Compactness Results for $\mathcal{H}-$holomorphic Curves in Symplectizations

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2. A. Doicu, U. Fuchs, *A Compactness Result for $\mathcal{H}-$holomorphic Curves in Symplectizations*
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Zu Nr. 1 Der Artikel fasst die Ergebnisse aus Appendix B, Appendix E und Appendix F der vorliegenden Arbeit zusammen. Mein eigener Anteil besteht aus Appendix B und Appendix E. Dr. Urs Fuchs gab hilfreiche Ideen und Komentare zu Appendix F.

Zu Nr. 2 Der Artikel setzt sich aus den Ergebnisse aus Kapitel 2 Teil I, Teil II sowie Appendix A, Appendix C und Appendix D der vorliegenden Arbeit zusammen. Mein eigener Anteil besteht aus Kapitel 2 Teil I, Appendix A, Appendix C, Appendix D, Kapitel 4 sowie Abschnitt 3.1. Dr. Urs Fuchs gab hilfreiche Ideen und Komentare zu Abschnitt 3.2.

Abstract

In [12] it is suggested that due to topological reasons, a suitable modification of the holomorphic curve equation is crucial for proving Weinstein conjecture in dimension three. In this regard, instead of the usual pseudoholomorphic curves, the following $\mathcal{H}-$*holomorphic curves* (here $\mathcal{H}$ stands for "harmonic") are considered. For a closed contact co-oriented $3-$manifold (M, α), where α is the contact form, a closed Riemann surface (S, j) with complex structure j, and a finite subset $\mathcal{P} \subset S$, a smooth map $u = (a, f) : S \backslash \mathcal{P} \to \mathbb{R} \times M$ is called a $\mathcal{H}-$holomorphic curve if

$$\begin{aligned} \pi_\alpha df \circ j &= J(f) \circ \pi_\alpha df, \\ f^*\alpha \circ j &= da + \gamma, \end{aligned}$$

holds, where $\pi_\alpha : TM \to \xi$ is the projection along the Reeb vector field X_α to the contact structure $\xi = \ker(\alpha)$, J is a $d\alpha-$compatible almost complex structure on ξ, and γ is a harmonic $1-$form on S with respect to the complex structure j, i.e. $d\gamma = d(\gamma \circ j) = 0$. Moreover, it is assumed that the energy of u, defined by

$$E(u, S\backslash\mathcal{P}) := \sup_{\varphi \in \mathcal{A}} \int_{\dot{S}} \varphi'(a) da \circ j \wedge da + \int_{\dot{S}} f^* d\alpha$$

is finite, where $\mathcal{A} = \{\varphi : \mathbb{R} \to [0, 1] \mid \varphi'(r) \geqslant 0, \forall r \in \mathbb{R}\}$. In [3], the proof of Weinstein conjecture in dimension three is reduced to a compactness problem of certain moduli spaces for the $\mathcal{H}-$holomorphic curve equation. The aim of the thesis is to analyze the compactness properties of the space of $\mathcal{H}-$holomorphic curves. As a matter of fact, we give a positive answer the following question. Given a sequence of $\mathcal{H}-$holomorphic curves $(u_n, S_n, j_n, \mathcal{P}_n, \gamma_n)$ with the properties:

- the cardinality of the set of punctures $\mathcal{P}_n$ and the genus of S_n is constant;
- the L^2-norm of γ_n, defined by
$$\|\gamma_n\|^2_{L^2(S)} := \int_{S_n} \gamma_n \circ j_n \wedge \gamma_n,$$
is uniformly bounded by a constant $C_0 > 0$;
- the energies $E(u_n; \dot{S})$ are uniformly bounded by a constant $E_0 > 0$;

is it possible to derive a notion of convergence and to describe the limit object? It should be pointed out that the classical convergence results of Symplectic Field Theory (SFT) established in [6] and [7] cannot be applied here; both versions rely on the monotonicity lemma, a result which is unknown for $\mathcal{H}-$holomorphic curves.

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Contents

Part I

Basic notions and main results

Chapter 1

Introduction

Let M be a closed, connected, $3-$dimensional manifold and let α be a $1-$form on M such that (M, α) is a contact manifold. Denote by X_α the Reeb vector field with respect to the contact form α on M. There is a major interest in describing the orbit structure of the dynamical system

$$\dot{x} = X_\alpha(x). \tag{1.0.1}$$

In general, this is a very hard problem, and in particular, the question on the existence of periodic orbits is relevant. A very influential conjecture on the existence of periodic orbits is due to A. Weinstein [22].

Conjecture 1. *(Weinstein conjecture) Every Reeb vector field X_α on a closed connected $3-$dimensional contact manifold (M, α) admits a periodic orbit.*

Actually, the Weinstein conjecture which is formulated for contact manifolds of arbitrary odd dimension, was proven by Taubes in dimension three [19]. There is however a strong version of the Weinstein conjecture [3], which is still an open problem. To solve it, one is hoping to apply pseudoholomorphic curve techniques.

Conjecture 2. *(Strong Weinstein conjecture) For every Reeb vector field X_α on a closed connected $3-$dimensional contact manifold (M, α), there exists finitely many periodic orbits $x_i : \mathbb{R}/T_i\mathbb{Z} \to M$ of period $T_i > 0$, for $i = 1, ..., n$, so that*

$$\sum_{i=1}^{n} [x_i] = 0,$$

where $[x_i]$ is the first homology class represented by the loop x_i.

An interesting feature of the Weinstein conjecture or the strong Weinstein conjecture is that it is closely related to pseudoholomorphic curve theory for contact manifolds. Let us make this more precise. Denote by $\xi = \ker(\alpha)$ the contact structure and let $\pi_\alpha : TM \to \xi$ be the canonical projection along the Reeb vector field X_α. Furthermore, choose $J : \xi \to \xi$ as a $d\alpha-$compatible almost complex structure. Denote by $\bar{J}$ the extension of J to a $\mathbb{R}-$invariant almost complex structure on $\mathbb{R} \times M$ by mapping $1 \in T\mathbb{R}$ to X_α and X_α to $-1 \in T\mathbb{R}$. Let (S, j) be a closed Riemann surface and denote by $\mathcal{P} \subset S$ a finite subset whose elements are called "punctures". The following definition is due to Hofer in [12].

Definition 3. A proper map $u = (a, f) : S \backslash \mathcal{P} \to \mathbb{R} \times M$ is called *pseudoholomorphic* if

$$\bar{J}(u) \circ d\bar{u} = d\bar{u} \circ j \text{ on } S\backslash\mathcal{P} \tag{1.0.2}$$

and

$$\int_{S\setminus\mathcal{P}} f^* d\alpha < \infty$$

is satisfied.

Remark 4. We have the following.

1. By projecting onto the contact structure through π_α, the pseudoholomorphic curve equation (1.0.2) can be written as
$$\begin{aligned} \pi_\alpha df \circ j &= J(u) \circ \pi_\alpha df \\ f^*\alpha \circ j &= da \end{aligned} \tag{1.0.3}$$
From the second equation of (1.0.3) it is apparent that $f^*\alpha \circ j$ defines the trivial cohomology class.

2. The quantity
$$\int_{S\setminus\mathcal{P}} f^* d\alpha$$
will be referred to as the $d\alpha$–energy and denoted by $E_{d\alpha}(u; S\setminus\mathcal{P})$. By local computation it can be shown that the integrand $f^*d\alpha$ is non-negative.

3. If $\mathcal{P} \neq \emptyset$ then the function a of a pseudoholomorphic curve $u = (a, f)$ is unbounded in a neighborhood of each puncture from $p \in \mathcal{P}$. To prove the unboundedness, assume that U is a closed neighborhood of p in $S\setminus\mathcal{P}$ such that $a(U) \subset [-K, K]$ for some $K > 0$. Because $u|_U$ is proper, $(u|_U)^{-1}([-K, K] \times M) = U$ has to be compact which is a contradiction. In this case, the function a tends either to $+\infty$ or $-\infty$ in a neighborhood of a puncture $p \in \mathcal{P}$. To show this, assume that this is not the case. Then there exists a point $p \in \mathcal{P}$ and two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in $S\setminus\mathcal{P}$ with the properties: $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = p$, $\lim_{n\to\infty} a(x_n) = \infty$, and $\lim_{n\to\infty} a(y_n) = -\infty$. By continuity, there exists a sequence of points $p_n \in S\setminus\mathcal{P}$ such that $p_n \to p$ and $a(p_n) = 0$ for all $n \in \mathbb{N}$, while by properness, $u^{-1}(\{0\} \times M) = a^{-1}(0)$ is a compact subset of $S\setminus\mathcal{P}$; this is a contradiction to the fact that $p_n \in a^{-1}(0)$ and $p_n \to p \in \mathcal{P}$. As a result, the set $\mathcal{P}$ can be written as $\mathcal{P} = \overline{\mathcal{P}} \amalg \underline{\mathcal{P}}$, where $\overline{\mathcal{P}}$ is the subset of punctures $p \in \mathcal{P}$ at which the function a tends to $+\infty$ in a neighborhood of p and $\underline{\mathcal{P}}$ is the subset of punctures at which a tends to $-\infty$.

4. For a non-constant pseudoholomorphic curve u, the set of punctures $\mathcal{P}$ is not empty. Assume that $\mathcal{P} = \emptyset$. Then by Stokes theorem, the $d\alpha$–energy is zero, and so, the image of f lies in a Reeb trajectory. By the maximum principle, the a coordinate is constant, and we have that $f(p) = x(h(p))$ for $p \in S$. Here, x is a Reeb trajectory, and $h : S \to S^1$ if x is periodic and $h : S \to \mathbb{R}$ if x is not periodic; in both cases, $dh = 0$. By local computation it follows that h has to be constant. Hence u is constant and we are led to a contradiction.

5. If u is a pseudoholomorphic curve and $\mathcal{P} \neq \emptyset$, then u is non-constant.

6. From the properness condition of Definition 3, the Hofer energy E_H of a pseudoholomorphic curve u, defined by
$$E_H(u; S\setminus\mathcal{P}) = \sup_{\varphi\in\mathcal{A}} \int_{S\setminus\mathcal{P}} u^* d(\varphi\alpha), \tag{1.0.4}$$
is finite, i.e.
$$E_H(u; S\setminus\mathcal{P}) < +\infty.$$
Here, the set $\mathcal{A}$ consists of all smooth maps $\varphi : \mathbb{R} \to [0, 1]$ with $\varphi'(r) \geqslant 0$ for all $r \in \mathbb{R}$. To prove this assertion we express the Hofer energy as
$$E_H(u; S\setminus\mathcal{P}) = \sup_{\varphi\in\mathcal{A}} \int_{S\setminus\mathcal{P}} u^* d(\varphi\alpha) = \sup_{\varphi\in\mathcal{A}} \left[\int_{S\setminus\mathcal{P}} \varphi'(a) da \circ j \wedge da + \int_{S\setminus\mathcal{P}} \varphi(a) f^* d\alpha\right], \tag{1.0.5}$$

and note that $\varphi'(a)da \circ j \wedge da$ is non-negative. Since the function φ is bounded by 1 and the $d\alpha$—energy is bounded, the term

$$\int_{S\setminus\mathcal{P}} \varphi(a) f^* d\alpha$$

is bounded. What is left to show is that

$$\sup_{\varphi\in\mathcal{A}} \int_{S\setminus\mathcal{P}} \varphi'(a) da \circ j \wedge da$$

is bounded. To prove this result we employ the same arguments as in Lemma 5.15 from [6]. More precisely, for a function $\varphi \in \mathcal{A}$ we compute

$$\int_{S\setminus\mathcal{P}} \varphi'(a) da \circ j \wedge da = \sum_{p\in\mathcal{P}} \lim_{n\to\infty} \int_{\partial D_{\frac{1}{n}}(p)} \varphi(a) f^*\alpha - \int_{S\setminus\mathcal{P}} \varphi(a) f^* d\alpha. \tag{1.0.6}$$

The second term on the right hand side of (1.0.6) is bounded since the $d\alpha$—energy of u is bounded. To estimate the first term on the right-hand side of (1.0.6) we proceed as follows. We assume that the function φ has the asymptotic $\varphi(r) \to c_\pm \in [0,1]$ as $r \to \pm\infty$. Let $M_{n,p} := a(D_{1/n}(p)) \subset \mathbb{R}$. For $p \in \overline{\mathcal{P}}$ set $r_{n,p} := \inf(M_{n,p})$, for $p \in \underline{\mathcal{P}}$ set $r_{n,p} := \sup(M_{n,p})$, and define accordingly $r_n^+ := \min_{p\in\overline{\mathcal{P}}}(r_{n,p})$ and $r_n^- := \max_{p\in\underline{\mathcal{P}}}(r_{n,p})$. Obviously, from the properness condition of Definition 3, $r_n^\pm \to \pm\infty$ as $n \to \infty$. Define now the following sequence of functions

$$\varphi_n(r) = \begin{cases} \varphi(r_n^+) & , r \geqslant r_n^+ \\ \varphi(r) & , r \in (r_n^-, r_n^+) \\ \varphi(r_n^-) & , r \leqslant r_n^- \end{cases}.$$

At the points $r_n^\pm$ we make this function smooth and still denote it by φ_n. Since $\varphi'(r) \geqslant 0$ for all $r \in \mathbb{R}$, we have $\varphi_n(r) \leqslant \varphi(r)$ for all $r \in \mathbb{R}$. Furthermore, for every $r \in \mathbb{R}$,

$$|\varphi(r) - \varphi_n(r)| \leqslant \epsilon_n,$$

where

$$\epsilon_n = \max\{|c^+ - \varphi(r_n^+)|, |c^- - \varphi(r_n^-)|\}.$$

Obviously, $\epsilon_n \to 0$ as $n \to \infty$, and so,

$$\begin{aligned} \int_{S\setminus\mathcal{P}} \varphi_n'(a) da \circ j \wedge da &= \sum_{p\in\mathcal{P}} \lim_{n\to\infty} \int_{\partial D_{\frac{1}{n}}(p)} \varphi_n(a) f^*\alpha - \int_{S\setminus\mathcal{P}} \varphi_n(a) f^* d\alpha \\ &= \sum_{p\in\overline{\mathcal{P}}} \lim_{n\to\infty} \int_{\partial D_{\frac{1}{n}}(p)} \varphi_n(a) f^*\alpha + \sum_{p\in\underline{\mathcal{P}}} \lim_{n\to\infty} \int_{\partial D_{\frac{1}{n}}(p)} \varphi_n(a) f^*\alpha \\ &\quad - \int_{S\setminus\mathcal{P}} \varphi_n(a) f^* d\alpha \\ &= \varphi(r_n^+) \sum_{p\in\overline{\mathcal{P}}} \lim_{n\to\infty} \int_{\partial D_{\frac{1}{n}}(p)} f^*\alpha + \varphi(r_n^-) \sum_{p\in\underline{\mathcal{P}}} \lim_{n\to\infty} \int_{\partial D_{\frac{1}{n}}(p)} f^*\alpha \\ &\quad - \int_{S\setminus\mathcal{P}} \varphi_n(a) f^* d\alpha. \end{aligned}$$

Moreover, by means of Stokes theorem,

$$\left|\int_{\partial D_{\frac{1}{n}}(p)} f^*\alpha\right| \leqslant \int_{S\backslash\mathcal{P}} f^*\alpha + \left|\int_{\partial D_1(p)} f^*\alpha\right|$$

for all $n \in \mathbb{N}$ and all $p \in \mathcal{P}$. Hence,

$$\int_{S\backslash\mathcal{P}} \varphi_n'(a)da \circ j \wedge da < \infty$$

for all $n \in \mathbb{N}$. Since φ_n' is a monotone sequence converging pointwise to φ', and the quantity $da \circ j \wedge da$ is non-negative, the monotone convergence theorem gives

$$\int_{S\backslash\mathcal{P}} \varphi_n'(a)da \circ j \wedge da \to \int_{S\backslash\mathcal{P}} \varphi'(a)da \circ j \wedge da$$

as $n \to \infty$.

The next result which is due to Hofer [11] shows that the Weinstein conjecture is equivalent to the existence of a non-constant pseudoholomorphic curve. For this reason, throughout this thesis, we assume that all periodic orbits are non-degenerate. This means that for every periodic orbit x of period T, the linear map $d\phi_T^\alpha(x(0)) : \xi_{x(0)} \to \xi_{x(T)}$ does not contain 1 in its spectrum.

Theorem 5. *For the closed,* $3-$*dimensional contact manifold* (M, α)*, the associated Reeb vector field* X_α *has a periodic orbit if and only if the nonlinear partial differential equation (1.0.2) has a non-constant solution of finite Hofer energy.*

Having a solution u of (1.0.2) with finite Hofer energy, a periodic orbit of the Reeb vector field X_α can be obtained by investigating the local behavior of u in a neighborhood of a puncture. In this regard, it has been shown that a non-constant solution of (1.0.2) with finite Hofer energy is asymptotic to a periodic orbit of the Reeb vector field X_α in a neighborhood of a puncture [13]. To explain how periodic orbits of X_α are related to pseudoholomorphic curves, let $u : S\backslash\mathcal{P} \to \mathbb{R} \times M$ be a pseudoholomorphic curve in the sense of Definition 3 and let $p \in \mathcal{P}$. A sufficiently small neighborhood of p in $S\backslash\mathcal{P}$ can be biholomorphically identified with $[0, \infty) \times S^1$ with respect to the standard complex structure i. Then there exists a periodic orbit x of period $|T| \neq 0$ of X_α, such that

$$\lim_{s\to\infty} f(s,t) = x(Tt), \text{ and } \lim_{s\to\infty} \frac{a(s,t)}{s} = T \text{ in } C^\infty(S^1),$$

where (s, t) are the coordinates on $[0, \infty) \times S^1$. It should be pointed out that the assumption on a non-empty set of punctures is essential for the existence of a non-constant solution of the nonlinear partial differential equation (1.0.2) and so of the existence of a periodic orbit of the Reeb vector field X_α.

There is one obvious question which should be addressed. Why does one replace the problem dealing with the behavior of an ordinary differential equation (i.e. finding periodic orbits) by the apparently much more sophisticated question about the existence of a certain solution for a nonlinear first order elliptic partial differential equation? The reason is the following. Due to Darboux theorem in the contact setting, periodic orbits of the Reeb vector field X_α are not completely contained in such a Darboux chart. Thus the reason for the existence of periodic orbits of the Reeb vector field X_α has to be global and linked with the topology of the manifold M and the Reeb condition of the vector field X_α. For the moment it is very promising to study the orbit structure of the dynamical system as described in (1.0.1) or more precisely the Weinstein conjecture in dimension three, with pseudoholomorphic curve methods. The pseudoholomorphic curve problem exhibits an enormous amount of structure and helps to view the Weinstein conjecture from a global point of view. So far, a proof of the Weinstein conjecture with pseudoholomorphic curve techniques is unknown. In the following we will sketch a strategy suggested by Hofer [12] and developed further by Abbas et al. [3].

In [12] Hofer suggested an interesting modification of equation (1.0.3), which depends on the genus of the domain; in the case of genus 0, the old equation is obtained. In this modified version, $f^*\alpha \circ j$ does not represent the trivial cohomology class, but rather some non-trivial cohomology class. Hence $f^*\alpha \circ j$ in the second equation of (1.0.3) can be replaced by

$$d(f^*\alpha \circ j) = 0.$$

It turns out that if we insist on keeping the specific behavior of the pseudoholomorphic curve near the punctures (which is essential for the existence of a periodic orbit of X_α) we have to require that the cohomology class of $f^*\alpha \circ j$ is trivial on a punctured neighborhood of each puncture. Thus in a neighborhood of each puncture we can still write $f^*\alpha \circ j = da$. As in [12] we will call (a, f) a local lift of f in a neighborhood of a puncture. This additional cohomology condition can be formulated as

$$[f^*\alpha \circ j] \in \tau^* H^1(S; \mathbb{R}),$$

where $\tau : S\backslash\mathcal{P} \hookrightarrow S$ is the inclusion. Hence we have replaced the second equation $f^*\alpha \circ j = da$ of (1.0.3) by the two requirements $d(f^*\alpha \circ j) = 0$ and $[f^*\alpha \circ j] \in \tau^* H^1(S; \mathbb{R})$. We point out that these two conditions do not involve the $\mathbb{R}-$coordinate a from $u = (a, f)$. Summing up, the modification of the partial differential equation (1.0.3) are

Definition 6. A smooth map $f : S\backslash\mathcal{P} \to M$ is called $\mathcal{H}-$*holomorphic* if

$$\text{the map } f \text{ is non-constant;} \tag{1.0.7}$$
$$\pi_\alpha df \circ j = J(u) \circ \pi_\alpha df \text{ on } S\backslash\mathcal{P}; \tag{1.0.8}$$
$$d(f^*\alpha \circ j) = 0 \text{ on } S\backslash\mathcal{P}; \tag{1.0.9}$$
$$[f^*\alpha \circ j] \in \tau^* H^1(S; \mathbb{R}); \tag{1.0.10}$$
$$\text{near each puncture a local lift } (a, f) \text{ is proper;} \tag{1.0.11}$$
$$\int_{S\backslash\mathcal{P}} f^* d\alpha < \infty. \tag{1.0.12}$$

Note that if S is a Riemann sphere (of genus 0) we have $H^1(S; \mathbb{R}) = 0$, and so, these equations are equivalent to the old ones and the local analysis of such a solution remains the same. Let us describe an equivalent definition of this modified pseudoholomorphic curve equation which is much more usable and will be used throughout this thesis. Conditions (1.0.9) and (1.0.10) imply that

$$[f^*\alpha \circ j] = \tau^*[\psi]$$

for a specific $[\psi] \in H^1(S; \mathbb{R})$. Here ψ is a closed $1-$form on S, and due to the Hodge theorem, which states that $H^1(S; \mathbb{R}) \cong \mathcal{H}^1_j(S)$ where $\mathcal{H}^1_j(S)$ is the vector space of harmonic $1-$forms with respect to the complex structure j on S, we can assume ψ to be a harmonic $1-$form on S. Hence we obtain $[f^*\alpha \circ j] = [\tau^*\psi]$, where $\tau^*\psi$ is a harmonic $1-$form on $S\backslash\mathcal{P}$. Consequently, there exists a function $a : S\backslash\mathcal{P} \to \mathbb{R}$ which is unique up to addition by a constant such that

$$f^*\alpha \circ j = da + \tau^*\psi \text{ on } S\backslash\mathcal{P},$$

where $\tau^*\psi$ is a harmonic $1-$form. In this regard, the following definition makes sense.

Definition 7. A smooth and proper map $u = (a, f) : S\backslash\mathcal{P} \to \mathbb{R} \times M$ with a bounded Hofer energy ($E_H(u; S\backslash\mathcal{P}) < +\infty$) is called $\mathcal{H}-$*holomorphic* if it satisfies the equations

$$\left.\begin{aligned} \pi_\alpha df \circ j &= J(u) \circ \pi_\alpha df \\ f^*\alpha \circ j &= da + \gamma \end{aligned}\right. \quad \text{on } S\backslash\mathcal{P} \tag{1.0.13}$$

for a harmonic $1-$form $\gamma \in \mathcal{H}^1_j(S)$.

Remark 8. From the above discussion it is apparent that equations (1.0.8)–(1.0.10) imply (1.0.13). Conversely, every smooth map $u = (a, f) : S\backslash\mathcal{P} \to \mathbb{R} \times M$ satisfying $f^*\alpha \circ j = da + \gamma$ for a harmonic 1−form $\gamma \in \mathcal{H}^1_j(S)$ also satisfies the conditions $d(f^*\alpha \circ j) = 0$ and $[f^*\alpha \circ j] \in \tau^*H^1(S;\mathbb{R})$. Thus, conditions (1.0.7)–(1.0.11) are equivalent to (1.0.13). It is also obvious that condition (1.0.11) is equivalent to the properness condition of Definition 7. The boundedness of the energy in Definitions 6 and 7 are equivalent. In the case $\mathcal{P} = \emptyset$ this is evident. For $\mathcal{P} \neq \emptyset$ this can be seen as follows. Assume that $f : S\backslash\mathcal{P} \to M$ is a $\mathcal{H}$−holomorphic curve in the sense of Definition 6. Applying the above procedure we obtain a smooth, proper map $u = (a, f) : S\backslash\mathcal{P} \to \mathbb{R} \times M$ satisfying equations (1.0.13) and having a finite $d\alpha$−energy. To prove that the Hofer energy of u is bounded, we argue as in Remark 4. A general representation for the Hofer energy of $\mathcal{H}$−holomorphic curves is

$$E_H(u; S\backslash\mathcal{P}) = \sup_{\varphi\in\mathcal{A}} \left[\int_{S\backslash\mathcal{P}} \varphi'(a)da \circ j \wedge da - \sum_{p\in\mathcal{P}} \lim_{r\to 0} \int_{\partial D_r(p)} \varphi(a)\gamma \circ j + \int_{S\backslash\mathcal{P}} \varphi(a)f^*d\alpha \right]. \tag{1.0.14}$$

Obviously, (1.0.14) is similar to (1.0.5) for usual pseudoholomorphic curves, excepting the term

$$\sum_{p\in\mathcal{P}} \lim_{r\to 0} \int_{\partial D_r(p)} \varphi(a)\gamma \circ j.$$

However, even in the case of $\mathcal{H}$−holomorphic curves, this additional term vanishes; from

$$\left| \int_{\partial D_r(0)} \varphi(a)\gamma \circ j \right| \leqslant \int_{\partial D_r(0)} |\varphi(a)||\gamma \circ j| \leqslant d(r) \, \|\gamma \circ j\|_{C^0(S)} \, ,$$

where $d(r)$ is the circumference of $\partial D_r(p)$ with respect to some Riemannian metric on S, and the fact that $d(r) \to 0$ as $r \to 0$ (the Riemannian metric is defined over the set of punctures $\mathcal{P}$), the conclusion readily follows. Hence, the Hofer energy of $\mathcal{H}$−holomorphic curves can also be computed by means of (1.0.5). As a result, the energy condition (1.0.12) implies the boundedness of the Hofer energy from Definition 7. Conversely, the boundedness of the Hofer energy trivially implies the boundedness of the $d\alpha$−energy. In the case $\mathcal{P} \neq \emptyset$ we deduce using Definition 7 that f is non-constant. Indeed, if f is constant, the Hofer energy vanishes, and from

$$0 = \sup_{\varphi\in\mathcal{A}} \int_{S\backslash\mathcal{P}} \varphi'(a)da \circ j \wedge da$$

we get $da = 0$; thus, a is constant. Consequently, u is constant, and so, the properness property is contradicted. Hence for $\mathcal{P} \neq \emptyset$, Definitions 6 and 7 are equivalent.

In our treatment, the $\mathcal{H}$−holomorphic curves are defined as in Definition 7. Note that the second equation of (1.0.13) has the same form as the old pseudoholomorphic curve equation up to addition by an element from $\mathcal{H}^1_j(S) \cong \mathbb{R}^{2g}$, where g is the genus of the Riemann surface (S, j). Therefore, such solutions are called $\mathcal{H}$−holomorphic curves ($\mathcal{H}$ standing for harmonic).

The modified pseudoholomorphic curve equation plays an important role in [3] and in particular, in [1]. In [3] the authors initiated a program of proving the general Weinstein conjecture in dimension three with methods of symplectic geometry, or more precisely with pseudoholomorphic curve techniques. Essentially, they reduced the proof of the general Weinstein conjecture to a compactness problem of the moduli space of solutions of the $\mathcal{H}$−holomorphic curve equation. One of the main tools in [3] is based on the so-called *Abbas solutions*, which have been constructed in [1]. Here the use of the $\mathcal{H}$−holomorphic curve equation is essential. To understand the main motivation for the use of the $\mathcal{H}$−holomorphic curve equation we explain briefly the main results of [1], and how the Abbas' solutions fit in the context of [3]. In this way the motivation of the $\mathcal{H}$−holomorphic curve equation will become apparent. We begin with some relevant definitions of [1].

Definition 9. (Open Book Decompositions) Assume $K \subset M$ is a link in M and that $\tau : M\backslash K \to S^1$ is a fibration so that the fibers $F_\vartheta := \tau^{-1}(\vartheta)$ are interiors of compact embedded surfaces $\overline{F_\vartheta}$ with boundary $\partial\overline{F_\vartheta} = K$, where ϑ is the coordinate along K. We also assume that K has a tubular neighborhood $K \times D$, $D \subset \mathbb{R}^2$ being the open unit disk, such that τ restricted to $K \times (D\backslash\{0\})$ is given by $\tau(\vartheta, r, \phi) = \phi$, where (r, ϕ) are polar coordinates on D. Then we call τ an *open book decomposition* of M, the link K is called the *binding* of the open book decomposition, and the surfaces F_ϑ are called the *pages* of the open book decomposition.

It is known that every closed, 3−dimensional, orientable manifold admits an open book decomposition. In particular, the notion of an open book decomposition on a contact, 3−dimensional manifold can be connected to the contact data.

Definition 10. (Supporting Open Book Decomposition) If (M, α) is a closed, 3−dimensional contact manifold and τ an open book decomposition with binding K we say that τ supports the contact structure $\xi = \ker(\alpha)$ if there exists a contact form α' representing the same contact structure as α so that $d\alpha'$ induces an area-form on each fiber F_ϑ with K consisting of closed orbits of the Reeb vector field X_α, and α' orients K as the boundary of $(F_\vartheta, d\alpha')$.

The above contact form α' will be referred to as the *Giroux contact form*. Every co-oriented contact, 3−manifold (M, α) is supported by some open book [10]. Now we will state the main result of [1].

Theorem 11. *Let (M, α) be a closed 3−dimensional contact manifold. Then there exists a contact form $\alpha' = f\alpha$ on M, where $f : M \to \mathbb{R}$ is a smooth positive function such that the following holds. There exists a smooth family $(S, j_\tau, \mathcal{P}_\tau, u_\tau = (a_\tau, f_\tau), \gamma_\tau)_{\tau\in S^1}$ of solutions of (1.0.13) for a suitable compatible complex structure $J : \ker(\alpha') \to \ker(\alpha')$ such that*

1. *all maps f_τ have the same asymptotic limit K at the punctures, where K is a finite union of periodic orbits of the Reeb vector field X_α;*
2. *for $\tau \neq \tau'$, $f_\tau(\dot{S}) \cap f_{\tau'}(\dot{S}) = \emptyset$;*
3. *$M\backslash K = \coprod_{\tau\in S^1} f_\tau(\dot{S})$;*
4. *the projection P onto S^1 defined by $p \in f_\tau(\dot{S}) \mapsto \tau$ is a fibration;*
5. *the open book decomposition given by (P, K) supports the contact structure $\ker(\alpha')$, and α' is a Giroux contact form.*

Practically, Abbas constructed a supporting open book decomposition whose pages are images of solutions of the $\mathcal{H}$−holomorphic curve equation. His construction is as follows. Starting with a supporting open book decomposition for the closed 3−dimensional contact manifold (M, α), which is possible due to Giroux [10], a Giroux contact form, which has a certain normal form near the binding, is constructed. By an argument established first by Chris Wendl in [21] and [20], the Giroux leaves are transformed to pseudoholomorphic curves by taking into account that one has a confoliation form ($\alpha \wedge d\alpha \geqslant 0$) instead of a contact form. Picking one Giroux leaf as starting point, a result which enables to perturb the Giroux leaf into a $\mathcal{H}$−holomorphic curve, while at the same time transforming the confoliation form into a contact form, is established. At this step the harmonic perturbation 1−form in the equation (1.0.13) plays an essential role. Actually, a 1−dimensional local family of solutions of the $\mathcal{H}$−holomorphic curve equation (and not just one) is constructed. Let us describe this step in more detail. Starting with a Giroux leaf which is a solution of the pseudoholomorphic curve equation, the problem of finding a local 1−dimensional family of leaves, which are solutions of the $\mathcal{H}$−holomorphic curve equation, is transformed into a transversality issue of a certain elliptic perturbed Cauchy-Riemann type operator and whose perturbation is a compact operator determined by the harmonic perturbation 1−form. This transformation is achieved by using the flow of the Reeb vector field in a similar way as we do in Appendix B. Having on honest transversality result, the index of the linearization of this operator has to be positive. If S has a genus different from 0, the index of this operator without considering the harmonic perturbation is $2-2g$; hence if $g \geqslant 1$, the index is non-positive and a transversality result cannot be established. By adding the harmonic perturbation, the index of the unperturbed linearized Cauchy-Riemann type operator changes by adding $\dim(\mathcal{H}^1_j(S)) = 2g$. Thus its index is 2, and by dividing out the $\mathbb{R}-$action in the first coordinate of $\mathbb{R}\times M$, the transversality theorem enables the construction of a 1−dimensional family of $\mathcal{H}-$holomorphic curves. At this stage, the $\mathcal{H}-$holomorphic curve equation plays an essential role. As a final step, a compactness result which extends the local 1−dimensional family of $\mathcal{H}-$holomorphic curves into a global S^1-family is proved; this in turn will serve as the foliation of the open book. The S^1-family of solutions is referred as Abbas solutions [3].

We explain now the use of Abbas solutions for proving the general Weinstein conjecture in the program described in [3]. Here, the generalized Weinstein conjecture is proved for a planar contact structure, i.e. when the pages of the open book decomposition have genus 0, using the classical SFT compactness result. The main idea of proving the general Weinstein conjecture is the following. Starting with a closed contact 3−dimensional manifold (M,α), a cobordism between α and the Giroux contact form α' is introduced. For the Giroux contact form α', Abbas solutions can be constructed following the guidelines above. By the local behaviour near punctures, we know that the $\mathcal{H}-$holomorphic curves are asymptotic to Reeb orbits; thus the generalized Weinstein conjecture for α' readily follows. In the next step, the cobordism and the classical SFT compactness result is used to deform the Abbas solutions into $\mathcal{H}-$holomorphic curves with respect to the initial contact form α. If a compactness result for $\mathcal{H}-$holomorphic curves is established, the program can be adapted to prove the generalized Weinstein conjecture for genus different that 0. In this thesis we describe a compactification of the moduli space of finite energy $\mathcal{H}-$holomorphic curves. However, we are only able to do this under certain conditions.

In the case of vanishing harmonic perturbation 1−form, there exists a canonical SFT compactness result which was established in [6], and in parallel, in [7]. Even though these works describe almost the same result, the techniques are different. In the following we sketch both techniques.

- In [6], the proof is based on the Deligne-Mumford convergence of stable Riemann surfaces, bubbling-off analysis, and the results of Hofer et al. [14]. First, the concept of a pseudoholomorphic building, which serves as the compactification of the moduli space of pseudoholomorphic curves in symplectizations, is introduced. Let us sketch this concept, while for a detailed analysis, we refer to [6] and [2]. By the behavior of pseudoholomorphic curves $u=(a,f): S\backslash\mathcal{P}\to\mathbb{R}\times M$ in a neighborhood of the punctures $\mathcal{P}$, i.e. its asymptotic, the set of punctures $\mathcal{P}$ can be divided into two disjoint subsets. One subset $\overline{\mathcal{P}}$ consists of positive punctures which correspond to positive asymptotics of u, and the other subset $\underline{\mathcal{P}}$ consists of negative punctures which correspond to negative asymptotics of u. To the punctured surface $S\backslash\mathcal{P}$, a compact surface with boundary $S^{\mathcal{P}}$ can be associated as follows. The compact surface with boundary $S^{\mathcal{P}}$ is obtained by blowing-up the punctures. Roughly speaking, a circle is attached to the corresponding puncture. The boundary Γ of $S^{\mathcal{P}}$ consists of a finite disjoint union of circles that can be divided into positive $\overline{\Gamma}$ and negative $\underline{\Gamma}$ boundary components corresponding to the charge of the blow-up. By the asymptotic behavior of u near the punctures, f can be continously extended to $S^{\mathcal{P}}$. This surface is referred to as the blow-up surface. Additionally, a finite number of pairs of points are chosen on the punctured surface $\mathcal{D}=\{d_1',d_1'',...,d_k',d_k''\}\subset S^{\mathcal{P}}$ and the pairs $d_i'\sim d_i''$ for $i=1,...,k$ are identified. The set $\mathcal{D}$ is called the set of *nodes* and the identified pair $d_i'\sim d_i''$ is called a *node*. The pseudoholomorphic curve u is called a *pseudoholomorphic building of height* 1 if in addition, $u(d_i')=u(d_i'')$ for all $i=1,...,k$. Hence a *pseudoholomorphic building of height* N is a collection of N nodal pseudoholomorphic buildings of height 1, such that the $j-$th pseudoholomorphic curve corresponds at the negative punctures to the same Reeb orbits, while the $(j-1)-$th pseudoholomorphic curve corresponds at the positive punctures. Note that the extended $M-$components at the blow-up surface of each nodal pseudoholomorphic building of height 1 glue togehter at the boundary circles according to their correspondence; hence a continous map is obtained. The main result of [6] is the following. Starting with a sequence of pseudoholomorphic curves with uniformly bounded Hofer energies, there exists a subsequence that converges in a certain way to a nodal pseudoholomorphic building of height N for some $N \in \mathbb{N}$. Here, the notion of convergence is defined on two subsets. Essentially, on the thick part, due to the Thick-Thin decomposition [2], the sequence of pseudoholomorphic curves is required to converge in C^∞_{loc} , while on the thin part, the results of [14] are used to describe a C^∞_{loc}, as well as a C^0 convergence. The idea of the proof is now the following. Using bubbling-off analysis, uniform gradient bounds are derived on the thick part of the surface, and so, elliptic regularity and application of Arzelà-Ascoli theorem yield a $C^\infty_{loc}-$convergence result on the thick part. The convergence on components of the thin part, which by methods of hyperbolic geometry are conformaly equivalent to cusps or hyperbolic cylinders, is performed using essentially the results of [14].

- While the analysis of the compactness in [6] is performed on the domain, the technique in [7] is different. Although the definition of the pseudoholomorphic buildings and the notion of the convergence are the same, the bubbling-off analysis is not performed on the domain; instead, the images $u_n(S_n\backslash\mathcal{P}_n)$ of the pseudoholomorphic curves in the symplectization $\mathbb{R}\times M$ are considered. These are divided into the so-called *essential regions* which can be regarded as compact manifolds with fixed boundaries, and *cylindrical regions* which are like the components of the thin part and are conformaly equivalent to long hyperbolic cylinders. The compactness on each of these components is then proved, and the results are "glued" together to obtain a global convergence result. The convergence of the essential regions is established by the Gromov convergence with free boundary, which essentially is the same as the Gromov convergence theorem for pseudoholomorphic curves. For cylindrical components, the result of [14] is used to prove convergence.

In the following we briefly describe the strategy which is used to derive a notion of compactness in the $\mathcal{H}-$holomorphic curve setting. The integrand of the Hofer energy for a $\mathcal{H}-$holomorphic curve is not always non-negative. This is a first difference to the classical SFT compactness. In order to have an honest version of the energy we slightly change the Hofer energy in order to make the integrands positive. For a $\mathcal{H}-$holomorphic curve $u=(a,f):S\backslash\mathcal{P}\rightarrow\mathbb{R}\times M$, defined on a punctured closed Riemann surface $S\backslash\mathcal{P}$, where $\mathcal{P}\subset S$ is the set of punctures, we define the energy of u as

$$E(u;S\backslash\mathcal{P})=\sup_{\varphi\in\mathcal{A}}\int_{S\backslash\mathcal{P}}\varphi'(a)da\circ j\wedge da+\int_{S\backslash\mathcal{P}}f^*d\alpha. \tag{1.0.15}$$

In the analysis of compactness for $\mathcal{H}-$holomorphic curves we will use (1.0.15) as the notion of energy instead the Hofer energy. Arguing as in Remark 8, it can be shown that

$$E(u;S\backslash\mathcal{P})=E_H(u;S\backslash\mathcal{P}). \tag{1.0.16}$$

However, if we restrict the domain of integration on subsets of $S\backslash\mathcal{P}$, then in general, the Hofer energy is different from the energy defined by (1.0.15). Also note that the integrand of the Hofer energy, when restricted to subsets of $S\backslash\mathcal{P}$, can be negative, wheras the integrand of the energy defined by (1.0.15) is non-negative. The first term in (1.0.15) is called the $\alpha-$*energy* of u on $S\backslash\mathcal{P}$ and will be denoted by $E_\alpha(u;S\backslash\mathcal{P})$, while the second term is called the $d\alpha-$*energy* of u on $S\backslash\mathcal{P}$ and will be denoted by $E_{d\alpha}(u;S\backslash\mathcal{P})$. Since u is $\mathcal{H}-$holomorphic, by straightforward calculation it can be shown that the integrands of the $\alpha-$ and $d\alpha-$energies are non-negative. It should be pointed out that in the case of pseudoholomorphic curves, (1.0.16) holds even on subsets of $S\backslash\mathcal{P}$. For the harmonic perturbation $1-$form γ of a $\mathcal{H}-$holomorphic curve defined on a Riemann surface (S,j), we define the L^2-norm of γ with respect to the complex structure j by

$$\|\gamma\|^2_{L^2(S)}=\int_S\gamma\circ j\wedge\gamma. \tag{1.0.17}$$

This quantity depends only on the complex structure j and the topology of the underlying surface S. In addition, for every isotopy class $[c]$ which is represented by a smooth loop c the *period* and *co-period* of γ over $[c]$ are

$$P_\gamma([c]) = \int_c \gamma \tag{1.0.18}$$

and

$$S_\gamma([c]) = \int_c \gamma \circ j, \tag{1.0.19}$$

respectively. Since γ is a harmonic $1-$form with respect to the complex structure j, the period and co-period do not depend on the specific choice of the representative of the isotopy class. Let $R_{[c]}$ be the conformal modulus of $[c]$ as defined in [7]. The *conformal period* of γ over c is defined by

$$\tau_{\gamma,[c]} = R_{[c]} P_\gamma([c]), \tag{1.0.20}$$

where the *conformal co-period* of γ over c is defined by

$$\sigma_{\gamma,[c]} = R_{[c]} S_\gamma([c]). \tag{1.0.21}$$

These two quantities connect the topology of the surface, the harmonic $1-$form γ and the conformal structure. The significance of these two quantities will become apparent in Section 2.3 when dealing with the convergence issue. The main result of this thesis, which is stated in Theorem 33 is the following. Starting with a sequence of $\mathcal{H}-$holomorphic curves $(S_n, j_n, \mathcal{P}_n, u_n = (a_n, f_n), \gamma_n)$ with uniformly bounded energies, uniformly bounded L^2-norms of the harmonic perturbation $1-$forms γ_n, and uniformly bounded conformal periods and co-periods, we will introduce a notion of convergence which is a generalization of the convergence of the classical SFT compactness theory. In this context, we will show that there exists a subsequence converging in the sense of Definition 31 to a limit $\mathcal{H}-$holomorphic curve which will be called stratified $\mathcal{H}-$holomorphic building (see Definition 27).
In the following we give an outline of this thesis and a rough description of the techniques used in the proof of the compactness result.
In Chapter 2 we review the basic concepts related to the compactness of $\mathcal{H}-$holomorphic curves. More precisely, Chapter 2 is organized as follows. In Section 2.1 we present the Deligne-Mumford convergence theorem for stable Riemann surfaces by following the analysis of [6] and [2]. We conclude this section by stating the Deligne-Mumford convergence. In Section 2.2 we provide the necessary information on contact manifolds, as well as a precise definition of $\mathcal{H}-$holomorphic curves. By Proposition 22, we recall a result similar to that established by Hofer et al. [13] stating that the behavior of $\mathcal{H}-$holomorphic curves in a neighborhood of the punctures is similar to that of usual pseudoholomorphic curves. This result will enable us to split the set of punctures into positive and negative punctures, which in turn are used in Section 2.3 to define a stratified $\mathcal{H}-$holomorphic building. This definition is similar to that of pseudoholomorphic buildings given in [6], [2], and [7]; the difference is that we allow two points, lying in the same level, to be connected by a finite length trajectory of the Reeb vector field. After defining this object, we formulate Theorem 33, which states that a sequence of $\mathcal{H}-$holomorphic curves with uniformly bounded energies, uniformly bounded L^2-norms of the harmonic perturbations, uniformly bounded conformal period and co-period posseses a subsequence that converges to a stratified $\mathcal{H}-$holomorphic building, in a C^∞_{loc} and a C^0 sense. Essentially, the $\mathcal{H}-$holomorphic curves converge in C^∞_{loc} away from the punctures and certain loops that degenerate to nodes, while the projections of the $\mathcal{H}-$holomorphic curves to M converge in C^0. In addition we derive a notion of level structure, which is similar to that from [6] and [7], and serves as a notion of C^0-convergence for the $\mathbb{R}-$coordinates.
The proof of the main compactness result on the thick part with certain points removed, and on the thin part and in a neighborhood of the removed points, are carried out in Sections 3.1 and 3.2 of Chapter 3, respectively. For the thick part, we use the Deligne-Mumford convergence and the thick-thin decomposition to show that the domains converge in the Deligne-Mumford sense to a punctured nodal Riemann surface. By using bubbling-off analysis and the results of Appendix D (to generate a sequence of holomorphic coordinates that behaves well

under Deligne-Mumford limit process) we prove, after introducing additional punctures, that the $\mathcal{H}-$holomorphic curves have uniformly bounded gradients in the complement of the special circles and certain marked points. By using the elliptic regularity theorem for pseudoholomorphic curves and Arzelà-Ascoli theorem we show that the $\mathcal{H}-$holomorphic curves together with the harmonic perturbations converge in C^{∞}_{loc} on the thick part with certain points removed to a $\mathcal{H}-$holomorphic curve with harmonic perturbation. This set of points is denoted by $\mathcal{Z}$. This result is similar to the bubbling-off analysis performed in [6]. However, in contrast to Lemma 10.7 of [6], we do not change the hyperbolic structure each time after adding the additional marked point generated by the bubbling-off analysis. The thin part is decomposed into cusps corresponding to neighborhoods of punctures and hyperbolic cylinders corresponding to nodes in the limit. As the perturbation harmonic $1-$forms are exact in a neighborhood of the punctures or the points that were removed in the first part, by means of a change of the $\mathbb{R}-$coordinate, the $\mathcal{H}-$holomorphic curves are turned into usual pseudoholomorphic curves on which the classical theory [6] or [7] is applicable. The case of hyperbolic cylinders is more interesting because the difference from the classical SFT compactness result is evident. Due to a lack of the monotonicity lemma, we cannot expect the $\mathcal{H}-$holomorphic curves to have uniformly bounded gradients, and so, to apply the classical SFT convergence theory. To deal with this problem we decompose the hyperbolic cylinder into a finite uniform number of smaller cylinders of two types:

- type ∞: cylinders having conformal modulus tending to infinity but $d\alpha-$energies strictly smaller than $\hbar$;
- type b_1: cylinders having bounded modulus but $d\alpha-$energies possibly larger than $\hbar$.

The cylinders of type ∞ and b_1 appear alternately, while here the constant $\hbar > 0$ is defined by

$$\hbar := \min\{|P_1 - P_2| \mid P_1, P_2 \in \mathcal{P}_{\alpha}, P_1 \neq P_2, P_1, P_2 \leqslant E_0\}, \tag{1.0.22}$$

where $\mathcal{P}_{\alpha}$ is the action spectrum of α as defined in [14] and $E_0 > 0$ is the uniform bound on the energy. Convergence results are derived for each cylinder type, and then glued together to obtain a convergence result on the whole hyperbolic cylinder. As cylinders of type ∞ have small $d\alpha-$energies, we prove by the classical bubbling-off analysis, that the $\mathcal{H}-$holomorphic curves have uniformly bounded gradients. To turn these maps into pseudoholomorphic curves, we perform a transformation by pushing them along the Reeb flow up to some specific time characterized by the uniformly bounded conformal period. These transformed curves are now pseudoholomorphic with respect to a domain-dependent almost complex structure on M, which due to the uniform boundedness of the conformal period varies in a compact set. In a final step, we use the results established in Appendices B and E to prove a convergence result (C^{∞}_{loc} and C^0) for cylinders of type ∞. In the case of cylinders of type b_1 we proceed as follows. Relying on a bubbling-off argument, as we did in the case of the thick part, we prove that the gradient blows up only in a finite uniform number of points and remains uniformly bounded on a compact complement of them. In this compact region we use Arzelà-Ascoli theorem to show that the $\mathcal{H}-$holomorphic curves together with the harmonic perturbations converge in C^{∞} to some $\mathcal{H}-$holomorphic curve. What is then left is the convergence in a neighborhood of the finitely many punctures where the gradient blows up. Here, a neighborhood of a puncture is a disc on which the harmonic perturbation can be made exact and can be encoded in the $\mathbb{R}-$coordinate of the $\mathcal{H}-$holomorphic curve. By this procedure we transform the $\mathcal{H}-$holomorphic curve into a usual pseudoholomorphic curve defined on a disc D. By the $C^{\infty}-$convergence established before on any compact complement of the punctures, we assume that the transformed curves converge on an arbitrary neighborhood of ∂D. Then we use the results of [7], especially Gromov compactness with free boundary, to obtain a convergence results for cylinders of type b_1. This part uses extensively the results established in Appendix A and Appendix E.

