_B \phi_\mu(|dh|) \leq c \fint_B \phi_\mu(|d\chi|) \quad \text{and} \quad \Big(\fint_B |V_\mu(d\chi) - V_\mu(dh)|^{2\theta}\Big)^{\frac{1}{\theta}} \leq \varepsilon \fint_B \phi_\mu(|d\chi|)$$

for a constant c depending only on p, ν *and* L.

Proof We proceed similarly to the proof of the ϕ-harmonic approximation lemma in [11] and start by $\gamma \geq 0$ via $\phi_\mu(\gamma) := \fint_B \phi_\mu(|d\chi|)$ which is possible due to the fact that ϕ_μ is a bijective on $[0, \infty)$. We next define $h \in \chi + W^{d,p}_T(B, \Lambda^{\ell-1})$ to the differential form satisfying $E_T(h; B) = E_T(\chi; B)$, $H(h; B) = H(\chi; B)$, and solving the Dirichlet problem

$$\int_B \langle a(x, dh), d\varphi \rangle = 0 \qquad \text{for every } \varphi \in C^\infty_T(B, \Lambda^{\ell-1}),$$

i.e. dh is in particular an a-harmonic form with the required boundary condition. We want to comment briefly on the existence of the differential form h: we first note that the space $dW^{1,p}_T(B, \Lambda^{\ell-1})$ is a Banach space, and its dual space can be identified with $dW^{1,q}_T(B, \Lambda^{\ell-1})$ for $1/p + 1/q = 1$ (see [30, Theorem 5.7]). We then observe from the assumptions (4.2) and (4.3) on the coefficients that the (nonlinear) map

$$d\chi \in dW^{1,p}_T(B, \Lambda^{\ell-1}) \mapsto E_T\left(a(x, d\chi + d\chi); B\right) \in dW^{1,q}_T(B, \Lambda^{\ell-1})$$

satisfies the prerequisites of the Browder-Minty Theorem, see [4, Theorem 2]. In particular, we find $\tau \in W^{1,p}_T(B, \Lambda^{\ell-1})$ satisfying $E_T(\tau; B) = H(\tau; B) = 0$ such that $E_T(a(x, d\chi + d\tau)) = 0$. Setting $h := \chi + \tau$, using the Hodge decomposition in (3.1) and integrating by parts formula, we hence end up with

$$\begin{aligned}\int_B \langle a(x, dh), d\varphi \rangle &= \int_B \langle E_T\left(a(x, dh); B\right), d\varphi \rangle + \int_B \langle E^*_N\left(a(x, dh); B\right), d\varphi \rangle \\ &\quad + \int_B \langle H\left(a(x, dh); B\right), d\varphi \rangle \\ &= \int_B \langle d^* E^*_N\left(a(x, dh); B\right), \varphi \rangle + \int_B \langle d^* H\left(a(x, dh); B\right), \varphi \rangle = 0\end{aligned}$$

for all $\varphi \in C^\infty_T(B, \Lambda^{\ell-1})$ (and thus for all $\varphi \in W^{d,p}_T(B, \Lambda^{\ell-1})$ by approximation). Since $h - \chi \in W^{d,p}_T(B)$, we get

$$\int_B \langle a(x, dh), dh \rangle = \int_B \langle a(x, dh), d\chi \rangle$$

Therefore, employing (4.4) we observe

$$\int_B \phi_\mu(|dh|) \leq c(p, \nu, L) \int_B \phi'_\mu(|dh|)|d\chi|.$$

As a consequence, we observe that if $d\chi = 0$ on B then $dh = 0$ follows and the statement of the lemma is trivially satisfied. Thus, recalling the definition of γ, we may assume $\gamma > 0$. We apply Young's inequality on the right-hand side of the previous inequality and, after absorbing one of the terms on the left-hand side, we get

$$\int_B \phi_\mu(|dh|) \le c(p, \nu, L) \int_B \phi_\mu(|d\chi|)\,, \tag{4.6}$$

where we have also used the fact $\Delta_2(\phi) = c(p)$. Therefore the convexity of ϕ_μ implies

$$\fint_B \phi_\mu(|d\chi - dh|) \le c \fint_B \phi_\mu(|d\chi|)\, dx \le c(p, \nu, L)\, \phi_\mu(\gamma). \tag{4.7}$$

Let $m_0 \in \mathbb{N}$ (to be fixed later). Then, by the Lipschitz truncation from Corollary 4.3 applied to the difference $\chi - h$, we find $\lambda \in [\gamma, 2^{m_0}\gamma]$ such that the Lipschitz truncation $(\chi - h)^\lambda$ satisfies

$$\|d(\chi - h)^\lambda\|_\infty \le c\,\lambda \qquad \text{and} \qquad \fint_B \phi_\mu\left(\lambda\, \mathbb{1}_{\{(\chi-h)^\lambda \neq \chi - h\}}\right) \le \frac{c\,\phi(\gamma)}{m_0} \tag{4.8}$$

with a constant c depending only on n, N and ℓ. We next compute

$$\fint_B \langle a(x, d\chi) - a(x, dh), d(\chi - h)^\lambda \rangle = \fint_B \langle a(x, d\chi), d(\chi - h)^\lambda \rangle$$

and define

$$\begin{aligned}
I :=& \fint_B \langle a(x, d\chi) - a(x, dh), d\chi - dh \rangle\, \mathbb{1}_{\{(\chi-h)^\lambda = \chi - h\}} \\
=& \fint_B \langle a(x, d\chi) - a(x, dh), d(\chi - h)^\lambda \rangle\, \mathbb{1}_{\{(\chi-h)^\lambda = \chi - h\}} \\
=& \fint_B \langle a(x, d\chi), d(\chi - h)^\lambda \rangle \\
& - \fint_B \langle a(x, d\chi) - a(x, dh), d(\chi - h)^\lambda \rangle \mathbb{1}_{\{(\chi-h)^\lambda \neq \chi - h\}} =: II + III.
\end{aligned}$$

Keeping in mind that ϕ_μ satisfies the Δ_2-condition, assumption (4.5), $\lambda \le 2^{m_0}\gamma$, and (4.7) yield

$$\begin{aligned}
\big|II\big| &\le \Big| \fint_B \langle a(x, d\chi), d(\chi - h)^\lambda \rangle \Big| \le \delta\Big(\fint_B \phi_\mu(|d\chi|) + c\,\phi_\mu(2^{m_0}\gamma)\Big) \\
&\le \delta\, c(n, N, \ell)\, c(p, m_0)\, \phi_\mu(\gamma).
\end{aligned}$$

Due to the growth condition (4.4)$_1$ and Young's inequality we get for $\delta_2 > 0$

$$
\begin{aligned}
|III| &\le c \fint_B \big(\phi_\mu'(|d\chi|) + \phi_\mu'(|dh|)\big)\,\lambda\,\mathbb{1}_{\{(\chi-h)^\lambda \neq \chi-h\}} \\
&\le \delta_2 \fint_B \big(\phi_\mu(|d\chi|) + \phi_\mu(|dh|)\big) + c_{\delta_2} \fint_B \phi_\mu(\lambda)\,\mathbb{1}_{\{(\chi-h)^\lambda \neq \chi-h\}} \\
&\le c(n, N, \ell, p, \nu, L)\left(\delta_2 + \frac{c_{\delta_2}}{m_0}\right)\phi_\mu(\gamma).
\end{aligned}
$$

We now apply Jensen's inequality and combine the estimates for II and III with the ellipticity condition to find

$$
\begin{aligned}
\Big(\fint_B |V_\mu(d\chi) - V_\mu(dh)|^{2\theta}\,\chi_{\{(\chi-h)^\lambda = \chi-h\}}\Big)^{\frac{1}{\theta}} &\le \fint_B |V_\mu(d\chi) - V_\mu(dh)|^{2}\,\chi_{\{(\chi-h)^\lambda = \chi-h\}} \\
&\le c(p,\nu)\, I = c(p,\nu)\,(II + III) \\
&\le c\left(\delta\, c(p, m_0) + \delta_2 + \frac{c_{\delta_2}}{m_0}\right)\phi_\mu(\gamma),
\end{aligned}
\tag{4.9}
$$

for a constant c depending only on n, N, ℓ, p, ν and L, and for every $\theta \in (0, 1)$. To finish the proof it still remains to bound the integral of $|V_\mu(d\chi) - V_\mu(dh)|^{2\theta}$ on the remaining set where $(\chi - h)^\lambda \neq \chi - h$. To this aim we define

$$
IV := \Big(\fint_B |V_\mu(d\chi) - V_\mu(dh)|^{2\theta}\,\mathbb{1}_{\{(\chi-h)^\lambda \neq \chi-h\}}\Big)^{\frac{1}{\theta}}.
$$

Then Hölder's inequality, (4.6)–(4.8) and $\gamma \le \lambda$ imply

$$
\begin{aligned}
IV &\le \Big(\fint_B |V_\mu(dh) - V_\mu(d\chi)|^2\Big)\Big(\fint_B \mathbb{1}_{\{(\chi-h)^\lambda \neq \chi-h\}}\Big)^{\frac{1-\theta}{\theta}} \\
&\le c\,\phi_\mu(\gamma)\left(\frac{|\{(\chi-h)^\lambda \neq \chi-h\}|}{|B|}\right)^{\frac{1-\theta}{\theta}} \\
&\le c\,\phi_\mu(\gamma)\left(\frac{\phi_\mu(\lambda)}{\phi_\mu(\gamma)}\,\frac{|\{(\chi-h)^\lambda \neq \chi-h\}|}{|B|}\right)^{\frac{1-\theta}{\theta}} \le c(n, N, \ell, p, \nu, L)\,\phi_\mu(\gamma)\, m_0^{\frac{\theta-1}{\theta}}.
\end{aligned}
$$

A combination of (4.9) with the estimate for (IV) gives

$$
\Big(\fint_B |V_\mu(d\chi) - V_\mu(dh)|^{2\theta}\Big)^{\frac{1}{\theta}} \le c\left(m_0^{\frac{\theta-1}{\theta}} + \delta\, c(p, m_0) + \delta_2 + \tfrac{c_{\delta_2}}{m_0}\right)\phi_\mu(\gamma)
$$

for c still depending only on n, N, ℓ, p, ν and L. Thus for every $\theta \in (0, 1)$ and every $\varepsilon > 0$, we can find first small $\delta_2 > 0$ (depending on these quantities and on ε), second large $m_0 > 0$ (depending additionally on the choice of θ), and third small $\delta > 0$ such that the right hand-side of the previous inequality can be made less than $\varepsilon\phi_\mu(\gamma)$, which by definition of γ proves the claim and thus finishes the proof of the lemma. □

Remark 4.6 It is possible to derive from the previous lemma and a Sobolev-Poincaré inequality (see [11, Theorem 2.3]) other approximation properties of χ by h. For example for given $\varepsilon > 0$ and $\theta \in (0, 1)$ we can choose $\delta > 0$ such that additionally

$$\fint_B \phi_\mu \left(\frac{|\chi - h|}{R} \right) \le \varepsilon \fint_B \phi_\mu(|d\chi|).$$

where R is the radius of the ball B. Hence, we also obtain the version in which the harmonic-type approximation lemmas were originally formulated. To prove this version, the so-called *shift-change* estimate in [8, Corollary 26] is employed, which allows to change from one index of ϕ to another in the following way: for every $\beta > 0$ there exists c_β (depending only on ϕ_μ and β) such that for all $a, b \in \mathbb{R}^k$ and $t \ge 0$ there holds $\phi_{\mu+|a|}(t) \le c_\beta \phi_{\mu+|b|}(t) + \beta |V_\mu(a) - V_\mu(b)|^2$. Since $\phi_{\mu+|a|}(|a-b|) \sim |V_\mu(a) - V_\mu(b)|^2$, and $\phi_\mu(|a|) \sim |V_\mu(a)|^2$, the choice $b = 0$ and $t = |a - b|$ implies

$$\phi_\mu(|a-b|) \le c_\beta |V_\mu(a) - V_\mu(b)|^2 + \beta\, c\, \phi_\mu(|a|).$$

Secondly, we state a suitable version of the $\mathcal{A}$-harmonic approximation lemma for differential forms for both the super- and the subquadratic case. This version is proved by adjusting the proof of [18, Lemma 3.3], [13, Lemma 6] and [38, Lemma 6.8], respectively, to differential forms, or in a similar way as in the proof of the a-harmonic approximation lemma presented above.