In Chapter 4 we discuss the condition imposed on the conformal period and co-period, that is, for a sequence of $\mathcal{H}-$holomorphic curves, the conformal period and co-period have to be uniformly bounded. The conformal period and co-period can be seen as a link between the conformal data and the topology on the Riemann surface as well as the harmonic perturbation $1-$form. Without these conditions, the transformation performed in Appendix B cannot be established. The reason is that the domain-dependent almost complex structure, which was constructed in order to change the $\mathcal{H}-$holomorphic curve into a usual pseudoholomorphic curve, does not vary in a compact space, and so, the results established in [14] cannot be applied. By means of a counterexample stated in Proposition 57 we show that the condition on the uniform bound of the conformal period is not always satisfied. It should be pointed out that Bergmann [5] claimed to have established a compactification of the space of $\mathcal{H}-$holomorphic curves by performing the same transformation as we did in Appendix B, i.e. by pushing the $M-$component of the $\mathcal{H}-$holomorphic curve by the Reeb flow up to some specific time determined by the conformal period, and then by assuming that the conformal period can be universally bounded by a quantity which depends only on the periods of the harmonic perturbation $1-$form (note that if the L^2-norm of a sequence of harmonic $1-$forms is uniformly bounded then their periods are also uniformly bounded). In this context, Proposition 57 contradicts his argument.

Chapter 2

Definitions and main results

In this chapter we present the basic concepts related to the compactness of $\mathcal{H}$−holomorphic curves. In particular, we provide the Deligne-Mumford compactness in order to describe the convergence of a sequence of Riemann surfaces, introduce the concept of a stratified $\mathcal{H}$−holomorphic buildings of height N, which serves as limit object, and discuss the convergence of such maps. The main result of this chapter is summarized in Theorem 33.

2.1 Deligne-Mumford convergence

In this section we review the Deligne-Mumford convergence following the analysis given in [6] and [2].
Consider the surface $(S, j, \mathcal{M} \amalg \mathcal{D})$, where (S, j) is a closed Riemann surface, and $\mathcal{M}$ and $\mathcal{D}$ are finite disjoint subsets of S. Assume that the cardinality of $\mathcal{D}$ is even. The points from $\mathcal{M}$ are called *marked points,* while the points from $\mathcal{D}$ are called *nodal points.* The points from $\mathcal{D}$ are organized in pairs, $\mathcal{D} = \{d_1', d_1'', d_2', d_2'', ..., d_k', d_k''\}$. A nodal surface $(S, j, \mathcal{M} \amalg \mathcal{D})$ is said to be *stable* if the stability condition $2g + |\mathcal{M} \cup \mathcal{D}| \geqslant 3$ is satisfied for each component of the surface S. In our analysis we do not deal with the stability of Riemann surfaces; this is only a technical condition and can always be achieved by adding additional marked points to $\mathcal{M}$. The stability ensures the convergence of the domains of $\mathcal{H}$−holomorphic curves; for more details we refer to [2]. With a nodal surface $(S, j, \mathcal{M} \amalg \mathcal{D})$ we can associate the following singular surface with double points,

$$\hat{S}_{\mathcal{D}} = S/\{d_i' \sim d_i'' \mid i = 1, ..., k\}.$$

The identified points $d_i' \sim d_i''$ are called *node*s (see Figures 2.1.1 and 2.1.2). The nodal surface $(S, j, \mathcal{M} \amalg \mathcal{D})$ is said to be *connected* if the singular surface $\hat{S}_{\mathcal{D}}$ is connected. For each $p \in \mathcal{M} \amalg \mathcal{D}$ of a stable nodal Riemann surface

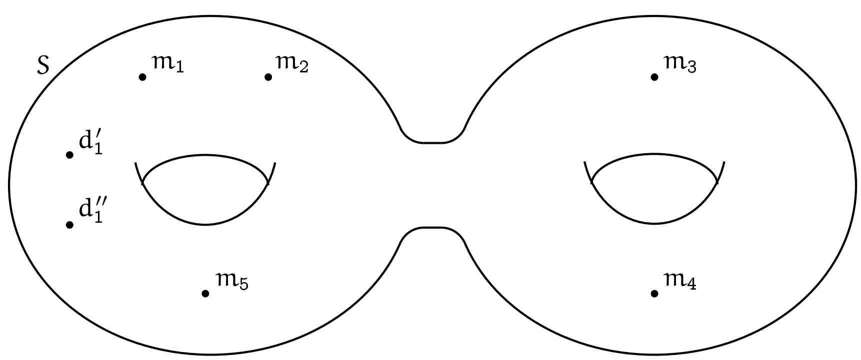

Figure 2.1.1: The surface S with marked points $\mathcal{M} = \{m_1, ..., m_5\}$ and nodal points $\mathcal{D} = \{d_1', d_1''\}$.

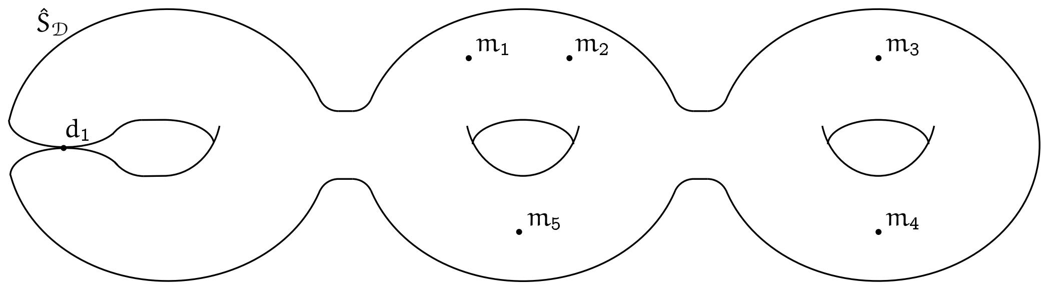

Figure 2.1.2: The singular surface $\hat{S}_{\mathcal{D}}$ with one node $d_1 = d_1' \sim d_1''$.

$(S, j, \mathcal{M} \amalg \mathcal{D})$, we define the surface S^p with boundary as the oriented blow-up of S at the point p. Thus S^p is the circle compactification of $S \backslash \{p\}$; it is a compact surface bounded by the circle $\Gamma_p = (T_p S \backslash \{0\})/\mathbb{R}_+$. The canonical projection $\pi : S^p \to S$ sends the circle Γ_p to the point p and maps $S^p \backslash \Gamma_p$ diffeomorphically to $S \backslash \{p\}$. Similarly, given a finite set $\mathcal{M}' = \{p_1, ..., p_k\} \subset \mathcal{M} \amalg \mathcal{D}$ of punctures, we consider a blow-up surface $S^{\mathcal{M}'}$ with k boundary components $\Gamma_1, ..., \Gamma_k$. It comes with the projection $\pi : S^{\mathcal{M}'} \to S$, which collapses the boundary circles $\Gamma_1, ..., \Gamma_k$ to points $p_1, ..., p_k$ and the maps $S^{\mathcal{M}'} \backslash \coprod_{i=1}^k \Gamma_i$ diffeomorphically to $\dot{S} = S \backslash \mathcal{M}'$.
The arithmetic genus g of a nodal surface $(S, j, \mathcal{M} \amalg \mathcal{D})$ is defined as

$$g = \frac{1}{2}|\mathcal{D}| - b_0 + \sum_{i=1}^{b_0} g_i + 1,$$

where $|\mathcal{D}| = 2k$ is the cardinality of $\mathcal{D}$, b_0 is the number of connected components of the surface S, and $\sum_{i=1}^{b_0} g_i$ is the sum of the genera of the connected components of S. The *signature* of a nodal curve $(S, j, \mathcal{M} \amalg \mathcal{D})$ is the pair (g, μ), where g is the arithmetic genus and $\mu = |\mathcal{M}|$. A stable nodal Riemann surface $(S, j, \mathcal{M} \amalg \mathcal{D})$ is called *decorated* if for each node there is an orientation reversing orthogonal map

$$r_i : \overline{\Gamma}_i = (T_{\overline{d}_i} S \backslash \{0\})/\mathbb{R}_+ \to \underline{\Gamma}_i = (T_{\underline{d}_i} S \backslash \{0\})/\mathbb{R}_+. \tag{2.1.1}$$

For the orthogonal orientation reversing map r_i, we must have that $r_i(e^{2\pi i\vartheta}p) = e^{-2\pi i\vartheta}r(p)$ for all $p \in \overline{\Gamma}_i$.
In the following we argue as in [6]. Consider the oriented blow-up $S^{\mathcal{D}}$ at the points of $\mathcal{D}$ as described above. The circles $\overline{\Gamma}_i$ and $\underline{\Gamma}_i$ defined by (2.1.1) are boundary circles for the points $d_i', d_i'' \in \mathcal{D}$. The canonical projection $\pi : S^{\mathcal{D}} \to S$, collapsing the circles $\overline{\Gamma}_i$ and $\underline{\Gamma}_i$ to the points d_i' and d_i'', respectively, induces a conformal structure on $S^{\mathcal{D}} \backslash \coprod_{i=1}^k \overline{\Gamma}_i \amalg \underline{\Gamma}_i$. The smooth structure of $S^{\mathcal{D}} \backslash \coprod_{i=1}^k \overline{\Gamma}_i \amalg \underline{\Gamma}_i$ extends to $S^{\mathcal{D}}$, while the extended conformal structure degenerates along the boundary circles $\overline{\Gamma}_i$ and $\underline{\Gamma}_i$ (see Figure 2.1.3). Let $(S, j, \mathcal{M} \amalg \mathcal{D}, r)$ be a decorated surface, where $r = (r_1, ..., r_k)$. By means of the mappings r_i, $i = 1, ..., k$, $\overline{\Gamma}_i$ and $\underline{\Gamma}_i$ can be glued together to yield a closed surface $S^{\mathcal{D},r}$. The genus of the surface $S^{\mathcal{D},r}$ is equal to the arithmetic genus of $(S, j, \mathcal{M} \amalg \mathcal{D})$. There exists a canonical projection $p : S^{\mathcal{D},r} \to \hat{S}_{\mathcal{D}}$ which projects the circle $\Gamma_i = \{\overline{\Gamma}_i, \underline{\Gamma}_i\}$ to the node $d_i = \{d_i', d_i''\}$. The projection p induces on the surface $S^{\mathcal{D},r}$ a conformal structure in the complement of the special circles Γ_i (see Figure 2.1.4); the conformal structure is still denoted by j. The continous extension of j to $S^{\mathcal{D},r}$ degenerates along the special circles Γ_i.
According to the uniformization theorem, for a stable surface $(S, j, \mathcal{M} \amalg \mathcal{D})$ there exists a unique complete hyperbolic metric of constant curvature -1 of finite volume, in the given conformal class j on $\dot{S} = S \backslash (\mathcal{M} \amalg \mathcal{D})$. For details see [2]. This metric is denoted by $h^{j,\mathcal{M}\amalg\mathcal{D}}$. Each point in $\mathcal{M} \amalg \mathcal{D}$ corresponds to a cusp of the hyperbolic metric $h^{j,\mathcal{M}\amalg\mathcal{D}}$. Assume that for a given stable Riemann surface $(S, j, \mathcal{M} \amalg \mathcal{D})$, the punctured surface $\dot{S} = S \backslash (\mathcal{M} \amalg \mathcal{D})$ is endowed with the uniformizing hyperbolic metric $h^{j,\mathcal{M}\amalg\mathcal{D}}$.

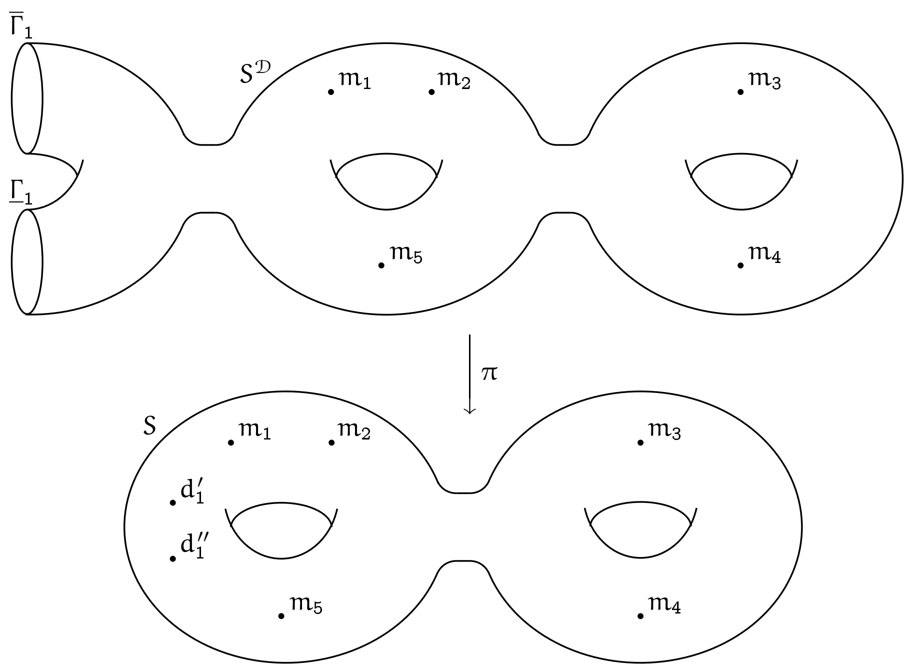

Figure 2.1.3: The surface $S^{\mathcal{D}}$ with boundary circles $\underline{\Gamma}_1$ and $\overline{\Gamma}_1$ and the projection $\pi : S^{\mathcal{D}} \to S$. π maps $S^{\mathcal{D}} \backslash (\underline{\Gamma}_1 \amalg \overline{\Gamma}_1)$ diffeomorphically to $S \backslash \{d_1', d_1''\}$.

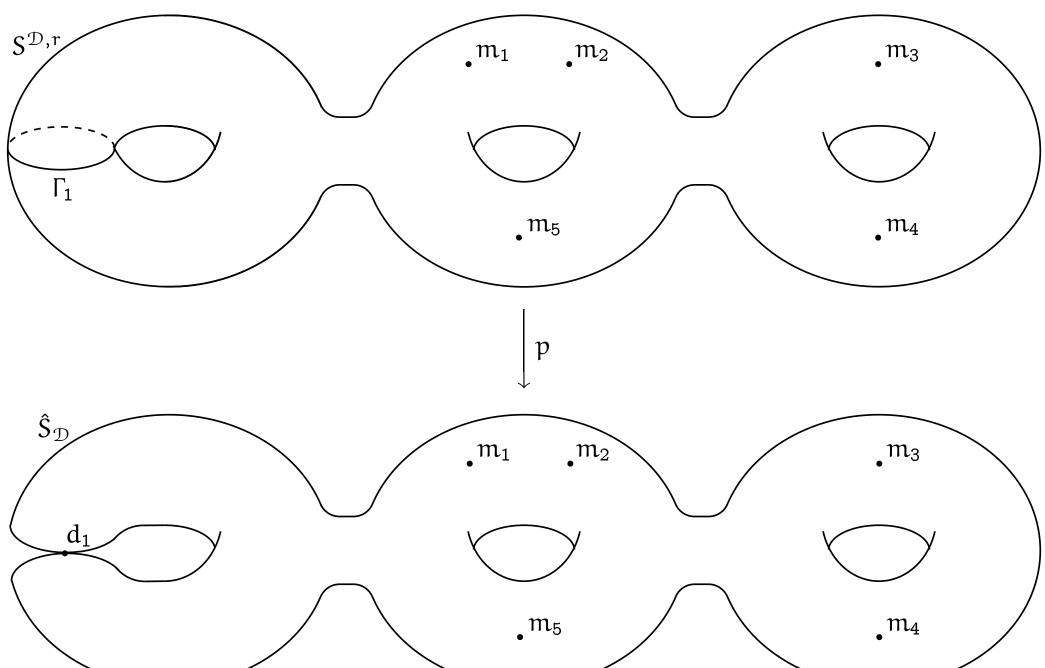

Figure 2.1.4: The surface $S^{\mathcal{D},r}$ and the projection $p : S^{\mathcal{D},r} \to \hat{S}_{\mathcal{D}}$. p maps $S^{\mathcal{D},r} \backslash \Gamma_1$ diffeomorphically to $\hat{S}_{\mathcal{D}} \backslash d_1$.

Fix $\delta > 0$, and denote by

$$\mathrm{Thick}_\delta(S, h^{j,\mathcal{M}\amalg\mathcal{D}}) = \left\{x \in \dot{S} \mid \rho(x) \geqslant \delta\right\}$$

and

$$\mathrm{Thin}_\delta(S, h^{j,\mathcal{M}\amalg\mathcal{D}}) = \overline{\left\{x \in \dot{S} \mid \rho(x) < \delta\right\}},$$

the δ–thick and δ–thin parts, respectively, where $\rho(x)$ is the injectivity radius of the metric $h^{j,\mathcal{M}\amalg\mathcal{D}}$ at the point $x \in \dot{S}$. A fundamental result of hyperbolic geometry states that there exists a universal constant $\delta_0 = \sinh^{-1}(1)$ such that for any $\delta < \delta_0$, $\dot{S}$ can be written as the disjoint union of $\mathrm{Thick}_\delta(S, h^{j,\mathcal{M}\amalg\mathcal{D}})$ and $\mathrm{Thin}_\delta(S, h^{j,\mathcal{M}\amalg\mathcal{D}})$, and each component C of $\mathrm{Thin}_\delta(S, h^{j,\mathcal{M}\amalg\mathcal{D}})$ is conformally equivalent either to a finite cylinder $[-R, R] \times S^1$ if the component C is not adjacent to a puncture, or to the punctured disk $D\backslash\{0\} \cong [0, \infty) \times S^1$ if it is adjacent to a puncture (see, for example, [15] and [2]). Each compact component C of the thin part contains a unique closed geodesic of length $2\rho(C)$ denoted by Γ_C, where $\rho(C) = \inf_{x\in C} \rho(x)$. When considering the δ–thick–thin decomposition we always assume that δ is chosen smaller than δ_0.
The uniformization metric $h^{j,\mathcal{M}\amalg\mathcal{D}}$ can be lifted to a metric $\overline{h}^{j,\mathcal{M}\amalg\mathcal{D}}$ on $\dot{S}^{\mathcal{D},r} := S^{\mathcal{D},r}\backslash\mathcal{M}$. The lifted metric degenerates along each circle Γ_i in the sense that the length of Γ_i is 0, and the distance of Γ_i to any other point in $\dot{S}^{\mathcal{D},r}$ is infinite. However, we can still speak about geodesics on $\dot{S}^{\mathcal{D},r}$ which are orthogonal to Γ_i, i.e., two geodesics rays, whose asymptotic directions at the cusps d_i' and d_i'' are related via the map r_i, and which correspond to a compact geodesic interval in $S^{\mathcal{D},r}$ intersecting orthogonally the circle Γ_i. It is convenient to regard $\mathrm{Thin}_\delta(S, h^{j,\mathcal{M}\amalg\mathcal{D}})$ and $\mathrm{Thick}_\delta(S, h^{j,\mathcal{M}\amalg\mathcal{D}})$ as subsets of $\dot{S}^{\mathcal{D},r}$. This interpretation provides a compactification of the non-compact components of $\mathrm{Thin}_\delta(S, h^{j,\mathcal{M}\amalg\mathcal{D}})$ not adjacent to points from $\mathcal{M}$. Any compact component C of $\mathrm{Thin}_\delta(S, h^{j,\mathcal{M}\amalg\mathcal{D}}) \subset \dot{S}^{\mathcal{D},r}$ is a compact annulus; it contains either a closed geodesic Γ_C, or one of the special circles, still denoted by Γ_C, which projects to a node (as described above).
Consider a sequence of decorated stable nodal marked Riemann surfaces $(S_n, j_n, \mathcal{M}_n \amalg \mathcal{D}_n, r_n)$ indexed by $n \in \mathbb{N}$.

Definition 12. The sequence $(S_n, j_n, \mathcal{M}_n\amalg\mathcal{D}_n, r_n)$ is said to converge in the Deligne-Mumford sense to a decorated stable nodal surface $(S, j, \mathcal{M}\amalg\mathcal{D}, r)$ if for sufficiently large n, there exists a sequence of diffeomorphisms $\varphi_n : S^{\mathcal{D},r} \to S_n^{\mathcal{D}_n,r_n}$ with $\varphi_n(\mathcal{M}) = \mathcal{M}_n$ such that the following are satisfied.

1. For any $n \geqslant 1$, the images $\varphi_n(\Gamma_i)$ of the special circles $\Gamma_i \subset S^{\mathcal{D},r}$ for $i = 1, ..., k$, are special circles or closed geodesics of the metrics $h^{j_n,\mathcal{M}_n\amalg\mathcal{D}_n}$ on $\dot{S}^{\mathcal{D}_n,r_n}$. All special circles on $S^{\mathcal{D}_n,r_n}$ are among these images.
2. $h_n \to \overline{h}$ in $C^\infty_{\mathrm{loc}}(\dot{S}^{\mathcal{D},r}\backslash \coprod_{i=1}^k \Gamma_i)$, where $h_n := \varphi_n^* h^{j_n,\mathcal{M}_n\amalg\mathcal{D}_n}$ and $\overline{h} := \overline{h}^{j,\mathcal{M}\amalg\mathcal{D}}$.
3. Given a component C of $\mathrm{Thin}_\delta(S, h^{j,\mathcal{M}\amalg\mathcal{D}}) \subset \dot{S}^{\mathcal{D},r}$ containing a special circle Γ_i, and given a point $c_i \in \Gamma_i$, let δ_i^n be the geodesic arc corresponding to the induced metric $h_n = \varphi_n^* h^{j_n,\mathcal{M}_n\amalg\mathcal{D}_n}$ for any $n \geqslant 1$, intersecting Γ_i orthogonally at the point c_i, and having the ends in the δ–thick part of the metric h_n. Then, in the limit $n \to \infty$, $(C \cap \delta_i^n)$ converge in C^0 to a continous geodesic for a metric $\overline{h}$ passing through the point c_i.

Remark 13. In view of the uniformization theorem, Condition 2 of Definition 12 is equivalent to the condition

$$\varphi_n^* j_n \to j \text{ in } C^\infty_{\mathrm{loc}}\left(\dot{S}^{\mathcal{D},r}\backslash\coprod_{i=1}^k \Gamma_i\right),$$

which in turn, by the removable singularity theorem, is equivalent to

$$\varphi_n^* j_n \to j \text{ in } C^\infty_{\mathrm{loc}}\left(S^{\mathcal{D},r}\backslash\coprod_{i=1}^k \Gamma_i\right).$$

In this context, a sequence $(S_n, j_n, \mathcal{M}_n \amalg \mathcal{D}_n)$ is said to converge in the Deligne-Mumford sense to $(S, j, \mathcal{M} \amalg \mathcal{D})$ if there exists a sequence of decorations r_n for $(S_n, j_n, \mathcal{M}_n \amalg \mathcal{D}_n)$ and a decoration r of $(S, j, \mathcal{M} \amalg \mathcal{D})$ such that $(S_n, j_n, \mathcal{M}_n \amalg \mathcal{D}_n, r_n)$ converges to $(S, j, \mathcal{M} \amalg \mathcal{D}, r)$ as Definition 12. We are now in the position to state the Deligne-Mumford convergence theorem.

Theorem 14. *(Deligne-Mumford) Any sequence of nodal stable Riemann surfaces* $(S_n, j_n, \mathcal{M}_n \amalg \mathcal{D}_n, r_n)$ *of signature* (g, μ) *has a subsequence which converges in the DeligneMumford sense to a decorated nodal stable Riemann surface* $(S, j, \mathcal{M} \amalg \mathcal{D}, r)$ *of signature* (g, μ).

Corollary 15. *Any sequence of stable Riemann surfaces* $(S_n, j_n, \mathcal{M}_n)$ *of signature* (g, μ) *has a subsequence which converges in the Deligne-Mumford sense to a decorated nodal stable Riemann surface* $(S, j, \mathcal{M} \amalg \mathcal{D}, r)$ *of signature* (g, μ).

2.2 $\mathcal{H}-$holomorphic curves

Let (M, α) be a 3−dimensional compact manifold equipped with a contact form α, which by definition, is a 1−form on M such that $\alpha \wedge d\alpha$ is a volume form. Associated to a pair (M, α) we have the contact structure $\xi = \ker(\alpha)$. The contact structure is a 2−dimensional subbundle of TM and $d\alpha|_\xi$ defines on any fiber a symplectic form. Hence $\xi \to M$ is a symplectic vector bundle with the symplectic form $d\alpha$. Furthermore, there exists a unique vector field X_α, called the Reeb vector field, defined by the two conditions

$$\iota_{X_\alpha}\alpha = 1 \text{ and } \iota_{X_\alpha} d\alpha = 0.$$

The vector field X_α spans a line bundle with global section X_α. Thus, a contact form α on M defines a natural splitting

$$TM = X_\alpha \mathbb{R} \oplus \xi$$

of the tangent bundle into a line bundle and a symplectic vector bundle $(\xi, d\alpha)$.
A compatible complex structure J for the contact structure $\xi \to M$ is a smooth fiber preserving fiberwise linear map $J : \xi \to \xi$ such that $J^2 = -\mathbb{1}$ and being compatible with the symplectic form $d\alpha$ on ξ. As a result

$$g_J(\cdot, \cdot) := d\alpha(\cdot, J\cdot)$$

defines a smooth fiberwise metric on the vector bundle $\xi \to M$ and

$$g(p)(v, w) := \alpha(p)(v)\alpha(p)(w) + d\alpha(p)(\pi_\alpha v, J(p)\pi_\alpha w)$$

for $p \in M$ and $v, w \in T_pM$ defines a smooth metric on M, where $\pi_\alpha : TM \to \xi$ is the projection along X_α. It is well known that the space of all such J's equipped with the $C^\infty-$topology is contractible.
Given J as above, there is an associated almost complex structure $\bar{J}$ and an associate Riemann metric $\bar{g}$ on $\mathbb{R} \times M$ defined by

$$\begin{aligned}\bar{J}(a, f)(h, k) &:= (-\alpha(f)(w), J(f)(\pi_\alpha w) + vX_\alpha(f)),\\ \bar{g}(a, f)((v, w), (v', w')) &:= vv' + \alpha(w)\alpha(w') + d\alpha(\pi_\alpha w, J(f)\pi_\alpha w'),\end{aligned} \tag{2.2.1}$$

where $(a, f) \in \mathbb{R} \times M$, $(v, w), (v', w') \in T_{(a,f)}(\mathbb{R} \times M)$.
In our treatment we assume that all periodic orbits are non-degenerate. This means that for every periodic orbit x of period T, the linear map $d\phi_T^\alpha(x(0)) : \xi_{x(0)} \to \xi_{x(T)}$ does not contain 1 in its spectrum. Consider now a

5−tuple $(S, j, \mathcal{P}, u, \gamma)$ consisting of a closed Riemann surface (S, j), a finite subset $\mathcal{P} \subset S$ called the set of punctures, a smooth map $u = (a, f) : \dot{S} \to \mathbb{R} \times M$, where $\dot{S} = S \backslash \mathcal{P}$, and a 1−form $\gamma \in \mathcal{H}^1_j(S)$, where $\mathcal{H}^1_j(S)$ represents the space of harmonic 1-forms on S with respect to j. The energy $E(u; \dot{S})$ of u is defined as in (1.0.15).

Definition 16. The 5−tuple $(S, j, \mathcal{P}, u, \gamma)$ is called a $\mathcal{H}-$*holomorphic* curve with harmonic perturbation γ if

$$\begin{aligned} \pi_\alpha df \circ j &= J \circ \pi_\alpha df \ \text{ on } \dot{S} \\ (f^*\alpha) \circ j &= da + \gamma \ \text{ on } \dot{S} \\ E(u; \dot{S}) &< +\infty. \end{aligned} \tag{2.2.2}$$

Here we consider a more general setting as in Definition 7 in the sense that the properness requirement is dismissed. The L^2-norm, period, co-period, conformal period and conformal co-period of the harmonic 1−form γ is defined as in (1.0.17), (1.0.18), (1.0.19), (1.0.20) and (1.0.21). Note that the closedness of γ and $\gamma \circ j$ implies that all these quantities depend only on the isotopy class of c.

Remark 17. Equation (2.2.2) can be also written as

$$\bar{\partial}_{\bar{J}} u = g$$

with

$$\bar{\partial}_{\bar{J}} u = \frac{1}{2}(du + \bar{J}(u) \circ du \circ j) \tag{2.2.3}$$

and

$$g = \frac{1}{2}\left(-\gamma \otimes \frac{\partial}{\partial r}, -(\gamma \circ j) \otimes X_\alpha\right)$$

being an anti-holomorphic section of the bundle $\mathrm{Hom}(u^*T(\mathbb{R} \times M)) \to \dot{S}$.

Locally, with respect to holomorphic coordinates $s + it$, Equation (2.2.2) takes the form

$$\begin{aligned} \pi_\alpha \partial_s f + J(u) \circ \pi_\alpha \partial_t f &= 0 \\ \alpha(\partial_s f) &= -\partial_t a - \gamma_t \\ \alpha(\partial_t f) &= \partial_s a + \gamma_s \end{aligned} \tag{2.2.4}$$

where $\gamma = \gamma_s ds + \gamma_t dt$. It is important to note that the integrands of the $\alpha-$ and $d\alpha-$energies are non-negative. Indeed, in the local holomorphic coordinates $s + it$, we have

$$\varphi'(a) da \circ j \wedge da = \varphi'(a)\left[(\partial_s a)^2 + (\partial_t a)^2\right] ds \wedge dt$$

and

$$f^* d\alpha = \left[\|\pi_\alpha \partial_s f\|^2_{g_J} + \|\pi_\alpha \partial_t f\|^2_{g_J}\right] ds \wedge dt.$$

Remark 18. If $E_{d\alpha}(u; \dot{S}) = 0$, then $f(\dot{S})$ is contained in some trajectory of the Reeb vector field X_α.

To describe the behavior of a $\mathcal{H}-$holomorphic curve near the puncture from $\mathcal{P}$ we need some auxiliary tools. One of these is the lemma about the removal of singularity. Consider a $\mathcal{H}-$holomorphic curve $(S, j, \mathcal{P}, u, \gamma)$, and assume that the set of punctures $\mathcal{P} \subset S$ is not empty. For $p \in \mathcal{P}$, consider a neighborhood $U(p) = U \subset S$, which is biholomorphic to the standard open disk $D \subset \mathbb{C}$, such that, under this biholomorphism, the point p is mapped to 0.

First we mention a removable singularity result for a harmonic 1−form γ defined on the punctured unit disk $D\backslash\{0\}$.

Lemma 19. *If γ is a harmonic 1−form defined on the punctured disk $D\backslash\{0\}$, and having a bounded L^2-norm with respect to the standard complex structure i on D, i.e. $\|\gamma\|^2_{L^2(D\backslash\{0\})} < \infty$ then γ can be extended across the puncture.*

Proof. With $z = s + it = (s,t)$ being the coordinates on D, we express γ as $\gamma = f(s,t)ds + g(s,t)dt$, where $f, g : D\backslash\{0\} \to \mathbb{R}$ are harmonic functions. As γ is harmonic with respect to the standard complex structure i, $F := f + ig : D\backslash\{0\} \to \mathbb{C}$ is a meromorphic function with a bounded L^2-norm, i.e.,

$$\int_{D\backslash\{0\}} |F(s,t)|^2 dsdt = \int_{D\backslash\{0\}} \left(|f(s,t)|^2 + |g(s,t)|^2\right) dsdt < \infty.$$

Consider the Laurent series of F,

$$F(z) = \sum_{n=-\infty}^{\infty} F_n z^n,$$

where $F_n \in \mathbb{C}$. Since the Laurent series converges in C^0_{loc} to F and $e^{2\pi in\theta}$ is an orthonormal system in $L^2(S^1)$, we infer that for every fixed $0 < \rho < 1$,

$$\int_0^1 |F(\rho e^{2\pi i\theta})|^2 d\theta = \sum_{n=-\infty}^{\infty} |F_n|^2 \rho^{2n}.$$

Consequently, due to Fubini's theorem,

$$\int_{D\backslash\{0\}} |F(z)|^2 dsdt = 2\pi \int_{(0,1]\times S^1} \rho |F(\rho e^{2\pi i\theta})|^2 d\theta d\rho = 2\pi \int_0^1 \sum_{n=-\infty}^{\infty} |F_n|^2 \rho^{2n+1} d\rho.$$

As the terms in the sum are all non-negative, it follows that

$$\int_{D\backslash\{0\}} |F(z)|^2 dsdt \geqslant 2\pi |F_n|^2 \int_0^1 \rho^{2n+1} d\rho$$

for all $n \in \mathbb{Z}$. However, for $n < 0$ and because of

$$\int_0^1 \rho^{2n+1} d\rho = \infty,$$

this yields a contradiction to the finiteness of the L^2-norm of F. Hence $F_{-n} = 0$ for all $n \geqslant 1$, and so, F can be extended to a holomorphic function on D. Therefore γ can be extended across the puncture.

□

A removable singularity result for $\mathcal{H}-$holomorphic curves is the following

Proposition 20. *Let* $(D, i, \{0\}, u, \gamma)$ *be a* $\mathcal{H}-$*holomorphic curve defined on* $D\backslash\{0\}$ *such that the image of* u *lies in a compact subset of* $\mathbb{R} \times M$. *Then* u *extends continously to a* $\mathcal{H}-$*holomorphic map on the whole disk* D.

Before proving Proposition 20 we state the following lemma.

Lemma 21. *Let* $u = (a, f) : [0,\infty) \times S^1 \to \mathbb{R} \times M$ *be a* $\mathcal{H}-$*holomorphic curve with harmonic perturbation* γ *with respect to the standard complex structure* i *on the half cylinder* $[0,\infty) \times S^1$. *Assume that* $E(u; [0,\infty) \times S^1) \leqslant E_0$ *and* $E_{d\alpha}(u; [0,\infty) \times S^1) \leqslant \hbar/2$, *where* $\hbar > 0$ *is the constant defined in (1.0.22) with respect to* E_0. *Then, for every* $\delta \in (0,1)$ *there exists a constant* $\kappa_\delta > 0$ *such that*

$$\|du(z)\| := \sup_{\|v\|_{eucl.}=1} \|du(z)v\|_{\overline{g}} < \kappa_\delta$$

for all $z \in [\delta, \infty) \times S^1$.

Proof. The proof is analogous to that of Lemma 37, by contradiction and using the standard bubbling-off analysis, hence omitted.

□

We come to the proof of Proposition 20.

Proof. (Proposition 20) Without loss of generality, we assume that the $d\alpha-$energy of u is less than $\hbar/2$. If this is not the case we can consider a smaller disk around 0. Since D is contractible and $d\gamma = d(\gamma \circ i) = 0$, the harmonic perturbation γ can be written as $\gamma = d\Gamma$, where $\Gamma : D \to \mathbb{R}$ is a harmonic function. Hence $\overline{u} = (\overline{a}, \overline{f}) := (a + \Gamma, f)$ is a pseudoholomorphic curve (unperturbed), which still has the property that its image lies in a (maybe larger) compact subset of $\mathbb{R} \times M$. By the biholomorphism $\psi : [0, \infty) \times S^1 \to D\backslash\{0\}$, $(s,t) \mapsto e^{-2\pi(s+it)}$, we consider the map $\hat{u} = (\hat{a}, \hat{f}) = \overline{u} \circ \psi : [0,\infty) \times S^1 \to \mathbb{R} \times M$. Obviously, $\hat{u}$ has a finite energy and a $d\alpha-$energy less than $\hbar/2$. The Hofer energy of $\overline{u}$ is bounded. Indeed, we have

$$\begin{aligned}
E_H(\overline{u}; D\backslash\{0\}) &= \sup_{\varphi\in\mathcal{A}} \int_{D\backslash\{0\}} \overline{u}^* d(\varphi\alpha) \\
&= \sup_{\varphi\in\mathcal{A}} \int_{[0,\infty)\times S^1} \hat{u}^* d(\varphi\alpha) \\
&= \sup_{\varphi\in\mathcal{A}} \lim_{R\to\infty} \int_{[0,R]\times S^1} \hat{u}^* d(\varphi\alpha) \\
&= \sup_{\varphi\in\mathcal{A}} \left[\int_{\{0\}\times S^1} \varphi(\hat{a})\hat{f}^*\alpha - \lim_{R\to\infty} \int_{\{R\}\times S^1} \varphi(\hat{a})\hat{f}^*\alpha \right].
\end{aligned}$$

From Lemma 21 it follows that $\overline{u}$ has a bounded energy. Application of the usual removable singularity theorem (see Lemma 5.5 of [6]) then finishes the proof of the proposition.

□

In a neighborhood of a puncture, the map a is either bounded or unbounded. In the first case, Proposition 20 can be used to extend the $\mathcal{H}-$holomorphic curve across the puncture. In the second case, in which $a : D\backslash\{0\} \to \mathbb{R}$ is unbounded, we have the following result.

Proposition 22. *Let $(D, i, \{0\}, u, \gamma)$ be a $\mathcal{H}-$holomorphic curve defined on $D\backslash\{0\}$ such that the image of u is unbounded in $\mathbb{R} \times M$. Then u is asymptotic to a trivial cylinder over a periodic orbit of X_α, i.e. after identifying $D\backslash\{0\}$ with the half open cylinder $[0, \infty) \times S^1$ there exists a periodic orbit x of period $|T|$ of X_α, where $T \neq 0$ such that*

$$\lim_{s\to\infty} f(s,t) = x(Tt) \textit{ and } \lim_{s\to\infty} \frac{a(s,t)}{s} = T \textit{ in } C^\infty(S^1)$$

where (s,t) denote the coordinates on $[0,\infty) \times S^1$.

Proof. As we restrict the curve to the disk, the harmonic perturbations γ are exact, i.e. there exists a harmonic function Γ defined on the unit open disk such that $\gamma = d\Gamma$. The new curve $\overline{u} = (\overline{a}, \overline{f}) = (a + \Gamma, f)$ is pseudoholomorphic. Let

$$\begin{aligned}
\psi : \mathbb{R}_+ \times S^1 &\to D\backslash\{0\} \\
(s,t) &\mapsto e^{-2\pi(s+it)}
\end{aligned}$$

be a biholomorphism, which maps $D\backslash\{0\}$ to the half open cylinder $\mathbb{R}_+ \times S^1$. We consider the pseudoholomorphic curve $\overline{u}$ as being defined on the half open cylinder $\mathbb{R}_+ \times S^1$ with finite energy and having an unbounded image in

$\mathbb{R} \times M$. Since the contact structure is non-degenerate, we obtain by Proposition 5.6 of [6], that there exist $T \neq 0$ and a periodic orbit x of X_α of period $|T|$ such that

$$\lim_{s \to +\infty} \overline{f}(s,t) = x(Tt) \text{ and } \lim_{s \to +\infty} \frac{\overline{a}(s,t)}{s} = T \text{ in } C^\infty(S^1).$$

By the boundedness of the harmonic function Γ, we have

$$\lim_{s \to +\infty} \frac{a(s,t)}{s} = T \text{ in } C^\infty(S^1).$$

Thus the proof of the proposition is finished.

□

The puncture $p \in \mathcal{P}$ is called *positive* or *negative* depending on the sign of the coordinate function a when approaching the puncture. Note that the holomorphic coordinates near the puncture affects only the choice of the origin on the orbit x of X_α; the parametrization of the asymptotic orbits induced by the holomorphic polar coordinates remains otherwise the same. Hence, the orientation induced on x by the holomorphic coordinates coincides with the orientation defined by the vector field X_α if and only if the puncture is positive.
Let $S^{\mathcal{P}}$ be the oriented blow-up of S at the punctures $\mathcal{P} = \{p_1, ..., p_k\}$ as defined in the previous section or in Section 4.3 of [6]. $S^{\mathcal{P}}$ is a compact surface with boundary circles $\Gamma_1, ..., \Gamma_k$. Noting that each of these circles is endowed with a canonical S^1-action and letting $\varphi_i : S^1 \to \Gamma_i$ be (up to a choice of the base point) the canonical parametrization of the boundary circle Γ_i, for $i = 1, ..., k$, we reformulate Proposition 22 as follows.

Proposition 23. *Let* $(S, j, \mathcal{P}, u, \gamma)$ *be a* $\mathcal{H}-$*holomorphic map without removable singularities. Then the map* $f : \dot{S} \to M$ *extends to a continous map* $\overline{f} : S^{\mathcal{P}} \to M$ *such that*

$$\overline{f}(\varphi_i(e^{2\pi i t})) = x_i(Tt), \tag{2.2.5}$$

where $x_i : S^1 = \mathbb{R}/\mathbb{Z} \to M$ *is a periodic orbit of the Reeb vector field* X_α *of period* $|T|$, *where* $T \neq 0$, *parametrized by the vector field* X_α. *The sign of* T *coincides with the sign of the puncture* $p_i \in \mathcal{P}$.

2.3 Stratified $\mathcal{H}-$holomorphic buildings

In this section we introduce the notion of a stratified $\mathcal{H}-$holomorphic building. These are the objects which are needed for the compactification of the moduli space of $\mathcal{H}-$holomorphic curves. In the first step of our analysis we define a $\mathcal{H}-$holomorphic building of height 1. Then we introduce the general notion of a $\mathcal{H}-$holomorphic building of height greater than 1, describe the notion of convergence of a sequence of $\mathcal{H}-$holomorphic curves to a stratified $\mathcal{H}-$holomorphic building, and finally, state the main result.
Let (S, j) be a Riemann surface, and $\underline{\mathcal{P}} \subset S$ and $\overline{\mathcal{P}} \subset S$ two disjoint unordered finite subsets called the sets of *negative* and *positive punctures,* respectively. Let $\underline{\mathcal{P}} = \{\underline{p}_1, ..., \underline{p}_l\}$, $\overline{\mathcal{P}} = \{\overline{p}_1, ..., \overline{p}_f\}$ and $\mathcal{P} = \underline{\mathcal{P}} \amalg \overline{\mathcal{P}}$. The set of *nodal points*, defined by

$$\mathcal{D} = \{d_1', d_1'', ..., d_k', d_k''\} \subset S,$$

is a finite subset of S, where the pair $\{d_i', d_i''\}$ will be called *node* (see Figure 2.3.1). Denote by $S^{\mathcal{P}}$ the blow-up of the surface $\dot{S} = S \backslash \mathcal{P}$ at the punctures $\mathcal{P}$. The surface $S^{\mathcal{P}}$ has $|\mathcal{P}|$ boundary components, which due to the splitting of $\mathcal{P}$, are denoted by $\underline{\Gamma} = \{\underline{\Gamma}_1, ..., \underline{\Gamma}_l\}$ and $\overline{\Gamma} = \{\overline{\Gamma}_1, ..., \overline{\Gamma}_f\}$ (see Figure 2.3.2).

Definition 24. $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma, \tau, \sigma)$, where $\tau = \{\tau_i\}_{i=1,...,|\mathcal{D}|/2}$, $\sigma = \{\sigma_i\}_{i=1,...,|\mathcal{D}|/2}$ and $\tau_i, \sigma_i \in \mathbb{R}$ for all $i = 1, ..., |\mathcal{D}|/2$ is called a *stratified* $\mathcal{H}-$*holomorphic building of height* 1 if the following conditions are satisfied.

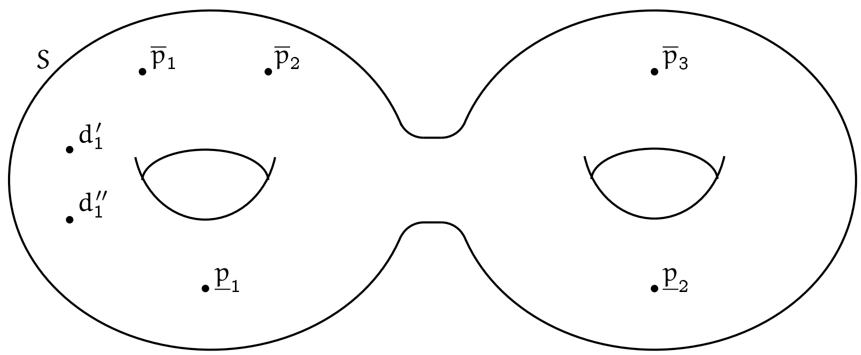

Figure 2.3.1: Surface S with punctures $\mathcal{P} = \{\overline{p}_1, \overline{p}_2, \overline{p}_3\} \amalg \{\underline{p}_1, \underline{p}_2\}$ and nodes $\mathcal{D} = \{d_1', d_1''\}$.

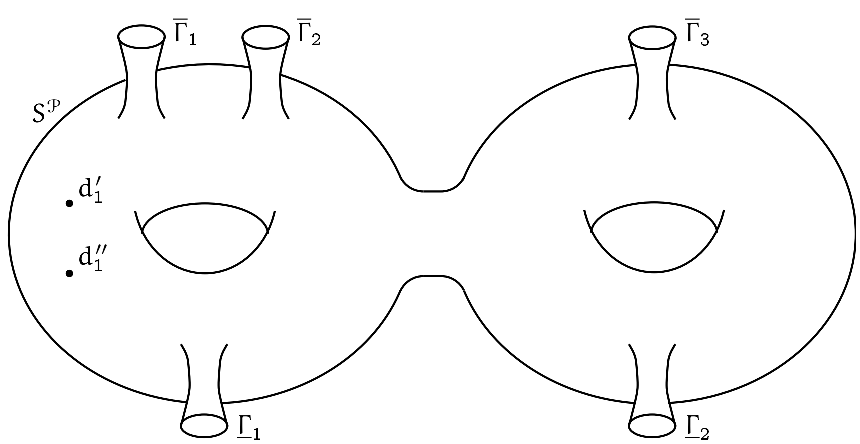

Figure 2.3.2: Blow-up surface $S^{\mathcal{P}}$ with boundary components $\Gamma = \{\overline{\Gamma}_1, \overline{\Gamma}_2, \overline{\Gamma}_3\} \amalg \{\underline{\Gamma}_1, \underline{\Gamma}_2\}$ and nodes $\mathcal{D} = \{d_1', d_1''\}$.

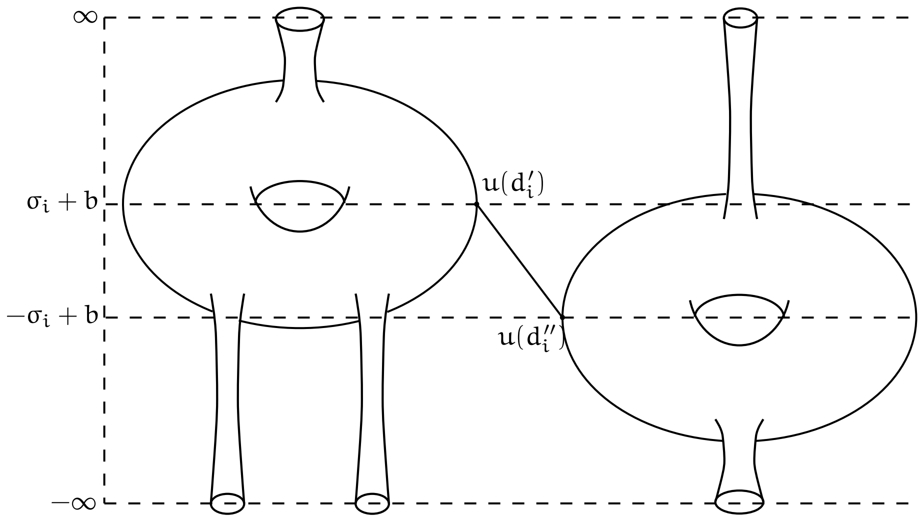

Figure 2.3.3: A stratified $\mathcal{H}-$holomorphic building of height 1.