Lemma 4.7 **($\mathcal{A}$-harmonic approximation)** *Let $\nu \le L$ be positive constants, $p \in (1, \infty)$ and $\ell \in \{0, \dots, n-1\}$. Then for every $\varepsilon > 0$ there exists a positive number $\delta \in (0, 1]$ depending only on $n, N, \ell, \nu/L$ and ε with the following property: whenever $\mathcal{A}$ is a bilinear form on Λ^ℓ which is elliptic in the sense of Legendre-Hadamard with ellipticity constant ν and upper bound L and whenever $\chi \in W^{d,p}(B_r, \Lambda^{\ell-1})$ is a differential form such that $\fint_{B_r} |V_1(d\chi)|^2 \le \varsigma^2 \le 1$ and such that $d\chi$ is approximately $\mathcal{A}$-harmonic in the sense that*

$$\left| \fint_{B_r} \mathcal{A}(d\chi, d\varphi) \right| \le \varsigma\, \delta \sup_{B_r} |d\varphi| \tag{4.10}$$

holds for all $\varphi \in C^1_T(B_r, \Lambda^{\ell-1})$, then there exists a form $h \in W^{d,p}(B_r, \Lambda^{\ell-1})$, $(h)_T = (\chi)_T$ on ∂B_r, such that dh is $\mathcal{A}$-harmonic and such that it satisfies

$$\sup_{B_{r/2}} |dh| + r \sup_{B_{r/2}} |Ddh| \le c \quad \text{and} \quad \fint_{B_{r/2}} \left| V_1\left(\frac{\chi - \varsigma h}{r} \right) \right|^2 \le \varsigma^2\, \varepsilon.$$

for a constant c depending only on n, N, p, ℓ, ν and L.

5 A Caccioppoli inequality

The first step in proving a regularity theorem for solutions of elliptic systems is to establish a suitable reverse-Poincaré or Caccioppoli-type inequality. In the situation under consideration in this paper, this means that a certain integral of ω on a ball is essentially controlled in terms of a differential form χ on a larger ball, and χ has the property that its exterior derivative $d\chi$ coincides with ω. As emphasized in [14, Sect. 3], the crucial point is that we actually prove a version for perturbations of ω in terms of V_μ, and the number μ depends only on the perturbation (when the perturbation converges to the identity this implies that μ approaches 0).

Hence, this version of the Caccioppoli inequality takes into account the possible degeneracy of the monotonicity condition (H1) (and therefore also of the ellipticity condition).

Lemma 5.1 *Let $p \in (1, \infty)$ and consider a weak solution $\omega \in L^p(B_r(x_0), \Lambda^\ell)$, $r < 1$, to the system (2.1) under the assumptions (H1), (H1) and (H1). Then, for every closed form $\xi \in L^p(B_r(x_0), \Lambda^{\ell-1}) \cap \ker d$ and $\zeta \in \Lambda^\ell$ there holds*

$$\fint_{B_{r/2}(x_0)} \left|V_{|\zeta|}(\omega - \zeta)\right|^2 \le c \fint_{B_r(x_0)} \left|V_{|\zeta|}\left(\frac{\chi - \xi - \zeta \cdot (x - x_0)}{r}\right)\right|^2 + c\,|\zeta|^p\, r^{2\beta} \tag{5.1}$$

for a constant c depending only on p, L and ν and an arbitrary form $\chi \in W^{1,p}(B_r(x_0), \Lambda^{\ell-1})$ satisfying $d\chi = \omega$ on $B_r(x_0)$.

Proof Let $\xi \in L^p(B_r(x_0), \Lambda^{\ell-1}) \cap \ker d$ and $\zeta \in \Lambda^\ell$ be fixed. Without loss of generality we may assume $x_0 = 0$. We choose an arbitrary differential form $\chi \in W^{1,p}(B_r, \Lambda^{\ell-1})$ satisfying $d\chi = \omega$ on $B_r(x_0)$ (which exists according to the Hodge decomposition, meaning that $d\chi$ is the exact projection of ω) and we consider a cut-off function $\eta \in C_0^\infty(B_r, [0, 1])$ such that $\eta \equiv 1$ on $B_{r/2}$ and $|D\eta| \le c/r$. We may take $\eta^p(\chi - \xi - \zeta \cdot x)$ as a test function in (2.1). Hence, since $A(x_0, \zeta)$ is constant, we obtain

$$\begin{aligned}
&\fint_{B_r} \langle A(x, \omega) - A(x, \zeta), \omega - \zeta \rangle\, \eta^p \\
&\quad = -p \fint_{B_r} \langle A(x, \omega) - A(x, \zeta), d\eta \wedge (\chi - \xi - \zeta \cdot x) \rangle\, \eta^{p-1} \\
&\qquad - \fint_{B_r} \langle A(x, \zeta) - A(x_0, \zeta), d\left(\eta^p(\chi - \xi - \zeta \cdot x)\right) \rangle.
\end{aligned} \tag{5.2}$$

Applying (H1) and Lemma 3.5 (applied on the space $(\Lambda^\ell(\mathbb{R}^n))^N$ instead of $\mathbb{R}^k$, this means that the constants will depend on n, N, ℓ instead of k) , we find for the first integral on the right-hand side of the last inequality

$$\begin{aligned}
&\left|p \fint_{B_r} \langle A(x, \omega) - A(x, \zeta), d\eta \wedge (\chi - \xi - \zeta \cdot x) \rangle\, \eta^{p-1}\right| \\
&\le c(p, L) \fint_{B_r} \left(|\zeta|^2 + |\omega|^2\right)^{\frac{p-2}{2}} |\omega - \zeta|\, r^{-1}\, |\chi - \xi - \zeta \cdot x|\eta^{p-1} \\
&\le c(p, L) \fint_{B_r} \left(|\zeta|^2 + |\omega - \zeta|^2\right)^{\frac{p-2}{2}} |\omega - \zeta|\, r^{-1}|\chi - \xi - \zeta \cdot x|\eta^{p-1} \\
&\le \varepsilon \fint_{B_r} \left|V_{|\zeta|}(\omega - \zeta)\right|^2 \eta^p + c(p, L) \fint_{B_r} \varepsilon^{1-p}\left|V_{|\zeta|}\left(\frac{\chi - \xi - \zeta \cdot x}{r}\right)\right|^2
\end{aligned} \tag{5.3}$$

for every $\varepsilon \in (0, 1)$. For the second integral on the right-hand side of (5.2) we infer similarly from (H1) concerning the Hölder continuity of the coefficients $A(\cdot, \cdot)$ with respect to the x-variable that

$$\Big| \fint_{B_r} \langle\, A(x,\zeta) - A(x_0,\zeta), d(\eta^p(\chi - \xi - \zeta\cdot x))\,\rangle \Big|$$

$$\le c(p,L)\, r^\beta \fint_{B_r} |\zeta|^{p-2}|\zeta| \left(|\omega - \zeta|\, \eta^p + \left| \frac{\chi - \xi - \zeta\cdot x}{r} \right| \eta^{p-1} \right)$$

$$\le \varepsilon \fint_{B_r} \big|V_{|\zeta|}(\omega - \zeta)\big|^2 \eta^p + c(p,L) \fint_{B_r} \left| V_{|\zeta|}\left(\frac{\chi - \xi - \zeta\cdot x}{r} \right) \right|^2$$

$$+ c(p,L)\, \varepsilon^{1-p}\, |\zeta|^p\, r^{2\beta}. \tag{5.4}$$

Finally, we estimate the left-hand side of (5.2) using the monotonicity assumption (H1) and we find

$$\fint_{B_r} \langle\, A(x,\omega) - A(x,\zeta), \omega - \zeta\,\rangle\, \eta^p \ge \nu \fint_{B_r} \left(|\zeta|^2 + |\omega|^2\right)^{\frac{p-2}{2}} |\omega - \zeta|^2\, \eta^p$$

$$\ge c(p)\, \nu \fint_{B_r} \big|V_{|\zeta|}(\omega - \zeta)\big|^2 \eta^p. \tag{5.5}$$

Choosing $\varepsilon = c(p)\nu/4$ we may then combine the estimates in (5.3), (5.4) and (5.5) together with (5.2). Keeping in mind the properties of the cut-off function η, we thus arrive at the desired inequality. □

6 Approximate $\mathcal{A}$- and a-harmonicity

Our next aim is to find a framework in which the $\mathcal{A}$-harmonic and the a-harmonic approximation lemma, respectively, can be applied. This means that we have to identify systems for which the smallness conditions in the sense of (4.10) and (4.5) hold true (provided that additional smallness assumptions are satisfied). This shall be accomplished in the non-degenerate case by linearization of the coefficients, whereas in the degenerate case assumption (H1) allows to define a suitable Uhlenbeck-type system.

To start with the non-degenerate case we first recall the definition of the excess: for every ball $B_r(x_0) \subset \mathbb{R}^n$, a fixed function $\omega \in L^p(B_r(x_0), \Lambda^\ell)$, $p \in (1,\infty)$, and every $\omega_0 \in \Lambda^\ell$ the excess of ω is defined via

$$\Phi(x_0, r, \omega_0) := \Phi(\omega; x_0, r, \omega_0) = \fint_{B_r(x_0)} \big|V_{|\omega_0|}(\omega - \omega_0)\big|^2.$$

Lemma 6.1 **(Approximate $\mathcal{A}$-harmonicity)** *Let $p \in (1,\infty)$. There exists a constant c_A depending only on p and L such that for every weak solution $\omega \in L^p(B_r(x_0), \Lambda^\ell)$, $r < 1$, to system (2.1) under the assumptions (H1) and (H1), and every $\omega_0 \in \Lambda^\ell$ such that $|\omega_0| \ne 0 \ne \Phi(x_0, r, \omega_0)$ we have*

$$\Big| \fint_{B_r(x_0)} \langle\, D_\omega A(x_0,\omega_0)\, |\omega_0|^{1-p}\, (\omega - \omega_0), d\varphi\,\rangle \Big|$$

$$\le c_A \left[\left(\frac{\Phi(r)}{|\omega_0|^p} \right)^{\frac{1}{2} + \frac{|p-2|}{2p}} + \left(\frac{\Phi(r)}{|\omega_0|^p} \right)^{\frac{1}{2} + \frac{\alpha}{2}} + r^\beta \right] \sup_{B_r(x_0)} |d\varphi|$$

for all $\varphi \in C^1_T(B_r(x_0), \Lambda^{\ell-1})$. Here we have abbreviated $\Phi(x_0, r, \omega_0)$ by $\Phi(r)$.