1. $(S, j, u, \mathcal{P}, \gamma)$ is a $\mathcal{H}-$holomorphic curve as in Definition 16.
2. For each $\{d_i', d_i''\} \in \mathcal{D}$, $\tau_i, \sigma_i \in \mathbb{R}$ the points $u(d_i')$ and $u(d_i'')$ are connected by the map $[-1/2, 1/2] \to \mathbb{R} \times M$, $s \mapsto (-2\sigma_i s + b, \phi^{\alpha}_{-2\tau_i s}(w_f))$ for some $b \in \mathbb{R}$ and $w_f \in M$ such that $u(d_i') = (\sigma_i + b, \phi^{\alpha}_{\tau_i}(w_f))$ and $u(d_i'') = (-\sigma_i + b, \phi^{\alpha}_{-\tau_i}(w_f))$.

See Figure 2.3.3.

Remark 25. The $M-$component $f : \dot{S} \to M$ of a stratified $\mathcal{H}-$holomorphic building $u = (a, f) : \dot{S} \to \mathbb{R} \times M$ of height 1 can be continously extended to $S^{\mathcal{P}}$. For the extension $\overline{f} : S^{\mathcal{P}} \to M$, it is apparent that $\overline{f}|_{\Gamma}$, where $\Gamma = \underline{\Gamma} \amalg \overline{\Gamma}$, defines parametrizations of Reeb orbits.

Remark 26. The energy of a $\mathcal{H}-$holomorphic building of height 1 is the sum of the $\alpha-$ and $d\alpha-$energies of the $\mathcal{H}-$holomorphic curve, as defined in (1.0.15).

In a second step we define a stratified $\mathcal{H}-$holomorphic building of height N. Let $(S_1, j_1), ..., (S_N, j_N)$ be closed (possibly disconected) Riemann surfaces, and for any $i \in \{1, ..., N\}$, let $\underline{\mathcal{P}}_i = \{\underline{p}_{ij}\} \subset S_i$ and $\overline{\mathcal{P}}_i = \{\overline{p}_{ij}\} \subset S_i$ be the sets of *negative* and *positive punctures on level* i, respectively. We further assume that there is a one-to-one correspondence between the elements $\overline{\mathcal{P}}_{i-1}$ and $\underline{\mathcal{P}}_i$ given by a bijective map $\varphi_i : \overline{\mathcal{P}}_{i-1} \to \underline{\mathcal{P}}_i$. A pair $\{\overline{p}_{i-1,j}, \underline{p}_{ij}\}$, where $\underline{p}_{ij} = \varphi_i(\overline{p}_{i-1,j})$, is called a *breaking point* between the levels S_{i-1} and S_i.
Let $\mathcal{P} = \coprod_{i=1}^{N} \underline{\mathcal{P}}_i \amalg \overline{\mathcal{P}}_i$ be the set of *punctures,* $\mathcal{P}_i = \underline{\mathcal{P}}_i \amalg \overline{\mathcal{P}}_i$ the set of *punctures at level* i,

$$\mathcal{D}_i = \{d_{i1}', d_{i1}'', ..., d_{ik_i}', d_{ik_i}''\}$$

the set of *nodes at level* i, and $\mathcal{D} = \coprod_{i=1}^{N} \mathcal{D}_i$ the set of all nodes (see Figure 2.3.4).
If $S_i^{\mathcal{P}_i}$ is the blow-up of S_i at the punctures $\mathcal{P}_i = \underline{\mathcal{P}}_i \amalg \overline{\mathcal{P}}_i$, then accounting of the splitting of the punctures $\mathcal{P}_i$, we denote the boundary components of $S_i^{\mathcal{P}_i}$ by $\underline{\Gamma}_i$ and $\overline{\Gamma}_i$; they correspond to the negative and positive punctures $\underline{\mathcal{P}}_i$ and $\overline{\mathcal{P}}_i$, respectively. There is a one-to-one correspondence between the elements of $\overline{\Gamma}_{i-1}$ and $\underline{\Gamma}_i$ given by an orientation reversing diffeomorphism $\Phi_i : \overline{\Gamma}_{i-1} \to \underline{\Gamma}_i$. A pair $\{\overline{\Gamma}_{i-1,j}, \underline{\Gamma}_{ij}\}$, where $\underline{\Gamma}_{ij} = \Phi_i(\overline{\Gamma}_{i-1,j})$, is called a *breaking orbit* for all $i = 2, ..., N$. This gives an identification of the boundary components $\overline{\Gamma}_{i-1}$ from $S_{i-1}^{\mathcal{P}_{i-1}}$ and the boundary components $\underline{\Gamma}_i$ from $S_i^{\mathcal{P}_i}$ (see Figure 2.3.5). Further on, let

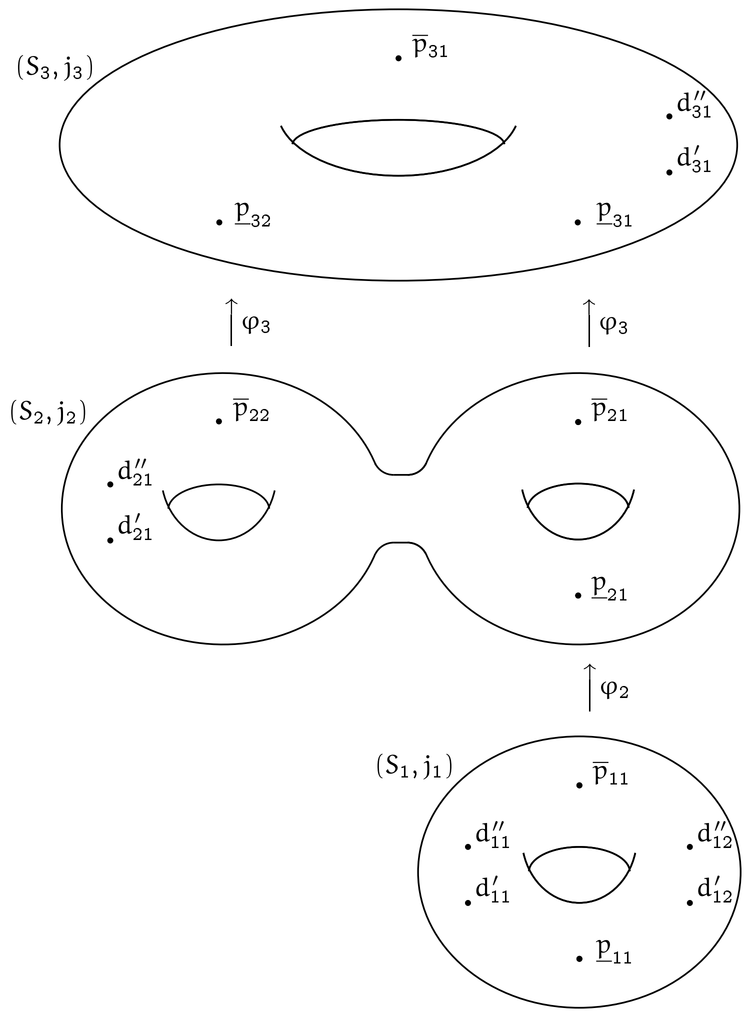

Figure 2.3.4: The Riemann surface $(S, j) = (S_1, j_1) \amalg (S_2, j_2) \amalg (S_3, j_3)$ with punctures $\underline{\mathcal{P}}_1 \amalg \overline{\mathcal{P}}_1 = \{\underline{\mathcal{P}}_{11}\} \amalg \{\overline{\mathcal{P}}_{11}\}$, $\underline{\mathcal{P}}_2 \amalg \overline{\mathcal{P}}_2 = \{\underline{p}_{21}\} \amalg \{\overline{p}_{21}, \overline{p}_{22}\}$ and $\underline{\mathcal{P}}_3 \amalg \overline{\mathcal{P}}_3 = \{\underline{p}_{31}, \underline{p}_{32}\} \amalg \{\overline{p}_{31}\}$, nodes $\mathcal{D}_1 = \{d'_{11}, d''_{11}, d'_{12}, d''_{12}\}$, $\mathcal{D}_2 = \{d'_{21}, d''_{21}\}$ and $\mathcal{D}_3 = \{d'_{31}, d''_{31}\}$ and the maps φ_2 and φ_3.

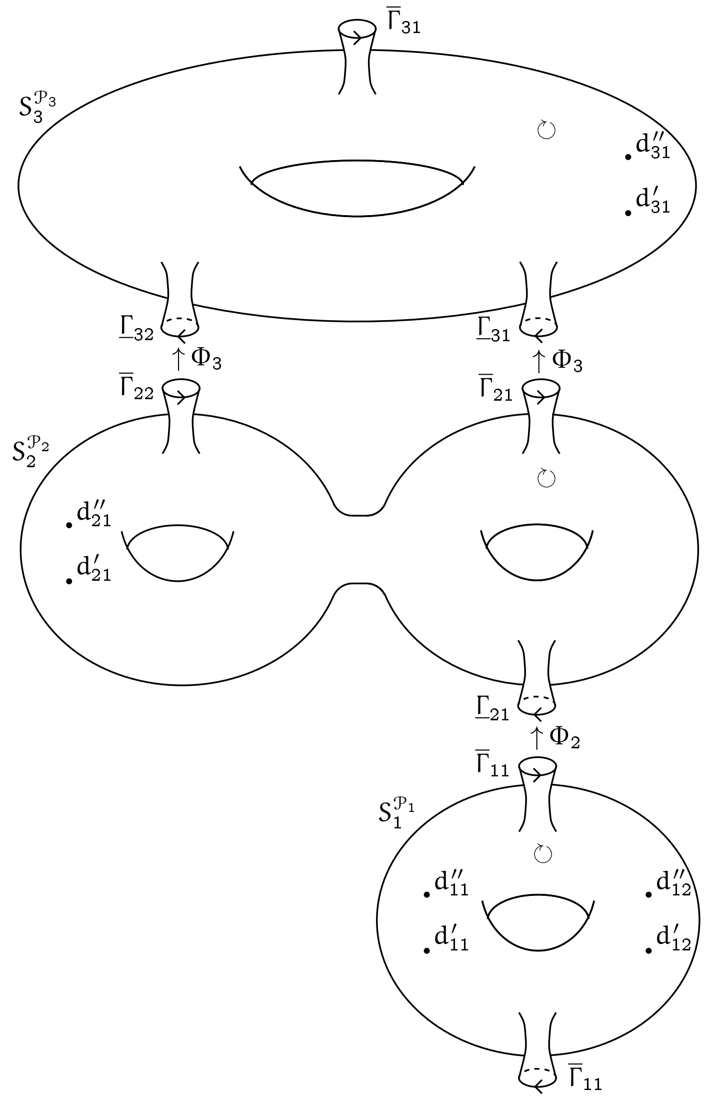

Figure 2.3.5: The surface $S^{\mathcal{P}} = S_1^{\mathcal{P}_1} \amalg S_2^{\mathcal{P}_2} \amalg S_3^{\mathcal{P}_3}$ with boundary components $\underline{\Gamma}_1 \amalg \overline{\Gamma}_1 = \{\underline{\Gamma}_{11}\} \amalg \{\overline{\Gamma}_{11}\}$, $\underline{\Gamma}_2 \amalg \overline{\Gamma}_2 = \{\underline{\Gamma}_{21}\} \amalg \{\overline{\Gamma}_{21}, \overline{\Gamma}_{22}\}$ and $\underline{\Gamma}_3 \amalg \overline{\Gamma}_3 = \{\underline{\Gamma}_{31}, \underline{\Gamma}_{32}\} \amalg \{\overline{\Gamma}_{31}\}$, nodes $\mathcal{D}_1 = \{d'_{11}, d''_{11}, d'_{12}, d''_{12}\}$, $\mathcal{D}_2 = \{d'_{21}, d''_{21}\}$ and $\mathcal{D}_3 = \{d'_{31}, d''_{31}\}$ and orientation reversing diffeomorphisms Φ_2 and Φ_3.

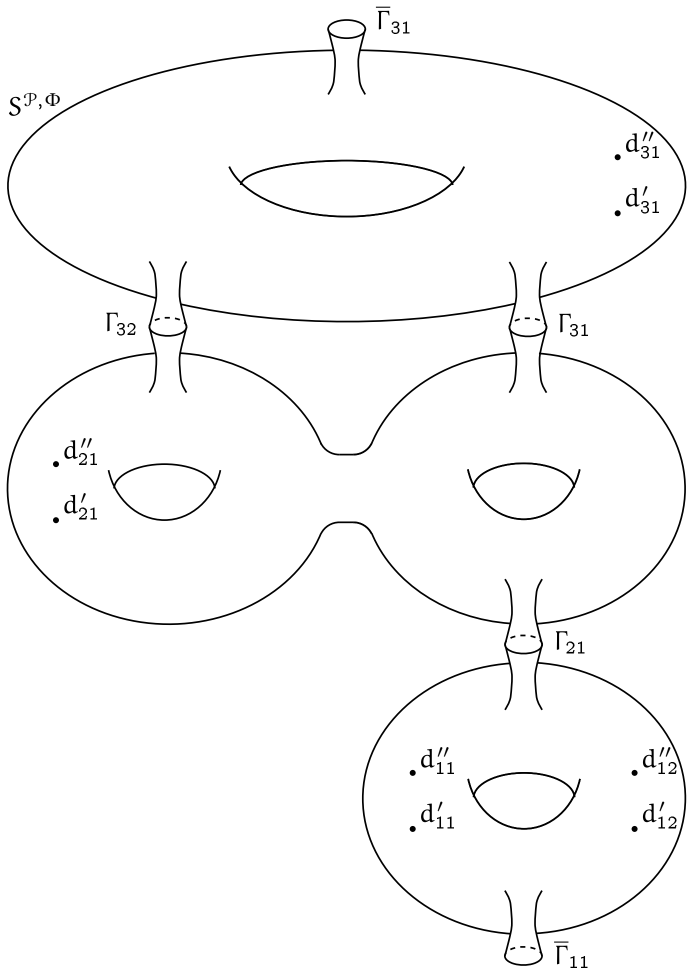

Figure 2.3.6: The surface $S^{\mathcal{P},\Phi}$ with nodes $\mathcal{D}_1 = \{d'_{11}, d''_{11}, d'_{12}, d''_{12}\}$, $\mathcal{D}_2 = \{d'_{21}, d''_{21}\}$ and $\mathcal{D}_3 = \{d'_{31}, d''_{31}\}$ and boundary circles $\underline{\Gamma}_{11}$ and $\overline{\Gamma}_{31}$.

$$S^{\mathcal{P},\Phi} := S_1^{\mathcal{P}_1} \cup_{\Phi_2} S_2^{\mathcal{P}_2} \cup_{\Phi_3} \dots \cup_{\Phi_N} S_N^{\mathcal{P}_N} := \left(\coprod_{i=1}^N S_i^{\mathcal{P}_i} \right) /\sim$$

where $\sim$ is defined by identifying the circles $\overline{\Gamma}_{i-1,j}$ and $\underline{\Gamma}_{ij}$ via the diffeomorphism for all $i = 2, \dots, N$ and $j = 1, \dots, |\underline{\mathcal{P}}_i|$. Obviously, $S^{\mathcal{P},\Phi}$ is a compact surface with $|\underline{\mathcal{P}}_1| + |\overline{\mathcal{P}}_N|$ boundary components. The equivalence class of $\overline{\Gamma}_{i-1,j}$ in $S^{\mathcal{P},\Phi}$, denoted by Γ_{ij} for all $i = 2, \dots, N$ and $j = 1, \dots, |\underline{\mathcal{P}}_i|$, is called a *special circle*; the collection of all special circles is denoted by Γ (see Figure 2.3.6). A tuple $(S, j, \mathcal{P}, \mathcal{D})$ with the properties described above will be called a *broken building of height* N.
We are now well prepared to introduce a stratified $\mathcal{H}-$homolomorphic building of height N.

Definition 27. A tuple $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma, \tau, \sigma)$, where $\tau = \{\hat{\tau}_{ij_i} \mid i = 1, \dots, N \text{ and } j_i = 1, \dots, |\mathcal{D}_i|/2\} \cup \{\tau_{ij_i} \mid i = 1, \dots, N-1 \text{ and } j_i = 1, \dots, |\overline{\Gamma}_i|\}$, $\sigma = \{\hat{\sigma}_{ij_i} \mid i = 1, \dots, N \text{ and } j_i = 1, \dots, |\mathcal{D}_i|/2\}$ and $(S, j, \mathcal{P}, \mathcal{D})$ is a broken building of height N, is called a *stratified $\mathcal{H}-$holomorphic building of height* N if the following are satisfied:

1. For any $i = 1, \dots, N$, $(S_i, j_i, u_i, \underline{\mathcal{P}}_i \amalg \overline{\mathcal{P}}_i, \mathcal{D}_i, \gamma_i, \{\hat{\tau}_{ij_i} \mid j_i = 1, \dots, |\mathcal{D}_i|/2\}, \{\hat{\sigma}_{ij_i} \mid j_i = 1, \dots, |\mathcal{D}_i|/2\})$ is a stratified $\mathcal{H}-$holomorphic building of height 1, where $u_i = u|_{S_i \setminus \mathcal{P}_i}$, and j_i is the complex structure on S_i.

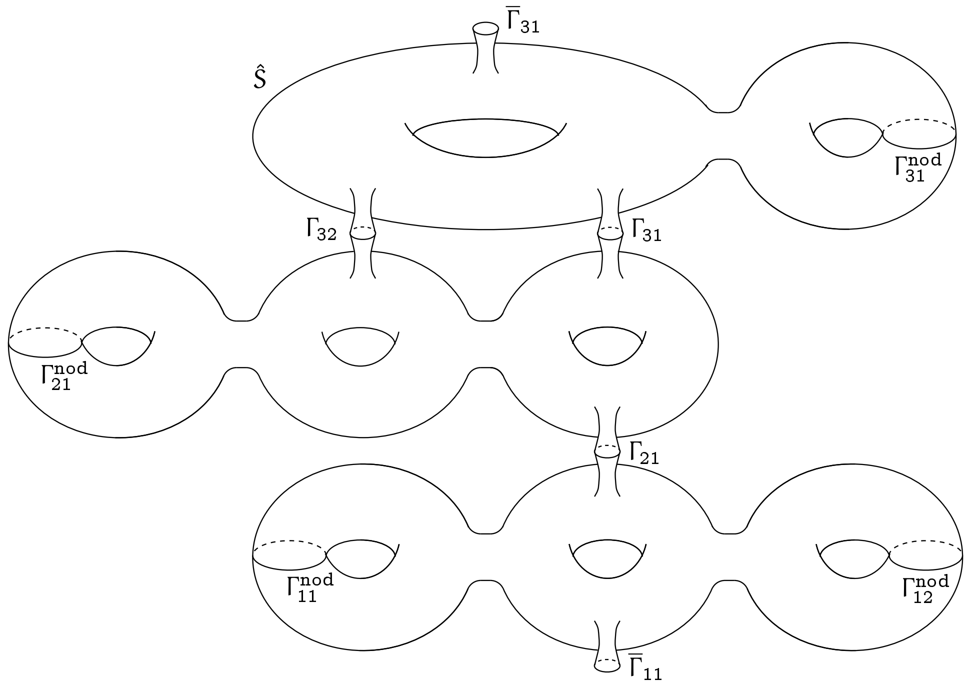

Figure 2.3.7: The surface $\hat{S}$ with boundary circles $\underline{\Gamma}_{11}$ and $\overline{\Gamma}_{31}$, special circles Γ_{21}, Γ_{31} and Γ_{32} and nodal special circles $\Gamma^{nod}_{11}, \Gamma^{nod}_{12}, \Gamma^{nod}_{21}$ and Γ^{nod}_{31}.

2. For all breaking points $\{\overline{p}_{i-1,j}, \underline{p}_{ij}\}$ and $\tau_{ij} \in \tau$, there exist $T_{ij} > 0$ such that the $\mathcal{H}-$holomorphic building of height 1, $u_{i-1} : \dot{S}_{i-1} \to \mathbb{R} \times M$ is asymptotic at $\overline{p}_{i-1,j}$ to a trivial cylinder over the Reeb orbit x_{ij} of period $T_{ij} > 0$, and $u_i : \dot{S}_i \to \mathbb{R} \times M$ is asymptotic at $\underline{p}_{ij}$ to the trivial cylinder over the Reeb orbit $x_{ij}(\cdot + \tau_{ij})$ of period $-T_{ij} < 0$.

Remark 28. The energy of a stratified $\mathcal{H}-$holomorphic buidling of height N is defined by

$$E(u) = \max_{1 \leqslant i \leqslant N} E_{\alpha}(u_i) + \sum_{i=1}^{N} E_{d\alpha}(u_i).$$

We come now to the convergence issue. Let let $S_i^{\mathcal{P}_i \cup \mathcal{D}_i}$ be the blow-up of S_i at the punctures $\mathcal{P}_i$ and nodes $\mathcal{D}_i$. To each pair of nodes $\{d'_{ij}, d''_{ij}\}$, the corresponding boundary of $S_i^{\mathcal{P}_i \cup \mathcal{D}_i}$ is denoted by $\{\Gamma'_{ij}, \Gamma''_{ij}\}$, and for each such pair of boundary circles, let $r_{ij} : \Gamma'_{ij} \to \Gamma''_{ij}$ be orientation reversing diffeomorphisms. The diffeomorphisms r_{ij} are used to glue the boundary circles Γ'_{ij} and Γ''_{ij} together. Consider the surface $\hat{S} := S^{\mathcal{P} \cup \mathcal{D}, \Phi \cup r}$ which is obtained from S by blowing-up the punctures $\mathcal{P}$ and the nodes $\mathcal{D}$, and by using the orientation reversing diffomorphisms Φ and r_{ij}. $\hat{S}$ is a compact surface with boundary components given by the sets $\underline{\Gamma}_1$ and $\overline{\Gamma}_N$. The equivalence class of Γ'_{ij} in $\hat{S}$ is denoted by Γ^{nod}_{ij} and is called *nodal special circles*; the set of all nodal special circles is denoted by Γ^{nod} (see Figure 2.3.7).

The *collar blow-up* $\overline{S}$ is a modification of the usual blow-up $\hat{S}$ defined in [6]. Essentially, we insert the cylinders $[-1/2, 1/2] \times S^1$ between the special circles $\overline{\Gamma}_{i-1,j}$ and $\underline{\Gamma}_{ij}$, and between the nodal special circles Γ'_{ij} and Γ''_{ij}. To obtain a surface with boundary components $\underline{\Gamma}_1$ and $\overline{\Gamma}_N$ that has the same topology as $\hat{S}$ we modify the orientation reversing the diffeomorphismsm Φ_{ij} and r_{ij} as follows:

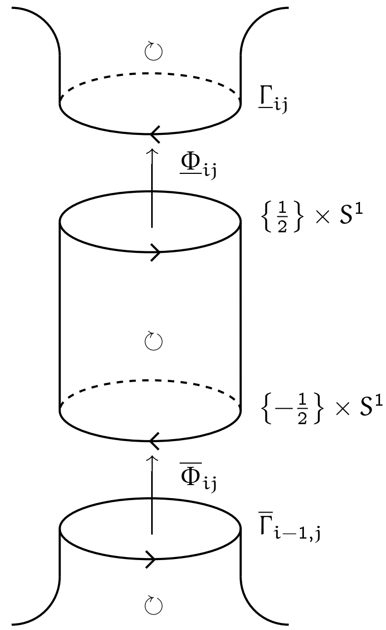

Figure 2.3.8: The glueing of $\overline{\Gamma}_{i-1,j}$, the cylinder $[-1/2, 1/2] \times S^1$ and $\underline{\Gamma}_{ij}$ via the orientation reversing diffeomorphisms $\overline{\Phi}_{ij} : \overline{\Gamma}_{i-1,j} \to \{-1/2\} \times S^1$ and $\underline{\Phi}_{ij} : \{1/2\} \times S^1 \to \underline{\Gamma}_{i,j}$.

A1 The orientation reversing diffeomorphisms Φ_{ij} correspond to two orientation reversing diffeomorphisms $\overline{\Phi}_{ij} : \overline{\Gamma}_{i-1,j} \to \{-1/2\} \times S^1$ and $\underline{\Phi}_{ij} : \{1/2\} \times S^1 \to \underline{\Gamma}_{ij}$ for all $i = 2, ..., N$ and $j = 1, ..., |\underline{\mathcal{P}}_i|$.

A2 Instead of glueing $\overline{\Gamma}_{i-1,j}$ and $\underline{\Gamma}_{ij}$ via the orientation reversing diffeomorphisms Φ_{ij}, we glue $\overline{\Gamma}_{i-1,j}$, the cylinder $[-1/2, 1/2] \times S^1$, and $\underline{\Gamma}_{ij}$ via the orientation reversing diffeomorphisms $\overline{\Phi}_{ij}$ and $\underline{\Phi}_{ij}$ (see Figure 2.3.8).

A3 For the nodal special circles Γ'_{ij} and Γ''_{ij}, we proceed analogously, and denote by $r'_{ij} : \Gamma'_{ij} \to \{-1/2\} \times S^1$ and $r''_{ij} : \{1/2\} \times S^1 \to \Gamma''_{ij}$ the orientation reversing diffeomorphisms that glue Γ'_{ij}, the cylinder $[-1/2, 1/2] \times S^1$ and Γ''_{ij} together.

Let $\overline{S}$ be the surface obtained by applying the above construction to all special and nodal special circles. The equivalence class of the cylinder $[-1/2, 1/2] \times S^1$ in $\overline{S}$ corresponding to the special circle Γ_{ij} is denoted by A_{ij}, and is called *special cylinder*. The equivalence class of the cylinder $[-1/2, 1/2] \times S^1$ in $\overline{S}$ corresponding to the nodal special circle Γ^{nod}_{ij} is denoted by A^{nod}_{ij}, and is called *nodal special cylinder*. The boundary circles of A_{ij} are still denoted by $\overline{\Gamma}_{i-1,j}$ and $\underline{\Gamma}_{ij}$, while the boundary circles of A^{nod}_{ij} are also still denoted by Γ'_{ij} and Γ''_{ij}. Finally, the collections of all special and nodal special cylinders are denoted by A and A^{nod}, respectively. Take notice that there exists a natural projection between the collar blow-up $\overline{S}$ and the blow-up surface $\hat{S}$, which is defined similarly to [6], i.e. it maps $\overline{S} \backslash (A \amalg A^{nod})$ diffeomorphically to $\hat{S} \backslash (\Gamma \amalg \Gamma^{nod})$ and the annuli A and A^{nod} are mapped to Γ and Γ^{nod}. This induces a conformal structure on $\overline{S} \backslash (A \amalg A^{nod})$. Let $\tilde{S}$ be the closed surface obtained from $\overline{S}$ by identifying the boundary components $\underline{\Gamma}_1$ and $\overline{\Gamma}_N$ to points, i.e. by reversing the blow-up.
Having now a stratified $\mathcal{H}-$holomorphic building $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma, \tau, \sigma)$ of height N, we define the continous extension $\overline{f}$ of f on the surface $\overline{S}$ and the continous extension $\overline{a}$ of a on $\overline{S} \backslash A$. The extension $\overline{f}$ may be defined on the clinders

A_{ij} and A_{ij}^{nod}, while the extension $\overline{a}$ is defined only on A_{ij}^{nod}. Set

$$\overline{f}(s,t) = \phi^{\alpha}_{-2s\hat{\tau}_{ij}}(w_f), \quad \text{for all } (s,t) \in A_{ij}^{nod} = [-1/2, 1/2] \times S^1,$$
$$\overline{f}(s,t) = \phi^{\alpha}_{-(s+\frac{1}{2})\tau_{ij}}(x_{ij}(T_{ij}t)), \quad \text{for all } (s,t) \in A_{ij} = [-1/2, 1/2] \times S^1$$

and

$$\overline{a}(s,t) = 2\hat{\sigma}_{ij}s + b, \quad \text{for all } (s,t) \in A_{ij}^{nod} = [-1/2, 1/2] \times S^1$$

for some $b \in \mathbb{R}$ and $w_f \in M$. Here x_{ij} is the Reeb orbit of period $T_{ij} > 0$.
We are now in the position to introduce the notion of convergence.

Definition 29. A sequence of $\mathcal{H}-$holomorphic curves $(S_n, j_n, u_n, \mathcal{P}'_n = \underline{\mathcal{P}}'_n \amalg \overline{\mathcal{P}}'_n, \gamma_n)$ converges in the C^∞_{loc} sense to a $\mathcal{H}-$holomorphic curve $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma)$, if the tuple $(S, j, \mathcal{P}, \mathcal{D})$ is a broken building of height N and there exists a sequence of diffeomorphisms $\varphi_n : \tilde{S} \to S_n$, where $\tilde{S}$ is the modified collar blow-up as defined above, such that $\varphi_n^{-1}(\overline{\mathcal{P}}'_n) = \overline{\mathcal{P}}_1$ and $\varphi_n^{-1}(\underline{\mathcal{P}}'_n) = \underline{\mathcal{P}}_N$ and such that the following conditions are satisfied:

1. The sequence of complex structures $(\varphi_n)_* j_n$ converges in C^∞_{loc} on $\tilde{S} \backslash (A \amalg A^{nod})$ to j.
2. The special circles of $(S_n, j_n, \mathcal{P}_n)$ are mapped by φ_n^{-1} bijectively onto $\{0\} \times S^1$ of A_{ij} or A_{ij}^{nod}. For every special cylinder A_{ij} there exists an annulus $\overline{A}_{ij} \cong [-1, 1] \times S^1$ such that $A_{ij} \subset \overline{A}_{ij}$ and $(\overline{A}_{ij}, (\varphi_n)_* j_n)$ and $(A_{ij}, (\varphi_n)_* j_n)$ are conformally equivalent to $([-R_n, R_n] \times S^1, i)$ and $([-R_n + h_n, R_n - h_n] \times S^1, i)$, respectively, where $R_n, h_n, R_n/h_n \to \infty$ as $n \to \infty$, i is the standard complex structure and the diffeomorphisms are of the form $(s,t) \mapsto (\kappa(s), t)$.
3. The $\mathcal{H}-$holomorphic curves $u_n \circ \varphi_n : \dot{\tilde{S}} := \tilde{S} \backslash (\overline{\mathcal{P}}_1 \amalg \underline{\mathcal{P}}_N) \to \mathbb{R} \times M$ together with the harmonic perturbation $(\varphi_n)^* \gamma_n$ which are defined on $\tilde{S}$ converge in C^∞_{loc} on $\dot{\tilde{S}} \backslash \left(A \amalg A^{nod}\right)$ to the $\mathcal{H}-$holomorphic curve u with harmonic perturbation γ. Note that $\dot{\tilde{S}} \backslash (A \amalg A^{nod})$ may be conformally identified with $S \backslash (\mathcal{P} \amalg \mathcal{D})$.

Next we describe the C^0-convergence. Let $(S_n, j_n, u_n, \mathcal{P}'_n, \gamma_n)$ be a sequence of $\mathcal{H}-$holomorphic curves. For any special circle Γ_{ij}, let $\tau^n_{ij} \in \mathbb{R}$ and $\sigma^n_{ij} \in \mathbb{R}$ be the conformal period of $\varphi_n^* \gamma_n$ on Γ_{ij} with respect to the complex structure $\varphi_n^* j_n$, and the conformal co-period of $\varphi_n^* \gamma_n$ on Γ_{ij} with respect to the complex structure $\varphi_n^* j_n$, respectively. For any nodal special circle Γ^{nod}_{ij} consider the numbers $\hat{\tau}^n_{ij} \in \mathbb{R}$ and $\hat{\sigma}^n_{ij} \in \mathbb{R}$, where $\hat{\tau}^n_{ij}$ is the conformal period of $\varphi_n^* \gamma_n$ on Γ^{nod}_{ij} with respect to the complex structure $\varphi_n^* j_n$, and $\hat{\sigma}^n_{ij}$ is the conformal co-period of $\varphi_n^* \gamma_n$ on Γ^{nod}_{ij} with respect to the complex structure $\varphi_n^* j_n$, respectively.

Remark 30. For a sequence $(S_n, j_n, u_n, \mathcal{P}'_n, \gamma_n)$ of $\mathcal{H}-$holomorphic curves that converges to a $\mathcal{H}-$holomorphic curve $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma)$ in the sense of Definition 29, the quantities τ^n_{ij}, σ^n_{ij}, $\hat{\tau}^n_{ij}$ and $\hat{\sigma}^n_{ij}$ can be unbounded (see, e.g, Chapter 4). If τ^n_{ij}, σ^n_{ij}, $\hat{\tau}^n_{ij}$ and $\hat{\sigma}^n_{ij}$ are bounded, then after going over to a further subsequence, and assuming that there exist the real numbers $\tau_{ij}, \sigma_{ij}, \hat{\tau}_{ij}, \hat{\sigma}_{ij} \in \mathbb{R}$ such that

$$\tau^n_{ij} \to \tau_{ij}, \tag{2.3.1}$$
$$\sigma^n_{ij} \to \sigma_{ij}, \tag{2.3.2}$$
$$\hat{\tau}^n_{ij} \to \hat{\tau}_{ij}, \tag{2.3.3}$$
$$\hat{\sigma}^n_{ij} \to \hat{\sigma}_{ij} \tag{2.3.4}$$

as $n \to \infty$, we are able to derive a C^0- convergence result.

The convergence of a sequence of $\mathcal{H}-$holomorphic curves to a stratified $\mathcal{H}-$holomorphic building of height N should be understood in the following sense:

Definition 31. A sequence of $\mathcal{H}-$holomorphic curves $(S_n, j_n, \mathcal{P}'_n, u_n, \gamma_n)$ converges in the C^0 sense to a statified $\mathcal{H}-$holomorphic building $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma, \tau, \sigma)$ of height N if the following conditions are satisfied.

1. The parameters τ^n_{ij}, σ^n_{ij}, $\hat{\tau}^n_{ij}$ and $\hat{\sigma}^n_{ij}$ converge as in (2.3.1)-(2.3.4).
2. The sequence $(S_n, j_n, \mathcal{P}'_n, u_n, \gamma_n)$ converges to the underlying $\mathcal{H}-$holomorphic curve $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma)$ in the sense of Definition 29 with respect to a sequence of diffeomorphisms $\varphi_n : \tilde{S} \to S_n$.
3. $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma, \tau, \sigma)$ is a stratified $\mathcal{H}-$holomorphic building of height N corresponding to the constants τ_{ij}, $\hat{\tau}_{ij}$ and $\hat{\sigma}_{ij}$, as in Definition 27.
4. The maps $u_n \circ \varphi_n$ converges in C^0_{loc} on $\dot{\tilde{S}} \backslash A$ to the blow-up map $\overline{u}$ defined on $\dot{\tilde{S}} \backslash A$.
5. The maps $f_n \circ \varphi_n$ converges in C^0 on $\overline{S}$ to the blow-up map $\overline{f}$ defined on $\overline{S}$.
6. $E(u_n; \dot{S}_n) \to E(u; \dot{S})$ as $n \to \infty$.

The compactness result will be established for finite energy $\mathcal{H}-$holomorphic curves with harmonic perturbation $1-$forms having uniformly bounded L^2-norms and uniformly bounded conformal periods and co-periods. Specifically, we will consider a sequence of $\mathcal{H}-$holomorphic curves $u_n = (a_n, f_n) : (S_n \backslash \mathcal{P}_n, j_n) \to \mathbb{R} \times M$ with harmonic perturbations γ_n, satisfying the following conditions:

B1 (S_n, j_n) are compact Riemann surfaces of the same genus and $\mathcal{P}_n \subset S_n$ is a finite set of punctures whose cardinality is independent of n.

B2 The energy of u_n, as well as the L^2-norm of γ_n are uniformly bounded by the constants $E_0 > 0$ and $C_0 > 0$, respectively.

Remark 32. For the sequence of punctured Riemann surfaces $(S_n, j_n, \mathcal{P}_n)$, the Deligne-Mumford convergence result implies that there exists a punctured nodal Riemann surface $(S, j, \mathcal{P}, \mathcal{D})$ and a sequence of diffeomorphisms $\varphi_n : S^{D,r} \to S_n$, such that $\varphi_n^* j_n$ converges outside certain circles in C^∞_{loc} to j. Here, $S^{D,r}$ is the surface obtained by blowing up the points from $\mathcal{D}$ and identifying them via the decoration r (see Section 2.1). Denote by Γ^{nod}_i, for $i = 1, ..., |\mathcal{D}|/2$, the equivalence classes of the boundary circles of $S^{\mathcal{D}}$in $S^{\mathcal{D},r}$. Let $\Gamma^{nod}_{n,i} = (\varphi_n)_* \Gamma^{nod}_i$ for all $n \in \mathbb{N}$ and $i = 1, ..., |\mathcal{D}|/2$.

The main result of our analysis is the following

Theorem 33. *Let $(S_n, j_n, u_n, \mathcal{P}_n, \gamma_n)$ be a sequence of $\mathcal{H}-$holomorphic curves in $\mathbb{R}\times M$ satisfying assumptions B1 and B2. Then there exists a subsequence that converges to a $\mathcal{H}-$holomorphic curve $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma)$ in the sense of Definition 29. Moreover, if there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ and all $1 \leqslant i \leqslant |\mathcal{D}|/2$ we have $|\tau_{[\Gamma^{nod}_{n,i}], \gamma_n}|, |\sigma_{[\Gamma^{nod}_{n,i}], \gamma_n}| < C$ then $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma)$ is a stratified broken $\mathcal{H}-$holomorphic building of height N and after going over to a subsequence the $\mathcal{H}-$holomorphic curves $(S_n, j_n, u_n, \mathcal{P}_n, \gamma_n)$ converges to $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma)$ in the sense of Definition 31.*

Part II

Proof of the compactness Theorem

Chapter 3

Proof of the Compactness Theorem

Let $(S_n, j_n, u_n, \mathcal{P}'_n, \gamma_n)$ be a sequence of $\mathcal{H}-$holomorphic curves satisfying Assumptions B1 and B2 from the end of Section 2.3. After introducing an additional finite set of points $\mathcal{M}_n$ disjoint from the set of punctures $\mathcal{P}'_n$ we assume that the domains $(S_n, j_n, \mathcal{P}'_n \amalg \mathcal{M}_n)$ of the sequence of $\mathcal{H}-$holomorphic curves are stable. This condition enables us to use the Deligne-Mumford convergence (see Section 2.1) which makes it possible to formulate a convergence result for the domains $(S_n, j_n, \mathcal{P}'_n \amalg \mathcal{M}_n)$. Note that $\mathcal{M}_n$ can be choosen in such a way that their cardinality is independent of the index n. As an additional structure, let h^{j_n} be the hyperbolic metric on $\dot{S}_n := S_n \backslash (\mathcal{P}'_n \amalg \mathcal{M}_n)$. By the Deligne-Mumford convergence result (Corollary 15) there exists a stable nodal decorated surface $(S, j, \mathcal{P} \amalg \mathcal{M}, \mathcal{D}, r)$ and a sequence of diffeomorphisms $\varphi_n : S^{\mathcal{D},r} \to S_n$, where $S^{\mathcal{D},r}$ is the closed surface obtained by blowing up the nodes and glueing pairs of nodal points according to the decoration r as described in Section 2.1, such that the following holds: Let h be the hyperbolic metric on $S \backslash (\mathcal{P} \amalg \mathcal{M} \amalg \mathcal{D})$. The diffeomorphisms φ_n map marked points into marked points and punctures into punctures, i.e. $\varphi_n(\mathcal{M}) = \mathcal{M}_n$ and $\varphi_n(\mathcal{P}) = \mathcal{P}'_n$. Via φ_n we pull-back the complex structures j_n and the hyperbolic metrics h^{j_n}, i.e. we define $j^{(n)} := \varphi_n^* j_n$ on $S^{\mathcal{D},r}$ and $h_n := \varphi_n^* h^{j_n}$ on $\dot{S}^{\mathcal{D},r} := S^{\mathcal{D},r} \backslash (\mathcal{M} \amalg \mathcal{P})$. By the Deligne-Mumford convergence, $h_n \to h$ in $C^\infty_{loc}(\dot{S}^{\mathcal{D},r} \backslash \coprod_j \Gamma_j)$ as $n \to \infty$, where Γ_j are the special circles in $S^{\mathcal{D},r}$ (see Section 2.1 for the definition of special circles) and by abuse of notation h denotes the hyperbolic metric on $\dot{S}^{\mathcal{D},r}$. This yields $j^{(n)} \to j$ in $C^\infty_{loc}(S^{\mathcal{D},r} \backslash \coprod_j \Gamma_j)$ as $n \to \infty$.

Let M be a closed contact manifold with co-oriented contact structure ξ given by the contact form α, i.e. $\xi = \ker(\alpha)$. Let X_α be the Reeb vector field associated with the contact form α and let $\pi_\alpha : TM \to \xi$ be the projection along the Reeb vector field. Furthermore, let J be a $d\alpha-$compatible almost complex structure on the contact structure ξ. Recall the metric g on M, defined by $g(\cdot,\cdot) = \alpha \otimes \alpha + d\alpha(\cdot, J\cdot)$ and the metric $\overline{g}$ on the symplectization $\mathbb{R} \times M$, defined by $\overline{g}(\cdot,\cdot) = dr \otimes dr + g$. Consider now the maps $\tilde{u}_n = (\tilde{a}_n, \tilde{f}_n) := u_n \circ \varphi_n : S^{\mathcal{D},r} \backslash \mathcal{P} \to \mathbb{R} \times M$ and $\tilde{\gamma}_n := \varphi_n^* \gamma_n \in \mathcal{H}^1_{j^{(n)}}(S^{\mathcal{D},r})$. Then $\tilde{u}_n$ is a $\mathcal{H}-$holomorphic curve with harmonic perturbation $\tilde{\gamma}_n$; it satisfies the equation

$$\left.\begin{aligned} \pi_\alpha d\tilde{f}_n \circ j^{(n)} &= J \circ \pi_\alpha d\tilde{f}_n \\ (\tilde{f}_n^* \alpha) \circ j^{(n)} &= d\tilde{a}_n + \tilde{\gamma}_n \end{aligned}\right. \quad \text{on } S^{\mathcal{D},r} \backslash \mathcal{P}$$

and has uniformly bounded energies, i.e. for $E_0 > 0$ and all $n \in \mathbb{N}$ we have $E(\tilde{u}_n; S^{\mathcal{D},r} \backslash \mathcal{P}) \leqslant E_0$. The L^2-norm of $\tilde{\gamma}_n$ goes over in

$$\|\tilde{\gamma}_n\|^2_{L^2(S^{\mathcal{D},r})} = \int_{S^{\mathcal{D},r}} \tilde{\gamma}_n \circ j^{(n)} \wedge \tilde{\gamma}_n = \int_{S^{\mathcal{D},r}} \varphi_n^* \gamma_n \circ \varphi_n^* j_n \wedge \varphi_n^* \gamma_n = \int_{S_n} \gamma_n \circ j_n \wedge \gamma_n = \|\gamma_n\|^2_{L^2(S_n)}$$

and it is apparent that the L^2-norm of $\tilde{\gamma}_n$ is uniformly bounded by the constant $C_0 > 0$. Hence B1 and B2 from the end of Section 2.3 are satisfied for $\tilde{u}_n$.

In the following, we first establish a convergence result on the thick part, i.e. on $S^{\mathcal{D},r}$ away from special circles, punctures and certain additional marked points, and then treat the components from the thin part.

3.1 The Thick Part

For the sequence $\tilde{u}_n : S^{\mathcal{D},r}\backslash\mathcal{P} \to \mathbb{R}\times M$ as defined above, we prove the C^∞_{loc}–convergence in the complement of the special circles and of a finite collection of points in $\dot{S}^{\mathcal{D},r} := S^{\mathcal{D},r}\backslash(\mathcal{P}\amalg\mathcal{M})$. Set $\dot{\hat{S}}^{\mathcal{D},r} := \dot{S}^{\mathcal{D},r}\backslash\coprod_j \Gamma_j$. To simplify the notation we continue to denote the maps $\tilde{u}_n$ by u_n and $\tilde{\gamma}_n$ by γ_n. The main result of this section is the following

Theorem 34. *There exists a subsequence of u_n, still denoted by u_n, a finite subset $\mathcal{Z}\subset\dot{\hat{S}}^{\mathcal{D},r}$, and a $\mathcal{H}$–holomorphic curve $u : \dot{\hat{S}}^{\mathcal{D},r}\backslash\mathcal{Z} \to \mathbb{R}\times M$ with harmonic perturbation γ defined on $S^{\mathcal{D},r}$ with respect to the complex structure j such that $u_n \to u$ in $C^\infty_{loc}(\dot{\hat{S}}^{\mathcal{D},r}\backslash\mathcal{Z})$ and $\gamma_n\to\gamma$ in $C^\infty_{loc}(\dot{\hat{S}}^{\mathcal{D},r})$.*

Before proving Theorem 34 we establish some preliminary results.
Assume that there exists a point $z^1\in\mathcal{K}\subset\dot{\hat{S}}^{\mathcal{D},r}$, where $\mathcal{K}$ is compact, and a sequence $z_n\in\mathcal{K}$ such that

$$z_n \to z^1 \quad \text{and} \quad \|du_n(z_n)\| \to \infty$$

as $n\to\infty$. The next lemma describing the convergence of conformal structures on Riemann surfaces is similar to Lemma 10.7 of [6].

Lemma 35. *There exist the open neighbourhoods $U_n(z^1) = U_n$ and $U(z^1) = U$ of z^1, and the diffeomorphisms*

$$\psi_n : D \to U_n,\ \ \psi : D \to U$$

such that

1. *ψ_n are $i - j^{(n)}$–biholomorphisms and ψ is a $i - j$–biholomorphism;*
2. *$\psi_n \to \psi$ in $C^\infty_{loc}(D)$ as $n\to\infty$ with respect to the Euclidean metric on D and the hyperbolic metric h on their images;*
3. *$\psi_n(0) = z^1$ for every n and $\psi(0) = z^1$;*
4. *$z_n \in U_n$ for every sufficiently large n;*
5. *$z^{(n)} := \psi_n^{-1}(z_n) \to 0$ as $n\to\infty$.*

Proof. Lemma 97 applied to the compact Riemann surface with boundary $\mathcal{K}$ and the interior point z^1, yields the diffeomorphisms $\psi_n : D \to U_n$ and $\psi : D\to U$ for which the first three assertions hold true. The fourth and fifth assertions are obvious since z_n converge to z^1.

□

Remark 36. The coordinate maps ψ_n and ψ have uniformly bounded gradients with respect to the Euclidian metric on D and the hyperbolic metric h on their images. This follows from the second assertion of Lemma 35.

Let $\hbar > 0$ be defined by (1.0.22). The next lemma essentially states that the $d\alpha$–energy concentrates around the point z^1 and is at least $\hbar/2 > 0$. The proof relies on bubbling-off analysis and proceeds as in Section 5.6 of [6].