Proof In what follows, we assume without loss of generality $x_0 = 0$ and that the test function $\varphi \in C^1_T(B_r, \Lambda^{\ell-1})$ satisfies $\sup_{B_r} |d\varphi| \le 1$. Since $A(0, \omega_0)$ is constant, we first observe

$$\fint_{B_r} \Big\langle \int_0^1 D_\omega A(0, \omega_0 + t(\omega - \omega_0))\, dt\, (\omega - \omega_0) d\varphi \Big\rangle = \fint_{B_r} \langle A(0, \omega) - A(x, \omega), d\varphi \rangle ,$$

and we then obtain

$$\begin{aligned} &\Big| \fint_{B_r} \langle D_\omega A(0, \omega_0)\, (\omega - \omega_0), d\varphi \rangle \Big| \\ &\quad \le \Big| \fint_{B_r} \langle \int_0^1 [D_\omega A(0, \omega_0) - D_\omega A(0, \omega_0 + t(\omega - \omega_0))]\, dt\, (\omega - \omega_0), d\varphi \rangle \Big| \\ &\qquad + \Big| \fint_{B_r} \langle A(0, \omega) - A(x, \omega), d\varphi \rangle \Big| = I + II. \end{aligned} \tag{6.1}$$

We are now going to estimate the two terms on the right-hand side distinguishing the super- and the subquadratic case. We start with $p \ge 2$ and apply assumption (H1) to term I. Since $(p-2-\alpha)/2 > 0$ by assumption, we may take advantage of the convexity of $t \mapsto t^{(p-2-\alpha)/2}$, and in view of Jensen's inequality we then find

$$\begin{aligned} I &\le c(p, L) \fint_{B_r} \left(|\omega_0|^2 + |\omega - \omega_0|^2\right)^{\frac{p-2-\alpha}{2}} |\omega - \omega_0|^{1+\alpha} \\ &\le c(p, L) \fint_{B_r} \left(|\omega - \omega_0|^{p-1} + |\omega_0|^{p-2-\alpha} |\omega - \omega_0|^{1+\alpha}\right) \\ &\le c(p, L) \Big(\fint_{B_r} |\omega - \omega_0|^p \Big)^{\frac{p-1}{p}} + c(p, L)\, |\omega_0|^{\frac{p-2}{2} - \frac{p\alpha}{2}} \Big(\fint_{B_r} |\omega_0|^{p-2} |\omega - \omega_0|^2 \Big)^{\frac{1+\alpha}{2}} \\ &\le c(p, L) \left(\Phi(r)^{\frac{p-1}{p}} + |\omega_0|^{\frac{p-2}{2} - \frac{p\alpha}{2}}\, \Phi(r)^{\frac{1+\alpha}{2}} \right) \end{aligned}$$

where in the last line we have also taken into account the definition of the excess $\Phi(r)$. For the second term in (6.1) we infer from $r < 1$:

$$\begin{aligned} II &\le L \fint_{B_r} |\omega|^{p-1} |x|^\beta \\ &\le c(p, L) \fint_{B_r} |\omega - \omega_0|^{p-1} + c(p, L)\, |\omega_0|^{p-1} r^\beta \\ &\le c(p, L)\, \Phi(r)^{\frac{p-1}{p}} + c(p, L)\, |\omega_0|^{p-1} r^\beta . \end{aligned}$$

In the subquadratic case instead, the application of the assumption (H1) gives

$$I \le L \fint_{B_r} \int_0^1 |\omega_0|^{p-2} |\omega + t(\omega - \omega_0)|^{p-2} \times \left(|\omega_0|^2 + |\omega + t(\omega - \omega_0)|^2\right)^{\frac{2-p-\alpha}{2}} \, dt \, |\omega - \omega_0|^{1+\alpha}.$$

We here note that $\omega_0 \neq 0$ by assumption. Hence, (H1) may be applied formally only for all $t \in [0, 1]$ such that $\omega + t(\omega - \omega_0) \neq 0$. However it is not difficult to see that the latter formula is justified in any case. We next distinguish the cases where $|\omega_0| \ge |\omega - \omega_0|$ or where the opposite inequality holds. Thus, using the technical Lemma 3.6, the growth of the V_μ-function and Jensen's inequality, we see

$$\begin{aligned}
I &\le c(L) \fint_{B_r} |\omega_0|^{-\alpha} \int_0^1 |\omega + t(\omega - \omega_0)|^{p-2} \, dt \, |\omega - \omega_0|^{1+\alpha} \\
&\quad + L \, |B_r|^{-1} \int_{\{x \in B_r : |\omega_0| < |\omega - \omega_0|\}} |\omega_0|^{p-2} \int_0^1 |\omega + t(\omega - \omega_0)|^{-\alpha} \, dt \, |\omega - \omega_0|^{1+\alpha} \\
&\le c(p, L) \fint_{B_r} |\omega_0|^{-\alpha} \left(|\omega_0|^2 + |\omega - \omega_0|^2\right)^{\frac{p-2}{2}} |\omega - \omega_0|^{1+\alpha} \\
&\quad + c(p, L) \, |B_r|^{-1} \int_{\{x \in B_r : |\omega_0| < |\omega - \omega_0|\}} |\omega_0|^{p-2} \left(|\omega_0| + |\omega - \omega_0|\right)^{-\alpha} |\omega - \omega_0|^{1+\alpha} \\
&\le c(p, L) \fint_{B_r} |\omega_0|^{\frac{p-2}{2} - \frac{p\alpha}{2}} \left|V_{|\omega_0|}(\omega - \omega_0)\right|^{1+\alpha} \\
&\quad + c(p, L) \, |B_r|^{-1} \fint_{\{x \in B_r : |\omega_0| < |\omega - \omega_0|\}} |\omega_0|^{p-2} \left|V_{|\omega_0|}(\omega - \omega_0)\right|^{\frac{2}{p}} \\
&\le c(p, L) \left(|\omega_0|^{\frac{p-2}{2} - \frac{p\alpha}{2}} \Phi(r)^{\frac{1+\alpha}{2}} + |\omega_0|^{p-2} \Phi(r)^{\frac{1}{p}}\right).
\end{aligned}$$

For the second integral we proceed similar to the superquadratic case and again distinguish the cases to finally obtain

$$\begin{aligned}
II &\le c(p, L) \, |B_r|^{-1} \int_{\{x \in B_r : |\omega_0| < |\omega - \omega_0|\}} |\omega - \omega_0|^{p-1} + c(p, L) \, |\omega_0|^{p-1} r^\beta \\
&\le c(p, L) \, |B_r|^{-1} \int_{\{x \in B_r : |\omega_0| < |\omega - \omega_0|\}} |\omega_0|^{p-2} \left|V_{|\omega_0|}(\omega - \omega_0)\right|^{\frac{2}{p}} + c(p, L) \, |\omega_0|^{p-1} r^\beta \\
&\le c(p, L) \left(|\omega_0|^{p-2} \Phi(r)^{\frac{1}{p}} + |\omega_0|^{p-1} r^\beta\right).
\end{aligned}$$

Inserting the inequalities for the terms I and II in the super- and subquadratic case in (6.1), we hence end up with

$$\Big| \fint_{B_r} \langle\, D_\omega A(x_0, \omega_0)\, (\omega - \omega_0), d\varphi\, \rangle \Big|$$
$$\le c(p, L)\, \Big(\Phi(r)^{\frac{1}{2} + \frac{|p-2|}{2p}}\, |\omega_0|^{\frac{p-2}{2}}\, |\omega_0|^{-\frac{|p-2|}{2}} + |\omega_0|^{\frac{p-2}{2} - \frac{p\alpha}{2}}\, \Phi(r)^{\frac{1}{2} + \frac{\alpha}{2}} + |\omega_0|^{p-1}\, r^\beta \Big)$$

which, divided by $|\omega_0|^{p-1}$ and after a rescaling argument in order to include general test functions $\varphi \in C^1_T(B_r, \Lambda^{\ell-1})$, immediately yields the desired result. □

To treat the degenerate case where the system is close to a possibly degenerate system of Uhlenbeck structure, we define analogously to [14] the excess via

$$\Psi(x_0, r) = \fint_{B_r(x_0)} |\omega|^p.$$

Then, the structure assumption (H1) allows us to prove the following

Lemma 6.2 (Approximate a**-harmonicity)** *Let* $p \in (1, \infty)$*. There exists a constant* c_H *depending only on* L *such that for every weak solution* $\omega \in L^p(B_r(x_0), \Lambda^\ell)$, $r < 1$, *to system* (2.1) *under the assumptions* (H1), (H1) *and* (H1) *and for every* $t > 0$ *we have*

$$\Big| \fint_{B_r(x_0)} \langle\, \rho_{x_0}(|\omega|)\, \omega, d\varphi\, \rangle \Big| \le c_H \left[t\, \Psi(r)^{\frac{p-1}{p}} + r^\beta\, \Psi(r)^{\frac{p-1}{p}} + \frac{\Psi(r)}{\widetilde{\mu}(t)} \right] \sup_{B_r(x_0)} |d\varphi|$$

for all $\varphi \in C^1_T(B_r(x_0), \Lambda^{\ell-1})$*. Here we have abbreviated* $\Psi(x_0, r)$ *by* $\Psi(r)$*.*

Proof We again assume without loss of generality $x_0 = 0$ and that the test function $\varphi \in C^1_T(B_r, \Lambda^{\ell-1})$ satisfies $\sup_{B_r} |d\varphi| \le 1$. We fix $t > 0$. Since ω is a weak solution to (2.1) we first observe

$$\Big| \fint_{B_r} \langle\, \rho_0(|\omega|)\, \omega, d\varphi\, \rangle \Big| = \Big| \fint_{B_r} \langle\, A(x, \omega) - \rho_0(|\omega|)\, \omega, d\varphi\, \rangle \Big|$$
$$\le \fint_{B_r} \big|A(x, \omega) - A(0, \omega)\big| + \Big| \fint_{B_r} \langle\, A(0, \omega) - \rho_0(|\omega|)\, \omega, d\varphi\, \rangle \Big|. \tag{6.2}$$

Using assumption (H1) on the Hölder continuity of the coefficients $A(\cdot, \cdot)$ with respect to the x-variable and Jensen's inequality, we easily find

$$\fint_{B_r} \big|A(x, \omega) - A(0, \omega)\big| \le L\, r^\beta\, \Psi(r)^{\frac{p-1}{p}}. \tag{6.3}$$

To estimate the second integral on the right-hand side of the previous inequality we now distinguish the cases where $|\omega| \le \widetilde{\mu}(t)$ and where $|\omega| > \widetilde{\mu}(t)$. In the first case, we may apply (H1) and see

$$|B_r|^{-1} \Big| \int_{B_r \cap \{|\omega| \le \widetilde{\mu}(t)\}} \langle\, A(0, \omega) - \rho_0(|\omega|)\, \omega, d\varphi\, \rangle \Big| \le t \fint_{B_r} |\omega|^{p-1} \le t \Big(\fint_{B_r} |\omega|^p \Big)^{\frac{p-1}{p}}.$$

In order to give an estimate for the integral on the remaining set $\{|\omega| > \widetilde{\mu}(t)\}$ we first recall the weak L^p-type estimate stating

$$\big|B_r \cap \{|\omega| > \widetilde{\mu}(t)\}\big| \leq \widetilde{\mu}(t)^{-p} \int_{B_r} |\omega|^p .$$

Thus, we infer from the upper bound (2.2) on the growth of $A(x, \omega)$ and Hölder's inequality that there holds

$$\begin{aligned}
&|B_r|^{-1} \Big| \int_{B_r \cap \{|\omega| > \widetilde{\mu}(t)\}} \langle\, A(0, \omega) - \rho_0(|\omega|)\, \omega, d\varphi \,\rangle \Big| \\
&\leq 2L\, |B_r|^{-1} \int_{B_r \cap \{|\omega| > \widetilde{\mu}(t)\}} |\omega|^{p-1} \\
&\leq 2L\, |B_r|^{-1} \big|B_r \cap \{|\omega| > \widetilde{\mu}(t)\}\big|^{\frac{1}{p}} \Big(\int_{B_r} |\omega|^p \Big)^{\frac{p-1}{p}} \leq \frac{2L}{\widetilde{\mu}(t)} \fint_{B_r} |\omega|^p .
\end{aligned}$$

Merging the previous estimates together, we finally arrive at the inequality

$$\Big| \fint_{B_r} \langle\, A(0, \omega) - \rho_0(|\omega|)\, \omega, d\varphi \,\rangle \Big| \leq t\, (\Psi(r))^{\frac{p-1}{p}} + \frac{2L}{\widetilde{\mu}(t)}\, \Psi(r)\,,$$

where we have used the definition of $\Psi(r)$. In combination with (6.3) the assertion of the lemma follows (after rescaling) immediately from (6.2). □

7 Excess decay estimates

In this section we take advantage of the results of the previous sections and deduce decay estimates for the excess of the solution on different balls in terms of the ratio of the radii. To this aim, the crucial ingredients in the non-degenerate and the degenerate situation—identified by a criterion involving the ratio of the excess to a suitable power of the meanvalue (which of course changes with the radius)—are the a priori estimates available for solutions to linear systems and to Uhlenbeck systems, respectively. In a second step these excess decay estimates have to be iterated. Once the non-degeneracy criterion is satisfied, the iteration proceeds in a standard way, but the criterion for degeneracy might fail as the radius decreases, i. e. at a certain radius the situation might become non-degenerate (and as we will see then remains non-degenerate for all smaller ones), and therefore, the two iterations finally have to be combined in a suitable iteration schemes.

Proposition 7.1 *Let* $p \in (1, \infty)$. *For every* $\beta' \in (0, 1)$ *there exist constants* $\theta = \theta(n, N, \ell, p, \nu, L, \beta') \in (0, 1/4]$, $\varepsilon_0 = \varepsilon_0(n, N, \ell, p, \nu, L, \alpha, \beta') \in (0, 1/2)$ *and* $r_0 = r_0(n, N, \ell, p, \nu, L, \alpha, \beta, \beta') \in (0, 1)$ *such that the following is true: for every weak solution* $\omega \in L^p(B_r(x_0), \Lambda^\ell)$, $r \leq r_0$, *to system* (2.1) *under the assumptions* (H1)–(H1) *which satisfies the smallness condition*

$$\Phi(x_0, r, (\omega)_{x_0,r}) < \varepsilon_0\, |(\omega)_{x_0,r}|^p\,, \tag{7.1}$$

we have the following growth condition:

$$\Phi(x_0, \theta r, (\omega)_{x_0,\theta r}) \leq \tfrac{1}{2}\, \theta^{2\beta'}\, \Phi(x_0, r, (\omega)_{x_0,r}) + c_0\, |(\omega)_{x_0,r}|^p\, (\theta r)^{2\beta}\,, \tag{7.2}$$

and the constant c_0 depends on n, N, ℓ, p, ν, L and β'.

Proof Without loss of generality we take $x_0 = 0$, and we shall further use the abbreviation $\Phi(r) = \Phi(0, r, (\omega)_{0,r})$. Moreover, we assume $\Phi(r) > 0$, otherwise $\Phi(\theta r) = 0$ and the assertion in (7.2) is trivially satisfied. Now let $\varepsilon > 0$ (to be determined later) and choose $\delta \in (0, 1]$ according to the $\mathcal{A}$-harmonic approximation Lemma 4.7. From (7.1) follows $|(\omega)_{0,r}| > 0$. Hence, denoting by $\chi \in W^{1,p}(B_r, \Lambda^{\ell-1}) \cap d^* W^{1,p}(B_r, \Lambda^\ell)$ the differential form which arises from the Hodge decomposition and satisfies $d\chi = \omega$, we define $\widetilde{\chi}$ via

$$\widetilde{\chi} = \frac{\chi - (\omega)_{0,r} \cdot x}{|(\omega)_{0,r}|} \qquad \text{on } B_r.$$

Then, by definition of $\widetilde{\chi}$ and $\Phi(r)$ there holds

$$\fint_{B_r} |V_1(d\widetilde{\chi})|^2 = |\omega_{0,r}|^{-p}\, \Phi(r) \le 1.$$

The approximate $\mathcal{A}$-harmonicity result from Lemma 6.1 further ensures

$$\begin{aligned}
&\Big| \fint_{B_r} \Big\langle \frac{D_\omega A(x_0, (\omega)_{0,r})}{|(\omega)_{0,r}|^{p-2}}, d\widetilde{\chi} \Big\rangle d\varphi \Big| \\
&\quad \le c_A \left(\frac{\Phi(r)}{|(\omega)_{0,r}|^p} + 2\,\delta^{-2}\, c_A^2\, r^{2\beta} \right)^{\frac{1}{2}} \left(\left(\frac{\Phi(r)}{|(\omega)_{0,r}|^p} \right)^{\frac{|p-2|}{p}} + \left(\frac{\Phi(r)}{|(\omega)_{0,r}|^p} \right)^{\alpha} + \frac{\delta^2}{2c_A^2} \right)^{\frac{1}{2}} \\
&\qquad \times \sup_{B_r} |d\varphi| \\
&\quad =: c_A \varsigma \left(\left(\frac{\Phi(r)}{|(\omega)_{0,r}|^p} \right)^{\frac{|p-2|}{p}} + \left(\frac{\Phi(r)}{|(\omega)_{0,r}|^p} \right)^{\alpha} + \frac{\delta^2}{2c_A^2} \right)^{\frac{1}{2}} \sup_{B_r} |d\varphi|
\end{aligned}$$

for all functions $\varphi \in C_T^1(B_r, \Lambda^{\ell-1})$ with the obvious abbreviation for ς. Now we assume

$$\left(\frac{\Phi(r)}{|(\omega)_{0,r}|^p} \right)^{\frac{|p-2|}{p}} + \left(\frac{\Phi(r)}{|(\omega)_{0,r}|^p} \right)^{\alpha} < \frac{\delta^2}{2c_A^2}. \tag{SC.1}$$

Then, provided that r is chosen sufficiently small (in dependency of the parameters c_A and δ) and that consequently ς is bounded from above by 1, we find that $d\widetilde{\chi}$ is approximately $\mathcal{A}$-harmonic with respect to $\mathcal{A} = |(\omega)_{0,r}|^{2-p} D_\omega A(x_0, (\omega)_{0,r})$, and $\mathcal{A}$ is elliptic with ellipticity constant ν and upper bound L, (see (2.4) and (2.3)). Hence, we infer the existence of a differential form $h \in W^{d,2}(B_r, \Lambda^\ell)$ such that dh is $\mathcal{A}$-harmonic and such that the estimates

$$\sup_{B_{r/2}} |dh| + r \sup_{B_{r/2}} |Ddh| \le c(n, N, p, \ell, \nu, L) \qquad \text{and} \qquad \fint_{B_{r/2}} \left| V_1\left(\frac{\widetilde{\chi} - \varsigma h}{r} \right) \right|^2 \le \varsigma^2\, \varepsilon \tag{7.3}$$

are satisfied. From the first inequality we obtain by Taylor expansion

$$\sup_{x\in B_{2\theta r}} |dh(x) - (dh)_{0,2\theta r}| \le (2\theta r) \sup_{B_{r/2}} |Ddh| \le c\,\theta$$

for c depending only on n, N, p, ℓ, ν and L as above. Hence, for $\theta \in (0, 1/4]$ (to be chosen later) we now use Lemma 3.5 on the properties of V_μ, Poincaré's inequality (for its application we denote by $h_0 = E_T(h - (dh)_{0,2\theta r} \cdot x;\, B_{2\theta r}) + H(h - (dh)_{0,2\theta r} \cdot x;\, B_{2\theta r})$ the closed form introduced in Lemmas 3.2) and (7.3). In this way we find

$$\begin{aligned}
&\fint_{B_{2\theta r}} \left|V_1\left(\frac{\widetilde{\chi} - \varsigma h_0 - \varsigma (dh)_{0,2\theta r}\cdot x}{2\theta r}\right)\right|^2 \\
&\le c(p) \fint_{B_{2\theta r}} \left|V_1\left(\frac{\widetilde{\chi} - \varsigma h}{2\theta r}\right)\right|^2 + c(p) \fint_{B_{2\theta r}} \left|V_1\left(\frac{\varsigma(h - h_0 - (dh)_{0,2\theta r}\cdot x)}{2\theta r}\right)\right|^2 \\
&\le c(p)\,\theta^{-n-\max\{2,p\}} \fint_{B_{r/2}} \left|V_1\left(\frac{\widetilde{\chi} - \varsigma h}{r}\right)\right|^2 \\
&\qquad + c(n, N, p) \fint_{B_{2\theta r}} \left(|\varsigma(dh - (dh)_{0,2\theta r})|^2 + |\varsigma(dh - (dh)_{0,2\theta r})|^{\max\{2,p\}}\right) \\
&\le c(p)\,\theta^{-n-\max\{2,p\}}\, \varsigma^2\, \varepsilon + c(n, N, p, \ell, \nu, L)\, \varsigma^2\, \theta^2 \\
&\le c(n, N, p, \ell, \nu, L)\, \varsigma^2 \left(\theta^{-n-\max\{2,p\}}\, \varepsilon + \theta^2\right).
\end{aligned}$$