Lemma 37. *For every open neighbourhood $U(z^1) = U \subset \dot{\hat{S}}^{\mathcal{D},r}$ we have*

$$0 < \hbar \leqslant \lim_{n\to\infty} E_{d\alpha}(u_n; U) \leqslant E_0.$$

In particular, for each open neighbourhood U *of* z^1 *there exists an integer* $N_1 \in \mathbb{N}$ *such that for all* $n \geqslant N_1$ *we have*

$$E_{d\alpha}(u_n; U) \geqslant \frac{\hbar}{2}.$$

Proof. Consider the maps $\hat{u}_n := u_n \circ \psi_n : D \to \mathbb{R} \times M$, where ψ_n are the biholomorphisms given by Lemma 35. They satisfy the $\mathcal{H}-$holomorphic equations

$$\begin{array}{ll} \pi_\alpha d\hat{f}_n \circ i & = J(\hat{f}_n) \circ \pi_\alpha d\hat{f}_n \\ (\hat{f}_n^* \alpha) \circ i & = d\hat{a}_n + \hat{\gamma}_n \end{array} \quad \text{on } D,$$

where $\hat{\gamma}_n := \psi_n^* \gamma_n$ is a harmonic 1$-$form on D with respect to i. The energy of $\hat{u}_n$ on D is uniformly bounded as $E(\hat{u}_n; D) \leqslant E_0$, while the L_2-norm of the i$-$harmonic 1$-$form $\hat{\gamma}_n$ is uniformly bounded on D as

$$\|\hat{\gamma}_n\|^2_{L^2(D)} = \int_D \hat{\gamma}_n \circ i \wedge \hat{\gamma}_n = \int_{U_n} \gamma_n \circ j^{(n)} \wedge \gamma_n \leqslant C_0.$$

by the constant C_0. Furthermore, for $z^{(n)} := \psi_n^{-1}(z_n)$, $\left\|d\hat{u}_n(z^{(n)})\right\| \to \infty$ as $n \to \infty$. This can be seen as follows. If $v_n \in T_{z^{(n)}}D$ with $\|v_n\|_{\text{eucl.}} = 1$ is such that

$$\left\| du_n(z_n) \frac{d\psi_n(z^{(n)})v_n}{\left\|d\psi_n(z^{(n)})v_n\right\|_{h_n}} \right\|_{\bar{g}} = \|du_n(z_n)\|,$$

then,

$$\begin{aligned} \left\|d\hat{u}_n(z^{(n)})v_n\right\|_{\tilde{g}} &= \left\| du_n(z_n) \frac{d\psi_n(z^{(n)})v_n}{\left\|d\psi_n(z^{(n)})v_n\right\|_{h_n}} \right\|_{\bar{g}} \left\|d\psi_n(z^{(n)})v_n\right\|_{h_n} \\ &= \|du_n(z_n)\| \left\|d\psi_n(z^{(n)})v_n\right\|_{h_n} \\ &\geqslant \|du_n(z_n)\| \frac{1}{2} \left\|d\psi_n(z^{(n)})\right\| \\ &\geqslant \|du_n(z_n)\| \frac{1}{4} \|d\psi(0)\| \to \infty \end{aligned}$$

as $n \to \infty$. The first inequality follows from the $i - j^{(n)}-$holomorphicity of ψ_n. Set $R'_n := \left\|d\hat{u}_n(z^{(n)})\right\|$ and note that $R'_n \to \infty$ as $n \to \infty$. Choose $\epsilon'_n > 0$ such that $\epsilon'_n \to 0$ and $R'_n\epsilon'_n \to \infty$ as $n \to \infty$, and consider

$$\epsilon''_n := \min\left\{\frac{1 - |z^{(n)}|}{4}, \epsilon'_n\right\}$$

for all $n \in \mathbb{N}$. Then, $\epsilon''_n \to 0$ and $R'_n\epsilon''_n \to \infty$ as $n \to \infty$, and $D_{2\epsilon''_n}(z^{(n)}) \subset D$ for all $n \in \mathbb{N}$. By Hofer's topological lemma (Lemma 2.39 of [2]) with respect to the sequences R'_n and ϵ''_n, there exist $\epsilon_n \in (0, \epsilon''_n]$ and $\tilde{z}^{(n)} \in D$ such that

1. $\epsilon_n \left\|d\hat{u}_n(\tilde{z}^{(n)})\right\| \geqslant \epsilon''_n R'_n$;
2. $|z^{(n)} - \tilde{z}^{(n)}| \leqslant 2\epsilon''_n$;
3. $\|d\hat{u}_n(z)\| \leqslant 2\left\|d\hat{u}_n(\tilde{z}^{(n)})\right\|$, for all $z \in D_{\epsilon_n}(\tilde{z}^{(n)})$.

For $R_n := \left\|d\hat{u}_n(\tilde{z}^{(n)})\right\|$, the first assertion yield $R_n \to \infty$, $R_n\epsilon_n \to \infty$ as $n \to \infty$. From $\epsilon_n \in (0, \epsilon''_n]$, we get

$\epsilon_n \to 0$, from the third assertion, we get
$$\|d\hat{u}_n(z)\| \leqslant 2R_n$$
for all $z \in D_{\epsilon_n}(\tilde{z}^{(n)})$, and finally, from the second assertion, we get $\tilde{z}^{(n)} \to 0$ as $n \to \infty$. Doing rescaling we define the maps
$$v_n(z) = (b_n(z), g_n(z)) := \left(\hat{a}_n\left(\tilde{z}^{(n)} + \frac{z}{R_n}\right) - \hat{a}_n(\tilde{z}^{(n)}), \hat{f}_n\left(\tilde{z}^{(n)} + \frac{z}{R_n}\right)\right)$$
for all $z \in D_{\epsilon_n R_n}(0)$. The maps $v_n = (b_n, g_n) : D_{\epsilon_n R_n}(0) \to \mathbb{R} \times M$ satisfy $\|dv_n(0)\| = 1$ and $\|dv_n(z)\| \leqslant 2$ for all $z \in D_{\epsilon_n R_n}(0)$, and we have
$$E_\alpha(v_n; D_{\epsilon_n R_n}(0)) = E_\alpha(\hat{u}_n; D_{\epsilon_n}(\tilde{z}^{(n)})) \leqslant E_\alpha(\hat{u}_n; D)$$
and
$$E_{d\alpha}(v_n; D_{\epsilon_n R_n}(0)) = E_{d\alpha}(\hat{u}_n; D_{\epsilon_n}(\tilde{z}^{(n)})) \leqslant E_{d\alpha}(\hat{u}_n; D)$$
giving $E(v_n; D_{\epsilon_n R_n}(0)) \leqslant E_0$. Moreover, v_n solves the $\mathcal{H}$—holomorphic equations
$$\begin{aligned} \pi_\alpha dg_n \circ i &= J \circ \pi_\alpha dg_n, \\ (g_n^* \alpha) \circ i &= db_n + \underline{\gamma}_n, \end{aligned}$$
where $\underline{\gamma}_n := \hat{\gamma}_n / R_n$. Because v_n has a bounded gradient, there exists a smooth map $v : \mathbb{C} \to \mathbb{R} \times M$ with a bounded energy (by E_0) such that $v_n \to v$ in $C^\infty_{loc}(\mathbb{C})$ as $n \to \infty$. Nevertheless, because $\hat{\gamma}_n$ is bounded in L^2—norm, $\underline{\gamma}_n \to 0$ as $n \to 0$. Thus $v = (b, g) : \mathbb{C} \to \mathbb{R} \times M$ is a pseudoholomorphic plane, i.e. it solves the pseudoholomorphic curve equation
$$\begin{aligned} \pi_\alpha dg \circ i &= J \circ \pi_\alpha dg, \\ (g^* \alpha) \circ i &= db. \end{aligned}$$
We prove now that the α— and $d\alpha$—energies of v are bounded. Let $R > 0$ be arbitrary and for some $\tau_0 \in \mathcal{A}$ consider
$$\begin{aligned} \int_{D_R(0)} \tau_0'(b) db \circ i \wedge db &= \lim_{n\to\infty} \int_{D_R(0)} \tau_0'(b_n) db_n \circ i \wedge db_n \\ &= \lim_{n\to\infty} \int_{D_{R/R_n}(\tilde{z}^{(n)})} \tau_0'(\hat{a}_n - \hat{a}_n(\tilde{z}^{(n)})) d\hat{a}_n \circ i \wedge d\hat{a}_n \\ &= \lim_{n\to\infty} \int_{D_{R/R_n}(\tilde{z}^{(n)})} \tau_n'(\hat{a}_n) d\hat{a}_n \circ i \wedge d\hat{a}_n \\ &\leqslant \lim_{n\to\infty} \sup_{\tau\in\mathcal{A}} \int_{D_{R/R_n}(\tilde{z}^{(n)})} \tau'(\hat{a}_n) d\hat{a}_n \circ i \wedge d\hat{a}_n \\ &= \lim_{n\to\infty} E_\alpha(\hat{u}_n; D_{R/R_n}(\tilde{z}^{(n)})), \end{aligned}$$
where $\tau_n = \tau_0(\cdot - \hat{a}_n(\tilde{z}^{(n)}))$ is a sequence of functions that belong to $\mathcal{A}$. Taking the supremum of the left-hand side over $\tau_0 \in \mathcal{A}$, we get
$$E_\alpha(v; D_R(0)) \leqslant \lim_{n\to\infty} E_\alpha(\hat{u}_n; D_{R/R_n}(\tilde{z}^{(n)})),$$
while picking some arbitrary $\epsilon > 0$, we obtain
$$E_\alpha(v; D_R(0)) \leqslant \lim_{n\to\infty} E_\alpha(\hat{u}_n; D_{R/R_n}(\tilde{z}^{(n)})) \leqslant \lim_{n\to\infty} E_\alpha(\hat{u}_n; D_\epsilon(0)).$$

For the $d\alpha-$energy, we proceed analogously: for $R > 0$ we have

$$E_{d\alpha}(v; D_R(0)) = \lim_{n\to\infty} \int_{D_R(0)} g_n^* d\alpha = \lim_{n\to\infty} \int_{D_{R/R_n}(\tilde{z}^{(n)})} \hat{f}_n^* d\alpha,$$

while picking some arbitrary $\epsilon > 0$, we find

$$E_{d\alpha}(v; D_R(0)) = \lim_{n\to\infty} \int_{D_{R/R_n}(\tilde{z}^{(n)})} \hat{f}_n^* d\alpha \leqslant \lim_{n\to\infty} \int_{D_\epsilon(0)} \hat{f}_n^* d\alpha \leqslant \lim_{n\to\infty} E_{d\alpha}(\hat{u}_n; D_\epsilon(0)).$$

Because the $\alpha-$ and $d\alpha-$energies are non-negative,

$$\begin{aligned} E(v; D_R(0)) &= E_\alpha(v; D_R(0)) + E_{d\alpha}(v; D_R(0)) \\ &\leqslant \lim_{n\to\infty} E_\alpha(\hat{u}_n; D_\epsilon(0)) + \lim_{n\to\infty} E_{d\alpha}(\hat{u}_n; D_\epsilon(0)) \\ &= \lim_{n\to\infty} E(\hat{u}_n; D_\epsilon(0)) \\ &\leqslant E_0, \end{aligned}$$

and since $R > 0$ was arbitrary, we obtain $E_{d\alpha}(v; \mathbb{C}) \leqslant E_0$. As v is a usual pseudoholomorphic curve, it follows that $E(v; \mathbb{C}) = E_H(v; \mathbb{C})$, where E_H is the Hofer energy defined by (1.0.4); thus $E_H(v; \mathbb{C}) \leqslant E_0$. Moreover, as v is non-constant we have by Remark 2.38 of [2], that for any $\epsilon > 0$,

$$0 < \hbar \leqslant E_{d\alpha}(v; \mathbb{C}) \leqslant \lim_{n\to\infty} E_{d\alpha}(\hat{u}_n; D_\epsilon(0)) \leqslant \lim_{n\to\infty} E_{d\alpha}(u_n; \psi_n(D_\epsilon(0))).$$

Choosing $\epsilon > 0$ such that $\psi_n(D_\epsilon(0)) \subset U$ for all n, we end up with

$$0 < \hbar \leqslant \lim_{n\to\infty} E_{d\alpha}(u_n; U) \leqslant E_0,$$

and the proof is finished.

□

The next proposition is proved by contradiction by means of Lemma 37.

Proposition 38. *There exists a subsequence of u_n, still denoted by u_n, and a finite subset $\mathcal{Z} \subset \dot{\hat{S}}^{\mathcal{D},r}$ such that for every compact subset $\mathcal{K} \subset \dot{\hat{S}}^{\mathcal{D},r} \backslash \mathcal{Z}$, there exists a constant $C_\mathcal{K} > 0$ such that*

$$\|du_n(z)\| := \sup_{v \in T_z S^{\mathcal{D},r}, \|v\|_{h_n}=1} \|du_n(z)v\|_{\overline{g}} \leqslant C_\mathcal{K}$$

for all $z \in \mathcal{K}$.

Proof. For the sequence u_n and any finite subset $\mathcal{Z} \subset \dot{\hat{S}}^{\mathcal{D},r}$, we define

$$\begin{aligned} \mathcal{Z}_{\{u_n\},\mathcal{Z}} := \Big\{ z \in \dot{\hat{S}}^{\mathcal{D},r} \backslash \mathcal{Z} \mid \ & \text{there exists a subsequence } u_{n_k} \text{ of } u_n \text{ and a} \\ & \text{sequence } z_k \in \dot{\hat{S}}^{\mathcal{D},r} \backslash \mathcal{Z} \text{ such that } z_k \to z \text{ and } \|du_{n_k}(z_k)\| \to \infty \text{ as } k \to \infty \Big\}. \end{aligned}$$

If $\mathcal{Z}_{\{u_n\},\emptyset}$ is empty then the assertion is fulfilled for the sequence u_n and the finite set $\mathcal{Z} = \emptyset$. Otherwise, we choose $z^1 \in \mathcal{Z}_{\{u_n\},\emptyset}$. In this case, there exists a sequence $z_n^1 \in \dot{\hat{S}}^{\mathcal{D},r}$ and a subsequence u_n^1 of u_n such that $z_n^1 \to z^1$ and $\|du_n^1(z_n^1)\| \to \infty$. Consider now the set $\mathcal{Z}_{\{u_n^1\},\{z^1\}}$. If $\mathcal{Z}_{\{u_n^1\},\{z^1\}}$ is empty then the assertion is fulfilled for the subsequence u_n^1 and the finite set $\mathcal{Z} = \{z^1\}$. Otherwise, we choose an element $z^2 \in \mathcal{Z}_{\{u_n^2\},\{z^1\}}$. In this case, by definition, there exists a sequence $z_n^2 \in \dot{\hat{S}}^{\mathcal{D},r}\backslash\{z^1\}$ and a subsequence u_n^2 of u_n^1 such that $z_n^2 \to z^2$ and $\|du_n^2(z_n^2)\| \to \infty$. Let us show that the set of points $\mathcal{Z} = \{z^1, z^2, ...\}$ constructed in this way is finite, or more precisely, that $|\mathcal{Z}| \leqslant 2E_0/\hbar$. Assume $|\mathcal{Z}| > 2E_0/\hbar$ and pick an integer $k > 2E_0/\hbar$ and pairwise different points $z^1, ..., z^k \in \mathcal{Z}$. Let $U_1, ..., U_k \subset \dot{\hat{S}}^{\mathcal{D},r}$ be some open pairwise disjoint neighborhoods of $z^1, ..., z^k$. Applying Lemma 37 inductively, we deduce that there exists a positive integer N such that for every $n \geqslant N$, $E_{d\alpha}(u_n; U_i) \geqslant \hbar/2$ for all $i = 1, ..., k$. Since the U_i are disjoint, we obtain

$$k\frac{\hbar}{2} \leqslant \sum_{i=1}^{k} E_{d\alpha}(u_n; U_i) \leqslant E_{d\alpha}(u_n; \dot{\hat{S}}^{\mathcal{D},r}) \leqslant E_0.$$

Thus $k \leqslant 2E_0/\hbar$ which is a contradiction to our assumption.

□

By means of Proposition 38 we can prove the convergence of the $\mathcal{H}-$holomorphic maps in a punctured thick part of the Riemann surface.

Proof. (of Theorem 34) For some sufficiently small $k \in \mathbb{N}$ we consider the subsets

$$\Omega_k := \mathrm{Thick}_{1/k}(\dot{\hat{S}}^{\mathcal{D},r}, h)\backslash \bigcup_{i=1}^{N} D_{1/k}^{h}(z^i),$$

where $\mathcal{Z} = \{z^1, ..., z^N\}$ is the subset in Proposition 38 and $D_{1/k}^{h}(z_i)$ is the open disk around z_i of radius $1/k$ with respect to the metric h. In order to keep the notation simple, the subsequence obtained by applying Proposition 38 is still denoted by u_n. Obviously, Ω_k build an exhaustion by compact sets of $\dot{\hat{S}}^{\mathcal{D},r}\backslash\mathcal{Z}$. These sets are compact surfaces with boundary. By Proposition 38, the maps u_n have uniformly bounded gradients on Ω_1. Thus after a suitable translation of the maps u_n in the $\mathbb{R}-$coordinate, there exists a subsequence u_n^1 of u_n that converges in $C^\infty(\Omega_1)$ to a map $u : \Omega_1 \to \mathbb{R}\times M$. Iteratively, at step $k+1$ there exists a subsequence u_n^{k+1} of u_n^k that converges in $C^\infty(\Omega_{k+1})$ to a map $u : \Omega_{k+1} \to \mathbb{R}\times M$ which is an extension from Ω_k to Ω_{k+1}. This procedure allows us to define a map $u : \dot{\hat{S}}^{\mathcal{D},r}\backslash\mathcal{Z} \to \mathbb{R}\times M$. After passing to some diagonal subsequene u_n^n, the maps u_n^n converge in $C^\infty_{loc}(\dot{\hat{S}}^{\mathcal{D},r}\backslash\mathcal{Z})$ to the map $u : \dot{\hat{S}}^{\mathcal{D},r}\backslash\mathcal{Z} \to \mathbb{R}\times M$. Since the L^2-norms of γ_n are uniformly bounded on $S^{\mathcal{D},r}$, they converge in $C^\infty_{loc}(\dot{\hat{S}}^{\mathcal{D},r})$ to some harmonic $1-$form γ with a bounded L^2-norm on $\dot{\hat{S}}^{\mathcal{D},r}$. This can be seen as follows. For each $p \in \mathrm{Thick}_{\rho/2}(\dot{\hat{S}}^{\mathcal{D},r}, h)$, consider the charts $\psi_n^p : D \to U_n^p$ and $\psi^p : D \to U^p$ as in Lemma 35 for a sufficiently small and fixed $\rho > 0$. As $\mathrm{Thick}_\rho(\dot{\hat{S}}^{\mathcal{D},r}, h)$ is compact, there exist finitely many $\{p_i\}_{i=1,...,N} \in \mathrm{Thick}_{\rho/2}(\dot{\hat{S}}^{\mathcal{D},r}, h)$ such that $U_n^{p_i}$, U^{p_i}, $U_{n,\delta}^{p_i} := \psi_n^{p_i}(D_{1-\delta}(0))$, and $U_\delta^{p_i} := \psi^{p_i}(D_{1-\delta}(0))$ cover the whole $\mathrm{Thick}_\rho(\dot{\hat{S}}^{\mathcal{D},r}, h)$ for a sufficiently small and fixed $\delta \ll \rho$. For some p_i, we pull-back the harmonic $1-$forms γ_n by $\psi_n^{p_i}$ to the harmonic $1-$form $\gamma'_{n,i}$ on D with uniformly bounded L^2-norms. By Lemmas 35 and 39, γ_n converges in $C^\infty(U_\delta^{p_i})$ to a harmonic $1-$form $\gamma^{(i)}$ on $U_\delta^{p_i}$ with respect to the hyperbolic metric h. Let l be an index such that $U_\delta^{p_l} \cap U_\delta^{p_i} \neq \emptyset$. On $U_\delta^{p_l}$ we go over to a further subsequence and arguing as above, we find that γ_n converges in $C^\infty(U_\delta^{p_l})$ to a harmonic $1-$form $\gamma^{(l)}$. The uniqueness of the limit implies that $\gamma^{(i)}$ and $\gamma^{(l)}$ agree on the overlaps $U_\delta^{p_l} \cap U_\delta^{p_i}$. Consequently, there exist a harmonic $1-$form γ^ρ on $\mathrm{Thick}_\rho(\dot{\hat{S}}^{\mathcal{D},r}, \overline{h})$ and a subsequence of γ_n, still denoted by γ_n, that converges in C^∞ to γ^ρ with respect to the hyperbolic metric h. Passing to a diagonal subsequence, we find that γ_n converges in C^∞_{loc} to a harmonic $1-$form γ defined on $\dot{\hat{S}}^{\mathcal{D},r}$ with respect to the hyperbolic metric h. What is left to show is that after projecting γ from $\dot{\hat{S}}^{\mathcal{D},r}$ to $S\backslash(\mathcal{M} \amalg \mathcal{P})$, γ can be extended across the punctures. This result follows from Lemma 19. Hence the map u is a $\mathcal{H}-$holomorphic curve on $\dot{\hat{S}}^{\mathcal{D},r}\backslash\mathcal{Z}$ with harmonic perturbation γ.

□

Lemma 39. *Let γ_n be a sequence of harmonic $1-$forms defined on the closed unit disk D and having uniformly bounded L^2-norms by the constant $C_0 > 0$. Then, for each $\delta > 0$ there exists a subsequence of γ_n, still denoted by γ_n, which converges in $C^\infty(D_{1-\delta}(0))$ to a harmonic $1-$form γ defined on $D_{1-\delta}(0)$.*

Proof. Let $\gamma_n = f_n dx + g_n dy$, where $f_n, g_n : D \to \mathbb{R}$ is a sequence of harmonic functions and x, y are the coordinates on D. Since γ_n has a uniformly bounded L^2-norm, f_n and g_n are uniformly bounded in $L^2(D)$. Let us show that the derivatives of f_n and g_n are uniformly bounded on $D_{1-\delta}(0)$. For $z \in D_{1-(\delta/2)}(0)$, the mean-value theorem for harmonic functions yields

$$\begin{aligned}|f_n(z)| &\leqslant \frac{16}{\pi\delta^2}\int_{D_{\frac{\delta}{2}}(0)} |f_n(x,y)|\,dxdy \\ &\leqslant \frac{4C_0}{\delta\sqrt{\pi}},\end{aligned}$$

and so, f_n is uniformly bounded in $D_{1-(\delta/2)}(0)$. Applying the same argument for the function g_n, we find that the holomorphic function $F_n := f_n + ig_n : D_{1-(\delta/2)}(0) \to \mathbb{C}$ is uniformly bounded. In view of the Cauchy integral formula we deduce that for $k \in \mathbb{N}$ and $z \in D_{1-\delta}(0)$, we have

$$|F_n^{(k)}(z)| = \frac{k!}{2\pi}\left|\int_{\partial D_{\frac{\delta}{2}}(z)} \frac{F_n(\xi)}{(\xi - z)^{k+1}} d\xi\right| = \frac{k!}{2\pi}\left|\int_0^{2\pi} 2^k i \frac{F_n(z+\delta e^{it})}{\delta^k e^{ikt}} dt\right| \leqslant \frac{2^{k+4}k!\sqrt{2C_0}}{\delta^{k+1}\sqrt{\pi}}.$$

Hence, for every $k \in \mathbb{N}_0$ the quantities $\|f_n\|_{C^k(D_{1-\delta}(0))}$ and $\|g_n\|_{C^k(D_{1-\delta}(0))}$ are uniformly bounded. From here we deduce by Arzelà-Ascoli theorem that f_n and g_n converge in $C^\infty(D_{1-\delta}(0))$ to the harmonic functions f and g defined on $D_{1-\delta}(0)$, respectively.

□

3.2 Convergence on the thin part and around the points from $\mathcal{Z}$

In this section we investigate the convergence of the $\mathcal{H}-$holomorphic curves u_n on the components of the thin part and in the neighborhood of the points from $\mathcal{Z}$ that were constructed in Theorem 34. For a sufficient small $\delta > 0$, the set $\mathrm{Thin}_\delta(\dot{S}^{\mathcal{D},r}, h_n)$ can be decomposed in two types of connected components: (I) the so called cusps, which are neighborhoods of punctures with respect to the hyperbolic metric, and (II) the components which are biholomorphic to the hyperbolic cylinders that mutate to nodes in the Deligne-Mumford limiting process. For more details we refer to Chapter 1 of [2]. This section is organized as follows. First, we analyze the convergence of u_n on components that can be identified with hyperbolic cylinders, and describe the limit object. Second, we treat the convergence of u_n on components that can be identified with cusps, and as before, describe the limit object. The convergence results established here can be used to describe the convergence of u_n in a neighborhood of the points from $\mathcal{Z}$. Third, we use the description of the convergence of the $\mathcal{H}-$holomorpic curves u_n on the thick part (established in Section 3.1), the thin part, and in the neighborhood of the points from $\mathcal{Z}$ (established in this section) to define a new surface by gluing the two parts together. On this surface we describe the convergence of u_n completely.

Before proceeding we emphasize that by techniques of hyperbolic geometry, the compact components of the thin part, called hyperbolic cylinders, can be biholomorphically identified, for a suitable $R > 0$, with the standard cylinders $[-R, R] \times S^1$ endowed with the standard complex structure i.

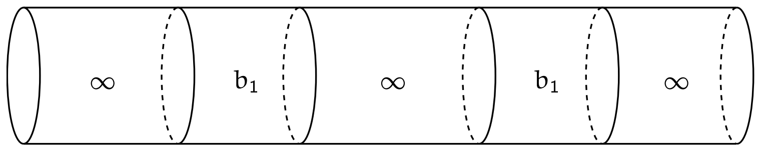

Figure 3.2.1: The component of the thin part, which is biholomorphic to a cylinder, is divided in cylinders of types b_1 and ∞ in an alternating order.

3.2.1 Cylinders

We analyze the convergence of u_n on compact components of the thin part which are biholomorphic to hyperbolic cylinders. When restricted to these cylinders, the curves u_n can have a $d\alpha-$energy larger than the constant $\hbar > 0$ defined in (1.0.22). Since we do not have a version of the monotonicity lemma in the $\mathcal{H}-$holomorphic case, the classical results on the asymptotic of holomorphic cylinders from [6] and [14] are not directly applicable. To deal with this problem we shift the maps by the Reeb flow to make them pseudoholomorphic. Actually we proceed as follows. We decompose the hyperbolic cylinder into a finite uniform number of smaller cylinders; some of them having conformal modulus tending to infinity but a $d\alpha-$energy strictly smaller than $\hbar$, and the rest of them having bounded modulus but a $d\alpha-$energy possibly larger than $\hbar$. We refer to these cylinders as cylinders of types ∞ and b_1, respectively. We consider an alternating appearance of these cylinders, as it can be seen in Figure 3.2.1.

The convergence and the description of the limit object are first treated for cylinders of type ∞, and then for cylinders of type b_1.
As cylinders of type ∞ have a small $d\alpha-$energy, we can assume, by the classical bubbling-off analysis, that the maps u_n have uniformly bounded gradients. To make the curves u_n pseudoholomorphic, we perform a transformation by pushing them along the Reeb flow up to some specific time. This procedure is made precise in Appendix B. As the gradients of these transformed curves still remain uniformly bounded, we can adapt the results of [14] to formulate a convergence result for the transformed curves (see Appendix B). Undoing the transformation we obtain a convergence result for the $\mathcal{H}-$holomorphic curves.
In the case of cylinders of type b_1 we proceed as follows. Relying on a bubbling-off argument, as we did in the case of the thick part (see Section 3.1), we assume that the gradients blow up only in a finite uniform number of points and remain uniformly bounded in a compact complement of them. In this compact region, the Arzelà-Ascoli theorem shows that the curves u_n together with the harmonic perturbations γ_n converge in C^∞ to some $\mathcal{H}-$holomorphic curve. What is then left is the convergence in a neighborhood of the finitely many punctures where the gradients blow up. Here, a neighborhood of a puncture is a disk on which the harmonic perturbation can be made exact and can be encoded in the $\mathbb{R}-$coordinate of the curve u_n. By this procedure we transform the $\mathcal{H}-$holomorphic curve into a usual pseudoholomorphic curve defined on a disk D. By the $C^\infty-$convergence of u_n on any compact complement of the punctures, we assume that the transformed curves converge on an arbitrary neighborhood of ∂D. This approach, which is described in detail in Section 3.2.3, uses a convergence result established in Appendix A. As for cylinders of type ∞, we undo the transformation and derive a convergence result for the $\mathcal{H}-$homolorphic curves on cylinders of type b_1. Finally, gluing all cylinders together, we are led to a convergence result for the entire component which is biholomorphic to a hyperbolic cylinder from the thin part.
Let C_n be a component of $\mathrm{Thin}_\delta(\dot{S}^{\mathcal{D},r}, h_n)$ which is conformally equivalent to the cylinder $[-\sigma_n^\delta, \sigma_n^\delta] \times S^1$. Observe that from the definition of Deligne-Mumford convergence, $\sigma_n^\delta \to \infty$ as $n \to \infty$. In the following, we drop the fixed, sufficiently small constant $\delta > 0$, and assume that the curves u_n are defined on $[-\sigma_n, \sigma_n] \times S^1$. Let $u_n = (a_n, f_n) : [-\sigma_n, \sigma_n] \times S^1 \to \mathbb{R} \times M$ be a sequence of $\mathcal{H}-$holomorphic curves with harmonic perturbations γ_n,

i.e.,

$$\pi_\alpha df_n \circ i = J(f_n) \circ \pi_\alpha df_n,$$
$$(f_n^*\alpha) \circ i = da_n + \gamma_n$$

on $[-\sigma_n, \sigma_n] \times S^1$, and let us assume that the energy of u_n, as well as the L^2–norm of γ_n on the cylinders are uniformly bounded, i.e. for the constants $E_0, C_0 > 0$ we have $E(u_n; [-\sigma_n, \sigma_n] \times S^1) \leqslant E_0$ and $\|\gamma_n\|^2_{L^2([-\sigma_n,\sigma_n]\times S^1)} \leqslant C_0$ for all $n \in \mathbb{N}$.

Before describing the decomposition of $[-\sigma_n, \sigma_n] \times S^1$ into cylinders of types ∞ and b_1 we give a proposition which states that the C^1–norm of the harmonic perturbation γ_n is uniformly bounded. This result will play an essential role in Section 3.2.3. We set $\gamma_n = f_n ds + g_n dt$, where f_n and g_n are harmonic functions defined on $[-\sigma_n, \sigma_n] \times S^1$ with coordinates (s, t) such that $f_n + ig_n$ is holomorphic. By the uniform L^2–bound of γ_n, we have

$$\|\gamma_n\|^2_{L^2([-\sigma_n,\sigma_n]\times S^1)} = \int_{[-\sigma_n,\sigma_n]\times S^1} \left(f_n^2 + g_n^2\right) dsdt \leqslant C_0$$

for all $n \in \mathbb{N}$. As a result, the L^2–norm of the holomorphic function $f_n + ig_n$ is uniformly bounded. Denote this function by $G_n = f_n + ig_n$.

Proposition 40. *For any $\delta > 0$ there exists a constant $C_\delta > 0$ such that*

$$\|G_n\|_{C^1([-\sigma_n+\delta,\sigma_n-\delta]\times S^1)} \leqslant C_\delta$$

for all $n \in \mathbb{N}$.

Proof. First, we prove that the sequence G_n is uniformly bounded in C^0–norm. As $G_n : [-\sigma_n, \sigma_n] \times S^1 \to \mathbb{C}$ is holomorphic, $f_n = \mathfrak{R}(G_n)$ and $g_n = \mathfrak{I}(G_n)$ are harmonic functions defined on $[-\sigma_n, \sigma_n] \times S^1$. For a sufficiently small $\delta > 0$ we establish C^0–bounds for f_n on the subcylinders $[-\sigma_n + (\delta/2), \sigma_n - (\delta/2)] \times S^1$. By the mean value theorem for harmonic function, we have

$$f_n(p) = \frac{16}{\pi\delta^2} \int_{D_{\frac{\delta}{4}}(p)} f_n(s,t) dsdt$$

for all $p \in [-\sigma_n + (\delta/2), \sigma_n - (\delta/2)] \times S^1$, where $D_{\delta/4}(p) \subset [-\sigma_n, \sigma_n] \times S^1$. Then Hölder's inequality yields

$$\begin{aligned}
|f_n(p)| &= \frac{16}{\pi\delta^2} \left| \int_{B_{\frac{\delta}{4}}(p)} f_n(s,t) dsdt \right| \\
&\leqslant \frac{16}{\pi\delta^2} \int_{B_{\frac{\delta}{4}}(p)} |f_n(s,t)| dsdt \\
&\leqslant \frac{16}{\pi\delta^2} \left(\int_{B_{\frac{\delta}{4}}(p)} |f_n(s,t)|^2 dsdt \right)^{\frac{1}{2}} \left(\int_{B_{\frac{\delta}{4}}(p)} dsdt \right)^{\frac{1}{2}} \\
&= \frac{4}{\sqrt{\pi}\delta} \left(\int_{B_{\frac{\delta}{4}}(p)} |f_n(s,t)|^2 dsdt \right)^{\frac{1}{2}} \\
&\leqslant \frac{4}{\sqrt{\pi}\delta} \sqrt{C_0}
\end{aligned}$$

for all $n \in \mathbb{N}$. As a result, we obtain

$$\|f_n\|_{C^0([-\sigma_n+\frac{\delta}{2},\sigma_n-\frac{\delta}{2}]\times S^1)} \leqslant \frac{4}{\sqrt{\pi}\delta}\sqrt{C_0},$$

and note that the same result holds for g_n.

By means of bubbling-off analysis we prove now that the gradient of G_n is uniformly bounded. Assume

$$\sup_{p\in[-\sigma_n+\delta,\sigma_n-\delta]\times S^1} |\nabla G_n(p)| \to \infty$$

as $n \to \infty$. Let $p_n \in [-\sigma_n+\delta, \sigma_n-\delta] \times S^1$ be such that

$$|\nabla G_n(p_n)| = \sup_{p\in[-\sigma_n+\delta,\sigma_n-\delta]\times S^1} |\nabla G_n(p)|;$$

then $R_n := |\nabla G_n(p_n)| \to \infty$ as $n \to \infty$. Set $\epsilon_n := R_n^{-\frac{1}{2}} \searrow 0$ as $n \to \infty$, and observe that $\epsilon_n R_n \to \infty$ as $n \to \infty$. Choose $n_0 \in \mathbb{N}_0$ sufficiently large such that $D_{10\epsilon_n}(p_n) \subset [-\sigma_n, \sigma_n] \times S^1$ for all $n \geqslant n_0$. By Hofer's topologial lemma there exist $\epsilon_n' \in (0, \epsilon_n]$ and $p_n' \in [-\sigma_n, \sigma_n] \times S^1$ satisfying:

1. $\epsilon_n' R_n' \geqslant \epsilon_n R_n$;
2. $p_n' \in D_{2\epsilon_n}(p_n) \subset D_{10\epsilon_n}(p_n)$;
3. $|\nabla G_n(p)| \leqslant 2R_n'$, for all $p \in D_{\epsilon_n'}(p_n') \subset D_{10\epsilon_n}(p_n)$,

where $R_n' := |du_n(p_n')|$. Via rescaling consider the maps $\tilde{G}_n : D_{\epsilon_n' R_n'}(0) \to \mathbb{C}$, defined by

$$\tilde{G}_n(w) := G_n\left(p_n' + \frac{w}{R_n'}\right)$$

for $w \in D_{\epsilon_n' R_n'}(0)$. Observe that $p_n' + (w/R_n') \in D_{\epsilon_n'}(p_n')$ for $w \in D_{\epsilon_n' R_n'}(0)$, and that for $\tilde{G}_n$ we have:

1. $|\nabla \tilde{G}_n(0)| = 1$;
2. $|\nabla \tilde{G}_n(w)| \leqslant 2$ for $w \in D_{\epsilon_n' R_n'}(0)$;
3. $\tilde{G}_n$ is holomorphic on $D_{\epsilon_n' R_n'}(0)$;
4. $\tilde{G}_n$ is uniformly bounded on $[-\sigma_n+\delta, \sigma_n-\delta] \times S^1$ (by Assertion 1).

By the usual regularity theory for pseudoholomorphic maps and Arzelà-Ascoli theorem, $\tilde{G}_n$ converge in $C^\infty_{\mathrm{loc}}(\mathbb{C})$ to a bounded holomorphic map $\tilde{G} : \mathbb{C} \to \mathbb{C}$ with $|\nabla \tilde{G}(0)| = 1$. By Liouville theorem this map can be only the constant map, and so, we arrive at a contradiction with $|\nabla \tilde{G}(0)| = 1$.

$\square$

For $\delta > 0$ we can replace the cylinder $[-\sigma_n+\delta, \sigma_n-\delta] \times S^1$ by $[-\sigma_n, \sigma_n] \times S^1$ if we consider $\mathrm{Thin}_\delta(\dot{S}^{\mathcal{D},r}, h_n)$ for a smaller $\delta > 0$. We come now to the decomposition of $[-\sigma_n, \sigma_n] \times S^1$ into cylinders of types ∞ and b_1. Consider the parameter-dependent function with parameter $h \in [-\sigma_n, \sigma_n]$ defined by

$$F_{n,h} : [h, \sigma_n] \to \mathbb{R},\ s \mapsto \int_{[h,s]\times S^1} f_n^* d\alpha.$$

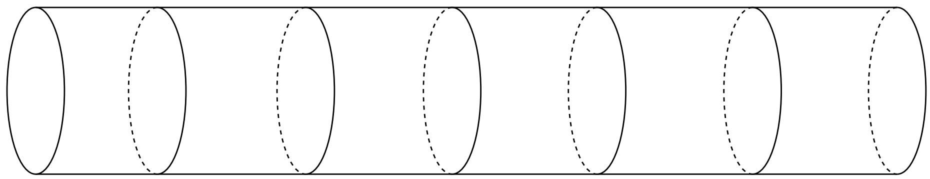

Figure 3.2.2: Decomposition of $[-\sigma_n, \sigma_n] \times S^1$ into smaller cylinders $[h_n^{(m)}, h_n^{(m+1)}] \times S^1$ having $d\alpha$–energy $\hbar/4$ or less.

As $f_n^* d\alpha$ is non-negative, $F_{n,h}$ is positive and increasing. For the constant $\hbar$ defined in (1.0.22), we set $h_n^{(0)} = -\sigma_n$, and define

$$h_n^{(m)} := \sup\left(F^{-1}_{n,h_n^{(m-1)}}\left(\left[0, \frac{\hbar}{4}\right]\right)\right).$$

Since $E_{d\alpha}(u_n; [-\sigma_n, \sigma_n] \times S^1) < E_0$, the sequence $\{h_n^{(m)}\}_{m \in \mathbb{N}_0}$ has to end after N_n steps, where $h_n^{(N_n)} = \sigma_n$. On the cylinder $[h_n^{(N_n-1)}, h_n^{(N_n)}] \times S^1$, the $d\alpha$–energy of u_n can be smaller than $\hbar/4$. Obviously, we have $-\sigma_n = h_n^{(0)} < h_n^{(1)} < ... < h_n^{(m)} < ... < h_n^{(N_n)} = \sigma_n$ giving $E_{d\alpha}(u_n; [h_n^{(m-1)}, h_n^{(m)}] \times S^1) = \hbar/4$ for $m = 1, ..., N_n - 1$ and $E_{d\alpha}(u_n; [h_n^{(N_n-1)}, h_n^{(N_n)}] \times S^1) \leqslant \hbar/4$. Hence the $d\alpha$–energy can be written as

$$E_{d\alpha}(u_n; [-\sigma_n, \sigma_n] \times S^1) = (N_n - 1)\frac{\hbar}{4} + E_{d\alpha}(u_n; [h_n^{(N_n-1)}, h_n^{(N_n)}] \times S^1),$$

which implies the following bound on N_n:

$$0 \leqslant N_n \leqslant \frac{4E_0}{\hbar} + 1.$$

After going over to a subsequence, we can further assume that N_n is also independent of n; for this reason, we set $N_n = N$. Thus the cylinders $[-\sigma_n, \sigma_n] \times S^1$ have been decomposed into N smaller subcylinders $[h_n^{(0)}, h_n^{(1)}] \times S^1, ..., [h_n^{(N-1)}, h_n^{(N)}] \times S^1$ on which we have $E_{d\alpha}(u_n; [h_n^{(m-1)}, h_n^{(m)}] \times S^1) = \hbar/4$ for $m \in \{1, ..., N-1\}$ and $E_{d\alpha}(u_n; [h_n^{(N-1)}, h_n^{(N)}] \times S^1) \leqslant \hbar/4$.

Definition 41. A sequence of cylinders $[a_n, b_n] \times S^1$, where $a_n, b_n \in \mathbb{R}$ and $a_n < b_n$ is called of type b_1 if $b_n - a_n$ is bounded from above, and of type ∞ if $b_n - a_n \to \infty$ as $n \to \infty$.

This is illustrated in Figure 3.2.2.

Lemma 42. *Let $[h_n^{(m-1)}, h_n^{(m)}] \times S^1$ be a cylinder of type ∞ and let $h > 0$ be chosen small enough such that $h_n^{(m)} - h_n^{(m-1)} - 2h = (h_n^{(m)} - h) - (h_n^{(m-1)} + h) > 0$ for all $n \in \mathbb{N}$. Then there exists a constant $C_h > 0$ such that*

$$\|du_n(z)\|_{C^0} = \sup_{\|v\|_{eucl}=1} \|du_n(z)v\| < C_h$$

for all $z \in [h_n^{(m-1)} + h, h_n^{(m)} - h] \times S^1$ and $n \in \mathbb{N}$.

Proof. The proof makes use of bubbling-off analysis. Assume that there exists $h > 0$ such that $h_n^{(m)} - h_n^{(m-1)} - 2h > 0$ and

$$\sup_{z \in [h_n^{(m-1)}+h, h_n^{(m)}-h] \times S^1} \|du_n(z)\|_{C^0} = \infty. \tag{3.2.1}$$

Then there exists a sequence $z_n \in (h_n^{(m-1)} + h, h_n^{(m)} - h) \times S^1$ with the property $R_n := \|du_n(z_n)\|_{C^0} \to \infty$ as $n \to \infty$. Let $\epsilon_n = R_n^{-\frac{1}{2}} \searrow 0$ as $n \to \infty$, and observe that $\epsilon_n R_n \to \infty$ as $n \to \infty$. Choose $n_0 \in \mathbb{N}$ sufficiently large such that $D_{10\epsilon_n}(z_n) \subset [h_n^{(m-1)}, h_n^{(m)}] \times S^1$ for all $n \geqslant n_0$. By Hofer's topological lemma, there exist $\epsilon'_n \in (0, \epsilon_n]$ and $z'_n \in [h_n^{(m-1)}, h_n^{(m)}] \times S^1$ satisfying:

1. $\epsilon'_n R'_n \geqslant \epsilon_n R_n$;
2. $z'_n \in D_{2\epsilon_n}(z_n) \subset D_{10\epsilon_n}(z_n)$;
3. $\|du_n(z)\|_{C^0} \leqslant 2R'_n$, for all $z \in D_{\epsilon'_n}(z'_n) \subset D_{10\epsilon_n}(z_n)$,

where $R'_n := \|du_n(z'_n)\|_{C^0}$. Applying rescaling consider the map $v_n : D_{\epsilon'_n R'_n}(0) \to \mathbb{R} \times M$, defined by

$$v_n(w) = (b_n(w), g_n(w)) := u_n\left(z'_n + \frac{w}{R'_n}\right) - a_n(z'_n)$$

for $w \in D_{\epsilon'_n R'_n}(0)$. Note that $z'_n + (w/R'_n) \in D_{\epsilon'_n}(z'_n)$ for $w \in D_{\epsilon'_n R'_n}(0)$, and that for v_n we have

1. $\|dv_n(0)\|_{C^0} = 1$;
2. $\|dv_n(w)\|_{C^0} \leqslant 2$ for $w \in D_{\epsilon'_n R'_n}(0)$;
3. $E_{d\alpha}(v_n; D_{\epsilon'_n R'_n}(0)) \leqslant \hbar/4$ (straightforward calculation shows that the $\alpha-$energy is also uniformly bounded);
4. v_n solves

   $$\begin{aligned} \pi_\alpha dg_n \circ i &= J \circ \pi_\alpha dg_n, \\ (g_n^*\alpha) \circ i &= db_n + \frac{\gamma_n}{R'_n} \end{aligned}$$

   on $D_{\epsilon'_n R'_n}(0)$.

As the gradients of v_n are uniformly bounded, v_n converge in $C^\infty_{loc}(\mathbb{C})$ to a finite energy plane $v = (b, g) : \mathbb{C} \to \mathbb{R} \times M$ characterized by:

1. $\|dv(0)\|_{C^0} = 1$;
2. $\|dv(w)\|_{C^0} \leqslant 2$ for $w \in \mathbb{C}$;
3. $E_{d\alpha}(v; \mathbb{C}) \leqslant \hbar/4$;
4. v is a finite energy holomorphic plane.

Assertion 3 follows from the fact that for an arbitrary $R > 0$ we have

$$E_{d\alpha}(v, D_R(0)) = \lim_{n\to\infty} E_{d\alpha}(v_n; D_R(0)) \leqslant \lim_{n\to\infty} E_{d\alpha}(v_n; D_{\epsilon'_n R'_n}(0)) \leqslant \frac{\hbar}{4},$$

while Assertion 4 follows from the fact that γ_n has a uniformly bounded L^2-norm. Note that by employing the above argument, a bound for the $\alpha-$energy can be also obtained. Now, as v is non-constant, Theorem 31 of [11] gives $E_{d\alpha}(v; \mathbb{C}) \geqslant \hbar$, which is a contradiction to Assertion 3. Thus Assumption (3.2.1) does not hold, and the gradient of u_n on cylinders of type ∞ is uniformly bounded.

□

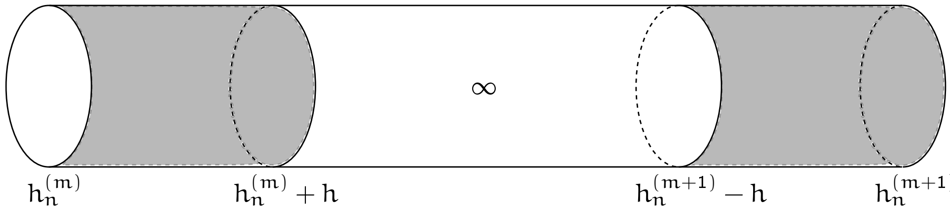

Figure 3.2.3: On the white surface, the pseudoholomorphic curves have uniformly bounded gradients.

Now we change the above decomposition so that the lengths of the cylinders of type b_1 are also bounded from below and describe the alternating appearance of cylinders of types ∞ and b_1. This process is necessary, because on the cylinders of type b_1 whose length tends to zero we cannot analyze the convergence behavior of the maps u_n and cannot describe their limit object. We proceed as follows.

Step 1. We consider a cylinder $[h_n^{(m)}, h_n^{(m+1)}] \times S^1$ of type ∞, on which we apply Lemma 42. When doing this we choose a sufficiently small constant $h > 0$, so that the gradients are uniformly bounded only on $[h_n^{(m)} + h, h_n^{(m+1)} - h] \times S^1$ by the constant $C_h > 0$, which in turn, is again a cylinder of type ∞. This can be seen in Figure 3.2.3. By this procedure, a cylinder $[h_n^{(m)}, h_n^{(m+1)}] \times S^1$ of type ∞ is decomposed into three smaller cylinders: two cylinders $[h_n^{(m)}, h_n^{(m)} + h] \times S^1$, $[h_n^{(m+1)} - h, h_n^{(m+1)}] \times S^1$ of type b_1 and one cylinder $[h_n^{(m)} + h, h_n^{(m+1)} - h] \times S^1$ of type ∞. The length of these two cylinders of type b_1 is $h > 0$. To any other cylinder of type ∞ we apply the same procedure with a fixed constant $h > 0$. Note that by Step 1, the gradients of u_n are uniformly bounded on the cylinders of type ∞ by the constant $C_h > 0$.

Step 2. We combine all cylinders of type b_1, which are next to each other, to form a bigger cylinder of type b_1. This can be seen in Figure 3.2.4. By this procedure, we guarantee that in a constellation consisting of three cylinders that lie next to each other, the type of the middle cylinder is different to the types of the left and right cylinders. Thus we got rid of the cylinders of type b_1 with length tending to zero, and make sure that the cylinders of types ∞ and b_1 appear alternately. We additionally assume that the first and last cylinders in the decomposition are of type ∞, since otherwise, we can glue the cylinder of type b_1 to the thick part of the surface and consider $\mathrm{Thin}_\delta(\dot{S}^{\mathcal{D},r}, h_n)$ for a smaller $\delta > 0$. By this procedure, we decompose $[-\sigma_n, \sigma_n] \times S^1$ into cylinders of types ∞ and b_1, while the first and last cylinders in the decomposition are of type ∞.

Step 3. For $\tilde{E}_0 = 2(E_0 + C_h)$ (see Remark 70 for the explanation of this choice) and in view of the non-degeneracy of the contact manifold (M, α), let the constant $\hbar_0$ be given by

$$\hbar_0 := \min\{|T_1 - T_2| \mid T_1, T_2 \in \mathcal{P}_\alpha, T_1 \neq T_2, T_1, T_2 \leqslant \tilde{E}_0\}. \tag{3.2.2}$$

Observe that because of $\tilde{E}_0 \geqslant E_0$, $\hbar_0 \leqslant \hbar$. If $[h_n^{(m-1)}, h_n^{(m)}] \times S^1$ is a cylinder of type ∞ for some $m \in \{1, ..., N\}$, we define the constant $\hbar_0$ as above and apply Step 1 and Step 2 to decompose this cylinder into cylinders of types ∞ and b_1, while the first and last cylinders in the decomposition are of type ∞. The cylinders of type ∞ have now a $d\alpha$−energy smaller than $\hbar_0/4$. We apply this procedure to all cylinders of type ∞. In summary, $[-\sigma_n, \sigma_n] \times S^1$ is decomposed into cylinders of type ∞ with a $d\alpha$−energy smaller than $\hbar_0/4$ and cylinders of type b_1, with the first and last cylinders being of type ∞.