Setting $\varepsilon = \theta^{n+2+\max\{2,p\}}$ and recalling the definitions of $\widetilde{\chi}$ and ς we hence find the preliminary decay estimate

$$\begin{aligned}
&\fint_{B_{2\theta r}} \left|V_{|(\omega)_{0,r}|}\left(\frac{\chi - (\omega)_{0,r}\cdot x - |(\omega)_{0,r}|\varsigma\,(h_0 + (dh)_{2\theta r}\cdot x)}{2\theta r}\right)\right|^2 \\
&\qquad\qquad \le c\,\theta^2 \left(\Phi(\rho) + \delta^{-2}\,|(\omega)_{0,r}|^p\, r^{2\beta}\right),
\end{aligned} \tag{7.4}$$

and the constant c depends only on n, N, ℓ, p, ν and L. In order to apply the Caccioppoli inequality from Lemma 5.1 we now have to pass from $V_\mu(\cdot)$ in the previous inequality with index $\mu_1 = |(\omega)_{0,r}|$ to a corresponding one with index $\mu_2 = |(\omega)_{0,r} + |(\omega)_{0,r}|\varsigma (dh)_{2\theta r}|$. This can be done if the indices are equivalent up to a constant. Therefore, since $|dh|$ is bounded in $B_{2\theta r}$ by a constant depending only on n, N, ℓ, p, ν and L, we now require an additional smallness condition on ς which guarantees the comparability of μ_1 and μ_2 in the sense that $\mu_1/2 \le \mu_2 \le 3\mu_1/2$. To this end we assume

$$c^2\, \tfrac{\Phi(r)}{|(\omega)_{0,r}|^p} \le \min\left\{\tfrac{1}{8}, \theta^n\right\}, \tag{SC.2}$$

$$c^2\, \delta^{-2}\, c_A^2\, r^{2\beta} \le \tfrac{1}{16} \tag{SC.3}$$

where c (without loss of generality we assume $c \ge 4$) is the constant appearing in (7.3) (the reason for requiring the smallness assumption with respect to θ^{-n} will become clear in the iteration immediately after this lemma). We now apply Lemma 3.7, the Caccioppoli inequality and the decay estimate (7.4) to find

$$
\begin{aligned}
\Phi(\theta r) &= \int_{B_{\theta r}} \left| V_{|(\omega)_{0,\theta r}|} \left(\omega - (\omega)_{0,\theta r}\right) \right|^2 \\
&\leq c(p) \int_{B_{\theta r}} \left| V_{|(\omega)_{0,r} + |(\omega)_{0,r}| \varsigma (dh)_{2\theta r}|} \left(\omega - (\omega)_{0,r} - |(\omega)_{0,r}| \varsigma (dh)_{2\theta r}\right) \right|^2 \\
&\leq c(p, L, \nu) \fint_{B_{2\theta r}} \left| V_{\mu_2} \left(\frac{\chi - (\omega)_{0,r} \cdot x - |(\omega)_{0,r}| \varsigma \left(h_0 + (dh)_{2\theta r} \cdot x\right)}{2\theta r} \right) \right|^2 \\
&\quad + c(p, L, \nu)\, \mu_2^p \, (\theta r)^{2\beta} \\
&\leq c(p, L, \nu) \fint_{B_{2\theta r}} \left| V_{\mu_1} \left(\frac{\chi - (\omega)_{0,r} \cdot x - |(\omega)_{0,r}| \varsigma \left(h_0 + (dh)_{2\theta r} \cdot x\right)}{2\theta r} \right) \right|^2 \\
&\quad + c(p, L, \nu)\, \mu_1^p \, (\theta r)^{2\beta} \\
&\leq c\, \theta^2 \left(\Phi(r) + \delta^{-2} \, |(\omega)_{0,r}|^p \, r^{2\beta} \right) + c\, |(\omega)_{0,r}|^p \, (\theta r)^{2\beta} =: c_1 \, \theta^2 \, \Phi(r) \\
&\quad + c_0 \, |(\omega)_{0,r}|^p \, (\theta r)^{2\beta} \,,
\end{aligned}
$$

and the constants c_1 depends only on n, N, ℓ, p, ν and L, and c_0 depends additionally on θ. Given $\beta' \in (0, 1)$ we now choose $\theta \in (0, 1)$ sufficiently small such that $2c_1\theta^2 \leq \theta^{2\beta'}$. For later purposes we also assume that $2^{\max\{2,p\}}\theta^{2\beta'} < 1$ is fullfilled. Note that this fixes θ in dependency of n, N, ℓ, p, ν, L and β' which in turn determines $\varepsilon = \theta^{n+2+\max\{2,p\}}$ and δ in dependency of the same quantities. Then we infer from the latter inequality the desired excess decay estimate stated in the proposition, provided that the smallness conditions (SC.1), (SC.2) and (SC.3) hold true. Taking into consideration the dependencies in (SC.1), (SC.2) on $\Phi(r)/|(\omega)_{0,r}|^p$, we observe that they are satisfied if $\Phi(r) \leq \varepsilon_0 |(\omega)_{0,r}|^p$ is required for a number ε_0 chosen sufficiently small in dependency of $n, N, \ell, p, \nu, L, \alpha$ and β'. For the iteration we will need an additional smallness condition $3^{p+3} c_0 \, r^{2\beta'} \leq \varepsilon_0$, thus, in view of the dependencies in the smallness condition (SC.3) on the radius, it suffices to choose $r < r_0$ for a number $r_0 > 0$ depending only on $n, N, \ell, p, \nu, L, \alpha, \beta$ and β', and the proof of the proposition is complete. □

Lemma 7.2 *Let $p \in (1, \infty)$, $\beta' \in (0, \beta]$ and $m \geq 1$. Then, with the numbers ε_0 and r_0 defined above, the following is true: for every weak solution $\omega \in L^p(B_R(x_0), \Lambda^\ell)$, $R \leq r_0$, to system* (2.1) *under the assumptions* (H1)–(H1) *which satisfies the smallness conditions*

$$
\Phi(x_0, R, (\omega)_{x_0,R}) < \varepsilon_0 \, |(\omega)_{x_0,R}|^p \quad \textit{and} \quad |(\omega)_{x_0,R}| < 2m \,, \tag{7.5}
$$

we have $|(\omega)_{x_0,r}| < 6m$ and

$$
\Phi(x_0, r, (\omega)_{x_0,r}) \leq c_{it} \left(\left(\frac{r}{R}\right)^{2\beta'} \Phi(x_0, R, (\omega)_{x_0,R}) + r^{2\beta'} \right) \tag{7.6}
$$

for all $r \leq R$, and the constant c_{it} depends only on $n, N, \ell, p, \nu, L, \beta'$ and m.

Proof The assertion follows by a more or less standard iteration procedure. However, for the convenience of the reader we give the main steps and refer to by now classical regularity papers for the details. In the first step one proves that the smallness condition (7.5) implies for every $k \in \mathbb{N}_0$:

(i) $\Phi(x_0, \theta^k R, (\omega)_{x_0,\theta^k R}) \leq 2^{-k} \, \theta^{2\beta' k} \, \Phi(x_0, R, (\omega)_{x_0,R}) + 3^{p+2} \, c_0 \, (\theta^k R)^{2\beta'} \, |(\omega)_{x_0,R}|^p$,

(ii) $\Phi(x_0, \theta^k R, (\omega)_{x_0,\theta^k R}) < \theta^{2\beta' k} \, \varepsilon_0 \, |(\omega)_{x_0,R}|^p$,

(iii) $|(\omega)_{x_0,R}| \leq 2^k \, |(\omega)_{x_0,\theta^k R}|$,

(iv) $\Phi(x_0, \theta^k R, (\omega)_{x_0,\theta^k R}) < \varepsilon_0 \, |(\omega)_{x_0,\theta^k R}|^p$,

(v) $|(\omega)_{x_0,\theta^k R}| \le 3\, |(\omega)_{x_0,R}|$,

and θ, c_0 are the constants appearing in the previous Proposition 7.1. These estimates are established by induction and essentially rely on Proposition 7.1 (sometimes one has to distinguish the sub- and superquadratic case and use Lemma 3.5).

In the second step we then derive a continuous version and consider $r \in (0, R]$ arbitrary. Then there exists a unique number $k \in \mathbb{N}_0$ such that $r \in (\theta^{k+1} R, \theta^k R]$, and exactly as above in (v), we find

$$|(\omega)_{x_0,r}| \le 3\, |(\omega)_{x_0,R}| < 6\, m.$$

Moreover, in view of (i) and Lemma 3.7, we get

$$\begin{aligned}
\Phi(x_0, r, (\omega)_{x_0,r}) &\le \left(\frac{\theta^k R}{r}\right)^n \fint_{B_{\theta^k R}(x_0)} \left|V_{|(\omega)_{x_0,r}|}\left(\omega - (\omega)_{x_0,r}\right)\right|^2 \\
&\le c(p)\, \theta^{-n} \fint_{B_{\theta^k R}(x_0)} \left|V_{|(\omega)_{x_0,\theta^k R}|}\left(\omega - (\omega)_{x_0,\theta^k R}\right)\right|^2 \\
&\le c(p)\, \theta^{-n} \left(2^{-k}\, \theta^{2\beta' k}\, \Phi(x_0, R, (\omega)_{x_0,R}) + 3^{p+2}\, c_0\, (\theta^k R)^{2\beta'}\, |(\omega)_{x_0,R}|^p\right) \\
&\le c_{it} \left(\left(\frac{r}{R}\right)^{2\beta'} \Phi(x_0, R, (\omega)_{x_0,R}) + r^{2\beta'}\right),
\end{aligned}$$

and due to the dependencies of θ we have $c_{it} = c_{it}(n, N, \ell, p, \nu, L, \beta', m)$. This completes the proof of the excess decay estimate (7.6) and thus of the lemma. □

As already mentioned before we derive an excess decay estimate for the degenerate situation where the mean value of ω on a ball $B_R(x_0)$ is "small" with respect to the excess (in some sense this assumption is equivalent to the system being degenerate). Duzaar and Mingione [14] had considered a degeneracy as the p-Laplace system, and they then concluded that approximate p-harmonicity allows to find an excess-decay estimate. We here argue similarly, namely we show that if the system exhibits a degeneracy as a system of Uhlenbeck-structure, then approximate a-harmonicity implies the desired excess-decay estimate. Nevertheless, our proof is slightly different in order to succeed in showing that also in the superquadratic situation one smallness condition on the mean value of ω (instead of an additional second condition on a smaller ball) is sufficient to prove the decay estimate. In what follows, we denote by $\gamma \in (0, 1)$ the exponent from the excess decay estimate in Proposition 3.9 for weak solutions of systems with Uhlenbeck structure (meaning that the weak solution has Hölder exponent $2\gamma/p$ in the superquadratic case and Hölder exponent γ in the subquadratic case).

Proposition 7.3 *Let $p \in (1, \infty)$. For every exponent $\gamma' \in (0, \min\{\gamma, \beta\})$ and every number $\kappa > 0$ there exist constants $\tau \in (0, 1/4]$ and $r_1 < 1$ depending on $n, N, \ell, p, \nu, L, \gamma, \gamma', \beta$ and κ, and a constant $\varepsilon_1 > 0$ depending additionally on $\widetilde{\mu}(\cdot)$ such that the following is true: Let $\omega \in L^p(B_r(x_0), \Lambda^\ell)$, $r \le r_1$, be a weak solution to system (2.1) under the assumptions (H1)–(H1). If*

$$\kappa\, |(\omega)_{x_0,r}|^p \le \Phi(x_0, r, (\omega)_{x_0,r}) < \varepsilon_1 \tag{7.7}$$

is fullfilled, then we have

$$\Phi(x_0, \tau r, (\omega)_{x_0,\tau r}) \le \tau^{2\gamma'}\, \Phi(x_0, r, (\omega)_{x_0,r}). \tag{7.8}$$

Proof We proceed similarly as in the proof of [14, Lemma 12], with some modifications mentioned in [17, Sect. 3.2]. Without loss of generality we take $x_0 = 0$, and we use the abbreviations $\Phi(r) = \Phi(0, r, (\omega)_{0,r})$ and $\Psi(r) = \Psi(0, r)$. From $|(\omega)_{0,r}|^p \leq \kappa^{-1}\Phi(r)$ we see for the superquadratic case $p \geq 2$

$$\Psi(r) \leq 2^{p-1} \fint_{B_r} |\omega - (\omega)_{0,r}|^p + 2^{p-1}\, |(\omega)_{0,r}|^p \leq 2^{p-1}(1+\kappa^{-1})\, \Phi(r),$$

whereas in the subquadratic case $p \in (1, 2)$ we distinguish the cases where $|\omega - (\omega)_{0,r}| \geq |(\omega)_{0,r}|$ and where the opposite inequality holds true, and we obtain

$$\begin{aligned}\Psi(r) &\leq 2^{p-1} \fint_{B_r} |\omega - (\omega)_{0,r}|^p + 2^{p-1}\, |(\omega)_{0,r}|^p \\ &\leq 2^{p-1}\, 2^{\frac{2-p}{2}} \fint_{B_r} \left|V_{|(\omega)_{0,r}|}\left(\omega - (\omega)_{0,r}\right)\right|^2 + 2^p\, |(\omega)_{0,r}|^p \leq 2^p\,(1+\kappa^{-1})\, \Phi(r).\end{aligned}$$

Hence, in any case we get

$$\Psi(r) \leq c_\Psi\, \Phi(r)\,, \tag{7.9}$$

where we have set $c_\Psi = 2^p(1+\kappa^{-1})$. We again denote by $\chi \in W^{1,p}(B_r, \Lambda^{\ell-1})$ a form satisfying $d\chi = \omega$. In view of Lemma 6.2 on approximate a-harmonicity we have for every $t > 0$ and every $\varphi \in C^1_T(B_r, \Lambda^{\ell-1})$:

$$\left| \fint_{B_r} \langle\, \rho_{x_0}(|d\chi|)\, d\chi, d\varphi\,\rangle \right| \leq c_H \left[t\, \Psi(r)^{\frac{p-1}{p}} + r^\beta\, \Psi(r)^{\frac{p-1}{p}} + \frac{\Psi(r)}{\widetilde{\mu}(t)} \right] \sup_{B_r} |d\varphi|$$

Now let $\tau \in (0, 1/4]$ a parameter to be specified later and define $\varepsilon = \tau^{p+\max\{1,p/2\}(n+2\gamma)}$. Furthermore, let $\delta = \delta(n, N, \ell, p, \nu, L, \varepsilon) \in (0, 1]$ be the constant according to the a-harmonic approximation with $\theta = n/(n+p)$: For all assumptions of Lemma 4.5 to be fullfilled it still remains to verify assumption (4.5). For this purpose we fix $t = t(L, \delta) > 0$ and a radius $r_1 = r_1(L, \delta, \beta) > 0$ such that $c_H t \leq \delta/3$ and $c_H\, r_1^\beta \leq \delta/3$, which in turn fixes $\widetilde{\mu}(t)$. If we assume that the smallness condition

$$c_H\, \frac{(c_\Psi\, \Phi(r))^{1/p}}{\widetilde{\mu}(t)} \leq \frac{\delta}{3} \tag{SC.4}$$

holds, then, after application of Young's inequality, χ satisfies all assumption of Lemma 4.5, provided that $r \leq r_1$. Consequently, there exists a differential form $h \in W^{d,p}(B_r, \Lambda^{\ell-1})$, $(h)_T = (\omega)_T$ on ∂B_r, such that dh is a-harmonic for $a(\bar{\omega}) = \rho_0(\bar{\omega})\, \bar{\omega}$ and which satisfies

$$\fint_{B_r} |dh|^p \leq c \fint_{B_r} |d\chi|^p \quad \text{and} \quad \left(\fint_{B_r} |d\chi - dh|^{\frac{np}{n+p}} \right)^{\frac{n+p}{n}} \leq c(n, N, \ell, p)\, \varepsilon \fint_{B_r} |d\chi|^p \tag{7.10}$$

for a constant c depending only on p, ν and L. We here have used the fact that for all possible choices of ρ satisfying the assumptions (G1)–(G1) the statement of Lemma 4.5 holds true with $\mu = 0$ (and hence $V_\mu(\bar{\omega}) = |\bar{\omega}|^{(p-2)/2}\bar{\omega}$) as well as a simple property of the V-function. We now use (7.10), the Sobolev-Poincaré and the Poincaré inequality (note that the first one can by applied immediately by construction of h and for the second one we denote by

$h_0 = E_T(h - (dh)_{0,2\tau r} \cdot x; B_{2\tau r}) + H(h - (dh)_{0,2\tau r} \cdot x; B_{2\tau r})$ the associated closed form on $B_{2\tau r}$). Due to the a priori estimate in Proposition 3.9 and (7.9) we thus find

$$\begin{aligned}
&(2\tau r)^{-p} \fint_{B_{2\tau r}} |\chi - h_0 - (dh)_{0,2\tau r} \cdot x|^p \\
&\le 2^{p-1} \Big[(2\tau r)^{-p} \fint_{B_{2\tau r}} |\chi - h|^p + (2\tau r)^{-p} \fint_{B_{2\tau r}} |h - h_0 - (dh)_{0,2\tau r} \cdot x|^p \Big] \\
&\le c \Big[\tau^{-n-p} \Big(\fint_{B_r} |d\chi - dh|^{\frac{np}{n+p}} \Big)^{\frac{n+p}{n}} + \fint_{B_{2\tau r}} |dh - (dh)_{0,2\tau r}|^p \Big] \\
&\le c \Big[\tau^{-n-p} \, \varepsilon \fint_{B_r} |d\chi|^p + \Phi(h; 2\tau r) \Big] \le c\, \tau^{2\gamma} \, [c_\Psi \, \Phi(r) + \Phi(h; r)] \qquad (7.11)
\end{aligned}$$

with a constant c depending only on n, N, ℓ, p, ν and L. The excess of the a-harmonic approximation h at radius r is estimated from above by a constant times the excess of ω, due to (7.10) and the inequality $|(dh)_{0,r}|^p \le \fint_{B_r} |dh|^p$. Similarly, introducing the abbreviation $P = h_0 + (dh)_{0,2\tau r} \cdot x$ and recalling that h_0 is closed, we infer from Proposition 3.9 the following estimate for $|dP|$ in terms of the excess

$$|dP| = |(dh)_{0,2\tau r}| \le c(n, N, \ell, p, \nu, L)\, (c_\Psi \, \Phi(r))^{1/p}. \qquad (7.12)$$

We now distinguish the cases $p \in (1, 2)$ and $p \ge 2$ in order to find a preliminary decay estimate for χ. In the subquadratic we note that $V_\mu(\xi)$ is decreasing in μ, and we hence arrive at

$$\fint_{B_{2\tau r}} \Big| V_{|dP|} \Big(\frac{\chi - P}{2\tau r} \Big) \Big|^2 \le (2\tau r)^{-p} \fint_{B_{2\tau r}} |\chi - P|^p \le c(n, N, \ell, p, L, \nu)\, c_\Psi \, \tau^{2\gamma} \, \Phi(r).$$

To obtain the analogous estimate in the superquadratic case we observe that due to the previous inequality (7.11), it only remains to estimate the quadratic term similarly to (7.11) via Jensen's inequality:

$$\begin{aligned}
(2\tau r)^{-2} \fint_{B_{2\tau r}} |dP|^{p-2} |\chi - P|^2 &= (2\tau r)^{-2} \fint_{B_{2\tau r}} |(dh)_{0,2\tau r}|^{p-2} |\chi - h_0 - (dh)_{0,2\tau r} \cdot x|^2 \\
&\le 2\, (2\tau r)^{-2} \, |(dh)_{0,2\tau r}|^{p-2} \fint_{B_{2\tau r}} |\chi - h|^2 \\
&\quad + 2\, (2\tau r)^{-2} |(dh)_{0,2\tau r}|^{p-2} \fint_{B_{2\tau r}} |h - h_0 - (dh)_{0,2\tau r} \cdot x|^2 \\
&\le c \Big[\tau^{-n-2} \big(c_\Psi \, \Phi(r) \big)^{\frac{p-2}{p}} \varepsilon^{\frac{2}{p}} \Big(\fint_{B_r} |d\chi|^p \Big)^{\frac{2}{p}} + \Phi(h; 2\tau r) \Big] \\
&\le c\, c_\Psi \, \tau^{2\gamma} \, \Phi(r)
\end{aligned}$$

with c depending only on n, N, ℓ, p, L and ν. Therefore, combining this inequality with the estimate for the integral over $|\chi - P|^p$ and taking into account the subquadratic case, we end up with the following preliminary decay estimate for every $p \in (1, \infty)$:

$$\fint_{B_{2\tau r}} \left| V_{|dP|}\left(\frac{\chi - P}{2\tau r}\right)\right|^2 \le c(n, N, \ell, p, L, \nu)\, c_\Psi\, \tau^{2\gamma}\, \Phi(r).$$

We next apply Lemma 3.7, the Caccioppoli inequality from Lemma 5.1, the previous inequality and the estimate (7.12). We then obtain similarly to the proof of the previous proposition (and recalling $d\chi = \omega$):

$$\begin{aligned}
\Phi(\tau r) &= \int_{B_{\tau r}} \left| V_{|(\omega)_{0,\tau r}|}\left(\omega - (\omega)_{0,\tau r}\right)\right|^2 \\
&\le c(p) \int_{B_{\tau r}} \left| V_{|dP|}\left(\omega - dP\right)\right|^2 \\
&\le c(p, L, \nu) \fint_{B_{2\tau r}} \left| V_{|dP|}\left(\frac{\chi - P}{2\tau r}\right)\right|^2 + c(p, L, \nu)\, |dP|^p\, (2\tau r)^{2\beta} \\
&\le c_3(n, N, \ell, p, L, \nu, \kappa)\, \left(\tau^{2\gamma} + (\tau r)^{2\beta}\right)\, \Phi(r).
\end{aligned} \tag{7.13}$$

For a given exponent $\gamma' \in (0, \min\{\gamma, \beta\})$ we now fix $\tau \in (0, 1/4]$ such that

$$c_3\, \tau^{\min\{2\gamma, 2\beta\}} \le \tau^{2\gamma'}. \tag{SC.5}$$

Hence, τ is fixed in dependency of $n, N, \ell, p, L, \nu, \gamma, \gamma', \beta$ and κ. The choice $\varepsilon = \tau^{p+\max\{1,p/2\}(n+2\gamma)}$ further fixes δ—and therefore also t and the radius r_1—with exactly the same dependencies as those appearing in τ. We now remark that (SC.4) may be rewritten as

$$\Phi(r) \le c_\Psi^{-1} \left(\frac{\delta\, \widetilde{\mu}(t)}{3c_H}\right)^p = 2^{-p}\, \frac{\kappa}{1+\kappa} \left(\frac{\delta\, \widetilde{\mu}(t)}{3c_H}\right)^p.$$

For later purposes, we additionally assume that

$$\Phi(r)^{\frac{1}{p}}\, \frac{\tau^{-n/2}}{1 - \tau^{\gamma'}}\, \kappa^{\frac{p-2}{2p}} + \Phi(r)^{\frac{1}{p}}\, \frac{\tau^{-n/p}}{1 - \tau^{2\gamma'/p}} \le 1. \tag{SC.6}$$

Hence, we observe that these smallness conditions are fullfilled if we choose ε_1 sufficiently small in dependency of the parameters stated in the proposition. This completes the proof. □

Remark We mention that the radius r appears in inequality (7.13) as a factor. Thus, we may replace (SC.5) by the following smallness condition concerning r and τ:

$$c_3\, r^\beta\, \tau^{2\beta} \le \frac{1}{2}\, \tau^{2\beta} \qquad \text{and} \qquad c_3\, \tau^{\frac{2\gamma}{p}} \le \frac{1}{2}\, \tau^{2\beta},$$

where $c_3 = c_3(n, N, \ell, p, L, \nu, \kappa)$. This enables us to state the excess decay estimate also with exponent $\gamma' = \beta$ when $\beta < \gamma$.