Step 4. We enlarge the cylinders of type b_1 without changing their type. Let $h > 0$ be as in Lemma 42 and pick $m \in \{1, ..., N\}$ such that $[h_n^{(m-1)}, h_n^{(m)}] \times S^1$ is of type b_1. For n sufficiently large, we replace the cylinder $[h_n^{(m-1)}, h_n^{(m)}] \times S^1$ by the bigger cylinder $[h_n^{(m-1)} - 3h, h_n^{(m)} + 3h] \times S^1$, and apply this procedure

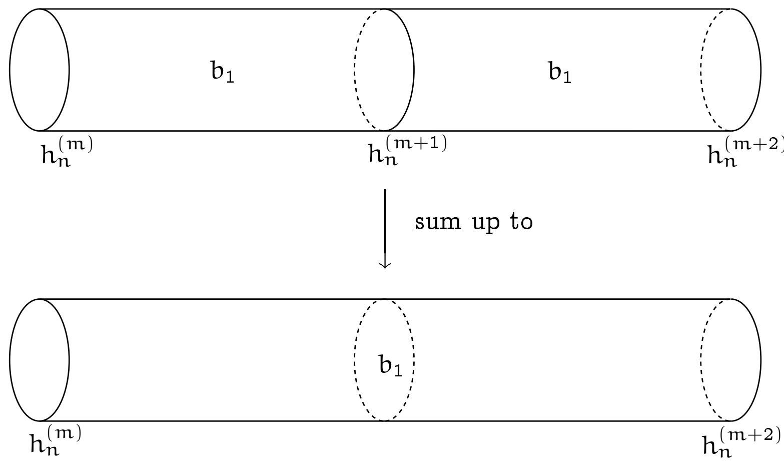

Figure 3.2.4: Two cylinders of type b_1 are combined to form a bigger cylinder of type b_1.

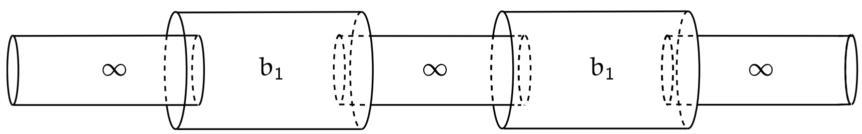

Figure 3.2.5: Decomposition of $[-\sigma_n, \sigma_n] \times S^1$ into cylinders of types ∞ and b_1 in an alternating order.

to all cylinders of type b_1. As a result, neighboring cylinders will overlap. Essentially, this means that if $[h_n^{(m-2)}, h_n^{(m-1)}] \times S^1$ is a cylinder of type ∞, which lies to the left of a cylinder $[h_n^{(m-1)} - 3h, h_n^{(m)} + 3h] \times S^1$ of type b_1, then their intersection is $[h_n^{(m-1)} - 3h, h_n^{(m-1)}] \times S^1$. This can be seen in Figure 3.2.5.

By the above procedure, the cylinder $[-\sigma_n, \sigma_n] \times S^1$ is decomposed into an alternating constellation of cylinders of types ∞ and b_1. On cylinders of type ∞, the $d\alpha-$energy is smaller than $\hbar_0/4$, while on cylinders of type b_1, the $d\alpha-$energy can be larger than $\hbar_0/4$. By Lemma 42, the gradients of the $\mathcal{H}-$holomorphic curves on the cylinders of type ∞ are uniformly bounded by the constant $C_h > 0$ with respect to the Euclidean metric on the domain, and to the metric described in (2.2.1) on the target space $\mathbb{R} \times M$. Finally, the cylinders of types ∞ and b_1 overlap.
We are now well prepared to analyse the convergence of the $\mathcal{H}-$holomorphic curves on cylinders of types ∞ and b_1. After obtaining separate convergence results, we glue the limit objects of these cylinders on the overlaps, and obtain a limit object on the whole cylinder $[-\sigma_n, \sigma_n] \times S^1$. Sections 3.2.2 and 3.2.3 deal with the convergence and the description of the limit object on cylinders of types ∞ and b_1, while in Section 3.2.4 we carry out the gluing of these two convergence results.

3.2.2 Cylinders of type ∞

We describe the convergence and the limit object of the sequence of $\mathcal{H}-$holomorphic curves u_n, defined on cylinders of type ∞. Let $m \in \{1, ..., N\}$ be such that $[h_n^{(m-1)}, h_n^{(m)}] \times S^1$ is a cylinder of type ∞ as described in Section 3.2.1, i.e. $h_n^{(m)} - h_n^{(m-1)} \to \infty$ as $n \to \infty$. Consider the diffeomorphism $\psi_n : [-R_n^{(m)}, R_n^{(m)}] \to [h_n^{(m)}, h_n^{(m+1)}]$ given by $\psi_n(s) = s + (h_n^{(m)} + h_n^{(m+1)})/2$ and the $\mathcal{H}-$holomorphic maps $u_n \circ \psi_n = (a_n \circ \psi_n, f_n \circ \psi_n) : [-R_n^{(m)}, R_n^{(m)}] \times S^1 \to \mathbb{R} \times M$ with harmonic perturbation $\psi_n^* \gamma_n$. For simplicity we continue to denote $u_n \circ \psi_n$ and $\psi_n^* \gamma_n$ by u_n and γ_n, respectively. For deriving a $C^\infty_{loc}-$convergence result we consider the following setting:

C1 $R_n^{(m)} \to \infty$ as $n \to \infty$.

C2 γ_n is a harmonic 1−form on $[-R_n^{(m)}, R_n^{(m)}] \times S^1$ with respect to the standard complex structure i, i.e. $d\gamma_n = d\gamma_n \circ i = 0$.

C3 The $d\alpha$−energy of u_n is uniformly small, i.e. $E_{d\alpha}(u_n; [-R_n^{(m)}, R_n^{(m)}] \times S^1) \leqslant \hbar_0/2$ for all n, where $\hbar_0$ is the constant defined in (3.2.2).

C4 The energy of u_n is uniformly bounded, i.e. for the constant $E_0 > 0$ we have $E(u_n; [-R_n^{(m)}, R_n^{(m)}] \times S^1) \leqslant E_0$ for all $n \in \mathbb{N}$.

C5 The map u_n together with the 1−form γ_n solve the $\mathcal{H}$−holomorphic curve equation

$$\begin{aligned} \pi_\alpha df_n \circ i &= J(f_n) \circ \pi_\alpha df_n, \\ (f_n^*\alpha) \circ i &= da_n + \gamma_n. \end{aligned} \quad \text{on } [-R_n^\delta, R_n^\delta] \times S^1$$

C6 The harmonic 1−form γ_n has a uniformly bounded L^2−norm, i.e. for the constant $C_0 > 0$ we have $\|\gamma_n\|^2_{L^2([-R_n^{(m)}, R_n^{(m)}] \times S^1)} \leqslant C_0$ for all n.

C7 The map u_n has a uniformly bounded gradient due to Lemma 42 and Step 4 of Section 3.2.1, i.e. for the constant $C_h > 0$ we have

$$\|du_n(z)\|_{C^0} = \sup_{\|v\|_{\text{eucl}}=1} \|du_n(z)v\| < C_h$$

for all $z \in [-R_n^{(m)}, R_n^{(m)}] \times S^1$ and all $n \in \mathbb{N}$.

C8 If $P_n := P_{\gamma_n}(\{0\} \times S^1)$ is the period of γ_n over the closed curve $\{0\} \times S^1$, as defined in (1.0.18), we assume that the sequence $R_n P_n$ is bounded by the constant $C > 0$. Moreover, after going over to some subsequence, we assume that $R_n P_n$ converges to some real number τ.

C9 If $S_n := S_{\gamma_n}(\{0\} \times S^1)$ is the co-period of γ_n over the curve $\{0\} \times S^1$ as defined in (1.0.19), we assume that $S_n R_n \to \sigma$ as $n \to \infty$.

Remark 43. The special circles Γ_i^{nod} in Remark 32 are of two types: contractible and non-contractible. In the contractible case, Γ_i^{nod} lies in the isotopy class of $(\rho_n \circ \psi_n)(\{0\} \times S^1)$, where ρ_n is the biholomorphism from a compact component C_n of the thin part to $[-\sigma_n, \sigma_n] \times S^1$ as described in Section 3.2.1, and the conformal periods and co-periods of the harmonic 1−forms γ_n vanish. Hence, conditions C1-C9 are satisfied on the sequence of degenerating cylinders $[-R_n^{(m)}, R_n^{(m)}] \times S^1$. In the non-contractible case, Γ_i^{nod} also lies in the isotopy class of $(\rho_n \circ \psi_n)(\{0\} \times S^1)$, and by the assumptions of Theorem 33, conditions C1-C9 are satisfied.

To simplify notation we drop the index m. By Theorem 72 and Remark 75 from Appendix B.1.2 we consider two cases. In Case 1, there exists a subsequence of u_n with vanishing center action, and we use Theorem 63 and Corollary 64 to describe the convergence of the $\mathcal{H}$−holomorphic curves with harmonic perturbations γ_n. In Case 2, each subsequence of u_n has a center action larger than $\hbar_0$, and we use Theorem 65 and Corollary 66 to describe the convergence.

Definition 44. For every sequence $h_n \in \mathbb{R}_+$ with $h_n < R_n$ and $h_n, R_n/h_n \to \infty$ as $n \to \infty$, consider a sequence of diffeomorphisms $\theta_n : [-R_n, R_n] \to [-1, 1]$ having the following properties:

1. The left and right shifts $\theta_n^+(s) := \theta_n(s + R_n)$ and $\theta_n^-(s) := \theta_n(s - R_n)$ defined on $[-h_n, 0] \to [1/2, 1]$ and $[0, h_n] \to [-1, -1/2]$, respectively, converge in C^∞_{loc} to the diffeomorphisms $\theta^- : [0, \infty) \to [-1, -1/2)$ and $\theta^+ : (-\infty, 0] \to (1/2, 1]$, respectively.

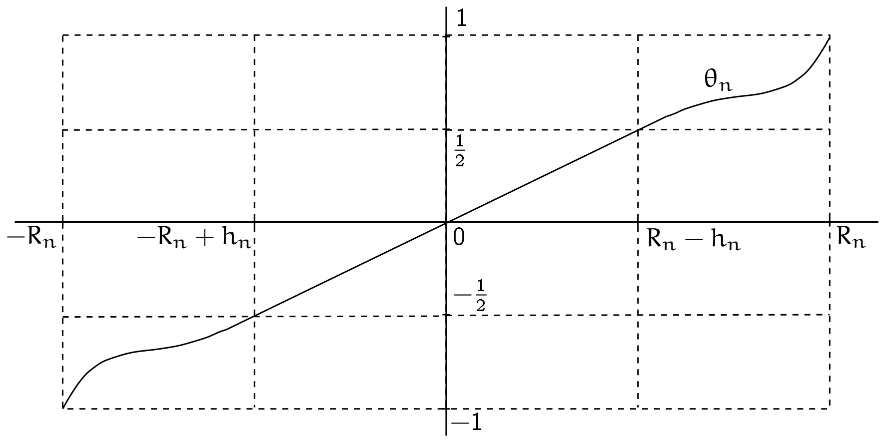

Figure 3.2.6: The diffeomorphism θ_n.

2. On $[-R_n + h_n, R_n - h_n]$ we define the diffeomorphism θ_n to be linear by requiring

$$\theta_n : \mathrm{Op}([-R_n + h_n, R_n - h_n]) \to \mathrm{Op}\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right), \ s \mapsto \frac{s}{2(R_n - h_n)},$$

where $\mathrm{Op}([-R_n + h_n, R_n - h_n])$ and $\mathrm{Op}([-1/2, 1/2])$ are sufficiently small neighborhoods of the intervals $[-R_n + h_n, R_n - h_n]$ and $[-1/2, 1/2]$, respectively.

See Figure 3.2.6.

Note that the diffeomorphism θ_n gives rise to a diffeomorphism between the cylinders $[-R_n, R_n] \times S^1$ and $[-1, 1] \times S^1$, according to $[-R_n, R_n] \times S^1 \to [-1, 1] \times S^1$, $(s, t) \mapsto (\theta_n(s), t)$. By abuse of notation these diffeomorphisms will be still denoted by θ_n. Denote by $u_n^{\pm}(s, t) := u_n(s \pm R_n, t)$ the left and right shifts of the maps u_n, and by $\gamma_n^{\pm} := \gamma_n(s \pm R_n, t)$ the left and right shifts of the harmonic perturbation, which are defined on $[0, h_n] \times S^1$ and $[-h_n, 0] \times S^1$, respectively. In both cases we use the diffeomorphisms θ_n to pull the structures back to the cylinder $[-1, 1] \times S^1$. Let $i_n := d\theta_n \circ i \circ d\theta_n^{-1}$ be the induced complex structure on $[-1, 1] \times S^1$. Then $u_n \circ \theta_n^{-1} : [-1, 1] \times S^1 \to \mathbb{R} \times M$ is a sequence of $\mathcal{H}-$holomorphic curves with harmonic perturbations $(\theta_n^{-1})^* \gamma_n$ with respect to the complex structure i_n on $[-1, 1] \times S^1$ and the cylindrical almost complex structure J on the target space $\mathbb{R} \times M$. From the result $\theta_n^{-1}(s) = (\theta_n^{-})^{-1}(s) - R_n$ and $\theta_n^{-1}(s) = (\theta_n^{+})^{-1}(s) + R_n$, and the fact that θ_n^{-} and θ_n^{+} converge in $C^{\infty}_{\mathrm{loc}}$ to θ^{-} on $[-1, -1/2)$ and θ^{+} on $(1/2, 1]$, respectively, it follows that the complex structures i_n converge in $C^{\infty}_{\mathrm{loc}}$ to a complex structure $\tilde{i}$ on $[-1, -1/2) \times S^1$ and $(1/2, 1] \times S^1$. First, we formulate the convergence in the case when there exists a subsequence of u_n, still denoted by u_n, with a vanishing center action (see Definition 74).

Theorem 45. *Let u_n be a sequence of $\mathcal{H}-$holomorphic cylinders with harmonic perturbations γ_n that satisfy C1-C9 and possessing a subsequence having vanishing center action. Then there exists a subsequence of u_n, still denoted by u_n, $\mathcal{H}-$holomorphic cylinders $u^{\pm}$ defined on $(-\infty, 0] \times S^1$ and $[0, \infty) \times S^1$, respectively, and a point $w = (w_a, w_f) \in \mathbb{R} \times M$ such that for every sequence $h_n \in \mathbb{R}_+$ and every sequence of diffeomorphisms $\theta_n : [-R_n, R_n] \to [-1, 1]$ constructed as in Remark 44 the following $C^{\infty}_{loc}-$ and C^0-convergence results hold (after a suitable shift of u_n in the $\mathbb{R}-$coordinate)*

$C^{\infty}_{loc}-$*convergence:*

1. *For any sequence $s_n \in [-R_n + h_n, R_n - h_n]$ there exists a constant $\tau_{\{s_n\}} \in [-\tau, \tau]$ (depending on the*

sequence $\{s_n\}$) such that after passing to a subsequence, the shifted maps $u_n(s+s_n,t)+S_ns_n$, defined on $[-R_n+h_n-s_n, R_n-h_n-s_n]\times S^1$, converge in C^∞_{loc} to $(w_a, \phi^\alpha_{-\tau_{\{s_n\}}}(w_f))$. The shifted harmonic perturbation 1−forms $\gamma_n(s+s_n,t)$ possess a subsequence converging in C^∞_{loc} to 0.

2. *The left shifts $u_n^-(s,t)-R_nS_n := u_n(s-R_n,t)-R_nS_n$, defined on $[0,h_n)\times S^1$, possess a subsequence that converges in C^∞_{loc} to a pseudoholomorphic half cylinder $u^-=(a^-,f^-)$, defined on $[0,+\infty)\times S^1$. The curve u^- is asymptotic to $(w_a, \phi^\alpha_\tau(w_f))$. The left shifted harmonic perturbation 1−forms γ_n^- converge in C^∞_{loc} to an exact harmonic 1−form $d\Gamma^-$, defined on $[0,+\infty)\times S^1$. Its asymptotics is 0.*
3. *The right shifts $u_n^+(s,t)+R_nS_n := u_n(s+R_n,t)+R_nS_n$, defined on $(-h_n,0]\times S^1$, possess a subsequence that converges in C^∞_{loc} to a pseudoholomorphic half cylinder $u^+=(a^+,f^+)$, defined on $(-\infty,0]\times S^1$. The curve u^+ is asymptotic to $(w_a, \phi^\alpha_{-\tau}(w_f))$. The right shifted harmonic perturbation 1−forms γ_n^+ converge in C^∞_{loc} to an exact harmonic 1−form $d\Gamma^+$, defined on $(-\infty,0]\times S^1$. Its asymptitics is 0.*

C^0−*convergence:*

1. *The maps $v_n : [-1/2,1/2]\times S^1 \to \mathbb{R}\times M$ defined by $v_n(s,t)=u_n(\theta_n^{-1}(s),t)$, converge in C^0 to $(-2\sigma s + w_a, \phi^\alpha_{-2\tau s}(w_f))$.*
2. *The maps $v_n^- - R_nS_n : [-1,-1/2]\times S^1 \to \mathbb{R}\times M$ defined by $v_n^-(s,t)=u_n((\theta_n^-)^{-1}(s),t)$, converge in C^0 to a map $v^- : [-1,-1/2]\times S^1\to\mathbb{R}\times M$ such that $v^-(s,t)=u^-((\theta^-)^{-1}(s),t)$ and $v^-(-1/2,t)=(w_a,\phi^\alpha_\tau(w_f))$.*
3. *The maps $v_n^+ + R_nS_n : [1/2,1]\times S^1 \to \mathbb{R}\times M$ defined by $v_n^+(s,t)=u_n((\theta_n^+)^{-1}(s),t)$, converge in C^0 to a map $v^+ : [1/2,1]\times S^1\to\mathbb{R}\times M$ such that $v^+(s,t)=u^+((\theta^+)^{-1}(s),t)$ and $v^+(1/2,t)=(w_a,\phi^\alpha_{-\tau}(w_f))$.*

An immediate corollary is

Corollary 46. *Under the same hypothesis of Theorem 45 the following C^∞_{loc}−convergence results hold.*

1. *The maps $v_n^- - R_nS_n$ converge in C^∞_{loc} to v^-, where v^- is asymptotic to $(w_a,\phi^\alpha_\tau(w_f))$ as $s\to -1/2$. The harmonic 1−forms $[(\theta_n^-)^{-1}]^*\gamma_n^-$ with respect to the complex structure $[(\theta_n^-)^{-1}]^*i$ converge in C^∞_{loc} to a harmonic 1−form $[(\theta^-)^{-1}]^*d\Gamma^-$ with respect to the complex structure $[(\theta^-)^{-1}]^*i$ which is asymptotic to some constant as $s\to -1/2$.*
2. *The maps $v_n^+ + R_nS_n$ converge in C^∞_{loc} to v^+, where v^+ is asymptotic to $(w_a,\phi^\alpha_{-\tau}(w_f))$ as $s\to 1/2$. The harmonic 1−forms $[(\theta_n^+)^{-1}]^*\gamma_n^-$ with respect to the complex structure $[(\theta_n^+)^{-1}]^*i$ converge in C^∞_{loc} to a harmonic 1−form $[(\theta^+)^{-1}]^*d\Gamma^+$ with respect to the complex structure $[(\theta^+)^{-1}]^*i$ which is asymptotic to some constant as $s\to 1/2$.*

Next we formulate the convergence in the case when there is no subsequence of u_n with a vanishing center action. This result follows from Theorem 65 of Appendix B.1.

Theorem 47. *Let u_n be a sequence of $\mathcal{H}$−holomorphic cylinders with harmonic perturbations γ_n satisfying C1-C9 and possessing no subsequence with vanishing center action. Then there exist a subsequence of u_n, still denoted by u_n, $\mathcal{H}$−holomorphic half cylinders $u^\pm$ defined on $(-\infty,0]\times S^1$ and $[0,\infty)\times S^1$, respectively, a periodic orbit x of period $|T|$, where $T\in\mathbb{R}\backslash\{0\}$, and sequences $\bar{r}_n^\pm\in\mathbb{R}$ with $|\bar{r}_n^+-\bar{r}_n^n|\to\infty$ as $n\to\infty$ such that for every sequence $h_n\in\mathbb{R}_+$ and every sequence of diffeomorphisms $\theta_n : [-R_n,R_n]\to[-1,1]$ as in Remark 44, the following convergence results hold (after a suitable shift of u_n in the $\mathbb{R}$−coordinate).*

C^∞_{loc}−*convergence:*

1. *For any sequence* $s_n \in [-R_n + h_n, R_n - h_n]$ *there exists a constant* $\tau_{\{s_n\}} \in [-\tau, \tau]$ *(depending on the sequence* $\{s_n\}$*) such that after passing to a subsequence, the shifted maps* $u_n(s + s_n, t) - s_n T - S_n s_n$*, defined on* $[-R_n + h_n - s_n, R_n - h_n - s_n] \times S^1$*, converge in* C^∞_{loc} *to* $(Ts + a_0, \phi^\alpha_{-\tau_{\{s_n\}}}(x(Tt)) = x(Tt + \tau_{\{s_n\}}))$*. The shifted harmonic perturbation* $1-$*forms* $\gamma_n(s + s_n, t)$ *possess a subsequence converging in* C^∞_{loc} *to* 0*.*
2. *The left shifts* $u_n^-(s,t) - R_n S_n$*, defined on* $[0, h_n) \times S^1$*, possess a subsequence that converges in* C^∞_{loc} *to a* $\mathcal{H}-$*holomorphic half cylinder* $u^- = (a^-, f^-)$*, defined on* $[0, +\infty) \times S^1$*. The curve* u^- *is asymptotic to* $(Ts + a_0, \phi^\alpha_\tau(x(Tt)) = x(Tt + \tau))$*. The left shifted harmonic perturbation* $1-$*forms* γ_n^- *converge in* C^∞_{loc} *to an exact harmonic* $1-$*form* $d\Gamma^-$*, defined on* $[0, +\infty) \times S^1$*. Their asymptotics are* 0*.*
3. *The right shifts* $u_n^+(s,t) + R_n S_n$*, defined on* $(-h_n, 0] \times S^1$ *possess a subsequence that converges in* C^∞_{loc} *to a* $\mathcal{H}-$*holomorphic half cylinder* $u^+ = (a^+, f^+)$*, defined on* $(-\infty, 0] \times S^1$*. The curve* u^+ *is asymptotic to* $(Ts + a_0, \phi^\alpha_{-\tau}(x(Tt)) = x(Tt - \tau))$*. The right shifted harmonic perturbation* $1-$*forms* γ_n^+ *converge in* C^∞_{loc} *to an exact harmonic* $1-$*form* $d\Gamma^+$*, defined on* $(-\infty, 0] \times S^1$*. Their asymptotics are* 0*.*

C^0-*convergence:*

1. *The maps* $f_n \circ \theta_n^{-1} : [-1/2, 1/2] \times S^1 \to M$ *converge in* C^0 *to* $\phi^\alpha_{-2\tau s}(x(Tt)) = x(Tt - 2\tau s)$*.*
2. *The maps* $f_n^- \circ (\theta_n^-)^{-1} : [-1, -1/2] \times S^1 \to M$ *converge in* C^0 *to a map* $f^- \circ (\theta^-)^{-1} : [-1, -1/2] \times S^1 \to M$ *such that* $f^-((\theta^-)^{-1}(-1/2), t) = \phi^\alpha_\tau(x(Tt)) = x(Tt + \tau)$*.*
3. *The maps* $f_n^+ \circ (\theta_n^+)^{-1} : [1/2, 1] \times S^1 \to M$ *converge in* C^0 *to a map* $f^+ \circ (\theta^+)^{-1} : [1/2, 1] \times S^1 \to M$ *such that* $f^+((\theta^+)^{-1}(1/2), t) = \phi^\alpha_{-\tau}(x(Tt)) = x(Tt - \tau)$*.*
4. *There exist* $C > 0$*,* $\rho > 0$ *and* $N \in \mathbb{N}$ *such that for any* $R > 0$*,* $a_n \circ \theta_n^{-1}(s,t) \in [\bar{r}_n^- + R - C, \bar{r}_n^+ - R + C]$ *for all* $n \geqslant N$ *and all* $(s,t) \in [-\rho, \rho] \times S^1$*.*

An immediate corollary is

Corollary 48. *Under the same hypothesis of Theorem 47 and the notations from Theorem 44 we have the following* $C^\infty_{loc}-$*convergence results.*

1. *The maps* $v_n^- - R_n S_n$ *converge in* C^∞_{loc} *to* v^- *where* $f^-((\theta^-)^{-1}(-1/2), t) = x(Tt + \tau)$*. The harmonic* $1-$*forms* $[(\theta_n^-)^{-1}]^*\gamma_n^-$ *with respect to the complex structure* $[(\theta_n^-)^{-1}]^* i$ *converge in* C^∞_{loc} *to a harmonic* $1-$*form* $[(\theta^-)^{-1}]^* d\Gamma^-$ *with respect to the complex structure* $[(\theta^-)^{-1}]^* i$ *which is asymptotic to some constant as* $s \to -1/2$*.*
2. *The maps* $v_n^+ + R_n S_n$ *converge in* C^∞_{loc} *to* v^+ *where* $f^+((\theta^+)^{-1}(1/2), t) = x(Tt - \tau)$*. The harmonic* $1-$*forms* $[(\theta_n^+)^{-1}]^*\gamma_n^-$ *with respect to the complex structure* $[(\theta_n^+)^{-1}]^* i$ *converge in* C^∞_{loc} *to a harmonic* $1-$*form* $[(\theta^+)^{-1}]^* d\Gamma^+$ *with respect to the complex structure* $[(\theta^+)^{-1}]^* i$ *which is asymptotic to some constant as* $s \to 1/2$*.*

Since $\theta^- : [0, \infty) \times S^1 \to [-1, -1/2) \times S^1$ is a biholomorphism with respect to the standard complex structure i on the domain and the pull-back structure $\tilde{i} := [(\theta^-)^{-1}]^* i$, we can identify $[-1, -1/2) \times S^1$ with the punctured disk equipped with the standard complex structure, that extends over the puncture.
We use now Theorems 45 and 47 to describe the limit object.
In Case 1, the "limit surface" in the symplectization consists of two disks which are connected by a straight line at the origin. The limit map $u = (a, f) : [-1, 1] \times S^1 \to \mathbb{R} \times M$ with the limit perturbation $1-$form γ can be described as follows (see Figure 3.2.7).

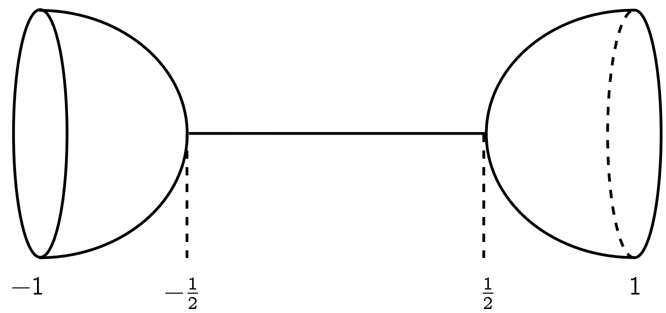

Figure 3.2.7: The limit surface consists of two cones connected by a straight line.

D1 On $[-1,-1/2)\times S^1$, u is a $\mathcal{H}-$holomorphic curve with harmonic perturbation γ such that at the puncture it is asymptotic to $(\sigma + w_a, \phi^\alpha_\tau(w_f))$, while the harmonic perturbation is asymptotic to a constant.

D2 On $(1/2,1]\times S^1$, u is a $\mathcal{H}-$holomorphic curve with harmonic perturbation γ such that at the puncture it is asymptotic to $(-\sigma + w_a, \phi^\alpha_{-\tau}(w_f))$, while the harmonic perturbation is asymptotic to a constant.

D3 On the middle part $[-1/2,1/2]\times S^1$, u is given by $u(s,t) = (-2\sigma s + w_a, \phi^\alpha_{-2\tau s}(w_f))$. On this part the $1-$form γ is not defined.

In Case 2, the limit surface is the disjoint union of the cylinders $[-1,-1/2)\times S^1$ and $(1/2,1]\times S^1$. The $\mathcal{H}-$holomorphic curve $u = (a,f) : ([-1,-1/2)\coprod(1/2,1])\times S^1 \to \mathbb{R}\times M$ with harmonic perturbation γ can be described as follows.

D1' u is asymptotic on $[-1,-1/2)\times S^1$ and $(1/2,1]\times S^1$ to a trivial cylinder over the Reeb orbit $x(Tt+\tau)$ or $x(Tt-\tau)$, respectively, while the harmonic perturbation is asymptotic to a constant.

D2' On the middle part $[-1/2,1/2]\times S^1$, the $M-$component f is given by $f(s,t) = x(Tt - 2\tau s)$.

3.2.3 Cylinders of type b_1

We analyze the convergence on cylinders of type b_1 by using the results of Appendix A. Let $m \in \{1,...,N\}$ be such that the cylinders $[h_n^{(m-1)} - 3h, h_n^{(m)} + 3h]\times S^1$ are of type b_1. By the construction described in the previous section and Lemma 42 and Step 1 from Section 3.2.1, the $\mathcal{H}-$holomorphic curves have uniform gradient bounds on the two boundary cylinders $[h_n^{(m-1)} - 3h, h_n^{(m-1)}]\times S^1$ and $[h_n^{(m)}, h_n^{(m)} + 3h]\times S^1$.
The convergence analysis is organized as follows. As in Section 3.1 we apply bubbling-off analysis on the cylinder $[h_n^{(m-1)}, h_n^{(m)}]\times S^1$ to show that on any compact set in the complement of a finite number of points $\mathcal{Z}^{(m)}$ in $[h_n^{(m-1)} - 3h, h_n^{(m)} + 3h]\times S^1$, the gradient of u_n is uniformly bounded. The points on which the gradient might blow up are located in $(h_n^{(m-1)} - h, h_n^{(m)} + h)\times S^1$. Each resulting puncture from $\mathcal{Z}^{(m)}$ lies in a disk D_r of radius r smaller than $h/2$. For a smaller radius r, we assume that all disks D_r are pairwise disjoint and that their union lies in $(h_n^{(m-1)} - h, h_n^{(m)} + h)\times S^1$ (see Figure 3.2.8).

Under these assumptions, the $\mathcal{H}-$holomorphic curves converge in C^∞ on the complement of the union of these disks (centered at the punctures) to a $\mathcal{H}-$holomorphic curve. What is left to prove is the convergence in each D_r;

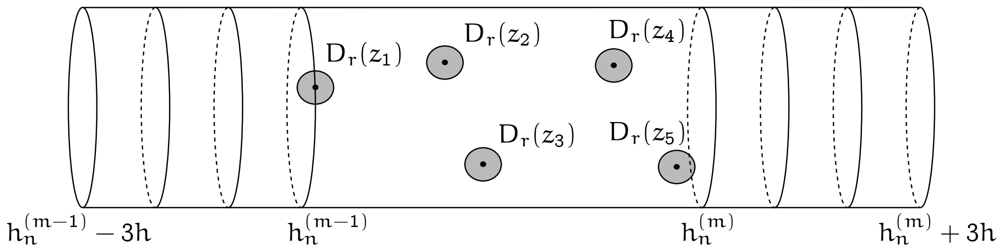

Figure 3.2.8: The gradient might blow up on the discs $D_r(z_i)$ contained in $(h_n^{(m-1)} - h, h_n^{(m)} + h) \times S^1$.

for this we use the results of Appendix A. In the final step, we glue the convergence results on the disks to the rest of the cylinder, and obtain the desired description on the entire cylinder of type b_1.
Under the biholomorphic map $[h_n^{(m-1)} - 3h, h_n^{(m)} + 3h] \times S^1 \to [0, H_n^{(m)}] \times S^1, (s,t) \mapsto (s - h_n^{(m-1)} + 3h, t)$, where $H_n^{(m)} := h_n^{(m)} - h_n^{(m-1)} + 6h$, assume that the $\mathcal{H}-$holomorphic curves u_n together with the harmonic perturbations γ_n are defined on $[0, H_n^{(m)}] \times S^1$. By going over to a subsequence, we have $H_n^{(m)} \to H^{(m)}$ as $n \to \infty$. Consider the translated $\mathcal{H}-$holomorphic curves $u_n - a_n(0,0) = (a_n, f_n) - a_n(0,0) : [0, H_n^{(m)}] \times S^1 \to \mathbb{R} \times M$ with harmonic perturbations γ_n. In order to keep the notation simple, let the curve $u_n - a_n(0,0)$ be still denoted by u_n. The analysis is performed in the following setting:

E1 The maps $u_n = (a_n, f_n)$ are $\mathcal{H}-$holomorphic curves with harmonic perturbation γ_n on $[0, H_n^{(m)}] \times S^1$ with respect to the standard complex structure i on the domain and the almost complex structure J on ξ.

E2 The maps u_n have uniformly bounded energies, while the harmonic perturbations γ_n have uniformly bounded L^2-norms, i.e., with the constants $E_0, C_0 > 0$ we have $E(u_n; [0, H_n^{(m)}] \times S^1) \leqslant E_0$ and $\|\gamma_n\|^2_{L^2([0,H_n^{(m)}] \times S^1)} \leqslant C_0$ for all $n \in \mathbb{N}$.

E3 The maps u_n have uniformly bounded gradients on $[0, 3h] \times S^1$ and $[H_n^{(m)} - 3h, H_n^{(m)}] \times S^1$ with respect to the Euclidean metric on the domain and the cylindrical metric on the target space $\mathbb{R} \times M$, i.e.

$$\|du_n(z)\| = \sup_{\|v\|_{\text{eucl.}}=1} \|du_n(z)v\|_{\overline{g}} < C_h$$

for all $z \in ([0, 3h] \cup [H_n^{(m)} - 3h, H_n^{(m)}]) \times S^1$ and $n \in \mathbb{N}$.

The next lemma states the existence of a finite set $\mathcal{Z}^{(m)}$ of punctures on which the gradient of u_n blows up.

Lemma 49. *There exists a finite set of points* $\mathcal{Z}^{(m)} \subset [3h, H_n^{(m)} - 3h] \times S^1$ *such that for any compact subset* $\mathcal{K} \subset ([0, H_n^{(m)}] \times S^1) \backslash \mathcal{Z}^{(m)}$ *there exists a constant* $C_{\mathcal{K}} > 0$ *such that*

$$\|du_n(z)\| = \sup_{\|v\|_{\textit{eucl.}}=1} \|du_n(z)v\|_{\overline{g}} < C_{\mathcal{K}}$$

for all $z \in \mathcal{K}$ *and* $n \in \mathbb{N}$.

Proof. The proof relies on the same arguments of bubbling-off analysis, which have been employed in Theorem 34 from Section 3.1 for the thick part.

□

Pick some $r > 0$ such that $r < h/2$, and let $D_r(\mathcal{Z}^{(m)})$ consists of $|\mathcal{Z}^{(m)}|$ pairwise disjoint closed disks of radius $r > 0$, centered at the punctures of $\mathcal{Z}^{(m)}$. Obviously, $D_r(\mathcal{Z}^{(m)}) \subset (2h, H_n^{(m)} - 2h) \times S^1$. Then by Lemma 49, u_n

has a uniformly bounded gradient on $([0, H_n^{(m)}] \times S^1)\backslash D_r(\mathcal{Z}^{(m)})$. As $([0, H_n^{(m)}] \times S^1)\backslash D_r(\mathcal{Z}^{(m)})$ is connected, we assume, after going over to some subsequence, that $u_n|_{([0,H_n^{(m)}]\times S^1)\backslash D_r(\mathcal{Z}^{(m)})}$ converge in C^∞ to some smooth map $u|_{([0,H^{(m)}]\times S^1)\backslash D_r(\mathcal{Z}^{(m)})} = (a, f)|_{([0,H^{(m)}]\times S^1)\backslash D_r(\mathcal{Z}^{(m)})}$. Before treating the convergence of the $\mathcal{H}-$holomorphic curves in a neighborhood of the punctures of $\mathcal{Z}^{(m)}$, we establish the convergence of the harmonic perturbations γ_n on $[0, H_n^{(m)}] \times S^1$, so that at the end

- $u_n|_{([0,H_n^{(m)}]\times S^1)\backslash D_r(\mathcal{Z}^{(m)})}$ converge in C^∞ to a $\mathcal{H}-$holomorphic curve $u|_{([0,H^{(m)}]\times S^1)\backslash D_r(\mathcal{Z}^{(m)})}$, and
- the harmonic perturbations γ_n have uniformly bounded C^k-norms on the disks $D_r(\mathcal{Z}^{(m)})$ for all $k \in \mathbb{N}_0$.

The latter result is needed to describe the convergence of the harmonic perturbations γ_n on the disks $D_r(\mathcal{Z}^{(m)})$. As in the previous section, we set $\gamma_n = f_n ds + g_n dt$, where f_n and g_n are harmonic functions defined on $[0, H_n^{(m)}] \times S^1$ such that $f_n + ig_n$ are holomorphic. By the uniform L^2-bound of γ_n it follows that

$$\|\gamma_n\|^2_{L^2([0,H_n^{(m)}]\times S^1)} = \int_{[0,H_n^{(m)}]\times S^1} \left(f_n^2 + g_n^2\right) ds dt \leqslant C_0$$

for all $n \in \mathbb{N}$, and so, that the L^2-norms of the holomorphic functions $f_n + ig_n$ are uniformly bounded. Letting $G_n = f_n + ig_n$ we state the following

Proposition 50. *There exists a subsequence of* G_n*, also denoted by* G_n*, that converges in* C^∞ *to some holomorphic map* G *defined on* $[0, H^{(m)}] \times S^1$*. Moreover, the harmonic perturbations* γ_n *converge in* C^∞ *to a harmonic map* γ*.*

Proof. By Proposition 40, G_n has a uniformly bounded C^1-norm, while by the standard regularity results from the theory of pseudoholomorphic curves (see, for example, Section 2.2.3 of [2]), the C^k derivatives of G_n are also uniformly bounded. Hence, in view of Arzelà-Ascoli theorem, we can extract a subsequence that converges to some holomorphic function G.

□

Let us analyze the convergence of the $\mathcal{H}-$holomorphic curves in a neighborhood of the punctures of $\mathcal{Z}^{(m)}$, which are given by Lemma 49. For $r > 0$ as above and $z \in \mathcal{Z}^{(m)}$, consider the closed disks $D_r(z)$ and the $\mathcal{H}-$holomorphic curves $u_n = (a_n, f_n) : D_r(z) \to \mathbb{R} \times M$ with harmonic perturbations γ_n that converge in C^∞ to some harmonic 1−form γ. According to the biholomorphism $D \to D_r(z)$, $p \mapsto rp + z$, where D is the standard closed unit disk, regard the $\mathcal{H}-$holomorphic curves u_n together with the harmonic perturbations as being defined on D instead of $D_r(z)$. The following setting is pertinent to our analysis:

F1 The maps $u_n = (a_n, f_n) : D \to \mathbb{R} \times M$ are $\mathcal{H}-$holomorphic curves with harmonic perturbations γ_n with respect to the standard complex structure i on D and the almost complex structure J on ξ.

F2 The maps $u_n = (a_n, f_n)$ and γ_n have uniformly bounded energies and L^2-norms.

F3 For any constant $1 > \tau > 0$, $u_n|_{A_{1,\tau}} = (a_n, f_n)|_{A_{1,\tau}}$ converge in C^∞ to a $\mathcal{H}-$holomorphic map u with harmonic perturbation γ, where $A_{1,\tau} = \{z \in D \mid \tau \leqslant |z| \leqslant 1\}$.

As the domain of definition D is simply connected, we infer that γ_n is exact, i.e. it can be written as $\gamma_n = d\tilde{\Gamma}_n$, where $\tilde{\Gamma}_n : D \to \mathbb{R}$ is a harmonic function. By Condition F2, $\tilde{\Gamma}_n$ has a uniformly bounded gradient $\nabla\tilde{\Gamma}_n$ in the L^2-norm, and it is apparent that the existence of $\tilde{\Gamma}_n$ is unique up to addition by a constant. Let us make some remarks on the choice of $\tilde{\Gamma}_n$ and discuss some of its properties. By using the mean value theorem for harmonic functions as in Proposition 40 we conclude (after eventually, shrinking D) that the gradient $\nabla\tilde{\Gamma}_n$ are uniformly bounded in C^0. Denote by $z = s + it$ the coordinates on D, and let

$$K_n = \frac{1}{\pi}\int_D \tilde{\Gamma}_n(s,t)dsdt$$

be the mean value of $\tilde{\Gamma}_n$, so that by the mean value theorem for harmonic functions, $K_n = \tilde{\Gamma}_n(0)$. Finally, define the map $\Gamma_n(z) := \tilde{\Gamma}_n(z) - \tilde{\Gamma}_n(0)$ which obviously satisfies $\gamma_n = d\Gamma_n$.

Remark 51. From Poincaré inequality it follows that $\|\Gamma_n\|_{L^2(D)} \leqslant c\left\|\nabla\tilde{\Gamma}_n\right\|_{L^2(D)}$ for some constant $c > 0$ and so, that Γ_n is uniformly bounded in L^2-norm. Again, by using the mean value theorem for harmonic functions, we deduce (after maybe shrinking D) that Γ_n has a uniformly bounded C^0-norm, and consequently, that Γ_n has a uniformly bounded C^1-norm. Because $\gamma_n = d\Gamma_n$ is a harmonic 1−form, $\partial_s\Gamma_n + i\partial_t\Gamma_n$ is a holomorphic function. In this context, by Proposition 40, Γ_n converge in C^∞ to a harmonic function $\Gamma : D \to \mathbb{R}$.

In the following we transform the $\mathcal{H}-$holomorphic curves defined on the disk in a usual pseudoholomorphic curve by encoding the harmonic perturbation $\gamma_n = d\Gamma_n$ in the $\mathbb{R}-$coordinate of the $\mathcal{H}-$holomorphic curve u_n. Specifically, we define the maps $\overline{u}_n = (\overline{a}_n, \overline{f}_n) = (a_n + \Gamma_n, f_n)$ which are obviously pseudoholomorphic. The transformation is usable if we ensure that the energy bounds are still satisfied. For an ordinary pseudoholomorphic curve, the sum of the $\alpha-$ and $d\alpha-$energies, that are both positive, yield the Hofer energy $E_H(\overline{u}_n; D)$. A uniform bound on the Hofer energy, which ensures a uniform bound on the $\alpha-$ and $d\alpha-$energies of $\overline{u}_n$, is

$$E_H(\overline{u}_n; D) = \sup_{\varphi\in\mathcal{A}}\int_D \overline{u}_n^* d(\varphi\alpha) = \sup_{\varphi\in\mathcal{A}}\int_{\partial D}\varphi(\overline{a}_n)\overline{f}_n^*\alpha \leqslant \int_{\partial D}|\overline{f}_n^*\alpha| \leqslant C_h.$$

Here, the last inequality follows from Condition F3, according to which, u_n converge in C^∞ in a fixed neighborhood of ∂D. Note that the constant C_h is guaranteed by Lemma 49.
In a next step we use the results of Appendix A to establish the convergence of the maps $\overline{u}_n$ and to describe their limit object. Then we undo the transformation in the $\mathbb{R}-$coordinate (more precisely, the encoding of γ_n in the $\mathbb{R}-$coordinate of the curve u_n) and give a convergence result together with a description of the limit object for u_n. Before proceeding we state the setting corresponding to the pseudoholomorphic curves $\overline{u}_n$.

G1 The maps $\overline{u}_n = (\overline{a}_n, \overline{f}_n) : D \to \mathbb{R}\times M$ solve the pseudoholomorphic curve equation

$$\pi_\alpha d\overline{f}_n \circ i = J(\overline{f}_n)\circ\pi_\alpha d\overline{f}_n,$$
$$\overline{f}_n^*\alpha\circ i = d\overline{a}_n$$

on D.

G2 The maps $\overline{u}_n$ have uniformly bounded energies.

G3 For any $\tau > 0$, $\overline{u}_n|_{A_{1,\tau}} = (\overline{a}_n, \overline{f}_n)|_{A_{1,\tau}}$ converge in C^∞ to a pseudoholomorphic map.

We consider two cases. In the first case, the $\mathbb{R}-$components of $\overline{u}_n$ are uniformly bounded, while in the second case they are not. Actually, the first case does not occur. We will prove this result in the next lemma by using standard bubbling-off analysis. Let $z_n \in D$ be the sequence choosen from the bubbling-off argument of Lemma 49, i.e. for which we have that

$$\|d\overline{u}_n(z_n)\| = \sup_{z\in D}\|d\overline{u}_n(z)\| \to \infty \tag{3.2.3}$$

as $n \to \infty$.

Lemma 52. *The $\mathbb{R}-$coordinates of the maps $\overline{u}_n$ are unbounded on* D.

Proof. We prove by contradiction using bubbling-off analysis. Assume that the $\mathbb{R}-$coordinates of the maps $\overline{u}_n$ are uniformly bounded. Employing the same arguments as in the proof of Lemma 42 for the sequence $R_n := \|d\overline{u}_n(z_n)\|$, we find that the maps $v_n : D_{\epsilon'_n R'_n}(0) \to \mathbb{R} \times M$ converge in $C^\infty_{loc}(\mathbb{C})$ to a non-constant finite energy holomorphic plane v. Note that the boundedness of $E_{d\alpha}(v;\mathbb{C})$ follows from the fact that for an arbitrary $R > 0$ we have

$$E_{d\alpha}(v, D_R(0)) = \lim_{n\to\infty} E_{d\alpha}(v_n; D_R(0)) \leqslant \lim_{n\to\infty} E_{d\alpha}(v_n; D_{\epsilon'_n R'_n}(0)) \leqslant C_h,$$

yielding $E_{d\alpha}(v;\mathbb{C}) \leqslant C_h$. As we have assumed that the $\mathbb{R}-$coordinates of $\overline{u}_n$ are uniformly bounded it follows that the $\mathbb{R}-$coordiantes of v_n, and so, of v, are also uniformly bounded. By singularity removal, v can be extended to a pseudoholomorphic sphere. Thus the $d\alpha-$energy vanishes and by the maximum principle, the function a is constant. For this reason, v must be constant and we are lead to a contradiction.

□

We consider now the second case in which the $\mathbb{R}-$coordinates of the maps $\overline{u}_n$ are unbounded, and make extensively use of the results of Appendix A. By the maximum principle, the function $\overline{a}_n$ tends to $-\infty$, while by Proposition 62, the maps $\overline{u}_n = (\overline{a}_n, f_n) : (D, i) \to \mathbb{R} \times M$ converge to a broken holomorphic curve $\overline{u} = (\overline{a}, \overline{f}) : (Z, j) \to \mathbb{R} \times M$. Here, Z is obtained as follows. Let Z be a surface diffeomorphic to D, and let $\Delta = \Delta_n \amalg \Delta_p \subset Z$ be a collection of finitely many disjoint loops away from ∂Z. Further on, let $Z\backslash\Delta_p = \coprod_{\nu=0}^{N+1} Z^{(\nu)}$ for some $N \in \mathbb{N}$ as described in Appendix A. For a loop $\delta \in \Delta_p$, there exists $\nu \in \{0, ..., N\}$ such that δ is adjacent to $Z^{(\nu)}$ and $Z^{(\nu+1)}$. Fix an embedded annuli

$$A^{\delta,\nu} \cong [-1, 1] \times S^1 \subset Z\backslash\Delta_n$$

such that $\{0\} \times S^1 = \delta$, $\{-1\} \times S^1 \subset Z^{(\nu)}$, and $\{1\} \times S^1 \subset Z^{(\nu+1)}$. In this context, there exist a sequence of diffeomorphism $\varphi_n : D \to Z$ and a sequence of negative real numbers $\min(a_n) = r_n^{(0)} < r_n^{(1)} < ... < r_n^{(N+1)} = -K-2$, where $K \in \mathbb{R}$ is the constant determined in Appendix A and $r_n^{(\nu+1)} - r_n^{(\nu)} \to \infty$ as $n \to \infty$ such that the following hold:

H1 $i_n := (\varphi_n)_* i \to j$ in C^∞_{loc} on $Z\backslash\Delta$.