Lemma 7.4 (Excess decay) *Let $p \in (1, \infty)$. For every exponent $\gamma' \in (0, \min\{\gamma, \beta\})$ and $m \ge 1$ there exist $\varepsilon_1 = \varepsilon_1(n, N, \ell, p, \nu, L, \gamma, \gamma', \alpha, \beta, \widetilde{\mu}(\cdot)) > 0$ and a radius*

$r_2 = r_2(n, N, \ell, p, \nu, L, \gamma, \gamma', \alpha, \beta) > 0$ *such that the following is true: Let* $\omega \in L^p(B_R(x_0), \Lambda^\ell)$, $R \le r_2$, *be a weak solution to system* (2.1) *under the assumptions* (H1)–(H1). *If the smallness conditions*

$$\Phi(x_0, R, (\omega)_{x_0,R}) < \varepsilon_1 \quad \textit{and} \quad |(\omega)_{x_0,R}| < m \tag{7.14}$$

are fullfilled, then we have

$$\Phi(x_0, r, (\omega)_{x_0,r}) \le c \left(\left(\frac{r}{R}\right)^{2\gamma'} \Phi(x_0, R, (\omega)_{x_0,R}) + r^{2\gamma'} \right) \quad \textit{for all } r \le R\,, \tag{7.15}$$

and c *depends on* $n, N, \ell, p, \nu, L, \gamma, \gamma', \beta, \alpha$ *and* m.

Proof We again take $x_0 = 0$ and use the abbreviation $\Phi(R) = \Phi(0, R, (\omega)_{0,R})$. Let $\gamma' \in (0, \min\{\gamma, \beta\})$, where γ is the exponent from Proposition 3.9, and choose $\beta' = \gamma'$ in Lemma 7.2. This fixes two positive constants

$$\varepsilon_0 = \varepsilon_0(n, N, \ell, p, \nu, L, \alpha, \gamma'),$$
$$r_0 = r_0(n, N, \ell, p, \nu, L, \alpha, \beta, \gamma').$$

Furthermore, we set $\kappa = \varepsilon_0$ and we find from Proposition 7.3 positive constants

$$\tau = \tau(n, N, \ell, p, \nu, L, \gamma, \gamma', \beta, \alpha),$$
$$r_1 = r_1(n, N, \ell, p, \nu, L, \gamma, \gamma', \beta, \alpha),$$
$$\varepsilon_1 = \varepsilon_1(n, N, \ell, p, \nu, L, \gamma, \gamma', \beta, \alpha, \widetilde{\mu}(\cdot)).$$

We define $r_2 := \min\{r_0, r_1\}$. We next observe that (7.14) ensures that the second inequality in the smallness assumption (7.7) required for the application of Proposition 7.3 is satisfied. We introduce the set of natural numbers

$$\mathbb{S} := \left\{ n \in \mathbb{N}_0 \colon \Phi(\tau^n R) \ge \varepsilon_0 \, |(\omega)_{0,\tau^n R}|^p \right\}$$

(we note that due to the different conditions in the excess-decay estimate [14, Proposition 4] we need - in contrast to [14, Lemma 13]—only one condition). In order to prove the desired excess decay estimate we have to distinguish the cases where the mean values of ω is always small (i. e. where the system is purely degenerate) and where the mean value for a certain radius (and then for every smaller radius) dominates the excess of ω:

Case $\mathbb{S} = \mathbb{N}$: By induction we prove for every $k \in \mathbb{N}_0$

$$\Phi(\tau^k R) < \varepsilon_1 \qquad \text{and} \qquad \Phi(\tau^k R) \le \tau^{2k\gamma'} \Phi(R). \tag{7.16}$$

For $k = 0$ these inequalities are trivially satisfied due to (7.14). Now, for a given $k \in \mathbb{N}_0$, we suppose $(7.16)_j$ for $j \in \{0, \ldots, k\}$. In view of $k \in \mathbb{S}$ we may apply Proposition 7.3 on the ball $B_{\tau^k R}$ and we find $\Phi(\tau^{k+1} R) \le \tau^{2\gamma'} \Phi(\tau^k R) \le \tau^{2(k+1)\gamma'} \Phi(R)$. Moreover, $\Phi(\tau^{k+1} R) < \varepsilon_1$ follows from $(7.16)_0$ and $\tau < 1$. This shows that (7.16) is valid for $k + 1$ and therefore, for every $k \in \mathbb{N}_0$. For proving the excess decay estimate (7.15) we first infer from Lemma 3.7

$$\begin{aligned} \Phi(r) &= \fint_{B_r} \left| V_{|(\omega)_{0,r}|} \left(\omega - (\omega)_{0,r}\right) \right|^2 \\ &\le c(p) \left(\frac{r}{R}\right)^{-n} \fint_{B_R} \left| V_{|(\omega)_{0,R}|} \left(\omega - (\omega)_{0,R}\right) \right|^2 = c(p) \left(\frac{r}{R}\right)^{-n} \Phi(R) \end{aligned} \tag{7.17}$$

for all $0 < r \le R$. For a continuous analogue of the decay estimate in (7.16) we consider $r \in (0, R]$ arbitrary. Then there exists a unique $k \in \mathbb{N}$ such that $r \in (\tau^{k+1} R, \tau^k R]$, and using (7.16) and (7.17) we conclude

$$\begin{aligned}\Phi(r) &\le c(p) \left(\frac{r}{\tau^k R}\right)^{-n} \Phi(\tau^k R) \le c(p)\, \tau^{-n}\, \tau^{2k\gamma'}\, \Phi(R) \\ &\le c(p)\, \tau^{-n-2\gamma'} \left(\frac{r}{R}\right)^{2\gamma'} \Phi(R)\,,\end{aligned} \tag{7.18}$$

and the statement of the lemma follows taking into account the dependencies of τ given above.

Case $\mathbb{S} \neq \mathbb{N}$: We define $k_0 := \min \mathbb{N} \setminus \mathbb{S}$. We obtain $\Phi(\tau^{k_0} R) < \varepsilon_0\, |(\omega)_{\tau^{k_0} R}|^p$ by definition of k_0, and the calculations leading to (7.16) reveal

$$\Phi(\tau^k R) \le \tau^{2k\gamma'}\, \Phi(R) \qquad \text{for every } k \le k_0. \tag{7.19}$$

Furthermore, we observe that (7.14) and (7.19) combined with the smallness condition (SC.6) ensure that the mean values of ω remain uniformly bounded in the sense that we have $|(\omega)_{0,\tau^k R}| < 2m$ for every $k \le k_0$: In the subquadratic case this can be seen as follows:

$$\begin{aligned}|(\omega)_{0,\tau^k R}| &\le |(\omega)_{0,R}| + \sum_{j=0}^{k-1} |(\omega)_{0,\tau^j R} - (\omega)_{0,\tau^{j+1} R}| \\ &< m + \tau^{-\frac{n}{p}} \sum_{j=0}^{k-1} \Phi(\tau^j R)^{\frac{1}{p}} + \tau^{-\frac{n}{2}} \sum_{j=0}^{k-1} \Phi(\tau^j R)^{\frac{1}{2}} |(\omega)_{0,\tau^j R}|^{\frac{2-p}{2}} \\ &\le m + \tau^{-\frac{n}{p}} (1-\tau^{2\gamma'/p})^{-1}\, \Phi(R)^{\frac{1}{p}} + \tau^{-\frac{n}{2}} (1-\tau^{\gamma'})^{-1}\, \Phi(R)^{\frac{1}{p}}\, \varepsilon_0^{\frac{p-2}{2p}} \le 2\,m.\end{aligned}$$

In the superquadratic case instead, we proceed analogously (but the third term in the sum does not appear) and get the same result. Hence, the assumptions of Lemma 7.2 are satisfied on the ball $B_{\tau^{k_0} R}$. In view of (7.19) we thus infer for every $r \in (0, \tau^{k_0} R]$

$$\Phi(r) \le c_{it} \left(\left(\frac{r}{\tau^{k_0} R}\right)^{2\gamma'} \Phi(\tau^{k_0} R) + r^{2\gamma'}\right) \le c_{it} \left(\left(\frac{r}{R}\right)^{2\gamma'} \Phi(R) + r^{2\gamma'}\right), \tag{7.20}$$

where c_{it} is the constant from Lemma 7.2 and depends only on $n, N, \ell, p, \nu, L, \gamma'$ and m. To finish the proof of the excess decay estimate (7.15) it still remains to consider radii $r \in (\tau^{k_0} R, R]$, but the assertion is then deduced easily from (7.19) following the line of arguments for the case $\mathbb{S} = \mathbb{N}$. Exactly as in the proof of the excess decay result stated in [14], the integer k_0 (which cannot be controlled and which depends on the point x_0 under consideration) is not reflected in the dependencies of the constant c appearing in (7.15). □

Remark As mentioned in the introduction, the proof presented here simplifies slightly the one of [14, Lemma 13]. The key point here is the definition of the set $\mathbb{S}$ which was previously defined in a way such that the condition was required to hold on two subsequent balls (see the different smallness assumptions (7.7) and [14, (5.25)] in the excess decay estimate).

8 Proofs of the main results

We finally come to the proof of the partial regularity results and the dimension reduction stated in Theorems 2.1 and 2.2.

Proof (of Theorem 2.1) We consider an arbitrary point $x_0 \in \Omega_0(\omega)$. Then, denoting by r_2 the radius from Lemma 7.4, we find $m \geq 1$ and $R \in (0, r_2)$ such that $B_R(x_0) \subset \Omega$, $\Phi(x_0, R, (\omega)_{x_0,R}) < \varepsilon_1$ and $|(\omega)_{x_0,R}| < m$, i.e. such that the assumptions (7.14) of Lemma 7.4 are fullfilled. Since (7.14) is an open condition and since the functions $x \mapsto (\omega)_{x,R}, x \mapsto \Phi(x, R, (\omega)_{x,R})$ are continuous, we observe that (7.14) is satisfied in a small neighborhood $B_s(x_0)$ of x_0. Hence, due to the equivalence of the excess $\Phi(x, R, (\omega)_{x,R})$ and the one given in (2.5), the excess decay estimate (7.15) and Campanato's characterization of Hölder continuous functions imply the local Hölder continuity of $V_0(\omega)$, from which in turn the local Hölder continuity of ω is obtained via [15, Lemma 3]. Finally, $|\Omega \setminus \Omega_0(\omega)| = 0$ follows from Lebesgue's differentiation theorem.