H2 The sequence $\overline{u}_n \circ \varphi_n^{-1}|_{Z^{(\nu)}} : Z^{(\nu)} \to \mathbb{R} \times M$ converges in C^∞_{loc} on $Z^{(\nu)}\backslash\Delta_n$ to a punctured nodal pseudoholomorphic curve $\overline{u}^{(\nu)} : (Z^{(\nu)}, j) \to \mathbb{R} \times M$, and in C^0_{loc} on $Z^{(\nu)}$.

H3 The sequence $\overline{f}_n \circ \varphi_n^{-1} : Z \to M$ converges in C^0 to a map $f : Z \to M$, whose restriction to Δ_p parametrizes the Reeb orbits and to Δ_n parametrizes points.

H4 For any $S > 0$, there exist $\rho > 0$ and $\tilde{N} \in \mathbb{N}$ such that $\overline{a}_n \circ \varphi_n^{-1}(s, t) \in [r_n^{(\nu)} + S, r_n^{(\nu+1)} - S]$ for all $n \geqslant \tilde{N}$ and all $(s, t) \in A^{\delta,\nu}$ with $|s| \leqslant \rho$.

To establish a convergence result for the $\mathcal{H}-$holomorphic curve u_n we undo the tranformation. The maps u_n are given by $u_n = \overline{u}_n - \Gamma_n$, where $\Gamma_n : D \to \mathbb{R}$ is the harmonic function defined in Remark 51. Observe that by Remark 51, the Γ_n converge in $C^\infty(D)$ to some harmonic function and are uniformly bounded in $C^0(D)$. Via the above diffeomorphisms $\varphi_n : D \to Z$, consider the functions $\mathscr{G}_n := \Gamma_n \circ \varphi_n^{-1} : Z \to \mathbb{R}$. Since Γ_n are harmonic functions with respect to i, $\mathscr{G}_n$ are harmonic functions on Z with respect to i_n. Moreover, their gradients and absolute values are bounded in L^2- and C^0-norms, respectively, i.e.

$$\int_Z d\mathscr{G}_n \circ i_n \wedge d\mathscr{G}_n \leqslant C_0 \tag{3.2.4}$$

and

$$\|\mathscr{G}_n\|_{C^0(Z)} \leqslant C_1 \tag{3.2.5}$$

for some constant $C_1 > 0$ and for all $n \in \mathbb{N}$, respectively.

Lemma 53. *For any compact subset $\mathcal{K} \subset (Z \backslash \Delta)$ there exists a subsequence of $\mathscr{G}_n$, also denoted by $\mathscr{G}_n$, such that $\mathscr{G}_n \to \mathscr{G}$ in $C^\infty(\mathcal{K})$ as $n \to \infty$, where $\mathscr{G}$ is a harmonic function defined on a neighborhood of $\mathcal{K}$.*

Proof. Let $\mathcal{K} \subset (Z\backslash\Delta)$ be a compact subset. By Lemma 97 there exists a finite covering of $\mathcal{K}$ by the charts $\psi_n^{(l)} : D \to U_n^{(l)}$ and $\psi^{(l)} : D \to U^{(l)}$, where $l \in \{1, ..., N\}$ and $N \in \mathbb{N}$. For some $r \in (0, 1)$, the following hold:

1. $\psi_n^{(l)}$ are $i - i_n-$biholomorphisms and $\psi^{(l)}$ is an $i - j-$biholomorphism;
2. $\psi_n^{(l)} \to \psi^{(l)}$ in $C^\infty_{loc}(D)$ as $n \to \infty$;
3. $\mathcal{K} \subset \bigcup_{l=1}^N \psi_n^{(l)}(D_r(0))$ for all $n \in \mathbb{N}$, and $\mathcal{K} \subset \bigcup_{l=1}^N \psi^{(l)}(D_r(0))$.

Consider the function $\mathscr{G}_n^{(l)} := \mathscr{G}_n \circ \psi_n^{(l)} : D \to \mathbb{R}$ for some $l \in \{1, ..., N\}$. Because $\psi_n^{(l)}$ are $i - i_n-$biholomorphisms, $\mathscr{G}_n^{(l)}$ is a harmonic function with respect to i. From (3.2.4) and (3.2.5), $\mathscr{G}_n^{(l)}$ satisfies

$$\int_D d\mathscr{G}_n^{(l)} \circ i \wedge d\mathscr{G}_n^{(l)} \leqslant C_0 \text{ and } \left\|\mathscr{G}_n^{(l)}\right\|_{C^0(D)} \leqslant C_1.$$

Relying on the compactness result for harmonic functions we assume that $\mathscr{G}_n^{(l)}$ converges in $C^0(D_{3r/2}(0))$ to a harmonic function $\mathscr{G}^{(l)}$ defined on $D_{3r/2}(0)$. By the mean value theorem for harmonic functions, there exists a constant $c > 0$ such that $\left\|\nabla\mathscr{G}_n^{(l)}\right\|_{C^0(D_{4r/3}(0))} \leqslant c$ for all $n \in \mathbb{N}$. Hence $\mathscr{G}_n^{(l)}$ is uniformly bounded in $C^1(D_{4r/3}(0))$. Because $d\mathscr{G}_n^{(l)}$ defines a harmonic $1-$form, $\partial_s\mathscr{G}_n^{(l)} + i\partial_t\mathscr{G}_n^{(l)}$ is a uniformly bounded holomorphic function defined on $D_{4r/3}(0)$, where s, t are the coordinates on $D_{4r/3}(0)$. By means of the Cauchy integral formula, all derivatives of $\partial_s\mathscr{G}_n^{(l)} + i\partial_t\mathscr{G}_n^{(l)}$ are uniformly bounded on $D_{5r/4}(0)$. From this and the fact that $\mathscr{G}_n^{(l)}$ converges uniformly to $\mathscr{G}^{(l)}$ we deduce that there exists a further subsequence, also denoted by $\mathscr{G}_n^{(l)}$, that converges in $C^\infty(D_{6r/5}(0))$ to a harmonic function $\mathscr{G}^{(l)} : D_{6r/5}(0) \to \mathbb{R}$. For n sufficiently large, $\psi^{(l)}(\overline{D_r(0)}) \subset \psi^{(l)}(D_{6r/5}(0))$ and $\psi^{(l)}(\overline{D_r(0)}) \subset \psi_n^{(l)}(D_{6r/5}(0))$. Hence the harmonic function $\mathscr{G}_n = \mathscr{G}_n^{(l)} \circ (\psi_n^{(l)})^{-1} : \psi^{(l)}(\overline{D_r(0)}) \to \mathbb{R}$ converges in $C^\infty(\psi^{(l)}(\overline{D_r(0)}))$ to a harmonic function $\tilde{\mathscr{G}}^{(l)} := \mathscr{G}^{(l)} \circ (\psi^{(l)})^{-1} : \psi^{(l)}(\overline{D_r(0)}) \to \mathbb{R}$. Obviously, if $l, l' \in \{1, ..., N\}$ are such that $\psi^{(l)}(D_r(0)) \cap \psi^{(l')}(D_r(0)) \neq \emptyset$, the uniqueness of the limit yields $\tilde{\mathscr{G}}^{(l)}|_{\psi^{(l)}(D_r(0))\cap\psi^{(l')}(D_r(0))} = \tilde{\mathscr{G}}^{(l')}|_{\psi^{(l)}(D_r(0))\cap\psi^{(l')}(D_r(0))}$. Hence all $\tilde{\mathscr{G}}^{(l)}$ glue together to a harmonic function defined in a neighborhood of $\mathcal{K}$. □

By Lemma 53 it is apparent that after going over to a diagonal subsequence, $\mathscr{G}_n$ converges in $C^\infty_{loc}(Z\backslash\Delta)$ to a harmonic function $\mathscr{G} : Z\backslash\Delta \to \mathbb{R}$ with respect to j. This shows that the $\mathcal{H}-$holomorphic curve $u_n \circ \varphi_n^{-1}|_{Z^{(\nu)}} : Z^{(\nu)} \to \mathbb{R} \times M$ with harmonic perturbation $d\mathscr{G}_n$ converges in C^∞_{loc} on $Z^{(\nu)}\backslash\Delta_n$ to a $\mathcal{H}-$holomorphic curve $u^{(\nu)} : (Z^{(\nu)}, j) \to \mathbb{R} \times M$ with harmonic perturbation $d\mathscr{G}$, where $u^{(\nu)} = \overline{u}^{(\nu)} - \mathscr{G}$ for all ν. What is left is the description of the convergence of the $\mathcal{H}-$holomorphic curves $u_n \circ \varphi_n^{-1}$ with harmonic perturbation $d\mathscr{G}_n$ in a neighborhood of the loops from Δ_n, i.e. across the nodes from Δ_n. Observe that, from (3.2.5), $\mathscr{G}_n$ is uniformly bounded on Z by the constant C_1 and the L^2-norm of $d\mathscr{G}_n$ is uniformly bounded by the constant C_0. A neighborhood C_n of a loop in Δ_n can be biholomorphically parametrized as $[-r_n, r_n] \times S^1$ by the biholomorphism $\psi_n : [-r_n, r_n] \times S^1 \to C_n$, where $r_n \to \infty$ as $n \to \infty$. From the C^0 bound of $\mathscr{G}_n$ on Z, the maps $u_n \circ \varphi_n^{-1}$ are uniformly bounded in C^0 on C_n (maybe after some shift in the $\mathbb{R}-$coordinate). Thus we consider the $\mathcal{H}-$holomorphic cylinder $u_n \circ \varphi_n^{-1} \circ \psi_n$ with harmonic

perturbation $\psi_n^* d\mathscr{G}_n$ defined on $[-r_n, r_n] \times S^1$. Note that the energy of $u_n \circ \varphi_n^{-1} \circ \psi_n$ is uniformly bounded by the constant E_0. As in Section 3.2 we divide the cylinder $[-r_n, r_n] \times S^1$ into cylinders of type ∞ with an energy less than $\hbar_0/2$ and cylinders of type b_1. We apply the result of Section 3.2.2 to cylinders of type ∞. Keep in mind that according to Remark 43), conditions C1-C9 are satisfied. For cylinders of type b_1, the maps $u_n \circ \varphi_n^{-1} \circ \psi_n$, after a specific shift in the $\mathbb{R}-$coordinate, are contained in a compact subset of $\mathbb{R} \times M$. By the usual bubbling-off analysis and the maximum principle, these maps together with the harmonic perturbation converge in C^∞ on cylinders of type b_1.
We "glue" the convergence result for the ∞-type subcylinders of $[-r_n, r_n] \times S^1$ introduced in Section 3.2.2 together with the $C^\infty-$convergence result for the cylinders of type b_1. This process is similar to that described in Section 3.2.4.

Remark 54. Around a puncture from $Z^{(\nu)}$, the $\mathcal{H}-$holomorphic curve $u_n \circ \varphi_n$ is asymptotic to a trivial cylinder over a Reeb orbit (see Section 2.2). This result is a consequence of the uniform C^0-bound of the harmonic functions $\mathscr{G}_n$.

We are now in the position to formulate the convergence result for the $\mathcal{H}-$holomorphic curves u_n with harmonic perturbation γ_n defined on the disk D.
There exist the diffeomorphisms $\varphi_n : D \to Z$ such that the following hold:

I1 $i_n \to j$ in C^∞_{loc} on $Z \backslash \Delta_p \amalg A^{nod}$.

I2 For every special cylinder A_{ij} of Z there exists an annulus $\overline{A}_{ij} \cong [-1,1] \times S^1$ such that $A_{ij} \subset \overline{A}_{ij}$ and $(\overline{A}_{ij}, i_n)$ and (A_{ij}, i_n) are conformally equivalent to $([-R_n, R_n] \times S^1, i)$ and $([-R_n + h_n, R_n - h_n] \times S^1, i)$, respectively, where $R_n - h_n, h_n \to \infty$ as $n \to \infty$, i is the standard complex structure and the diffeomorphisms are of the form $(s,t) \mapsto (\kappa(s), t)$.

I3 The sequence of $\mathcal{H}-$holomorphic curves (D, i, u_n, γ_n) with boundary converges to a stratified $\mathcal{H}-$holomorphic building $(Z, j, u, \mathcal{P}, D, \gamma)$ in the sense of Definition 31 from Section 2.3. Note that the periods and conformal periods of γ vanish. Moreover, the curves converge in C^∞ in a neighborhood of the boundary ∂D.

This convergence result can be applied to disks such as neighborhoods of all points of $\mathcal{Z}^{(m)}$. To deal with the entire cylinder of type b_1, we glue the obtained convergence result on disks centered at points of $\mathcal{Z}^{(m)}$ to the complement of disk neighborhoods of $\mathcal{Z}^{(m)}$. During the convergence description of the $\mathcal{H}-$holomorphic curves u_n restricted to disk neighborhoods of the points of $\mathcal{Z}^{(m)}$, the diffeomorphism φ_n, describing the convergence, have the property that in a neighborhood of ∂D they are independent of n (see Appendix A). Coming back to the puncture $z \in \mathcal{Z}^{(m)}$ we focus on the neighborhood $D_r(z)$. Considering the translation and stretching diffeomorphism $D \to D_r(z)$, $p \mapsto z + rp$, we see that $\varphi_n : D_r(z) \backslash D_{r\tau}(z) \hookrightarrow Z$ is independent of n; hereafter, we drop the index n and denote it by $\varphi : D_r(z) \backslash D_{r\tau}(z) \hookrightarrow Z$. This map is used to glue Z and $([0, H^{(m)}] \times S^1) \backslash D_{r\tau}(z)$ along the collar $D_r(z) \backslash D_{r\tau}(z)$. Consider the surface

$$C^{(m)} = (([0, H^{(m)}] \times S^1) \backslash D_{r\tau}(z)) \amalg Z \big/ \sim,$$

where $x \sim y$ if and only if $x \in D_r(z) \backslash D_{r\tau}(z)$, $y \in \varphi(D_r(z) \backslash D_{r\tau}(z))$ and $\varphi(x) = y$. This gives rise to the diffeomorphism $\psi_n^{(m)} : [0, H_n^{(m)}] \times S^1 \to C^{(m)}$, defined by

$$\psi_n^{(m)}(x) = \begin{cases} x, & x \in C^{(m)} \backslash D_r(z) \\ \varphi_n(x), & x \in D_r(z). \end{cases}$$

We are now able to describe the convergence on cylinders of type b_1. Let Δ_n, Δ_p and A^{nod} be the collection of loops from $C^{(m)}$ obtained by the above convergence process for each point of $\mathcal{Z}^{(m)}$. Take notice that the complex structure $j^{(m)}$ on $C^{(m)}$ is given by

$$j^{(m)}(p) := \begin{cases} i, & p \in C^{(m)} \backslash D_r(\mathcal{Z}^{(m)}) \\ j, & p \in Z \end{cases}$$

and that it is well-defined since φ is a biholomorphism. There exists a sequence of diffeomorphisms $\psi_n^{(m)}$: $[0, H_n^{(m)}] \times S^1 \to C^{(m)}$ such that the following hold:

J1 $(\psi_n^{(m)})_* i \to j^{(m)}$ in C^∞_{loc} on $C^{(m)} \backslash \Delta_p \amalg A^{nod}$.

J2 For every special cylinder A_{ij} of $C^{(m)}$ there exists an annulus $\overline{A}_{ij} \cong [-1, 1] \times S^1$ such that $A_{ij} \subset \overline{A}_{ij}$ and $(\overline{A}_{ij}, i_n)$ and (A_{ij}, i_n) are conformally equivalent to $([-R_n, R_n] \times S^1, i)$ and $([-R_n + h_n, R_n - h_n] \times S^1, i)$, respectively, where $R_n, R_n - h_n \to \infty$ as $n \to \infty$, i is the standard complex structure and the diffeomorphisms are of the form $(s, t) \mapsto (\kappa(s), t)$.

J3 The $\mathcal{H}-$holomorphic curves $([0, H_n^{(m)}] \times S^1, i, u_n, \gamma_n)$ with boundary converges to a stratified $\mathcal{H}-$holomorphic building $(C^{(m)}, j, u, \mathcal{P}, \mathcal{D}, \gamma)$ with boundary in the sense of Definition 31.

3.2.4 Gluing cylinders of type ∞ with cylinders of type b_1

By a modified version of the diffeomorphisms θ_n we identify the cylinders of type ∞ with the cylinder $[-1-2h, 1+2h] \times S^1$ where $h > 0$ is the constant from Lemma 42, so that after the gluing process, we end up with a bigger cylinder of finite length and a sequence of diffeomorphisms. Let us make this procedure more precise.
Let $[h_n^{(m-1)}, h_n^{(m)}] \times S^1$ and $[h_n^{(m)} - 2h, h_n^{(m+1)} + 2h] \times S^1$ be cylinders of types ∞ and b_1, respectively. First we consider the cylinders $[h_n^{(m-1)}, h_n^{(m)}] \times S^1$ of type ∞. With the constant $h > 0$ defined in Section 3.2.1, let $[h_n^{(m-1)} + 3h, h_n^{(m)} - 3h] \times S^1$ be a subcylinder. By the uniform gradient bounds of u_n on cylinders of type ∞, we conclude that the $\mathcal{H}-$holomorphic curves u_n together with the harmonic perturbations γ_n converge in C^∞ on $[h_n^{(m)} - 3h, h_n^{(m)}] \times S^1$ to a $\mathcal{H}-$holomorphic curve u with harmonic perturbation γ. For the subcylinders $[h_n^{(m-1)} + 3h, h_n^{(m)} - 3h] \times S^1$ we perform the same analysis as in Theorems 45 and 47. After going over to a subsequence we obtain a sequence of diffeomorphisms

$$\theta_n : [h_n^{(m-1)} + 3h, h_n^{(m)} - 3h] \times S^1 \to [-1, 1] \times S^1,$$

so that Theorems 45 and 47 hold for the cylinders $[h_n^{(m-1)} + 3h, h_n^{(m)} - 3h] \times S^1$. Next we extend the diffeomorphisms θ_n to $[h_n^{(m-1)} + h, h_n^{(m)} - h] \times S^1$, such that

$$\theta_n|_{([h_n^{(m-1)}+h, h_n^{(m-1)}+2h] \times S^1) \amalg ([h_n^{(m)}-2h, h_n^{(m)}-h] \times S^1)} = \mathrm{id}.$$

By this procedure, we have obtained a diffeomorphism $\theta_n : [h_n^{(m-1)} + h, h_n^{(m)} - h] \times S^1 \to [-1 - 2h, 1 + 2h] \times S^1$ which is the identity near the boundary. We consider now the cylinders of type b_1 and note that the diffomorphisms

$$\psi_n : [h_n^{(m)} - 2h, h_n^{(m+1)} + 2h] \times S^1 \to C^{(m)}$$

have the property that

$$\psi_n|_{([h_n^{(m)}-2h, h_n^{(m)}-h] \times S^1) \amalg ([h_n^{(m+1)}+h, h_n^{(m+1)}+2h] \times S^1)} = \mathrm{id}.$$

In this regard we consider the surface

$$\left(([-1-2h, 1+2h] \times S^1) \amalg C^{(m)}\right)/_\sim$$

where $x \sim y$ if and only if $x \in [h, 2h] \times S^1$ and $y \in [h_n^{(m)} - 2h, h_n^{(m)} - h] \times S^1$ such that $\theta_n(y) = x$.
By this procedure we glue all cylinders of types ∞ and b_1, and obtain a bigger cylinder C_n together with a sequence of diffeomorphisms $\Phi_n : [-\sigma_n, \sigma_n] \times S^1 \to C_n$, where $[-\sigma_n, \sigma_n] \times S^1$ is the parametrization of the $\delta-$thin part, i.e. of $\mathrm{Thin}_\delta(\dot{S}^{\mathcal{D},r}, h_n)$. Let $\varphi_n : \mathcal{C}_n \to [-\sigma_n, \sigma_n] \times S^1$ be the conformal parametrization of the cylindrical component of $\mathrm{Thin}_\epsilon(\dot{S}^{\mathcal{D},r}, h_n)$. Since both ends of $[-\sigma_n, \sigma_n] \times S^1$ contain cylinders of type ∞, we infer by the above construction, that Φ_n is identity near the boundary. Specifically, with the constant $h > 0$ we have

$$\Phi_n|_{([-\sigma_n, -\sigma_n+h] \times S^1) \amalg ([\sigma_n - h, \sigma_n] \times S^1)} = \mathrm{id}.$$

Then we consider the surface

$$\left(\left(\dot{S}^{D,r} \backslash \varphi_n^{-1}([-\sigma_n + h, \sigma_n - h] \times S^1)\right) \amalg C_n\right)/_\sim$$

where $x \sim y$ if and only if $x \in \dot{S}^{\mathcal{D},r} \backslash \varphi_n^{-1}([-\sigma_n + h, \sigma_n - h] \times S^1)$ and $y \in C_n$ such that $\Phi_n \circ \varphi_n(x) = y$.
In this way we handle all components of $\mathrm{Thin}_\delta(\dot{S}^{\mathcal{D},r}, h_n)$ that are conformal equivalent to hyperbolic cylinders.

3.2.5 Punctures and elements of $\mathcal{Z}$

We analyze the convergence of u_n on components of the thin part which are biholomorphic to cusps, as well as, in a neighborhood of the points from $\mathcal{Z}$. Recall that cusps correspond to neighborhoods of punctures. Let $p \in S^{\mathcal{D},r}$ be a puncture or an element from $\mathcal{Z}$. By Lemma 97 of Appendix D, there exist the open neighborhoods U_n and U of p, and the biholomorphisms $\psi_n : D \to U_n$ and $\psi : D \to U$ such that ψ_n converge in C^∞ to ψ. We consider the sequence of $\mathcal{H}-$holomorphic curves u_n with harmonic perturbations γ_n restricted to U_n. By the convergence of u_n on the thick part, for every open neighbourhoods U and V of p, such that $V \Subset U$, the $\mathcal{H}-$holomorphic curves u_n together with the harmonic perturbations γ_n converge in C^∞ on $\overline{U \backslash V}$ to some $\mathcal{H}-$holomorphic curve u with harmonic perturbation γ. Via the biholomorphisms ψ_n and ψ, we consider the $\mathcal{H}-$holomorphic curves u_n and the harmonic perturbations γ_n as being defined on $D\backslash\{0\}$. Actually, we consider the following setup: For the sequence of $\mathcal{H}-$holomorphic curves $u_n = (a_n, f_n) : D\backslash\{0\} \to \mathbb{R} \times M$ with the harmonic perturbations γ_n defined on the whole disk D, the following are satisfied:

K1 The energy of u_n is uniformly bounded, i.e. with the constant $E_0 > 0$ we have $E(u_n; D\backslash\{0\}) \leqslant E_0$ for all $n \in \mathbb{N}$.

K2 The L^2-norms of γ_n are uniformly bounded, i.e. with the constant $C_0 > 0$ we have $\|\gamma_n\|^2_{L^2(D\backslash\{0\})} \leqslant C_0$ for all $n \in \mathbb{N}$.

K3 For every open neighborhoods U and V of p such that $V \Subset U$, the $\mathcal{H}-$holomorphic curves u_n with harmonic perturbations γ_n converge in C^∞ on $\overline{U \backslash V}$ to a $\mathcal{H}-$holomorphic curve u with harmonic perturbation γ.

We consider two cases. In the first case there exists a subsequence of u_n for which the singularity at 0 is removable, i.e. the $\mathbb{R}-$coordinate a_n is bounded in a neighborhood of 0, but not necessarily uniformly bounded. In particular, this case is typically for neighborhoods of points from $\mathcal{Z}$. Hence the sequence of $\mathcal{H}-$holomorphic curves u_n can be defined across the puncture 0 and we end up with a sequence of $\mathcal{H}-$holomorphic disks with fixed boundary. To describe the compactness we use the results of Section 3.2.3.
In the second case, there exists no subsequence of the u_n that has a bounded $\mathbb{R}-$coordinate a_n near 0. Since D is simply connected, there exists a harmonic function $\tilde{\Gamma}_n : D \to \mathbb{R}$ such that $\gamma_n = d\tilde{\Gamma}_n$. By the second condition from

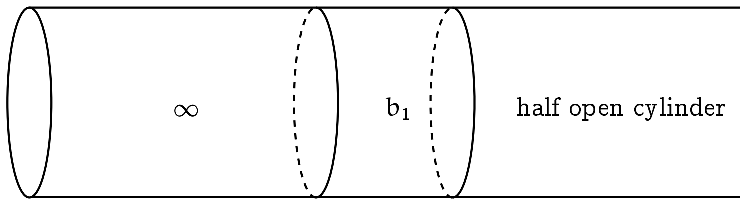

Figure 3.2.9: Decomposition of a punctured neighbourhood into cylinders of type ∞, b_1 and a half open cylinder.

above, the gradients $\nabla\tilde{\Gamma}_n$ are uniformly bounded in L^2-norm by the constant $C_0 > 0$. Denote by

$$K_n = \frac{1}{\pi}\int_D \tilde{\Gamma}_n(x,y)dxdy$$

the mean value of $\tilde{\Gamma}_n$ on the disk D. Furthermore, define $\Gamma_n := \tilde{\Gamma}_n - K_n$; Γ_n is a harmonic function on the disk with vanishing average and satisfying $\gamma_n = d\Gamma_n$, while the gradients $\nabla\Gamma_n$ have uniformly bounded L^2-norms. By Poincaré inequality, the L^2-norm of Γ_n is uniformly bounded, i.e. with the constant $C_0 > 0$ we have $\|\Gamma_n\|_{L^2(D)} \leqslant C_0$ for all $n \in \mathbb{N}$. Pick $\tau \in (0,1)$ and denote by D_τ the disk around 0 of radius τ. From the mean value inequality for harmonic functions, Γ_n is uniformly bounded in $C^0(D_\tau)$. Via the biholomorphism $[0,\infty) \times S^1 \to D\backslash\{0\}$, $(s,t) \mapsto e^{-2\pi(s+it)}$ we consider the $\mathcal{H}-$holomorphic maps u_n together with the harmonic perturbations γ_n as being defined on the half open cylinder $[0,\infty) \times S^1$. Specifically we consider the following setup: For the sequence $u_n = (a_n, f_n) : [0,\infty) \times S^1 \to \mathbb{R} \times M$ of $\mathcal{H}-$holomorphic half cylinders with harmonic perturbations γ_n the following are satisfied:

L1 The energy of u_n and the L^2-norm of the harmonic perturbations γ_n are uniformly bounded, i.e. with the constants $E_0, C_0 > 0$ we have $E(u_n; [0,\infty) \times S^1) \leqslant E_0$ and $\|\gamma_n\|^2_{L^2(D\backslash\{0\})} \leqslant C_0$ for all $n \in \mathbb{N}$.

L2 The $\mathcal{H}-$holomorphic curves u_n converge in C^∞_{loc} to a $\mathcal{H}-$holomorphic curve u with harmonic perturbation γ.

L3 The harmonic perturbations γ_n satisfy $\gamma_n = d\Gamma_n$, where $\Gamma_n : [0,\infty) \times S^1 \to \mathbb{R}$ is a harmonic function with a uniformly bounded gradient $\nabla\Gamma_n$ in L^2-norm. Furthermore, Γ_n is uniformly bounded in $C^0([0,\infty) \times S^1)$.

By using the decomposition discussed in Section 3.2.1 we split the half cylinder into smaller cylinders with $d\alpha-$energies smaller than $\hbar_0/2$. As described in Section 3.2.1 we end up with a sequence of finitely many cylinder of types ∞ and b_1, and a half cylinder with a $d\alpha-$energy less than $\hbar_0/2$. The appearance of the cylinders of types b_1 and ∞ is alternating; the decomposition starts with a cylinder of type ∞ and ends with a cylinder of type b_1 followed by the half cylinder (see Figure 3.2.9).
For the cylinders of types ∞ and b_1 we formulate the convergence results as in Sections 3.2.3 and 3.2.2. Since the harmonic 1$-$forms γ_n are defined over the puncture p, the period of the harmonic perturbation γ_n over each cylinder (either of type ∞ or type b_1) is 0. Hence, the converge properties of the cylinders of type ∞ are the same as in the classical theory of Hofer (see [14]), and we are left with the half cylinder having a $d\alpha-$energy smaller than $\hbar_0/2$. We have the following setup:

M1 $u_n = (a_n, f_n) : [0,\infty) \times S^1 \to \mathbb{R} \times M$ is a $\mathcal{H}-$holomorphic curve with harmonic perturbation γ_n.

M2 The energy of u_n and the L^2-norm of γ_n are uniformly bounded by the constants E_0 and C_0, respectively, while the $d\alpha-$energy of u_n is smaller than $\hbar_0/2$.

M3 The harmonic perturbations γ_n satisfy $\gamma_n = d\Gamma_n$, where $\Gamma_n : [0,\infty)\times S^1 \to \mathbb{R}$ is a harmonic function with a uniformly bounded gradient $\nabla\Gamma_n$ in L^2−norm. Furthermore, Γ_n is uniformly bounded in $C^0([0,\infty)\times S^1)$.

M4 The gradients of u_n are uniformly bounded, i.e. with the constant $C_h > 0$ from Lemma 42 we have

$$\|du_n(z)\| = \sup_{\|v\|_{\mathrm{eucl.}}=1} \|du_n(z)(v)\|_{\overline{g}} \leqslant C_h \tag{3.2.6}$$

for all $z \in [0,\infty)\times S^1$ and all $n \in \mathbb{N}$.

By bubbling-off analysis and in view of the uniformly small $d\alpha$−energy, Assumption (3.2.6) is also valid. Moreover, by the mean value thorem for harmonic functions and the uniformly boundedness of the L^2−norms of $\nabla\Gamma_n$, the harmonic perturbation γ_n is uniformly bounded in C^0 on $[0,\infty)\times S^1$ with respect to the standard Euclidean metric. We turn the $\mathcal{H}$−holomorphic curve u_n with harmonic perturbation γ_n into a usual pseudoholomorphic curve $\overline{u}_n$ by setting $\overline{u}_n = (\overline{a}_n, \overline{f}_n) = (a_n + \Gamma_n, f_n)$ as in Section 3.2.3. In the following we show that the α− and $d\alpha$−energies of $\overline{u}_n$ are uniformly bounded. As $\overline{f}_n = f_n$ we have

$$E_{d\alpha}(\overline{u}_n; [0,\infty)\times S^1) = E_{d\alpha}(u_n; [0,\infty)\times S^1) \leqslant \frac{\hbar_0}{2}$$

and therefore the $d\alpha$−energy is uniformly small. By definition and accounting on the uniform bound on the gradients (3.2.6) and the uniform C^0−bound of the harmonic 1−forms γ_n, we obtain

$$\begin{aligned}
E_\alpha(\overline{u}_n; [0,+\infty)\times S^1) &= \sup_{\varphi\in\mathcal{A}} \int_{[0,+\infty)\times S^1} \varphi'(\overline{a}_n) d\overline{a}_n \circ i \wedge d\overline{a}_n \\
&= -\sup_{\varphi\in\mathcal{A}} \int_{[0,+\infty)\times S^1} d(\varphi(\overline{a}_n) d\overline{a}_n \circ i) - \varphi(\overline{a}_n) d(d\overline{a}_n \circ i) \\
&= -\sup_{\varphi\in\mathcal{A}} \int_{[0,+\infty)\times S^1} d(\varphi(\overline{a}_n) d\overline{a}_n \circ i) + \varphi(\overline{a}_n) f_n^* d\alpha \\
&= -\sup_{\varphi\in\mathcal{A}} \left[\int_{[0,+\infty)\times S^1} d(\varphi(\overline{a}_n) d\overline{a}_n \circ i) + \int_{[0,+\infty)\times S^1} \varphi(\overline{a}_n) f_n^* d\alpha \right] \\
&\leqslant -\sup_{\varphi\in\mathcal{A}} \int_{[0,+\infty)\times S^1} d(\varphi(\overline{a}_n) d\overline{a}_n \circ i) + E_{d\alpha}(u_n) \\
&= -\sup_{\varphi\in\mathcal{A}} \left[\lim_{r\to\infty} \int_{\{r\}\times S^1} \varphi(\overline{a}_n) d\overline{a}_n \circ i - \int_{\{0\}\times S^1} \varphi(\overline{a}_n) d\overline{a}_n \circ i \right] + E_{d\alpha}(u_n) \\
&\leqslant \lim_{r\to\infty} \int_{\{r\}\times S^1} |d\overline{a}_n \circ i| + \int_{\{0\}\times S^1} |d\overline{a}_n \circ i| + E_{d\alpha}(u_n) \\
&\leqslant 2C_h + \frac{\hbar_0}{2}.
\end{aligned}$$

Thus the α−energy is uniformly bounded. From the definition of $\tilde{E}_0$ (see Section 3.2.1) we have $E(\overline{u}_n; [0,\infty)\times S^1) \leqslant \tilde{E}_0$ for all $n \in \mathbb{N}$. In this regard, we consider the following setup:

N1 $\overline{u}_n = (\overline{a}_n, f_n) : [0,\infty)\times S^1 \to \mathbb{R}\times M$ is a pseudoholomorphic curve.

N2 The energy of $\overline{u}_n$ is uniformly bounded by $\tilde{E}_0$, while the $d\alpha$−energy of $\overline{u}_n$ is uniformly smaller than $\hbar_0/2$.

Using the diffeomorphism θ defined above together with the notation (C.0.2), and employing Theorem 96 of Appendix C we have the following

Theorem 55. *There exists a subsequence of $\overline{u}_n$, still denoted by $\overline{u}_n$ such that the following is satisfied.*

1. *$\overline{u}_n$ is asymptotic to the same Reeb orbit, i.e. there exists a Reeb orbit x of period $|T| \neq 0$ with $|T| \leqslant \tilde{E}_0$ and a sequence $c_n \in S^1$ such that*
$$\lim_{s\to\infty} \overline{f}_n(s,t) = x(T(t+c_n)) \quad \textit{and} \quad \lim_{s\to\infty} \frac{\overline{a}_n(s,t)}{s} = T$$
for all $n \in \mathbb{N}$.

2. *$\overline{u}_n$ converge in C^∞_{loc} to a pseudoholomorphic half cylinder $\overline{u} : [0,\infty) \times S^1 \to \mathbb{R} \times M$ having a bounded energy and a $d\alpha-$energy smaller than $\hbar_0/2$. Moreover, there exists $c^* \in S^1$ such that*
$$\lim_{s\to\infty} \overline{f}(s,t) = x(T(t+c^*)) \quad \textit{and} \quad \lim_{s\to\infty} \frac{\overline{a}(s,t)}{s} = T.$$

3. *The maps $g_n = \overline{f}_n \circ \theta^{-1} : [0,1] \times S^1 \to M$, where $g_n(1,t) = x(T(t+c_n))$ converge in C^0 to a map $g : [0,1] \times S^1 \to M$, that satisfy $g(1,t) = x(T(t+c^*))$, where x is the same Reeb orbit of period $|T| \neq 0$ as in Part 1 of the theorem.*

With this result we are in the position to formulate the convergence of the sequence of $\mathcal{H}-$holomorphic half cylinders u_n with harmonic perturbations γ_n.

Theorem 56. *There exists a subsequence u_n still denoted by u_n such that the following is satisfied.*

1. *u_n is asymptotic to the same Reeb orbit, i.e. there exists a Reeb orbit x of period $|T| \neq 0$ with $|T| \leqslant \tilde{E}_0$ and a sequence $c_n \in S^1$ such that*
$$\lim_{s\to\infty} f_n(s,t) = x(T(t+c_n)) \quad \textit{and} \quad \lim_{s\to\infty} \frac{a_n(s,t)}{s} = T$$
for all $n \in \mathbb{N}$.

2. *u_n converge in C^∞_{loc} to a $\mathcal{H}-$holomorphic half cylinder $u : [0,\infty) \times S^1 \to \mathbb{R} \times M$ with harmonic perturbation γ having a bounded energy and a $d\alpha-$energy smaller than $\hbar_0/2$. Moreover, there exists $c^* \in S^1$ such that*
$$\lim_{s\to\infty} f(s,t) = x(T(t+c^*)) \quad \textit{and} \quad \lim_{s\to\infty} \frac{a(s,t)}{s} = T.$$

3. *The maps $g_n = f_n \circ \theta^{-1} : [0,1] \times S^1 \to M$, where $g_n(1,t) = x(T(t+c_n))$ converge in C^0 to a map $g : [0,1] \times S^1 \to M$, and satisfy $g(1,t) = x(T(t+c^*))$, where x is the same Reeb orbit of period $|T| \neq 0$ as in Part 1 of the theorem.*

Proof. Since the Γ_n are uniformly bounded in C^0-norm, the first assertion is obvious. Employing the same arguments as in Appendix **E**, i.e. the mean value theorem for harmonic functions and Cauchy integral formula, we deduce that Γ_n have uniformly bounded derivatives, and so, converge in C^∞_{loc} on $[0,\infty) \times S^1$ to a harmonic function $\Gamma : [0,\infty) \times S^1 \to \mathbb{R}$ with a gradient bounded in L^2-norm. Let us show that $\Gamma : [0,\infty) \times S^1 \to \mathbb{R}$ is bounded in C^0. Via the conformal diffomorphism $[0,\infty) \times S^1 \to D\backslash\{0\}, (s,t) \mapsto e^{-2\pi(s+it)}$ we assume that the harmonic functions Γ_n and Γ are defined on the disk D. Then, since the Γ_n are uniformly bounded in C^0 and have gradients with uniformly bounded L^2-norms, it follows that $\Gamma_n \to \Gamma$ in $C^\infty(\overline{D_\rho(0)})$ for some $0 < \rho < 1$. This shows that

Γ is uniformly bounded on D and hence, via the conformal map $[0,\infty) \times S^1 \to D\backslash\{0\}$ it is uniformly bounded on $[0,\infty) \times S^1$. Thus, the second assertion is proved, and by means of $\bar{f}_n = f_n$, the third assertion is evident.

□

By cutting a small piece of finite length from the infinite half cylinder, we can make the cylinder preceding the infinite half cylinder to be of type b_1. Assuming that the infinite half cylinder is of type ∞, we glue all cylinders of types ∞ and b_1 together (as described in the previous section). Via the map $[0,1) \times S^1 \to D\backslash\{0\}$, $(s,t) \mapsto (1-s)e^{2\pi i t}$, we identify the cylinder $[0,1) \times S^1$, which is diffeomorphic with the infinite half open cylinder, with a punctured disk $D\backslash\{0\}$. In this way the upper half open cylinder $[0,1) \times S^1$ can be identified with a neighbourhood of a puncture.

Chapter 4

Discussion on conformal period

In this section we analyze Condition C8 of Section 3.2.2 dealing with the boundedness of the sequence R_nP_n, and which can be regarded as a connection between the conformal data of the Riemann surface and the harmonic $1-$forms γ_n. Without this additional condition the convergence result from Appendix B cannot be established. The reason is that the almost complex structure constructed on the contact manifold M might not vary in a compact interval. We show that this condition is not automatically satisfied by giving a counterexample. It should be pointed out that this example contradicts Lemma A.2 of [5]. Essentially, we will construct a sequence of harmonic $1-$forms γ_n on a sequence of stable Riemann surfaces that degenerate along a single circle, have uniformly bounded L^2-norms but unbounded P_n/ℓ_n, where P_n denotes the period of γ_n along the degenerating circle and ℓ_n its length with respect to the hyperbolic metric. Observe that the quantity $1/\ell_n$ is similar to R_n .
Let $(S_n, j_n, \mathcal{M}_n)$ be a sequence of stable Riemann surfaces of genus g, where $\mathcal{M}_n \subset S_n$ are finite sets of marked points with the same cardinality. Choose a basis $c_1, ..., c_{2g} \in H_1(S_n;\mathbb{Z})$ which is independent of n. This choice is possible because all S_n have genus g and are closed (they are topologically the same). By the Deligne-Mumford convergence,

$$(S_n, j_n, \mathcal{M}_n) \to (S, j, \mathcal{M}, \mathcal{D}, r),$$

where $(S, j, \mathcal{M}, \mathcal{D}, r)$ is a decorated nodal Riemann surface. Again, according to the definition of the Deligne-Mumford convergence, there exist diffeomorphisms $\varphi_n : S^{\mathcal{D},r} \to S_n$ such that $j_n \to j$ on $S^{\mathcal{D},r}\backslash \coprod_{j=1}^{l} \Gamma_j$ or equivalently, $h_n \to h$ on $\dot{S}^{\mathcal{D},r}\backslash \coprod_{j=1}^{l} \Gamma_j$ where Γ_j are special circles, and h_n and j_n are the pull-back of the complex structure and the hyperbolic metric from S_n and $\dot{S}_n$ via the diffeomorphism φ_n. Assume that $l = 1$, i.e. that there exists only one degenerating geodesic in the Deligne-Mumford convergence. Denote this geodesic by Γ. Furthermore, assume that $\Gamma = c_1$ (Γ lies in the homology class of c_1). The main result of this section is the following

Proposition 57. *There exists a sequence of harmonic $1-$forms $\gamma_n \in \mathcal{H}^1_{j_n}(S_n)$ with uniformly bounded L^2-norms, periods, and co-periods, but unbounded conformal periods.*

Proof. Choose a sequence of harmonic $1-$forms $\gamma_n \in \mathcal{H}^1_{j_n}(S^{\mathcal{D},r})$ with vanishing periods except on Γ (on all of c_i with $i \neq 1$ except on $c_1 = \Gamma$). By normalization, assume that $\|\gamma_n\|_{L^2(S^{\mathcal{D},r})} = 1$. The uniform bounds on the L^2-norms imply that the periods P_n of γ_n over Γ converge to 0 (Lemma 98). Thus, by the second part of the proof of Theorem 34, γ_n converge in C^∞_{loc} to γ on $S^{\mathcal{D},r}\backslash\Gamma$ which can be seen as a harmonic $1-$form on a closed, smooth Riemann surface S of genus one less, with vanishing periods. By Hodge theory, we have $\gamma = 0$. For n sufficiently large, the L^2-norms of γ_n concentrate in the collar neighborhood around Γ. Indeed, from

$$1 = \|\gamma_n\|^2_{L^2(S^{\mathcal{D},r})} = \|\gamma_n\|^2_{L^2(\mathcal{C}_n)} + \|\gamma_n\|^2_{L^2(S^{\mathcal{D},r}\backslash\mathcal{C}_n)},$$

where $\mathcal{C}_n$ is the cylindrical component of the $\delta-$thin part for some sufficiently small but fixed $\delta > 0$, it follows that $S^{\mathcal{D},r}\backslash\mathcal{C}_n$ is contained in a compact subset of $S^{\mathcal{D},r}\backslash\Gamma$, and so, that $\|\gamma_n\|^2_{L^2(S^{\mathcal{D},r}\backslash\mathcal{C}_n)}$ converge to 0, and for n

sufficiently large we have $\|\gamma_n\|_{L^2(\mathcal{C}_n)} \leqslant 1$ and $\|\gamma_n\|_{L^2(\mathcal{C}_n)} \to 1$ as $n \to \infty$. If F_n is the unique holomorphic 1–form with $\mathrm{Re}(F_n) = \gamma_n$,

$$\|F_n\|^2_{L^2(S^{\mathcal{D},r})} = \frac{i}{2}\int_{S^{\mathcal{D},r}} F_n \wedge \overline{F}_n.$$

The collar $\mathcal{C}_n$ is conformaly equivalent to $[-R_n, R_n] \times S^1$, where $R_n \sim 1/\ell_n$ and ℓ_n is the length of Γ with respect to h_n. On $\mathcal{C}_n$ we write $\gamma_n = f_n ds + g_n dt$, where f_n and g_n are harmonic functions on the cylinder $[-R_n, R_n] \times S^1$ (s is the coordinate in $[-R_n, R_n]$ and t is the coordinate on S^1), express the holomorphic 1–form F_n as $F_n = (f_n - ig_n)dz = (f_n - ig_n)(ds + idt)$, and note that $\|F_n\|_{L^2(\mathcal{C}_n)} = \|\gamma_n\|_{L^2(\mathcal{C}_n)}$. Consider the quantity $|\,\|F_n\|_{L^2(\mathcal{C}_n)} - |b_0|\,\|dz\|_{L^2(\mathcal{C}_n)}\,|$, where $b_0 = -\tilde{S}_n - iP_n$ and $\tilde{S}_n$ is the co-period defined by

$$\tilde{S}_n = \int_\Gamma \gamma_n \circ j_n = -\int_{\{0\}\times S^1} f_n(0,t)dt.$$

Recalling that

$$P_n = \int_\Gamma \gamma_n = \int_{\{0\}\times S^1} g(0,t)dt,$$

we obtain

$$\begin{aligned}\Big|\|F_n\|_{L^2(\mathcal{C}_n)} - |b_0|\,\|dz\|_{L^2(\mathcal{C}_n)}\Big| &= \Big\|\big[(f_n + \tilde{S}_n) - i(g_n - P_n)\big]\,dz\Big\|_{L^2(\mathcal{C}_n)}\\ &\leqslant \Big\|(f_n + \tilde{S}_n)dz\Big\|_{L^2(\mathcal{C}_n)} + \|(g_n - P_n)dz\|_{L^2(\mathcal{C}_n)}.\end{aligned}$$

Further calculation gives $\|dz\|_{L^2(\mathcal{C}_n)} = \sqrt{2R_n}$, $\Big\|(f_n + \tilde{S}_n)dz\Big\|_{L^2(\mathcal{C}_n)} = \Big\|f_n + \tilde{S}_n\Big\|_{L^2([-R_n,R_n]\times S^1)}$ and similarly $\|(g_n - P_n)dz\|_{L^2(\mathcal{C}_n)} = \|g_n - P_n\|_{L^2([-R_n,R_n]\times S^1)}$ with respect to the standard Euclidean metric on the cylinder $[-R_n, R_n] \times S^1$. Application of Lemma 58 yields

$$\begin{aligned}\Big\|f_n + \tilde{S}_n\Big\|^2_{L^2([-R_n,R_n]\times S^1)} &= \int_{-R_n}^{R_n} \Big\|f_n(s) + \tilde{S}_n\Big\|^2_{L^2(S^1)} ds\\ &\leqslant \left(36\int_{-R_n}^{R_n} \rho^2(s)ds\right) \max\left\{\Big\|f_n(-R_n) + \tilde{S}_n\Big\|^2_{L^2(S^1)}, \Big\|f_n(+R_n) + \tilde{S}_n\Big\|^2_{L^2(S^1)}\right\}\end{aligned}$$

and

$$\|g_n - P_n\|^2_{L^2([-R_n,R_n]\times S^1)} \leqslant \left(36\int_{-R_n}^{R_n} \rho^2(s)ds\right) \max\left\{\|g_n(-R_n) - P_n\|^2_{L^2(S^1)}, \|g_n(+R_n) - P_n\|^2_{L^2(S^1)}\right\},$$

where ρ is the function from Lemma 58. Using

$$\int_{-R_n}^{R_n} \rho^2(s)ds = 4(1 - e^{-4R_n}) \leqslant 4,$$

we obtain

$$\begin{aligned}\Big\|f_n + \tilde{S}_n\Big\|^2_{L^2([-R_n,R_n]\times S^1)} &\leqslant 144\max\left\{\Big\|f_n(-R_n) + \tilde{S}_n\Big\|^2_{L^2(S^1)}, \Big\|f_n(+R_n) + \tilde{S}_n\Big\|^2_{L^2(S^1)}\right\},\\ \|g_n - P_n\|^2_{L^2([-R_n,R_n]\times S^1)} &\leqslant 144\max\left\{\|g_n(-R_n) - P_n\|^2_{L^2(S^1)}, \|g_n(+R_n) - P_n\|^2_{L^2(S^1)}\right\}.\end{aligned}$$

Because the harmonic 1–forms γ_n converge to 0 in $C^\infty_{loc}(S^{\mathcal{D},r}\backslash\Gamma)$, $f_n(\pm R_n)$, $g_n(\pm R_n)$, $\tilde{S}_n$, and P_n converge to zero. Hence

$$\Big|\|F_n\|_{L^2(\mathcal{C}_n)} - \sqrt{2}|b_0|\sqrt{R_n}\Big| \to 0$$

as $n \to \infty$. As $\|F_n\|_{L^2(\mathcal{C}_n)}$ is almost 1, there exists the constants $C_0, C_1 > 0$ such that

$$C_0 \frac{1}{\sqrt{2R_n}} \quad \leqslant \quad |b_0| \leqslant C_1 \frac{1}{\sqrt{2R_n}}$$

giving

$$C_0 \frac{1}{\sqrt{2R_n}} \leqslant \sqrt{P_n^2 + \tilde{S}_n^2} \leqslant C_1 \frac{1}{\sqrt{2R_n}},$$

or equivalently,

$$C_0 \sqrt{\frac{R_n}{2}} \leqslant \sqrt{(P_n R_n)^2 + (\tilde{S}_n R_n)^2} \leqslant C_1 \sqrt{\frac{R_n}{2}}.$$

These inequalities show that either $P_n R_n$ or $\tilde{S}_n R_n$ tend to ∞, although P_n and $\tilde{S}_n$ stay uniformly bounded (γ_n have uniformly bounded L^2–norms). If $P_n R_n$ remains uniformly bounded then we replace γ_n by $\gamma_n \circ j_n$.

□

Lemma 58. *For any harmonic functions* f *and* g *on the cylinder* $[-R, R] \times S^1$ *such that* $\eta = f ds + g dt$ *is a harmonic* 1*–form on* $[-R, R] \times S^1$ *we have*

$$\begin{aligned} \left\|f(s) + \tilde{S}\right\|_{L^2(S^1)} &\leqslant 6\rho(s) \max\left\{\left\|f(-R) + \tilde{S}\right\|_{L^2(S^1)}, \left\|f(+R) + \tilde{S}\right\|_{L^2(S^1)}\right\} \\ \|g(s) - P\|_{L^2(S^1)} &\leqslant 6\rho(s) \max\left\{\|g(-R) - P\|_{L^2(S^1)}, \|g(+R) - P\|_{L^2(S^1)}\right\} \end{aligned}$$

for all $s \in [-R, R]$. *Here* $\tilde{S}$ *and* P *are the co-period and the period of* η*, respectively, and*

$$\rho(s)^2 = 8e^{-2R} \cosh(2s).$$

Proof. Any harmonic 1–form η defined on the cylinder $[-R, R] \times S^1$ can be written as $\eta = (-\tilde{S} ds + P dt) + \tilde{f}(s,t) ds + \tilde{g}(s,t) dt$ where $\tilde{f}$ and $\tilde{g}$ are harmonic functions on $[-R, R] \times S^1$ with vanishing average. Note that the average of f corresponds to the co-period $\tilde{S}$ and the average of g corresponds to $-P$. To show this, write η in the form $\eta = f(s,t) ds + g(s,t) dt$ and compute the averages of f and g as

$$\frac{1}{2R} \int_{[-R,R]\times S^1} f(s,t) ds \wedge dt = \frac{1}{2R} \int_{[-R,R]\times S^1} \eta \wedge dt = \int_{\{0\}\times S^1} \eta \circ j = -\tilde{S}$$

and

$$\frac{1}{2R} \int_{[-R,R]\times S^1} g(s,t) ds \wedge dt = \frac{1}{2R} \int_{[-R,R]\times S^1} ds \wedge \eta = -\frac{1}{2R} \int_{-R}^{R} \left(\int_{\{s\}\times S^1} \eta \right) ds = P,$$

respectively. Hence the 1–form $\eta - (-\tilde{S} ds + P dt) = \tilde{f}(s,t) ds + \tilde{g}(s,t) dt$ has vanishing average twist and vanishing periods. The Fourier series of $\tilde{f}$ and $\tilde{g}$ in the t variable are

$$\begin{aligned} \tilde{f}(s,t) &= \frac{a_0(s)}{2} + \sum_{k=1}^{\infty} a_k(s) \cos(kt) + b_k(s) \sin(kt), \\ \tilde{g}(s,t) &= \frac{\alpha_0(s)}{2} + \sum_{k=1}^{\infty} \alpha_k(s) \cos(kt) + \beta_k(s) \sin(kt). \end{aligned}$$

Since $\tilde{f}$ and $\tilde{g}$ are harmonic, the Fourier expansion coefficients solve $a_k'' = k^2 a_k$, $b_k'' = k^2 b_k$, $\alpha_k'' = k^2 \alpha_k$ and

$\beta_k'' = k^2\beta_k$ for $k \in \mathbb{N}_0$. The solutions to these ordinary differential equations are of the form

$$\begin{aligned}
a_0(s) &= c_0 + s d_0,\\
a_k(s) &= c_k \cosh(ks) + d_k \sinh(ks),\\
b_k(s) &= e_k \cosh(ks) + f_k \sinh(ks),\\
\alpha_0(s) &= \delta_0 + \epsilon_0 s,\\
\alpha_k(s) &= \delta_k \cosh(ks) + \epsilon_k \sinh(ks),\\
\beta_k(s) &= \eta_k \cosh(ks) + \theta_k \sinh(ks).
\end{aligned}$$

Since $d\eta = d(\eta \circ j) = 0$ we obtain $\partial_t \tilde{f} = \partial_s \tilde{g}$ and $\partial_s \tilde{f} = -\partial_t \tilde{g}$, giving $a_0(s) = c_0$ and $\alpha_0(s) = \delta_0$. As $\tilde{f}ds + \tilde{g}dt$ has vanishing co-period and vanishing period, we find $a_0(s) = \alpha_0(s) = 0$, and the following relations relating the coefficients a_k, b_k, α_k, and β_k for $k \in \mathbb{N}$: $\delta_k = f_k$, $\epsilon_k = e_k$, $\eta_k = -d_k$ and $\theta_k = -c_k$. Consequently, a_k, b_k, α_k, and β_k can be written as

$$\begin{aligned}
a_k(s) &= c_k \cosh(ks) + d_k \sinh(ks),\\
b_k(s) &= e_k \cosh(ks) + f_k \sinh(ks),\\
\alpha_k(s) &= f_k \cosh(ks) + e_k \sinh(ks),\\
\beta_k(s) &= -d_k \cosh(ks) - c_k \sinh(ks).
\end{aligned}$$

Let us express $\tilde{f}$ and $\tilde{g}$ as

$$\begin{aligned}
\tilde{f}(s,t) &= \sum_{k=1}^{\infty} a_k(s)\cos(kt) + b_k(s)\sin(kt) = \sum_{k\in\mathbb{Z}\setminus\{0\}} F_k(s) e^{2\pi i k t},\\
\tilde{g}(s,t) &= \sum_{k=1}^{\infty} \alpha_k(s)\cos(kt) + \beta_k(s)\sin(kt) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \Gamma_k(s) e^{2\pi i k t},
\end{aligned}$$

where $F_k = \frac{1}{2}(a_k - ib_k)$, $F_{-k} = \frac{1}{2}(a_k + ib_k)$, $\Gamma_k = \frac{1}{2}(\alpha_k - i\beta_k)$, and $\Gamma_{-k} = \frac{1}{2}(\alpha_k + i\beta_k)$ for $k \geqslant 1$. From

$$\frac{\cosh(ks)}{\cosh(kR)} \leqslant 3e^{-R}\cosh(s) \leqslant 3\rho(s) \text{ and } \frac{|\sinh(ks)|}{|\sinh(Rs)|} \leqslant 3e^{-R}\cosh(s) \leqslant 3\rho(s),$$

where $\rho(s)^2 = 8e^{-2R}\cosh(2s)$, it follows that

$$\cosh(ks) \leqslant 3\rho(s)\cosh(kR) \text{ and } |\sinh(ks)| \leqslant 3\rho(s)\sinh(Rs).$$

Define the functions

$$K(k) = \begin{cases} +1, & c_k \text{ and } d_k \text{ have the same parity} \\ -1, & \text{otherwise} \end{cases}$$

and

$$G(k) = \begin{cases} +1, & e_k \text{ and } f_k \text{ have the same parity} \\ -1, & \text{otherwise.} \end{cases}$$

For $s \in [0,R] \times S^1$ we then have

$$\left\|\tilde{f}(s)\right\|^2_{L^2(S^1)} = \sum_{k\in\mathbb{Z}\setminus\{0\}} |F_k(s)|^2$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{k=1}^{\infty}(c_k\cosh(ks)+d_k\sinh(ks))^2+\frac{1}{2}\sum_{k=1}^{\infty}(e_k\cosh(ks)+f_k\sinh(ks))^2\\
&= \frac{1}{2}\sum_{k=1,K(k)=1}^{\infty}(c_k\cosh(ks)+d_k\sinh(ks))^2+\frac{1}{2}\sum_{k=1,K(k)=-1}^{\infty}(c_k\cosh(ks)+d_k\sinh(ks))^2\\
&+ \frac{1}{2}\sum_{k=1,G(k)=1}^{\infty}(e_k\cosh(ks)+f_k\sinh(ks))^2+\frac{1}{2}\sum_{k=1,G(k)=-1}^{\infty}(e_k\cosh(ks)+f_k\sinh(ks))^2\\
&= \frac{1}{2}\sum_{k=1,K(k)=1}^{\infty}c_k^2\cosh^2(ks)+d_k^2\sinh^2(ks)+2c_kd_k\cosh(ks)\sinh(ks)\\
&+ \frac{1}{2}\sum_{k=1,K(k)=-1}^{\infty}c_k^2\cosh^2(ks)+d_k^2\sinh^2(ks)-(-2c_kd_k)\cosh(ks)\sinh(ks)\\
&+ \frac{1}{2}\sum_{k=1,G(k)=1}^{\infty}e_k^2\cosh^2(ks)+f_k^2\sinh^2(ks)+2e_kf_k\cosh(ks)\sinh(ks)\\
&+ \frac{1}{2}\sum_{k=1,G(k)=-1}^{\infty}e_k^2\cosh^2(ks)+f_k^2\sinh^2(ks)-(-2e_kf_k)\cosh(ks)\sinh(ks)\\
&\leqslant \frac{1}{2}9\rho(s)^2\sum_{k=1,K(k)=1}^{\infty}c_k^2\cosh^2(kR)+d_k^2\sinh^2(kR)+2c_kd_k\cosh(kR)\sinh(kR)\\
&+ \frac{1}{2}9\rho(s)^2\sum_{k=1,K(k)=-1}^{\infty}c_k^2\cosh^2(kR)+d_k^2\sinh^2(kR)+2c_kd_k\cosh(kR)\sinh(-kR)\\
&+ \frac{1}{2}9\rho(s)^2\sum_{k=1,G(k)=1}^{\infty}e_k^2\cosh^2(kR)+f_k^2\sinh^2(kR)+2e_kf_k\cosh(kR)\sinh(kR)\\
&+ \frac{1}{2}9\rho(s)^2\sum_{k=1,G(k)=-1}^{\infty}e_k^2\cosh^2(Rk)+f_k^2\sinh^2(Rk)+2e_kf_k\cosh(Rk)\sinh(-kR)\\
&= \frac{9}{2}\rho(s)^2\sum_{k=1,K(k)=1}^{\infty}(c_k\cosh(kR)+d_k\sinh(kR))^2+\frac{9}{2}\rho(s)^2\sum_{k=1,K(k)=-1}^{\infty}(c_k\cosh(-kR)+d_k\sinh(-kR))^2\\
&+ \frac{9}{2}\rho(s)^2\sum_{k=1,G(k)=1}^{\infty}(e_k\cosh(kR)+f_k\sinh(kR))^2+\frac{9}{2}\rho(s)^2\sum_{k=1,G(k)=-1}^{\infty}(e_k\cosh(-kR)+f_k\sinh(-kR))^2\\
&\leqslant \frac{9}{2}\rho(s)^2\sum_{k=1}^{\infty}(c_k\cosh(kR)+d_k\sinh(kR))^2+\frac{9}{2}\rho(s)^2\sum_{k=1}^{\infty}(c_k\cosh(-kR)+d_k\sinh(-kR))^2\\
&+ \frac{9}{2}\rho(s)^2\sum_{k=1}^{\infty}(e_k\cosh(kR)+f_k\sinh(kR))^2+\frac{9}{2}\rho(s)^2\sum_{k=1}^{\infty}(e_k\cosh(-kR)+f_k\sinh(-kR))^2\\
&= \frac{9}{2}\rho(s)^2\sum_{k=1}^{\infty}a_k(R)^2+\frac{9}{2}\rho(s)^2\sum_{k=1}^{\infty}a_k(-R)^2+\frac{9}{2}\rho(s)^2\sum_{k=1}^{\infty}b_k(R)^2+\frac{9}{2}\rho(s)^2\sum_{k=1}^{\infty}b_k(-R)^2\\
&= 9\rho(s)^2\left(\left\|\tilde{f}(R)\right\|^2_{L^2(S^1)}+\left\|\tilde{f}(-R)\right\|^2_{L^2(S^1)}\right).
\end{aligned}
$$

The same inequality holds for negative s, and a similar estimate can be derived for the harmonic function $\tilde{g}$. Thus

$$\begin{aligned}\left\|\tilde{f}(s)\right\|^2_{L^2(S^1)} &\leqslant 9\rho(s)^2\left(\left\|\tilde{f}(R)\right\|^2_{L^2(S^1)} + \left\|\tilde{f}(-R)\right\|^2_{L^2(S^1)}\right),\\ \|\tilde{g}(s)\|^2_{L^2(S^1)} &\leqslant 9\rho(s)^2\left(\|\tilde{g}(R)\|^2_{L^2(S^1)} + \|\tilde{g}(-R)\|^2_{L^2(S^1)}\right),\end{aligned}$$

and from $\tilde{f}(s,t) := f(s,t) + \tilde{S}$ and $\tilde{g}(s,t) := g(s,t) - P$, we end up with

$$\begin{aligned}\left\|f(s) + \tilde{S}\right\|^2_{L^2(S^1)} &\leqslant 9\rho(s)^2\left(\left\|f(R) + \tilde{S}\right\|^2_{L^2(S^1)} + \left\|f(-R) + \tilde{S}\right\|^2_{L^2(S^1)}\right)\\ &\leqslant 18\rho(s)^2 \max\left\{\left\|f(R) + \tilde{S}\right\|^2_{L^2(S^1)}, \left\|f(-R) + \tilde{S}\right\|^2_{L^2(S^1)}\right\},\\ \|g(s) - P\|^2_{L^2(S^1)} &\leqslant 9\rho(s)^2\left(\|g(R) - P\|^2_{L^2(S^1)} + \|g(-R) - P\|^2_{L^2(S^1)}\right)\\ &\leqslant 18\rho(s)^2 \max\left\{\|g(R) - P\|^2_{L^2(S^1)}, \|g(-R) - P\|^2_{L^2(S^1)}\right\}.\end{aligned}$$

□

Remark 59. In [5], a notion of convergence for $\mathcal{H}-$holomorphic curves is derived by using a result (Lemma A.2) which states that the conformal co-period of a harmonic $1-$form on a Riemann surface can be universally controlled by its periods. Proposition 57 gives a counterexample to this statement.

Part III

Appendix

Appendix A

Holomorphic disks with fixed boundary

This appendix is devoted to the description of the convergence of pseudoholomorphic disks with fixed boundaries in a symplectization, as well as, of their limit object. The results are used for proving the convergence of a cylinder of "finite length", i.e. of type b_1 as discussed in Section 3.2.3.

Let $u_n = (a_n, f_n) : D \to \mathbb{R} \times M$ be a sequence of pseudoholomorphic curves in the symplectization $\mathbb{R} \times M$ of the contact manifold (M, α), and being defined on the open unit disk D with respect to the standard complex structure i and the cylindrical almost complex structure $\bar{J}$ on $\mathbb{R} \times M$. Fix some $\tau > 0$ (to be defined later) and assume that there exists a subsequence of u_n, also denoted by u_n, such that

$$u_n \to u \tag{A.0.1}$$

as $n \to \infty$ in $C^\infty(D \backslash \overline{D_\tau(0)})$. Furthermore, we assume that the Hofer energy $E_H(u_n; D)$ of u_n is uniformly bounded. In the following we analyze the convergence of u_n.

The functions a_n can be supposed to be not uniformly bounded. If this is not the case, we may deduce using standard bubbling-off analysis that the gradients of u_n are uniformly bounded on all of D, which in turn, implies that u_n converge in $C^\infty(D)$ to a pseudoholomorphic disk with finite Hofer energy.

To describe the convergence and the limit object we use the results from [7] and [9]. However, since the arguments in [7] and [9] can be almost carried out line by line, we drop the details and explain only the strategy, point out the differences and mention the convergence result. As we have assumed that the $\mathbb{R}-$coordinates of u_n are unbounded, the maximum principle for subharmonic functions gives $a_n \to -\infty$. By (A.0.1) we have the $C^\infty-$convergence of u_n on an arbitrary neighborhood of ∂D, and by a specific choice of this neighborhood, we assume that the $\mathbb{R}-$components of u_n, when restricted to this neighborhood, do not leave a fixed interval $[-K, K]$ for some $K \in \mathbb{R}$ with $K > 0$. Thus from level $-K-2$ we start with the decomposition of $a_n^{-1}((-\infty, -K-2])$ into cylindrical, essential and one "bottom" boundary components. This decomposition which is identical to the decomposition done in [7] and [9] is illustrated in Figure A.0.1. From [7] and [9] we know that there are at most $N_0 \in \mathbb{N}$ cylindrical components.

In addition to the above decomposition, we add one more boundary components, namely the "upper" boundary component. In the following we investigate the convergence of the upper boundary component. This component is given by

$$\mathcal{B}_n := a_n^{-1}([-K-2-R_0, \infty))$$

where $R_0 > 0$ is the constant from Section 5.4 of [10]. By the above considerations, $\mathcal{B}_n$ is contained in a compact region $X = [-R, R] \times M \subset \mathbb{R} \times M$ for all $n \in \mathbb{N}$ and a sufficiently large $R > 0$. This surface has two types of boundaries. The first one is the boundary ∂D which lies in a specific neighborhood such that its image under u_n belongs to $[-K, K] \times M$. The second one is the boundary which connects certain cylindrical components from the

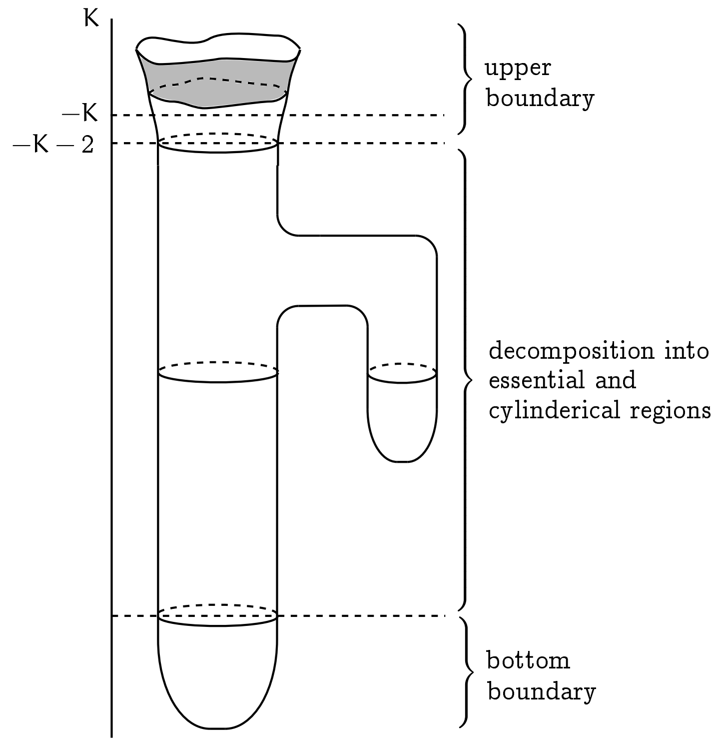

Figure A.0.1: Decomposition of $a_n^{-1}((-\infty, -K-2])$

decomposition of $a_n^{-1}((-\infty, -K-2])$. For the cylindrical, essential and bottom boundary components we use the results established in [7] and [9] to describe the convergence and the limit object. The arguments by be applied line by line. For the "upper" boundary component we use Theorem 3.2 of [7], also known as the Gromov compactness with free-boundary theorem (hereafter simply called Gromov compactness theorem).

Before stating the Gromov compactness theorem we explain the notion of convergence by considering a general setting as in **[7]**. Let $\overline{\Sigma}$ be a compact surface of genus g with m smooth boundary components and q distinct marked points $\mathcal{M} = \{z^1, ..., z^q\} \subset \mathrm{int}(\overline{\Sigma})$ in the interior of $\overline{\Sigma}$. Here g is by definition, the genus of the surface obtained by filling in a disk at each boundary component. Consider a finite collection Δ of disjoint simple loops in $\mathrm{int}(\overline{\Sigma})$. Denote by Σ the nodal surface obtained by collapsing the loops in Δ. Thus, Σ is a finite disjoint union of smooth surfaces with finitely many pairs of points identified. Denote by Δ the image of Δ under the projection $\pi : \overline{\Sigma} \to \Sigma$. A conformal structure j on Σ is a conformal structure on each component of Σ. We call the pair (Σ, j) a nodal Riemann surface. A continous map $u : (\Sigma, j) \to (X, \overline{J})$ is called a nodal holomorphic curve if its restriction to each component of Σ is holomorphic. Moreover, we require that there is no sphere with less than three nodal or marked points on which u is constant. We will refer to this property as stability. Denote by $u : \overline{\Sigma} \to X$ its left, which is constant on each component of Δ.

Definition 60. We say that a sequence of pseudoholomorphic curves with q marked points $u_n : (\Sigma_n, j_n, \mathcal{M}_n) \to (X, \overline{J})$ converges to a nodal holomorohic curve $u : (\Sigma, j, \mathcal{M}) \to (X, \overline{J})$ if there exists a sequence of diffeomorphisms $\phi_n : \Sigma_n \to \Sigma$ such that

1. $(\phi_n)_* j_n \to \pi^* j$ in C^∞_{loc} on $\Sigma \backslash \Delta$ and $\phi_n(z^l_n) = z^l$ for all $l = 1, ..., q$,
2. $u_n \circ \phi_n^{-1} \to u$ in C^∞_{loc} on $\Sigma \backslash (\Delta \cup \partial\Sigma)$,

3. $u_n \circ \phi_n^{-1} \to u$ in C^0_{loc} on $\Sigma \backslash \partial\Sigma$,
4. $\text{area}_{\overline{g}}(u_n) \to \text{area}_{\overline{g}}(u)$,

where $\text{area}_{\overline{g}}$ of a pseudoholomorphic curve is defined as in Section 2.2 of [15].

The Gromov compactness theorem will be formulated for pseudoholomorphic curves $u : (\Sigma, j) \to (X, \overline{J})$ satisfying the following conditions.

O1 (Σ, j) is a compact Riemann surface of genus g with m boundary components and q distinct marked points $\mathcal{M}$ in the interior.

O2 The area of u with respect to $\overline{g}$ is bounded by the constant $C > 0$.

O3 The image of u is contained in a compact subset $K \subset X$.

O4 At the boundary components Γ of (Σ, j) there exists mutually disjoint conformal embeddings

$$\beta^\Gamma : [0, 5L] \times S^1 \hookrightarrow \Sigma \backslash \mathcal{M}$$

mapping $\{0\} \times S^1$ onto Γ for some $L \geqslant L_0(g, m, q, C, K) \geqslant 1$.

O5 For each boundary component Γ, the differential of $u \circ \beta^\Gamma$ satisfies

$$\frac{1}{D} \leqslant \left\| d(u \circ \beta^\Gamma)(z) \right\| \leqslant D$$

for all $z \in [0, 5L] \times S^1$ with respect to the Euclidean metric on $[0, 5L] \times S^1$ and the cylinderical metric $\overline{g}$ on X, for some constant $D > 0$.

Theorem 61. *(Gromov compactness with free boundary) Let $u_n : (\Sigma_n, j_n, \mathcal{M}_n) \to (X, \overline{J})$ be a sequence of pseudoholomorphic curves with q marked points satisfying (O1)-(O5) with g, m, q, C, K, L, D independent of n. Then, a subsequence of u_n converges to a nodal pseudoholomorphic curve $u : (\Sigma, j, \mathcal{M}) \to (X, \overline{J})$. Moreover, we can choose the maps ϕ_n such that the restricted maps*

$$\phi_n \circ \beta_n^\Gamma : [0, L] \times S^1 \to \Sigma \backslash \Delta$$

are independent of n and Γ.

For a proof we refer to [10]. The Gromov convergence result will be applied to the maps

$$u_n|_{\mathcal{B}_n} = (a_n, f_n)|_{\mathcal{B}_n} : \mathcal{B}_n \to (X, \overline{J}),$$

where the choice of the neighborhood of ∂D, on which the $\mathbb{R}-$components of u_n lie in $[-K, K]$, plays an essential role. The existence of a special parametrization of a neighborhood of ∂D will enable us to apply "Gromov compactness with free boundary" in the analysis of the convergence property of the upper boundary component. Essentially, the application of "Gromov compactness with free boundary", requires that the properties (O4) and (O5) under Definition 3.1 of [7] are satisfied. The following considerations ensure these conditions: Choose $L_0' \geqslant 1$ as in Remark 3.3 after Theorem 3.2 of [7]. More precisely, L_0' depends only on the genus g of the surface, the number of boundary components m, the number of marked points q, the uniform bound C on the area of the considered pseudoholomorphic curves, the constant ϵ_0 from Remark II.4.3 of [15], and the constant C_{ML} from Lemma 3.17 of [7] (the classical monotonicity lemma). For this L_0' we write $L_0'(g, m, q, C, \epsilon_0, C_{ML})$. Further on, choose L_0 as

$$L_0 := \max\{L_0'(0,1,2,C,\epsilon_0,C_{ML}), L_0'(0,2,1,C,\epsilon_0,C_{ML}),\\ L_0'(0,3,0,C,\epsilon_0,C_{ML}), ..., L_0'(0,2N_0,0,C,\epsilon_0,C_{ML})\}.$$

Note that when determining the constant L_0' in the first two cases, we introduce one and two artificial punctures, i.e. $q = 2$ or $q = 1$, in order to make our surface stable. Set $\tau_0 = e^{-10\pi L_0}$ and choose $\tau < \tau_0$. In view of (A.0.1), assume that there exists a constant $K > 0$ such that $u_n(D\backslash\overline{D_\tau(0)}) \subset [-K, K] \times M$ for all $n \in \mathbb{N}$. Hence the boundary is fixed in the symplectization. The boundary region can be conformaly parametrized as follows. Consider the map $\beta_{\partial D,0} : [0, 5L_0] \times S^1 \to D\backslash D_{\tau_0}(0)$, $(s,t) \mapsto e^{-2\pi(s+it)}$. This map is obviously a conformal parametrization of the boundary region. Let now $L = -\ln(\tau)/10\pi$. Obviously, $L \geqslant L_0$ and the map $\beta_{\partial D} : [0, 5L] \times S^1 \to D\backslash D_\tau(0)$, $(s,t) \mapsto e^{-2\pi(s+it)}$ is a conformal parametrization of a neighborhood of the boundary circle ∂D. Fix this boundary. This conformal parametrization is obviously independent of n and will be used in conjunction with "Gromov compactness with free boundary". Finally, glue the upper boundary component to the rest of the surface, and obtain the resulting limit surface together with the convergence description.
To formulate the convergence result we introduce some notations. Let Z be an oriented surface diffeomorphic to the standard unit disk D and $\Delta = \Delta_n \amalg \Delta_p \subset Z$ a collection of finitely many disjoint simple loops divided into two disjoint sets. Denote by Z_{Δ_n} the surface obtained by collapsing the curves in Δ_n to points. Write

$$Z^* := Z_{\Delta_n}\backslash\Delta_p =: Z^{(0)} \amalg \coprod_{\nu=1}^{N} Z^{(\nu)} \amalg Z^{(N+1)}$$

as a disjoint union of components $Z^{(\nu)}$. Here $Z^{(0)}$ is the bottom boundary component which is the disjoint union of finitely many disks, while $Z^{(N+1)}$ is the upper boundary component whose boundary is of two types. One type is the boundary of the disk D and the other boundary components are certain loops from Δ_p. Let j be a conformal structure on $Z\backslash\Delta$ such that $(Z\backslash\Delta, j)$ is a punctured Riemann surface together with an identification of distinct pairs of punctures given by the elements of Δ. This shows that Z^* has the structure of a nodal punctured Riemann surface with a remaining identification of punctures given by the loops $\{\delta^i\}_{i\in I} = \Delta_p$, for some index set I. A broken pseudoholomorphic curve (with $N+2$ levels) is a map $F = (F^{(0)}, F^{(1)}, ..., F^{(N+1)}) : (Z^*, j) \to X$, where $X = \coprod_{\nu=0}^{N+1}(\mathbb{R}\times M)$ such that $F^{(\nu)} : (Z^{(\nu)}, j) \to \mathbb{R}\times M$ is a punctured pseudoholomorphic curve with the additional property that F extends to a continous map $\overline{F} : Z \to \overline{X}$. Here $\overline{X}$ is obtained as follows. The negative end of the compactification of $\mathbb{R}\times M$ of the $\nu-$th copy is glued to the positive end of the compactification of $\mathbb{R}\times M$ of the copy $\nu+1$. This procedure is done for $\nu = 0, ..., N$. For a loop $\delta \in \Delta_p$, there exists $\nu \in \{0, ..., N\}$ such that δ is adjacent to $Z^{(\nu)}$ and $Z^{(\nu+1)}$. Fix an embedded annuli $A^{\delta,\nu} \cong [-1, 1] \times S^1 \subset Z\backslash\Delta_n$ such that $\{0\} \times S^1 = \delta$, $\{-1\} \times S^1 \subset Z^{(\nu)}$ and $\{1\} \times S^1 \subset Z^{(\nu+1)}$.
In this context, we state a convergence result which has been established in [7] and [9].

Proposition 62. *The sequence of pseudoholomorphic disks* $u_n = (a_n, f_n) : (D, i) \to \mathbb{R}\times M$ *satisfying (A.0.1) and having a uniformly bounded Hofer energy has a subsequence that converges to a broken pseudoholomorphic curve* $u = (a, f) : (Z, j) \to \mathbb{R}\times M$ *with* $N+2$ *levels in the following sense: There exists a sequence of diffomorphisms* $\varphi_n : D \to Z$ *and a sequence of negative real numbers* $\min(a_n) = r_n^{(0)} < r_n^{(1)} < ... < r_n^{(N+1)} = -K-2$ *with* $K \in \mathbb{R}$ *and* $r_n^{(\nu+1)} - r_n^{(\nu)} \to \infty$ *as* $n \to \infty$ *such that the following hold:*

1. *Z with the circles Δ collapsed to points is a nodal Riemann surface (in the sense of the above discussion, but with boundary). $i_n := (\varphi_n)_* i \to j$ in C^∞_{loc} on $Z\backslash\Delta$. For every $i \in I$, the annulus $(A^i, (\varphi_n)_* i)$ is conformally equivalent to a standard annulus $[-R_n, R_n] \times S^1$ by a diffeomorphism of the form $(s,t) \mapsto (\kappa(s), t)$ with $R_n \to \infty$ as $n \to \infty$.*

2. *The sequence* $u_n \circ \varphi_n^{-1}|_{Z^{(\nu)}} : Z^{(\nu)} \to \mathbb{R} \times M$ *converges in* C^∞_{loc} *on* $Z^{(\nu)} \backslash \Delta_n$ *to a punctured nodal pseudo-holomorphic curve* $u^{(\nu)} : (Z^{(\nu)}, j) \to \mathbb{R} \times M$*, and in* C^0_{loc} *on* $Z^{(\nu)}$*.*

3. *The sequence* $f_n \circ \varphi_n^{-1} : Z \to M$ *converges in* C^0 *to a map* $f : Z \to M$*, whose restriction to* Δ_p *parametrizes the Reeb orbits and to* Δ_n *parametrizes points.*

4. *For any* $S > 0$ *, there exist* $\rho > 0$ *and* $K \in \mathbb{N}$ *such that* $a_n \circ \varphi_n^{-1}(s, t) \in [r_n^{(\nu)} + S, r_n^{(\nu+1)} - S]$ *for all* $n \geqslant K$ *and all* $(s, t) \in A^{\delta,\nu}$ *with* $|s| \leqslant \rho$*.*

5. *The diffeomorphisms* $\varphi_n \circ \beta_{\partial D} : [0, 5L] \times S^1 \hookrightarrow Z$ *are independent of* n*.*

Appendix B

$\mathcal{H}-$holomorphic cylinders of small area

In this appendix we describe the convergence of a sequence of $\mathcal{H}-$holomorhic cylinders $u_n = (a_n, f_n) : [-R_n, R_n] \times S^1 \to \mathbb{R} \times M$ with harmonic perturbations γ_n. As before, we denote by $\mathcal{P}_\alpha \subset \mathbb{R}$ the set defined by

$$\mathcal{P}_\alpha = \{0\} \cup \{T > 0 \mid \text{there exists a } T-\text{priodic orbit of } X_\alpha\}.$$

We assume that all periodic orbits of the Reeb vector field X_α are non-degenerate, in the sense that the linearized flow along any periodic orbit restricted to the contact structure has no eigenvalues equal to 1. We will refer to this case as the *Morse case*. As shown in [2], a non-degenerate $T-$periodic orbit is isolated among periodic orbits having periods close to T. Thus we define the constant $\hbar_0 > 0$, introduced in Section 3.1, by

$$\hbar_0 = \min\{|T_1 - T_2| \mid T_1, T_2 \in \mathcal{P} \text{ with } T_1, T_2 \leqslant \tilde{E}_0 \text{ and } T_1 \neq T_2\}$$

where $\tilde{E}_0 = 2(C_1 + E_0)$ is defined in Section 3.2.5 Step 3. Note that $E_0 \leqslant \tilde{E}_0$. For a sequence of $\mathcal{H}-$holomorphic cylinders $u_n = (a_n, f_n) : [-R_n, R_n] \times S^1 \to \mathbb{R} \times M$ with harmonic perturbations γ_n the analysis is performed in the following setting.

P1 $R_n \to \infty$ as $n \to \infty$.

P2 There exist constants $\delta_0 > 0$ and $C_1 > 0$ such that $\|df_n(z)\| := \sup_{\|v\|_{\text{eucl.}}=1} \|df_n(z)v\|_g < C_1$ for all $z \in ([-R, -R + \delta_0] \amalg [R - \delta_0, R]) \times S^1$.

P3 The energy of u_n, as well as the L^2-norm of γ_n are uniformly bounded by the constants $E_0 > 0$ and $C_0 > 0$, respectively.

P4 For $\tilde{E}_0$, the $d\alpha-$energy of u_n is uniformly bounded by $\hbar_0/2$.

P5 There exists a constant $C > 0$ such that for all $n \in \mathbb{N}$, we have $|\tau_n|, |\sigma_n| < C$, where τ_n is the conformal period of γ_n on $[-R_n, R_n] \times S^1$, i.e. $\tau_n = R_n P_n$, and σ_n is the conformal co-period of γ_n on $[-R_n, R_n] \times S^1$, i.e. $\sigma_n = R_n S_n$. Here, P_n and S_n are the period and co-period of γ_n on the cylinder $[-R_n, R_n] \times S^1$. After going over to a subsequence, we assume that $\tau_n \to \tau$ and $\sigma_n \to \sigma$ as $n \to \infty$ for some $\tau, \sigma \in \mathbb{R}$ and $\tau, \sigma \geqslant 0$.

The task is to describe the asymptotic behavior of such cylinders. More precisely, we will derive the following results. For a finite energy $\mathcal{H}-$holomorphic cylinder $u = (a, f) : [-R, R] \times S^1 \to \mathbb{R} \times M$ with harmonic perturbation γ we introduce the notion of center action (Definition 74) as in [14] which may be defined as the unique element $A(u) \in \mathcal{P}_\alpha$ which is sufficiently close to

$$\int_{S^1} u(0)^* \alpha.$$

For more details the reader might consult Section B.1. By Theorem 72 it follows that $A(u)$ is either 0 or strictly greater than some positive constant which will be determined in Section B.1.2. We distinguish between two cases; the first case is when there exists a subsequence of u_n with a vanishing center action and the second case is when there is no subsequence of u_n with this property. In this regard, Theorem 63 deals with the asymptotic behavior in the case of vanishing center action, while Theorem 65 deals with the asymptotic behavior in the case of positive center action.
Before stating the main result we recall the construction of a sequence of diffeomorphisms $\theta_n : [-R_n, R_n] \to [-1, 1]$ introduced in Definition 44. The construction is similar to that given in [7] and will enable us to describe the C^0-convergence.

Theorem 63. *Let u_n be a sequence of $\mathcal{H}-$holomorphic cylinders with harmonic perturbations γ_n that satisfy P1-P5 and possessing a subsequence having vanishing center action. Then there exists a subsequence of u_n, still denoted by u_n, the $\mathcal{H}-$holomorphic cylinders $u^\pm$ defined on $(-\infty, 0] \times S^1$ and $[0, \infty) \times S^1$, respectively, and a point $w = (w_a, w_f) \in \mathbb{R} \times M$ such that for every sequence $h_n \in \mathbb{R}_+$ and every sequence of diffeomorphisms $\theta_n : [-R_n, R_n] \to [-1, 1]$ constructed as in Remark 44 the following $C^\infty_{loc}-$ and C^0-convergence results hold (after a suitable shift of u_n in the $\mathbb{R}-$coordinate)*

$C^\infty_{loc}-$convergence:

1. *For any sequence $s_n \in [-R_n + h_n, R_n - h_n]$ there exists a constant $\tau_{\{s_n\}} \in [-\tau, \tau]$ (depending on the sequence $\{s_n\}$) such that after passing to a subsequence, the shifted maps $u_n(s + s_n, t) + S_n s_n$, defined on $[-R_n + h_n - s_n, R_n - h_n - s_n] \times S^1$, converge in C^∞_{loc} to $(w_a, \phi^\alpha_{-\tau_{\{s_n\}}}(w_f))$. The shifted harmonic perturbation $1-$forms $\gamma_n(s + s_n, t)$ possess a subsequence converging in C^∞_{loc} to 0.*
2. *The left shifts $u^-_n(s, t) - R_n S_n := u_n(s - R_n, t) - R_n S_n$, defined on $[0, h_n) \times S^1$, possess a subsequence that converge in C^∞_{loc} to a pseudoholomorphic half cylinder $u^- = (a^-, f^-)$, defined on $[0, +\infty) \times S^1$. The curve u^- is asymptotic to $(w_a, \phi^\alpha_\tau(w_f))$. The left shifted harmonic perturbation $1-$forms γ^-_n converge in C^∞_{loc} to an exact harmonic $1-$form $d\Gamma^-$, defined on $[0, +\infty) \times S^1$. Their asymptotics are 0.*
3. *The right shifts $u^+_n(s, t) + R_n S_n := u_n(s + R_n, t) + R_n S_n$, defined on $(-h_n, 0] \times S^1$, possess a subsequence that converge in C^∞_{loc} to a pseudoholomorphic half cylinder $u^+ = (a^+, f^+)$, defined on $(-\infty, 0] \times S^1$. The curve u^+ is asymptotic to $(w_a, \phi^\alpha_{-\tau}(w_f))$. The right shifted harmonic perturbation $1-$forms γ^+_n converge in C^∞_{loc} to an exact harmonic $1-$form $d\Gamma^+$, defined on $(-\infty, 0] \times S^1$. Their asymptotics are 0.*

C^0-convergence:

1. *The maps $v_n : [-1/2, 1/2] \times S^1 \to \mathbb{R} \times M$ defined by $v_n(s, t) = u_n(\theta_n^{-1}(s), t)$, converge in C^0 to $(-2\sigma s + w_a, \phi^\alpha_{-2\tau s}(w_f))$.*
2. *The maps $v^-_n - R_n S_n : [-1, -1/2] \times S^1 \to \mathbb{R} \times M$ defined by $v^-_n(s, t) = u_n((\theta^-_n)^{-1}(s), t)$, converge in C^0 to a map $v^- : [-1, -1/2] \times S^1 \to \mathbb{R} \times M$ such that $v^-(s, t) = u^-((\theta^-)^{-1}(s), t)$ and $v^-(-1/2, t) = (w_a, \phi^\alpha_\tau(w_f))$.*
3. *The maps $v^+_n + R_n S_n : [1/2, 1] \times S^1 \to \mathbb{R} \times M$ defined by $v^+_n(s, t) = u_n((\theta^+_n)^{-1}(s), t)$, converge in C^0 to a map $v^+ : [1/2, 1] \times S^1 \to \mathbb{R} \times M$ such that $v^+(s, t) = u^+((\theta^+)^{-1}(s), t)$ and $v^+(1/2, t) = (w_a, \phi^\alpha_{-\tau}(w_f))$.*

An immediate corollary is

Corollary 64. *Under the same hypothesis of Theorem 63 the following $C^\infty_{loc}-$convergence results hold.*

1. *The maps $v_n^- - R_n S_n$ converge in C^∞_{loc} to v^-, where v^- is asymptotic to $(w_a, \phi^\alpha_\tau(w_f))$ as $s \to -1/2$. The harmonic 1−forms $[(\theta_n^-)^{-1}]^*\gamma_n^-$ with respect to the complex structure $[(\theta_n^-)^{-1}]^* i$ converge in C^∞_{loc} to a harmonic 1−form $[(\theta^-)^{-1}]^* d\Gamma^-$ with respect to the complex structure $[(\theta^-)^{-1}]^* i$ which is asymptotic to some constant as $s \to -1/2$.*

2. *The maps $v_n^+ + R_n S_n$ converge in C^∞_{loc} to v^+, where v^+ is asymptotic to $(w_a, \phi^\alpha_{-\tau}(w_f))$ as $s \to 1/2$. The harmonic 1−forms $[(\theta_n^+)^{-1}]^*\gamma_n^-$ with respect to the complex structure $[(\theta_n^+)^{-1}]^* i$ converge in C^∞_{loc} to a harmonic 1−form $[(\theta^+)^{-1}]^* d\Gamma^+$ with respect to the complex structure $[(\theta^+)^{-1}]^* i$ which is asymptotic to some constant as $s \to 1/2$.*

Proof. To show that v_n^- converge in C^∞_{loc} to v^- we recall that

$$\begin{aligned} v_n^-(s,t) - S_n R_n = (&\overline{a}_n^-((\theta_n^-)^{-1}(s), t) - \Gamma_n^-((\theta_n^-)^{-1}(s), t) \\ &- S_n R_n, \phi^\alpha_{-P_n(\theta_n^-)^{-1}(s) + P_n R_n}(\overline{f}_n^-((\theta_n^-)^{-1}(s), t))) \end{aligned}$$

and that $\theta_n^- \to \theta^-$ in C^∞_{loc}. The convergence of the harmonic perturbations γ_n follows from Corollary 107, while the convergence of v_n^+ is proved in an analogous manner.

□

In the case of positive center action we have the following.

Theorem 65. *Let u_n be a sequence of $\mathcal{H}$−holomorphic cylinders with harmonic perturbations γ_n satisfy P1-P5 and possessing no subsequence with vanishing center action. Then there exist a subsequence of u_n, still denoted by u_n, $\mathcal{H}$−holomorphic half cylinders $u^\pm$ defined on $(-\infty, 0] \times S^1$ and $[0, \infty) \times S^1$, respectively, a periodic orbit x of period $T \in \mathbb{R}\backslash\{0\}$, and sequences $\overline{r}_n^\pm \in \mathbb{R}$ with $|\overline{r}_n^+ - \overline{r}_n^n| \to \infty$ as $n \to \infty$ such that for every sequence $h_n \in \mathbb{R}_+$ and every sequence of diffeomorphisms $\theta_n : [-R_n, R_n] \to [-1, 1]$ as in Remark 44, the following convergence results hold (after a suitable shift of u_n in the $\mathbb{R}$−coordinate).*

C^∞_{loc}−convergence:

1. *For any sequence $s_n \in [-R_n + h_n, R_n - h_n]$ there exists a constant $\tau_{\{s_n\}} \in [-\tau, \tau]$ (depending on the sequence $\{s_n\}$) such that after passing to a subsequence, the shifted maps $u_n(s + s_n, t) - s_n T - S_n s_n$, defined on $[-R_n + h_n - s_n, R_n - h_n - s_n] \times S^1$, converge in C^∞_{loc} to $(Ts + a_0, \phi^\alpha_{-\tau_{\{s_n\}}}(x(Tt)) = x(Tt + \tau_{\{s_n\}}))$. The shifted harmonic perturbation 1−forms $\gamma_n(s + s_n, t)$ possess a subsequence converging in C^∞_{loc} to 0.*

2. *The left shifts $u_n^-(s,t) - R_n S_n$, defined on $[0, h_n) \times S^1$, possess a subsequence that converges in C^∞_{loc} to a $\mathcal{H}$−holomorphic half cylinder $u^- = (a^-, f^-)$, defined on $[0, +\infty) \times S^1$. The curve u^- is asymptotic to $(Ts + a_0, \phi^\alpha_\tau(x(Tt)) = x(Tt + \tau))$. The left shifted harmonic perturbation 1−forms γ_n^- converge in C^∞_{loc} to an exact harmonic 1−form $d\Gamma^-$, defined on $[0, +\infty) \times S^1$. Their asymptotics are 0.*

3. *The right shifts $u_n^+(s,t) + R_n S_n$, defined on $(-h_n, 0] \times S^1$ possess a subsequence that converges in C^∞_{loc} to a $\mathcal{H}$−holomorphic half cylinder $u^+ = (a^+, f^+)$, defined on $(-\infty, 0] \times S^1$. The curve u^+ is asymptotic to $(Ts + a_0, \phi^\alpha_{-\tau}(x(Tt)) = x(Tt - \tau))$. The right shifted harmonic perturbation 1−forms γ_n^+ converge in C^∞_{loc} to an exact harmonic 1−form $d\Gamma^+$, defined on $(-\infty, 0] \times S^1$. Their asymptotics are 0.*

C^0−convergence:

1. *The maps $f_n \circ \theta_n^{-1} : [-1/2, 1/2] \times S^1 \to M$ converge in C^0 to $\phi^\alpha_{-2\tau s}(x(Tt)) = x(Tt - 2\tau s)$.*

2. *The maps* $f_n^- \circ (\theta_n^-)^{-1} : [-1,-1/2] \times S^1 \to M$ *converge in* C^0 *to a map* $f^- \circ (\theta^-)^{-1} : [-1,-1/2] \times S^1 \to M$ *such that* $f^-((\theta^-)^{-1}(-1/2), t) = \phi_\tau^\alpha(x(Tt)) = x(Tt+\tau)$.

3. *The maps* $f_n^+ \circ (\theta_n^+)^{-1} : [1/2, 1] \times S^1 \to M$ *converge in* C^0 *to a map* $f^+ \circ (\theta^+)^{-1} : [1/2,1] \times S^1 \to M$ *such that* $f^+((\theta^+)^{-1}(1/2), t) = \phi_{-\tau}^\alpha(x(Tt)) = x(Tt-\tau)$.

4. *There exist* $C > 0$, $\rho > 0$ *and* $N \in \mathbb{N}$ *such that for any* $R > 0$, $a_n \circ \theta_n^{-1}(s,t) \in [\overline{r}_n^- + R - C, \overline{r}_n^+ - R + C]$ *for all* $n \geqslant N$ *and all* $(s,t) \in [-\rho, \rho] \times S^1$.

An immediate corollary is

Corollary 66. *Under the same hypothesis of Theorem 65 and the notations from Theorem 63 we have the following* C^∞_{loc}*–convergence results.*

1. *The maps* $v_n^- - R_n S_n$ *converge in* C^∞_{loc} *to* v^- *where* $f^-((\theta^-)^{-1}(-1/2), t) = x(Tt+\tau)$. *The harmonic* 1*–forms* $[(\theta_n^-)^{-1}]^* \gamma_n^-$ *with respect to the complex structure* $[(\theta_n^-)^{-1}]^* i$ *converge in* C^∞_{loc} *to a harmonic* 1*–form* $[(\theta^-)^{-1}]^* d\Gamma^-$ *with respect to the complex structure* $[(\theta^-)^{-1}]^* i$ *which is asymptotic to some constant as* $s \to -1/2$.

2. *The maps* $v_n^+ + R_n S_n$ *converge in* C^∞_{loc} *to* v^+ *where* $f^+((\theta^+)^{-1}(1/2), t) = x(Tt-\tau)$. *The harmonic* 1*–forms* $[(\theta_n^+)^{-1}]^* \gamma_n^-$ *with respect to the complex structure* $[(\theta_n^+)^{-1}]^* i$ *converge in* C^∞_{loc} *to a harmonic* 1*–form* $[(\theta^+)^{-1}]^* d\Gamma^+$ *with respect to the complex structure* $[(\theta^+)^{-1}]^* i$ *which is asymptotic to some constant as* $s \to 1/2$.