We now consider $x_0 \in \Omega_0(\omega)$ such that additionally the assumption (2.6) is satisfied. Then we choose ε_0 and r_0 according to Lemma 7.2 (with $\beta' = \beta$ and an appropriate number $m \geq 1$). We observe that (2.6) guarantees that the assumptions in (7.5) are fullfilled for x_0. Since this is also an open condition we find a small neighborhood $B_s(x_0)$ of x_0 such that it is satisfied for all $y \in B_s(x_0)$, and therefore, we end up with the decay estimate (7.6) for all $y \in B_s(x_0)$. Consequently, Campanato's characterization of Hölder continuous functions yields that $V_0(\omega)$ is locally Hölder continuous with exponent β, which implies that ω is Hölder continuous with exponent $\min\{\beta, 2\beta/p\}$. Moreover, if $\omega(x_0) \neq 0$, this result may still be improved in the superquadratic case: since ω is already continuous, we may assume $|(\omega)_{y,R}| \neq 0$ in $B_s(x_0)$ (after possibly choosing s smaller if necessary), and we conclude that the excess $\Phi(y, R, (\omega)_{y,R})$ is dominated by the quadratic term for every $y \in B_s(x_0)$. This immediately yields the improved local Hölder regularity result with exponent β and finishes the proof. □

We shall now address the estimate on the Hausdorff dimension for the singular set stated in Theorem 2.2. To this aim we proceed as Mingione in [35,34] and differentiate the system in a fractional sense. With this reasoning we first come up with the following fractional differentiability result which states that weak solutions actually belongs to a suitable fractional Sobolev space (these spaces can be viewed as interpolation spaces between Lebesgue spaces and the classical Sobolev spaces of integer order; for the relevant definitions we refer to [3, Chapter 7]).

Lemma 8.1 *Let $p \in (1, \infty)$ and consider a weak solution $\omega \in L^p(B_R(x_0), \Lambda^\ell)$ to the system (2.1) under the assumptions (H1), (H1) and (H1). Then we have $V(\omega) \in W^{\beta',2}_{\mathrm{loc}}(B_R(x_0), \Lambda^\ell)$ for all $\beta' < \beta$.*

Proof We start by proving an estimate for finite differences of $V(\omega)$. To this aim we denote by $\chi \in W^{1,p}(B_R(x_0), \Lambda^{\ell-1}) \cap d^* W^{1,p}(B_R(x_0), \Lambda^\ell)$ the differential form from the Hodge decomposition which satisfies $d\chi = \omega$. Furthermore, consider $B_r(y) \subset B_R(x_0)$ and let $\eta \in C_0^\infty(B_{3r/4}(y), [0, 1])$ be a cut-off function with $\eta \equiv 1$ on $B_{r/2}(y)$ and $|D\eta| \leq c/r$. We then introduce the finite difference operator $\tau_{s,h}$ via $\tau_{s,h}\tau(x) := \tau(x + he_s) - \tau(x)$ for an arbitrary form τ, every real number $h \in \mathbb{R}$ and $s \in \{1, \ldots, n\}$. Analogously as in [35, proof of Proposition 3.1] we then choose $\tau_{s,-h}(\eta^2 \tau_{s,h}\chi)$ with $s \in \{1, \ldots, n\}$ and h sufficiently small as a test function in the weak formulation of (2.1). Then taking into account the assumptions (H1), (H1) and (H1) it follows

$$\fint_{B_{r/2}(y)} |\tau_{s,h} V(\omega)|^2 \leq c\, |h|^{2\beta} \fint_{B_r(y)} |V(\omega)|^2$$

for all $s \in \{1, \dots, n\}$ and a constant c depending only on p, L, ν and r (but independently of h). Due to the uniform estimate in h and the fact that $B_r(y) \subset B_R(x_0)$ was chosen arbitrarily, the assertion then follows from [3, 7.73]. □

Proof (of Theorem 2.2) As a consequence of a measure density result going back to Giusti, see [35, Sect. 4], the previous lemma implies that the singular set of every weak solution ω to (2.1) is actually not only of Lebesgue measure zero, but that its Hausdorff dimension is not greater than $n - 2\beta$. This finishes the dimension reduction. □

9 Modifications for inhomogeneous systems

In this section we briefly describe the modifications which are necessary in order to treat inhomogeneous systems of the form

$$d^* A(\,\cdot\,, \omega) = B(\,\cdot\,, \omega) \quad \text{and} \quad d\omega = 0 \tag{9.1}$$

on a bounded domain Ω, where the coefficients satisfy the assumptions (H1)–(H1) and where the inhomogeneity $B\colon \Omega \times \Lambda^\ell \to \mathbb{R}^N$ satisfies for all $x \in \Omega$ and $\omega \in \Lambda^\ell$

(H6) a controllable growth condition, i. e. $|B(x, \omega)| \le L\,(1 + |\omega|^2)^{\frac{p-1}{2}}$.

The regularity proof now has to be adapted slightly. For convenience of the reader we here collect the major changes. Firstly, an additional term of the form

$$\fint_{B_r} \langle\, B(x, \omega), \chi - \xi - \zeta \cdot x \,\rangle\, \eta^p \le c(p) \fint_{B_r} \left(1 + |\omega|^2 + |\omega - \zeta|^2\right)^{\frac{p-1}{2}} \left|\frac{\chi - \xi - \zeta \cdot x}{r}\right| r\, \eta^p$$

appears in the proof of Caccioppoli inequality in Lemma 5.1. Distinguishing the cases where $|\omega - \zeta|^2 \ge 1 + |\zeta|^2$ and where the opposite inequality holds, it is then easy to deduce from the technical Lemma 3.5 that the statement continues to hold with $|\zeta|^p$ replaced by $(1 + |\zeta|)^p$ on the right-hand side of (5.1).

Next, the statements and proofs of the approximate $\mathcal{A}$- and the approximate a-harmonicity (see Lemmas 6.1 and 6.2, respectively) have to be adjusted. According to Remark 3.3, we may first pass from an arbitrary test function $\varphi \in C^1_T(B_r(x_0), \Lambda^{\ell-1})$ to the related function $\varphi - \varphi_0 \in W^{1,p}_T(B_r(x_0), \Lambda^{\ell-1})$, where φ_0 is a closed $(\ell - 1)$-form satisfying $(\varphi_0)_T = 0$. We observe that $\varphi - \varphi_0$ is also admissible as a test function. Thus, in inequalities (6.1) and (6.2), respectively, there appears an additional term arising from the inhomogeneity:

$$III := \left| \fint_{B_r(x_0)} \langle\, B(x, \omega), \varphi - \varphi_0 \,\rangle \right|.$$

Using Jensen's and Poincaré's inequality we gain one r-power and we find

$$III \le \left[(1 + |\omega_0|)^{p-1} + \Phi(r)^{\frac{p-1}{p}} \right] r \sup_{B_r(x_0)} |d\varphi|$$

Then, for $\omega_0 \in \Lambda^\ell$ such that $|\omega_0| \ne 0 \ne \Phi(x_0, r, \omega_0)$ we obtain approximate $\mathcal{A}$-harmonicity, and for $\omega_0 = 0$ (and hence $\Phi(r) = \Psi(r)$) we get instead approximate a-harmonicity, with the following obvious modifications: in the statement of Lemma 6.1 the term r^β has to be replaced by $r^\beta + |\omega_0|^{1-p} r$, whereas in the one of Lemma 6.2 there appears on the right-hand side $c_H r \sup_{B_r(x_0)} |d\varphi|$ as additional term.

In Proposition 7.1 we now require a smallness assumption stronger than (7.1), namely

$$\varepsilon_0\, r^{\min\left\{\beta,\, \frac{p(1-\beta)}{(p-1)}\right\}} + \Phi(x_0, r, (\omega)_{x_0,r}) < \varepsilon_0\, |(\omega)_{x_0,r}|^p\,,$$

which helps to get in the position to control the (possibly diverging) term $|\omega_0|^{2-p} r$ from above. The proof of the proposition is then done as before, and the only change is that again $|\omega_0|$ on the right-hand side of the assertion (7.2) is replaced by $1+|\omega_0|$. Accordingly, changing the assumption (7.5) in the iteration and requiring some further smallness conditions on θ, namely

$$2^{\max\{2,p\}} \theta^{\min\left\{\beta',\, \frac{p(1-\beta)}{(p-1)}\right\}} < 1\,,$$

we may iterate in the non-degenerate situation similarly to before in Lemma 7.2, and as an intermediate result we still obtain estimate (7.6). Accordingly, the smallness assumption (7.7) in Proposition 7.3 has to be replaced by

$$\kappa\, |(\omega)_{x_0,r}|^p \leq \Phi(x_0, r, (\omega)_{x_0,r}) + \kappa\, r^{\min\left\{\beta,\, \frac{p(1-\beta)}{(p-1)}\right\}}\,,$$

and its proof changes slightly as follows. We first note that $\Psi(r) \leq c_\Psi \Phi(r) + c(p) r^{\min\{\beta, p(1-\beta)/(p-1)\}}$. The approximate a-harmonicity condition is then verified as before, ending up with nearly the same assertion, but finding now instead of (7.8) the estimate

$$\Phi(x_0, \tau r, (\omega)_{x_0,\tau r}) \leq 2^{-1}\, \tau^{2\gamma'}\, \Phi(x_0, r, (\omega)_{x_0,r}) + \widetilde{c}_0\, (\tau r)^{\min\{\beta, p(1-\beta)/(p-1)\}}.$$

Assuming an additional smallness assumption on r_1 (in dependency of ε_1 which results in an additional dependency of $\mu(\cdot)$ for r_1), an excess decay estimate—which combines the non-degenerate and the degenerate situation—can be proved as in Lemma 7.4 (with the obvious modification of the set $\mathbb{S}$). It should be noted that as a final regularity result we again obtain partial regularity, and the Hölder exponent in regular points with (2.6) is optimal, whereas in the remaining points the Hölder exponent might be smaller as in the corresponding homogeneous situation, due to the radius-related term $r^{\min\{\beta, p(1-\beta)/(p-1)\}}$ appearing on the right-hand side of the previous inequality.

With these modifications we then obtain the extension of Theorem 2.1 to weak solutions of inhomogeneous systems. Furthermore, establishing a variant of Lemma 8.1 for the inhomogeneous case (with $\beta' < \min\{\beta, p/(2p-2)\}$), we also achieve a dimension reduction of the singular set. The final result can then be stated as follows:

Theorem 9.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $p \in (1,\infty)$ and consider a weak solution $\omega \in L^p(\Omega, \Lambda^\ell)$ to the inhomogeneous system (9.1) under the assumptions (H1)–(H6). Then there exists $\sigma = \sigma(n, N, p, \ell, L, \nu, \beta)$ and an open subset $\Omega_0(\omega) \subset \Omega$ such that*

$$\omega \in C^{0,\sigma}_{\mathrm{loc}}(\Omega_0, \Lambda^\ell) \quad \textit{and} \quad \dim_{\mathcal{H}}(\Omega \setminus \Omega_0(\omega)) \leq n - \min\left\{2\beta, \frac{p}{p-1}\right\}.$$

Acknowledgements L.B. was financially supported by GNAMPA junior visiting program 2008. She greatly appreciated the warm and friendly atmosphere at the Department of Mathematics of Universitá degli Studi di Napoli "Federico II" during her stay. B.S. was supported by MIUR/Cofin "Calcolo delle Variazioni e Teoria geometrica della Misura" 2008. She would like to thank the pleasant hospitality of the Mathematical Institute of the University of Erlangen-Nürnberg where part of this project was carried out.

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