In order to establish this, we need to make use of a modified version of the results from [14].

Remark 67. If the sequence of $\mathcal{H}$–holomorphic curves u_n together with the harmonic perturbations γ_n satisfy conditions P1-P4 we can conclude that the left and right shifts $u_n^\pm$ together with the harmonic perturbations $\gamma_n^\pm$ defined on $[0, h_n] \times S^1$ and $[-h_n, 0] \times S^1$, respectively, converge after a suitable shift in the $\mathbb{R}$–coordinate in C^∞_{loc}, to $\mathcal{H}$–holomorphic half cylinders $u^\pm$ with harmonic perturbations $d\Gamma^\pm$ defined on $[0,\infty) \times S^1$ and $(-\infty, 0] \times S^1$, respectively. The $\mathcal{H}$–holomorphic curves $u^\pm$ are asymptotic to points $w^\pm = (w_a^\pm, w_f^\pm) \in \mathbb{R} \times M$ or trivial cylinders over Reeb orbits $(x^\pm, T)$. Without the assumption P5, the asymptotic data of u^- and u^+ cannot be described as in Theorems 63 and 65. In fact, dropping assumption P5 it is not possible to connect the asymptotic data w^- or $x^-(T\cdot)$ of the left shifted $\mathcal{H}$–holomorphic curve u^- to the asymptotic data w^+ of $x^+(T\cdot)$ of the right shifted $\mathcal{H}$–holomorphic curve u^+ by a compact cylinder as in Theorems 63 and 65. In the proof of these theorems it will become apparent that P5 is a necessary condition for the C^0–convergence result.

We begin this Appendix by considering a general $\mathcal{H}$–holomorphic cylinder $u = (a, f) : [-R, R] \times S^1 \to \mathbb{R} \times M$ with harmonic perturbation γ and having the following properties:

Q1 $E(u; [-R,R] \times S^1) \leqslant E_0$ and $\|\gamma\|^2_{L^2([-R,R]\times S^1)} \leqslant C_0$.

Q2 $E_{d\alpha}(u; [-R,R] \times S^1) \leqslant \hbar_0/2$.

Q3 The conformal period $\tau = PR$, where P is the period of γ over the cylinder $[-R,R] \times S^1$ is bounded, i.e. for the constant $C > 0$ from Assumption A5, we have $|\tau| \leqslant C$.

Q4 There exist constants $\delta_0 > 0$ and $C_1 > 0$ such that $\|df(z)\| < C_1$ for all $z \in ([-R, -R+\delta_0] \amalg [R-\delta_0, R]) \times S^1$.

In Section B.1, this $\mathcal{H}$−holomorphic curve is transformed, as in [20], by the flow $\phi^\alpha : \mathbb{R}\times M \to M$ of the Reeb vector field X_α into a usual pseudoholomorphic curve with respect to a domain dependent almost complex structure that varies in a compact set. Here, condition Q3 is essential. The transformed curve is a $\bar{J}_{Ps}$−holomorphic curve. The lower index Ps, where P is the period of the harmonic perturbation and s the coordinate in $[-R, R]$, describes the variation of the complex structure $\bar{J}_{Ps}$; we have $|Ps| \leqslant C$ for all $s \in [-R, R]$. The conditions imposed on the energy are transferred to the $\bar{J}_{Ps}$−holomorphic curves. We then derive a notion of center action for the $\bar{J}_{Ps}$−holomorphic curves by employing the same arguments as in Theorem 1.1 of [14]; here, we distinguish the cases when the center action vanishes and is greater than $\hbar_0$.

In Section B.2 we consider the case of vanishing center action. First, we derive a result for $\bar{J}_{Ps}$−holomorphic curves, which is similar to that established in Theorem 1.2 of [14], and which basically states that a finite energy $\bar{J}_{Ps}$−holomorphic curve with uniformly small $d\alpha$−energy and having vanishing center action, is close to a point in $\mathbb{R}\times M$. This is done by using a version of monotonicity lemma for $\bar{J}_{Ps}$−holomorphic curves given in Appendix F. Then we describe the asymptotic behavior of $\bar{J}_{Ps}$−holomorphic curves, and finally, by using the inverse transformation with the flow of the Reeb vector field, we translate these results in the language of $\mathcal{H}$−holomorphic cylinders and prove Theorem 63.

In Section B.3 we formulate the above findings in the case of positive center action. We prove a result which is similar to that stated by Theorem 1.3 of [14] for $\bar{J}_{Ps}$−holomorphic curves, and then Theorem 65.

In order to prove Theorems 63 and 65 we use a compactness result for a sequence of harmonic functions defined on cylinders and possessing certain properties; this is established in Appendix E.

B.1 $\bar{J}_{Ps}$−holomorphic curves and center action

In this section we transform a $\mathcal{H}$−holomorphic curve into a pseudoholomorphic curve with domain-dependent almost complex structure on the target space $\mathbb{R}\times M$, and introduce a notion of center action for this curve.

B.1.1 $\bar{J}_{Ps}$−holomorphic curves

We consider a $\mathcal{H}$−holomorphic curve $u = (a, f) : [-R, R] \times S^1 \to \mathbb{R}\times M$ with harmonic perturbation γ satisfying Assumptions Q1-Q4, and construct a new map $\bar{u} = (\bar{a}, \bar{f}) : [-R, R] \times S^1 \to \mathbb{R}\times M$ as follows. Let $\phi^\alpha_t : M \to M$ be the Reeb flow on M. Defining

$$\bar{f}(s,t) := \phi^\alpha_{Ps}(f(s,t)) \tag{B.1.1}$$

we find by straightforward calculation that

$$\pi_\alpha d\bar{f} = d\phi^\alpha_{Ps}\pi_\alpha df \quad \text{and} \quad \bar{f}^*\alpha = Pds + f^*\alpha$$

giving

$$\bar{f}^*\alpha\circ i = -Pdt + f^*\alpha\circ i = -Pdt + da + \gamma. \tag{B.1.2}$$

Remark 68. Obviously, as γ is a harmonic 1−form, the 1−form $-Pdt+\gamma$ is harmonic with vanishing period over $[-R, R]\times S^1$. Thus $-Pdt+\gamma$ is globally exact, i.e. there exists a harmonic function $\Gamma : [-R, R]\times S^1 \to \mathbb{R}$ which is unique up to addition of a constant such that $-Pdt+\gamma = d\Gamma$. By technical reasons, which will become apparent later on, we choose Γ such that it has vanishing mean value over $[-R, R]\times S^1$, i.e.

$$\frac{1}{2R}\int_{[-R,R]\times S^1}\Gamma(s,t)ds\wedge dt = 0.$$

Set
$$\overline{a} := a + \Gamma \tag{B.1.3}$$
where Γ was chosen as in Remark 68.
Define the domain-dependent almost complex structure $\overline{J} : [-C, C] \times M \to \mathrm{End}(\xi)$ by
$$\overline{J}_\rho(p) = d\phi^\alpha_\rho(\phi^\alpha_{-\rho}(p)) \circ J_\xi(\phi^\alpha_{-\rho}(p)) \circ d\phi^\alpha_{-\rho}(p) \tag{B.1.4}$$
for all $\rho \in [-C, C]$ and all $p \in M$, where $C > 0$ is the constant from Assumption Q3. Note that for a $\mathcal{H}-$holomorphic curve $u : [-R, R] \times S^1 \to \mathbb{R} \times M$ satisfying Assumptions Q1-Q4, $Ps \in [-C, C]$ for all $s \in [-R, R]$.

Proposition 69. *The curve $\overline{u} = (\overline{a}, \overline{f}) : [-R, R]\times S^1 \to \mathbb{R}\times M$, where $\overline{a}$ and $\overline{f}$ are the maps defined by (B.1.3) and (B.1.1), respectively, is pseudoholomorphic with respect to the domain-dependent almost complex structure $\overline{J}$ varying in a compact space of almost complex structures, i.e.*
$$\pi_\alpha d\overline{f}(s, t) \circ i = \overline{J}_{Ps}(\overline{f}(s, t)) \circ \pi_\alpha d\overline{f}(s, t), \tag{B.1.5}$$
$$(\overline{f}^*\alpha) \circ i = d\overline{a} \tag{B.1.6}$$
for all $(s, t) \in [-R, R] \times S^1$. Moreover, for the $\alpha-$ and $d\alpha-$energies we have
$$E_{d\alpha}(\overline{u}; [-R, R] \times S^1) = E_{d\alpha}(u; [-R, R] \times S^1),$$
$$E_\alpha(\overline{u}; [-R, R] \times S^1) \leqslant \int_{\{R\}\times S^1} |f^*\alpha| + \int_{\{-R\}\times S^1} |f^*\alpha| + E_{d\alpha}(u; [-R, R] \times S^1).$$

Proof. By Remark 68 it is obvious that Equation (B.1.6) holds. Let us consider Equation (B.1.5). The left-hand side of this equation goes over in
$$\pi_\alpha d\overline{f}(s, t) \circ i = d\phi^\alpha_{Ps}\pi_\alpha df \circ i,$$
while the right-hand side goes over in
$$\begin{aligned}
&\overline{J}_{Ps}(\overline{f}(s, t)) \circ \pi_\alpha d\overline{f}(s, t)\\
&= \overline{J}_{Ps}(\phi^\alpha_{Ps}(f(s, t))) \circ d\phi^\alpha_{Ps}(f(s, t))\pi_\alpha df(s, t)\\
&= d\phi^\alpha_{Ps}(\phi^\alpha_{-Ps}(\phi^\alpha_{Ps}(f(s, t)))) \circ J(\phi^\alpha_{-Ps}(\phi^\alpha_{Ps}(f(s, t))))\\
&\qquad \circ d\phi^\alpha_{-Ps}(\phi^\alpha_{Ps}(f(s, t))) \circ d\phi^\alpha_{Ps}(f(s, t))\pi_\alpha df(s, t)\\
&= d\phi^\alpha_{Ps}(f(s, t)) \circ J(f(s, t)) \circ \pi_\alpha df(s, t).
\end{aligned}$$
Hence
$$\begin{aligned}
&\pi_\alpha d\overline{f}(s, t) \circ i - \overline{J}_{Ps}(\overline{f}(s, t)) \circ \pi_\alpha d\overline{f}(s, t)\\
&= d\phi^\alpha_{Ps}\pi_\alpha df \circ i - d\phi^\alpha_{Ps}(f(s, t)) \circ J(f(s, t)) \circ \pi_\alpha df(s, t)\\
&= d\phi^\alpha_{Ps}\,[\pi_\alpha df \circ i - J(f(s, t)) \circ \pi_\alpha df(s, t)]\\
&= 0.
\end{aligned}$$
Thus $\overline{u} = (\overline{a}, \overline{f}) : [-R, R] \times S^1 \to \mathbb{R} \times M$ is an $i - \overline{J}-$holomorphic curve, where $\overline{J}$ is a domain-dependent almost complex structure. The energies transform as follows. The $d\alpha-$energy remains unchanged. Indeed, by definition we have
$$E_{d\alpha}(\overline{u}; [-R, R] \times S^1) \quad = \quad \int_{[-R,R]\times S^1} \overline{f}^* d\alpha,$$

and by noticing that

$$\overline{f}^{*}d\alpha = d\alpha(d\phi^{\alpha}_{P_s}\pi_\alpha df\cdot, d\phi^{\alpha}_{P_s}\pi_\alpha df\cdot) = f^* d\alpha,$$

we obtain

$$E_{d\alpha}(\overline{u};[-R,R]\times S^1) = \int_{[-R,R]\times S^1} \overline{f}^{*}d\alpha = \int_{[-R,R]\times S^1} f^* d\alpha = E_{d\alpha}(u;[-R,R]\times S^1).$$

For the $\alpha-$energy we start from the definition and obtain

$$\begin{aligned}
E_\alpha(\overline{u};[-R,R]\times S^1) &= \sup_{\varphi\in\mathcal{A}} \int_{[-R,R]\times S^1} \varphi'(\overline{a})d\overline{a}\circ i\wedge d\overline{a}\\
&= \sup_{\varphi\in\mathcal{A}} \left[-\int_{[-R,R]\times S^1} d(\varphi(\overline{a})d\overline{a}\circ i) - \varphi(\overline{a})d(d\overline{a}\circ i)\right]\\
&= \sup_{\varphi\in\mathcal{A}} \left[-\int_{[-R,R]\times S^1} d(\varphi(\overline{a})d\overline{a}\circ i) + \int_{[-R,R]\times S^1} \varphi(\overline{a})d(d\overline{a}\circ i)\right]\\
&= \sup_{\varphi\in\mathcal{A}} \left[-\int_{\partial([-R,R]\times S^1)} \varphi(\overline{a})d\overline{a}\circ i - \int_{[-R,R]\times S^1} \varphi(\overline{a})f^*d\alpha\right]\\
&\leqslant \sup_{\varphi\in\mathcal{A}} \left[\left|\int_{\partial([-R,R]\times S^1)} \varphi(\overline{a})d\overline{a}\circ i\right| + \int_{[-R,R]\times S^1} \varphi(\overline{a})f^*d\alpha\right]\\
&\leqslant \sup_{\varphi\in\mathcal{A}} \left[\left|\int_{\{R\}\times S^1} \varphi(\overline{a})d\overline{a}\circ i\right| + \left|\int_{\{-R\}\times S^1} \varphi(\overline{a})d\overline{a}\circ i\right| + E_{d\alpha}(u;[-R_n,R_n]\times S^1)\right]\\
&\leqslant \sup_{\varphi\in\mathcal{A}} \left[\int_{\{R\}\times S^1} \varphi(\overline{a})|d\overline{a}\circ i| + \int_{\{-R\}\times S^1} \varphi(\overline{a})|d\overline{a}\circ i| + E_{d\alpha}(u;[-R_n,R_n]\times S^1)\right]\\
&\leqslant \left[\int_{\{R\}\times S^1} |f^*\alpha| + \int_{\{-R\}\times S^1} |f^*\alpha| + E_{d\alpha}(u;[-R,R]\times S^1)\right].
\end{aligned}$$

□

Remark 70. The $\alpha-$energy of $\overline{u}_n$, that was constructed from u_n (satisfying Assumptions A1-A5), is uniformly bounded. To show this we argue as follows. Due to Assumption P2, the quantities

$$\int_{\{R_n\}\times S^1} |f_n^*\alpha| \text{ and } \int_{\{-R_n\}\times S^1} |f_n^*\alpha|$$

are uniformly bounded by the constant $C_1 > 0$. Hence, according to the definition of $\tilde{E}_0$ we obtain

$$E(\overline{u}_n;[-R_n,R_n]\times S^1) = E_\alpha(\overline{u}_n;[-R_n,R_n]\times S^1) + E_{d\alpha}(\overline{u}_n;[-R_n,R_n]\times S^1) \leqslant \tilde{E}_0.$$

For this reason it makes sense to assume, by Proposition 69, that the energy of $\overline{u}_n$ is uniformly bounded.

To analyze the properties of the transformed pseudoholomorphic curve $\overline{u}$, we consider the following additional structure on M: On the contact structure $\xi = \ker(\alpha)$, let $\overline{J}: [-C,C]\times M \to \mathrm{End}(\xi)$ be the parameter-dependent almost complex structure defined by (B.1.4) having the property $\overline{J}_\rho(p)^2 = -\mathbb{1}$ for all $\rho\in[-C,C]$ and all $p\in M$. On $\mathbb{R}\times M$ we use the following family of Riemannian metrics:

$$\overline{g}_{\rho,p}(v,w) = dr\otimes dr(v,w) + \alpha\otimes\alpha(v,w) + d\alpha(v,\overline{J}_\rho(p)w) \tag{B.1.7}$$

for all $\rho\in[-C,C]$ and all $p\in M$, where r is the coordinate on the $\mathbb{R}-$component of $\mathbb{R}\times M$.

Definition 71. A triple $(\overline{u},R,P)$ is called a $\overline{J}_{P_s}-$*holomorphic curve* if $P,R\in\mathbb{R}$ with $R>0$, and $\overline{u} = (\overline{a},\overline{f})$:

$[-R, R] \times S^1 \to \mathbb{R} \times M$ satisfy the following assumptions:

1. For the constant $C > 0$ from Assumption Q3 we have $|PR| \leqslant C$.
2. $\overline{u}$ solves the $i - \overline{J}_{Ps}-$holomorphic curve equation

$$\pi_\alpha d\overline{f}(s,t) \circ i = \overline{J}_{Ps}(\overline{f}(s,t)) \circ \pi_\alpha d\overline{f}(s,t),$$
$$\overline{f}^* \alpha \circ i = d\overline{a}.$$

3. The energy $E(\overline{u}; [-R, R] \times S^1)$ of $\overline{u}$ is bounded by the constant $\tilde{E}_0$.
4. The $d\alpha-$energy of $\overline{u}$ is smaller than $\hbar_0/2$.
5. For the constant $\delta_0 > 0$ from Assumption Q4 we have $\|d\overline{f}(z)\| < \tilde{C}_1$ for all $z \in ([-R, -R+\delta_0] \amalg [R-\delta_0, R]) \times S^1$, for some constant $\tilde{C}_1 \geqslant C_1$.

B.1.2 Center action

In the following we apply the results established in [14] to this new curve, and introduce the notion of the center action for the $\overline{J}_{Ps}-$holomorphic curve $(\overline{u}, R, P)$.
The next result is similar to Theorem 1.1 of [14].

Theorem 72. *For all ψ such that $0 < \psi < \hbar_0/2$, there exists $h_0 > 0$ such that for any $R > h_0$ and any $\overline{J}_{Ps}-$holomorphic curve $(\overline{u}, R, P)$ there exists a unique element $T \in \mathcal{P}$ such that $T \leqslant \tilde{E}_0$ and*

$$\left| \int_{S^1} \overline{u}(0)^* \alpha - T \right| < \frac{\psi}{2}.$$

To prove the theorem we need the following lemma.

Lemma 73. *For any $\delta > 0$ there exists a constant $C_1' > 0$ such that the gradients of all $\overline{J}_{Ps}-$holomorphic curves $(\overline{u}, R, P)$ with $R > \delta$, are uniformly bounded on $[-R+\delta, R-\delta] \times S^1$ by the constant C_1', i.e.*

$$\sup_{(s,t) \in [-R+\delta, R-\delta] \times S^1} \|d\overline{u}(s,t)\|_{g_{eucl.}, \overline{g}_{Ps}} \leqslant C_1'.$$

Proof. We prove this lemma by using bubbling-off analysis. Let us assume that the assertion is not true. Then we find $\delta_0 > 0$ such that for any $C_{1,n} = n$ there exist the $\overline{J}_{P_n s}-$holomorphic curves $(\overline{u}_n, R_n, P_n)$ with $R_n > \delta_0$ such that

$$\sup_{(s,t) \in [-R_n+\delta_0, R_n-\delta_0] \times S^1} \|d\overline{u}_n(s,t)\|_{g_{eucl.}, \overline{g}_{P_n s}} \geqslant C_{1,n} = n.$$

Consequently, there exists the points $(s_n, t_n) \in [-R_n + \delta_0, R_n + \delta_0] \times S^1$ for which

$$\|d\overline{u}_n(s_n, t_n)\|_{g_{eucl.}, \overline{g}_{P_n s_n}} = \sup_{(s,t) \in [-R_n+\delta_0, R_n-\delta_0] \times S^1} \|d\overline{u}_n(s,t)\|_{g_{eucl.}, \overline{g}_{P_n s}} \geqslant n.$$

Set $\mathcal{R}_n := \|d\overline{u}_n(s_n, t_n)\|_{g_{eucl.}, \overline{g}_{P_n s_n}}$ and note that $\mathcal{R}_n \to \infty$. Choose a sequence ϵ_n such that $\epsilon_n > 0$, $\epsilon_n \to 0$ and $\epsilon_n \mathcal{R}_n \to +\infty$. Now, apply Hofer's topological lemma [1] to the continous sequence of functions $\|d\overline{u}_n(s,t)\|_{g_{eucl.}, \overline{g}_{P_n s}}$ defined on $[-R_n, R_n] \times S^1$. For each (s_n, t_n) and ϵ_n, there exist $(s_n', t_n') \in [-R_n + \delta_0, R_n - \delta_0] \times S^1$ and $\epsilon_n' \in (0, \epsilon_n]$ with the properties:

1. $\epsilon_n' \left\| d\overline{u}_n(s_n', t_n') \right\|_{g_{\text{eucl.}}, \overline{g}_{P_n s_n'}} \geqslant \epsilon_n \left\| d\overline{u}_n(s_n, t_n) \right\|_{g_{\text{eucl.}}, \overline{g}_{P_n s_n}}$;
2. $|(s_n, t_n) - (s_n', t_n')|_{g_{\text{eucl.}}} \leqslant 2\epsilon_n$;
3. $\left\| d\overline{u}_n(s, t) \right\|_{g_{\text{eucl.}}, \overline{g}_{P_n s}} \leqslant 2 \left\| d\overline{u}_n(s_n', t_n') \right\|_{g_{\text{eucl.}}, \overline{g}_{P_n s_n'}}$ for all (s, t) such that $|(s, t) - (s_n', t_n')| \leqslant \epsilon_n'$.

Thus we have found the points (s_n', t_n') and a sequence ϵ_n' such that:

1. $\epsilon_n' > 0$, $\epsilon_n' \to 0$, $\mathcal{R}_n' := \left\| d\overline{u}_n(s_n', t_n') \right\|_{g_{\text{eucl.}}, \overline{g}_{P_n s_n'}} \to \infty$ and $\epsilon_n' \mathcal{R}_n' \to \infty$;
2. $\left\| d\overline{u}_n(s, t) \right\|_{g_{\text{eucl.}}, \overline{g}_{P_n s}} \leqslant 2\mathcal{R}_n'$ for all (s, t) such that $|(s, t) - (s_n', t_n')| \leqslant \epsilon_n'$.

We do now rescaling. Setting $z_n' = (s_n', t_n')$ and defining the maps

$$\tilde{u}_n(s, t) := \left(\overline{a}_n \left(z_n' + \frac{z}{\mathcal{R}_n'} \right) - \overline{a}_n(z_n'), \overline{f}_n \left(z_n' + \frac{z}{\mathcal{R}_n'} \right) \right) = (\tilde{a}(z), \tilde{f}(z))$$

for $z = (s, t) \in B_{\epsilon_n' \mathcal{R}_n'}(0)$, we obtain

$$d\tilde{u}_n(z) = \frac{1}{\mathcal{R}_n'} d\overline{u}_n \left(z_n' + \frac{z}{\mathcal{R}_n'} \right)$$

and

$$\left\| d\tilde{u}_n(z) \right\|_{g_{\text{eucl.}}, \overline{g}_{P_n\left(s_n' + \frac{s}{\mathcal{R}_n'}\right)}} = \frac{1}{\mathcal{R}_n'} \left\| d\overline{u}_n \left(z_n' + \frac{z}{\mathcal{R}_n'} \right) \right\|_{g_{\text{eucl.}}, \overline{g}_{P_n\left(s_n' + \frac{s}{\mathcal{R}_n'}\right)}}.$$

Thus, for all $z = (s, t) \in B_{\epsilon_n' \mathcal{R}_n'}(0)$ we have that

$$\left\| d\tilde{u}_n(z) \right\|_{g_{\text{eucl.}}, \overline{g}_{P_n\left(s_n' + \frac{s}{\mathcal{R}_n'}\right)}} \leqslant 2 \tag{B.1.8}$$

and $\left\| d\tilde{u}_n(0) \right\|_{g_{\text{eucl.}}, \overline{g}_{P_n s_n'}} = 1$, and moreover, that $\tilde{u} = (\tilde{a}, \tilde{f})$ solves

$$\pi_\alpha d\tilde{f}_n(z) \circ i = \overline{J}_{P_n\left(s_n' + \frac{s}{\mathcal{R}_n'}\right)}(\tilde{f}_n(z)) \circ \pi_\alpha d\tilde{f}_n(z),$$
$$\tilde{f}_n^* \alpha \circ i = d\tilde{a}_n.$$

As $P_n s_n'$ is bounded by C, we go over to some convergent subsequence, i.e., $P_n s_n' \to \rho$ as $n \to \infty$. From the uniform gradient bound (B.1.8) it follows that there exists a subsequence converging in C^∞_{loc} to some curve $\tilde{u} = (\tilde{a}, \tilde{f}) : \mathbb{C} \to \mathbb{R} \times M$ such that:

1. $\tilde{u}$ solves
$$\pi_\alpha d\tilde{f}(z) \circ i = \overline{J}_\rho(\tilde{f}(z)) \circ \pi_\alpha d\tilde{f}(z) \text{ and } \tilde{f}^* \alpha \circ i = d\tilde{a};$$
2. the gradient bounds go over in $\left\| d\tilde{u}(z) \right\|_{g_{\text{eucl.}}, \overline{g}_{P s'}} \leqslant 2$ and $\left\| d\tilde{u}(0) \right\|_{g_{\text{eucl.}}, \overline{g}_{P s'}} = 1$.

From the last two results, $\tilde{u}$ is a usual non-constant pseudoholomorphic plane with bounded energy by the constant $\tilde{E}_0$ (finite energy plane). As the $d\alpha-$energy is smaller than $\hbar_0$ we arrive at a contradiction (see [13]).

□

Proof. (of Theorem 72) We prove Theorem 72 by contradiction. Assume that we find $0 < \tilde{\psi} < \hbar_0/2$ such that for any constant $h_{0,n} = n$, there exist $R_n > h_{0,n} = n$ and the $\bar{J}_{Ps}$−holomorphic curves $(\bar{u}_n, R_n, P_n)$ satisfying

$$\left| \int_{S^1} \bar{u}_n(0)^* \alpha - T \right| \geqslant \frac{\tilde{\psi}}{2}$$

for any $T \in \mathcal{P}$ with $T \leqslant \tilde{E}_0$. By Lemma 73, we have for $\delta = 1$,

$$\sup_{(s,t)\in[-R_n+1, R_n-1]\times S^1} \|d\bar{u}_n(s,t)\|_{g_{\text{eucl.}}, \bar{g}_{P_n s}} \leqslant \tilde{C}_1.$$

As $|P_n R_n| \leqslant C$ and $R_n \to \infty$ it follows that $P_n \to 0$ as $n \to \infty$. Furthermore, by the boundedness of $P_n R_n$, the metrics $\bar{g}_{P_n s}$ are equivalent for all $s \in [-R_n, R_n]$ and all n (the almost complex structures $\bar{J}_\tau$ varies in a compact set). Hence there exists a constant $C_2 > 0$ such that

$$\frac{1}{C_2} \|\cdot\|_0 \leqslant \|\cdot\|_{P_n s} \leqslant C_2 \|\cdot\|_0$$

for all n and all $s \in [-R_n, R_n]$. By making the constant C_1' larger (eventually by multiplying it with C_2) we obtain

$$\sup_{(s,t)\in[-R_n+1, R_n-1]\times S^1} \|d\bar{u}_n(s,t)\|_{g_{\text{eucl.}}, \bar{g}_0} \leqslant C_1'.$$

Thus the maps $\bar{u}_n$ converge in C^∞_{loc} to some usual $\bar{J}_0$−holomorphic curve $\bar{u} = (\bar{a}, \bar{f}) : \mathbb{R} \times S^1 \to \mathbb{R} \times M$, for which we have:

1. $\bar{u}$ solves
$$\pi_\alpha d\bar{f}(z) \circ i = \bar{J}_0(\bar{f}(z)) \circ \pi_\alpha d\bar{f}(z) \text{ and } \bar{f}^* \alpha \circ i = d\bar{a};$$
2. $E(\bar{u}; \mathbb{R} \times S^1) \leqslant \tilde{E}_0$, $E_{d\alpha}(\bar{u}; \mathbb{R} \times S^1) \leqslant \hbar_0/2$ and
$$\left| \int_{S^1} \bar{u}(0)^* \alpha - T \right| \geqslant \frac{\tilde{\psi}}{2}$$
for all $T \in \mathcal{P}$ with $T \leqslant \tilde{E}_0$.

The rest of the proof proceeds as in the proof of Theorem 1.1 from [14]. For the sake of completeness we present this proof in detail. The map $\bar{u}$ can be regarded as a finite energy map defined on a 2−punctured Riemannian sphere. A puncture is removable or has a periodic orbit on the Reeb vector field as asymptotic limit. In both cases, the limits

$$\lim_{s\to\pm\infty} \int_{S^1} \bar{u}(s)^* \alpha \in \mathbb{R}$$

exist. The limit is equal to 0 if the puncture is removable, and equal to the period of the asymptotic limit if this is not the case. As a result and by means of Stoke's theorem, the $d\alpha$−energy of $\bar{u}$ can be written as

$$\int_{\mathbb{R}\times S^1} \bar{u}^* d\alpha = T_2 - T_1,$$

with $T_2 \geqslant T_1$, where $T_1, T_2 \in \mathcal{P}$ and $T_1, T_2 \leqslant \tilde{E}_0$. By the energy estimates, $E_{d\alpha}(\bar{u}; \cdot) \leqslant \hbar_0/2$, and from the definition of the constant $\hbar_0$ we conclude that that $T_1 = T_2$. Set $T := T_1 = T_2$. If $T = 0$, both punctures are removable, $\bar{u}$ has an extension to a $\bar{J}_0-$holomorphic finite energy sphere $S^2 \to \mathbb{R} \times M$, and so, the map $\bar{u}$ must be constant; hence

$$\int_{S^1} \bar{u}(0)^*\alpha = 0 = T,$$

which contradicts the assumption on the center action. If $T > 0$, the finite energy cylinder $\bar{u}$ is non-constant, has a vanishing $d\alpha-$energy., and so, $\bar{u}$ must be a cylinder over a periodic orbit $x(t)$ of the form $\bar{u}(s,t) = (Ts+c, x(Tt+d))$ for some constants c and d, and with a period $T \leqslant \tilde{E}_0$; hence

$$\int_{S^1} u(0)^*\alpha - T = 0 < \frac{\tilde{\psi}}{2},$$

which again contradicts the assumption on the center action. Thus, there exists a constant $h_0 > 0$ such that for any $\bar{J}_{P_S}-$holomorphic curve $(\bar{u}, R, P)$ with $R > h_0$ satisfying the energy estimates, the center loop $\bar{u}(0,\cdot)$ has an action close to an element $T \in \mathcal{P}$ with $T \leqslant \tilde{E}_0$, i.e.

$$\left| \int_{S^1} \bar{u}(0)^*\alpha - T \right| < \frac{\psi}{2}. \tag{B.1.9}$$

To deal with the uniqueness issue, we consider two elements $T_1, T_2 \in \mathcal{P}$ with $T_1, T_2 \leqslant \tilde{E}_0$ satisfying the above estimate. Then we have

$$|T_1 - T_2| < \frac{\psi}{2} + \frac{\psi}{2} = \psi.$$

By assumption, $\psi < \hbar_0/2$, and from the definition of $\hbar_0$ it follows that $T_1 = T_2$. Therefore the element $T \in \mathcal{P}$ satisfying $T \leqslant \tilde{E}_0$ and the estimate (B.1.9) is unique.

□

Definition 74. The unique element $T \in \mathcal{P}_\alpha$ associated with the $\bar{J}_{P_S}-$holomorphic curve $(\bar{u}, R, P)$ satisfying the assumptions of Theorem 72 is called the *center action* of $\bar{u}$ and is denoted by

$$A(\bar{u}) = T.$$

If the curve $\bar{u} = (\bar{a}, \bar{f}) : [-R, R] \times S^1 \to \mathbb{R} \times M$ fulfills the assumptions of Theorem 72, the actions of all loops are estimated by

$$\left| \int_{S^1} \bar{u}(s)^*\alpha - T \right| < \frac{\psi}{2} + \frac{\hbar_0}{2} < \hbar_0$$

for all $s \in [-R, R]$.

Remark 75. From the definition of the constant $\hbar_0$, the center action $A(\bar{u})$ of a curve $\bar{u}$ fulfilling the assumptions of Theorem 72 satisfies $A(\bar{u}) = 0$ or $A(\bar{u}) \geqslant \hbar_0$.

Before going any further we make a remark about the metrics involved.

Remark 76. For any ρ, the norms induced by the parameter-dependent metrics $\bar{g}_\rho$ on $\mathbb{R} \times M$ that are defined by (B.1.7) are equivalent, i.e. there exists a positive constant $\bar{C}_1 > 0$ such that

$$\frac{1}{\bar{C}_1} \|\cdot\|_{\bar{g}_\rho} \leqslant \|\cdot\|_{\bar{g}_0} \leqslant \bar{C}_1 \|\cdot\|_{\bar{g}_\rho}. \tag{B.1.10}$$

This follows from the fact that the parameter-dependent almost complex structure $\bar{J}_\rho$ varies in a compact set.

B.2 Vanishing center action

In view of Remark 75 and Theorem 72 we consider the case in which there exists a subsequnce of $\overline{u}_n$ with vanishing center action. We use a version of the monotonicity lemma (Corollary 118) to characterize the behavior of a $\bar{J}_{P_S}$−holomorphic curve $(\overline{u}, P, R)$ (Theorem 78). Using these results we describe the convergence of a sequence of $\bar{J}_{P_S}$−holomorphic cylinders (Theorem 80) and then prove Theorem 63.

Lemma 77. *Choose $0 < \psi < \hbar_0/2$, and let $h_0 > 0$ be the corresponding constant from Theorem 72. For all $\delta > 0$ there exists $h \geqslant h_0$ such that for any $R > h$ and any $\bar{J}_{P_S}$−holomorphic curve $(\overline{u}, R, P)$ fulfilling the assumptions of Theorem 72 and having vanishing center action, the loops $\overline{u}(s)$ satisfy*

$$\mathit{diam}_{\overline{g}_0}(\overline{u}(s)) \leqslant \delta \quad \mathit{and} \quad |\alpha(\partial_t\overline{u}(s))| \leqslant \delta \tag{B.2.1}$$

for all $s \in [-R+h, R-h]$.

Proof. The proof is similar to that given in [14]. Nevertheless, for the sake of completeness it is sketched here. We consider (B.2.1). Arguing indirectly we find a constant $\delta_0 > 0$, a sequence $R_n \geqslant h_n := n + h_0$, and a sequence of $\bar{J}_{P_S}$−holomorphic curves $(\overline{u}_n, R_n, P_n)$ such that

$$\begin{aligned} E(\overline{u}_n; [-R_n, R_n] \times S^1) &\leqslant \tilde{E}_0, \\ E_{d\alpha}(\overline{u}_n; [-R_n, R_n] \times S^1) &\leqslant \frac{\hbar_0}{2}, \\ \left| \int_{S^1} \overline{u}_n(0)^*\alpha \right| &\leqslant \frac{\psi}{2}, \\ \mathrm{diam}_{\overline{g}_0}(\overline{u}_n(s_n)) &\geqslant \delta_0 \end{aligned}$$

for a sequence $s_n \in [-R_n + n + h_0, R_n - n - h_0]$. By Stoke's theorem, we have

$$\left| \int_{S^1} \overline{u}_n(s_n)^*\alpha \right| < \hbar_0.$$

Define now the maps $\tilde{u}_n = (\tilde{a}_n, \tilde{f}_n) : [-R_n - s_n, R_n + s_n] \times S^1 \to \mathbb{R} \times M$ by

$$\tilde{u}_n(s,t) := (\overline{a}_n(s+s_n,t), \overline{f}_n(s+s_n,t)),$$

for which, the above assumptions go over in

$$\begin{aligned} E(\tilde{u}_n; [-R_n, R_n] \times S^1) &\leqslant \tilde{E}_0, \\ E_{d\alpha}(\tilde{u}_n; [-R_n, R_n] \times S^1) &\leqslant \frac{\hbar_0}{2}, \\ \left| \int_{S^1} \tilde{u}_n(0)^*\alpha \right| &< \hbar_0, \\ \mathrm{diam}_{\overline{g}_0}(\tilde{u}_n(0)) &\geqslant \delta_0. \end{aligned}$$

As $s_n \in [-R_n + n + h_0, R_n - n - h_0]$, we see that $|R_n + s_n| \to \infty$ and $|R_n - s_n| \to \infty$ as $n \to \infty$. Moreover, $\tilde{u}_n$ satisfies the pseudoholomorphic curve equation

$$\begin{aligned} \pi_\alpha d\tilde{f}_n(s,t) \circ i &= \bar{J}_{-P_n(s+s_n)}(\tilde{f}_n(s,t)) \circ \pi_\alpha d\tilde{f}_n(s,t), \\ \tilde{f}_n^*\alpha \circ i &= d\tilde{a}_n. \end{aligned}$$

For the new sequence

$$\tilde{v}_n(s,t) = (\tilde{b}_n(s,t), \tilde{v}_n(s,t)) = (\tilde{a}_n(s,t) - \tilde{a}_n(0,0), \tilde{f}_n(s,t)),$$

the $\mathbb{R}$–invariance of $\bar{J}_\tau$ and of $\bar{g}_0$, yields

$$\begin{aligned} E(\tilde{v}_n; [-R_n - s_n, R_n - s_n] \times S^1) &\leqslant \tilde{E}_0, \\ E_{d\alpha}(\tilde{v}_n; [-R_n - s_n, R_n - s_n] \times S^1) &\leqslant \frac{\hbar_0}{2}, \\ \left| \int_{S^1} \tilde{v}_n(0)^* \alpha \right| &< \hbar_0, \\ \mathrm{diam}_{\bar{g}_0}(\tilde{v}_n(0)) &\geqslant \delta_0 \end{aligned}$$

and

$$\begin{aligned} \pi_\alpha d\tilde{v}_n(s,t) \circ i &= \bar{J}_{-P_n(s+s_n)}(\tilde{v}_n(s,t)) \circ \pi_\alpha d\tilde{v}_n(s,t), \\ \tilde{v}_n^* \alpha \circ i &= d\tilde{b}_n. \end{aligned}$$

By the same bubbling-off argument as in the proof of Theorem 72, a subsequence of $\tilde{v}_n$ converges in C^∞_{loc} to a usual $\bar{J}_\tau$–holomorphic cylinder $\tilde{v} = (b, v) : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ for some fixed $\tau \in [-C, C]$ (after going to a subsequence, this the limit of $P_n s_n$) characterized by

$$\begin{aligned} E_\alpha(\tilde{v}; \mathbb{R} \times S^1) + E_{d\alpha}(\tilde{v}; \mathbb{R} \times S^1) &\leqslant \tilde{E}_0, \\ E_{d\alpha}(\tilde{v}; \mathbb{R} \times S^1) &= 0, \\ \left| \int_{S^1} \tilde{v}(0)^* \alpha \right| &< \hbar_0, \\ \mathrm{diam}_{\bar{g}_0}(\tilde{v}(0)) &\geqslant \delta_0. \end{aligned}$$

In particular, $\tilde{v}$ is a non-constant finite energy cylinder having a vanishing $d\alpha$–energy. Hence $\tilde{v}$ is a cylinder over a periodic orbit of period $0 < T \leqslant \tilde{E}_0$.Consequently, we obtain

$$\int_{S^1} v(0)^* \alpha = T \geqslant \hbar_0,$$

meaning that v is constant. This contradicts our assumptions, and therefore, $\mathrm{diam}_{\bar{g}_0}(\bar{u}(s)) \leqslant \delta$ for all $s \in [-R + h, R - h]$. For $|\alpha(\partial_t \bar{u}(s))| \leqslant \delta$ we proceed analogously, and the proof is finished.

□

The next theorem characterizes the behavior of a $\bar{J}_{Ps}$–holomorphic curve $(\bar{u}, R, P)$ with vanishing center action.

Theorem 78. *Let ψ be as in Theorem 72 and let $h_0 > 0$ be the constant from Theorem 72. For any $\epsilon > 0$ there exists $h_1 \geqslant h_0$ such that for any $R > h_1$ and any $\bar{J}_{Ps}$–holomorphic curve $(\bar{u}, R, P)$ satisfying $A(\bar{u}) = 0$ we have $\bar{u}([-R + h_1, R - h_1] \times S^1) \subset B^{\bar{g}_0}_\epsilon(\bar{u}(0,0))$.*

Proof. In the first part of the proof we employ exactly the same arguments as in the proof of Theorem 1.2 from [14]. With $\epsilon > 0$ as in the statement of the theorem, we choose $\delta > 0$ and $0 < r \leqslant \epsilon$ sufficiently small such that

$$6\delta < C_8 r^2 \text{ and } 4\delta + r \leqslant \frac{\epsilon}{2}. \tag{B.2.2}$$

For the $\bar{J}_{Ps}$–holomorphic curve $(\bar{u}, R, P)$ with $R > h$ and h as in the Lemma 77, and satisfying the assumptions of Theorem 78, we have $\mathrm{diam}_{\bar{g}_0}(\bar{u}(s)) \leqslant \delta$ and $|\alpha(\partial_t \bar{f}(s))| \leqslant \delta$ for all $s \in [-R + h, R - h]$. The definition of the energy and Stoke's theorem give

$$E(\overline{u}|_{[-R+h,R-h]\times S^1};[-R+h,R-h]\times S^1)\leqslant 6\delta. \tag{B.2.3}$$

If the conclusion of Theorem 78 is not true for this h, we find a point $(s_0,t_0)\in[-R+h,R-h]\times S^1$ for which

$$\mathrm{dist}_{\overline{g}_0}(\overline{u}(s_0,t_0),\overline{u}(0,0))\geqslant\epsilon.$$

From $\mathrm{diam}_{\overline{g}_0}(\overline{u}(s))\leqslant\delta$ we obtain

$$\mathrm{dist}_{\overline{g}_0}(\overline{u}(s_0,t),\overline{u}(0,t'))\geqslant\epsilon-2\delta$$

for all $t,t'\in S^1$. Choosing a point s_1 between 0 and s_0 such that

$$\mathrm{dist}_{\overline{g}_0}(\overline{u}(s_1,t),\overline{u}(s_0,t'))\geqslant\frac{\epsilon}{2}-4\delta \ \text{ and } \ \mathrm{dist}_{\overline{g}_0}(\overline{u}(s_1,t),\overline{u}(0,t'))\geqslant\frac{\epsilon}{2}-4\delta$$

for all $t,t'\in S^1$, using $r\leqslant\epsilon/2-4\delta$, and applying the monotonicity Lemma 118 to the open ball $B^{\overline{g}_0}_r(\overline{u}(s_1,t_1))$, we conclude that $E(\overline{u}|_{[-R+h,R-h]\times S^1};[-R+h,R-h]\times S^1)\geqslant C_8r^2$. In view of (B.2.3), this implies that $C_8r^2\leqslant 2\delta$, which is in contradiction to the choice in (B.2.2). Hence $\overline{u}(s,t)\in B^{\overline{g}_0}_\epsilon(\overline{u}(0,0))$ for all $(s,t)\in[-R+h,R-h]\times S^1$ as claimed by Theorem 78.

□

B.2.1 Proof of Theorem 63

We are now well prepared to describe the convergence and the limit object of the $\mathcal{H}-$holomorphic cylinders u_n with harmonic perturbations γ_n. Consider a sequence of $\mathcal{H}-$holomorphic cylinders $u_n=(a_n,f_n):[-R_n,R_n]\times S^1\to\mathbb{R}\times M$ with harmonic perturbation 1−forms γ_n satisfying Assumptions P1-P5. As in Section B.1 we transform the map u_n into a $\overline{J}_{Ps}-$holomorphic curve $\overline{u}_n$ with respect to the domain-dependent almost complex structure $\overline{J}_\rho$. We consider the new sequence of maps $\overline{f}_n$ defined by $\overline{f}_n(s,t):=\phi^\alpha_{P_ns}(f_n(s,t))$ for all $n\in\mathbb{N}$. Thus $\overline{u}_n=(\overline{a}_n,\overline{f}_n):[-R_n,R_n]\times S^1\to\mathbb{R}\times M$ is a $\overline{J}_{P_ns}-$holomorphic curve. Due to Remark 70 the triple (u_n,R_n,P_n) is a $\overline{J}_{P_ns}-$holomorphc curve as in Definition 71. After shifting $\overline{u}_n$ by $-a_n(0,0)$ in the $\mathbb{R}-$coordinate, we assume by Proposition 102 that $\overline{a}_n(0,0)$ is bounded. Hence, after going over to a subsequence, we assume that $\overline{u}_n(0,0)\to w=(w_a,w_f)\in\mathbb{R}\times M$ as $n\to\infty$.
By Theorem 78 applied to the sequence of $\overline{J}_{P_ns}-$holomorphic curves $(\overline{u}_n,R_n,P_n)$ we have the following

Corollary 79. *For every sequence $h_n\in\mathbb{R}_+$ satisfying $h_n<R_n$ and $h_n,R_n/h_n\to\infty$ and every $\epsilon>0$ there exists $N\in\mathbb{N}$ such that*

$$\overline{u}_n([-R_n+h_n,R_n-h_n]\times S^1)\subset B^{\overline{g}_0}_\epsilon(w)$$

for all $n\geqslant N$. Moreover, for the period P_n and co-period S_n we have that $h_nP_n,h_nS_n\to 0$ as $n\to\infty$.

Proof. Consider a sequence $h_n\in\mathbb{R}_+$ such that $h_n<R_n$ and $h_n,R_n/h_n\to\infty$ as $n\to\infty$ and let $\epsilon>0$ be given. From Theorem 78 there exists $h_\epsilon>0$ and $N_\epsilon\in\mathbb{N}$ such that for all $n\geqslant N_\epsilon$, we have $R_n>h_\epsilon$ and $\overline{u}_n([-R_n+h_\epsilon,R_n-h_\epsilon]\times S^1)\subset B^{\overline{g}_0}_\epsilon(w)$. By making N_ϵ sufficiently large and accounting of $h_n\to\infty$, we may assume that for all $n\geqslant N_\epsilon$, we have that $R_n>h_n>h_\epsilon$, which in turns, gives $\overline{u}_n([-R_n+h_n,R_n-h_n]\times S^1)\subset B^{\overline{g}_0}_\epsilon(w)$. The second statement follows from the fact that $R_nP_n\to\tau$, $R_nS_n\to\sigma$ and $h_nR_n\to\infty$ as $n\to\infty$.

□

To describe the C^0-convergence of the maps u_n we define a sequence of diffeomorphisms, which is similar to that constructed in Section 4.4 of [7]. For a sequence $h_n\in\mathbb{R}_+$ with $h_n<R_n$ and $h_n,R_n/h_n\to\infty$ as $n\to\infty$, let

$\theta_n : [-R_n, R_n] \to [-1, 1]$ be a sequence of diffeomorphisms defined as in Remark 44. We define the maps

$$\begin{aligned}
\bar{v}_n(s,t) &= \bar{u}_n(\theta_n^{-1}(s),t), \ s \in [-1,1], \\
\bar{v}_n^-(s,t) &= \bar{u}_n^-((\theta_n^-)^{-1}(s),t), \ s \in [-1,1/2], \\
\bar{v}_n^+(s,t) &= \bar{u}_n^+((\theta_n^+)^{-1}(s),t), \ s \in [1/2,1], \\
\bar{v}^-(s,t) &= \bar{u}^-((\theta^-)^{-1}(s),t), \ s \in [-1,-1/2), \\
\bar{v}^+(s,t) &= \bar{u}^+((\theta^+)^{-1}(s),t), \ s \in (1/2,1],
\end{aligned} \tag{B.2.4}$$

and

$$\begin{aligned}
v_n(s,t) &= u_n(\theta_n^{-1}(s),t), \ s \in [-1,1], \\
v_n^-(s,t) &= u_n^-((\theta_n^-)^{-1}(s),t), \ s \in [-1,-1/2], \\
v_n^+(s,t) &= u_n^+((\theta_n^+)^{-1}(s),t), \ s \in [1/2,1], \\
v^-(s,t) &= u^-((\theta^-)^{-1}(s),t), \ s \in [-1,-1/2), \\
v^+(s,t) &= u^+((\theta^+)^{-1}(s),t), \ s \in (1/2,1],
\end{aligned} \tag{B.2.5}$$

where, $\bar{u}_n^\pm(s,t) = \bar{u}_n(s \pm R_n, t)$ and $u_n^\pm(s,t) = u_n(s \pm R_n, t)$ are the left and right shifts of the maps $\bar{u}_n$ and u_n, respectively.
The next theorem states a $C^\infty_{\text{loc}}-$ and a C^0-convergence result for the maps $\bar{u}_n$.

Theorem 80. *There exist a subsequence of the sequence of $\bar{J}_{P_n s}-$holomorphic curves $(\bar{u}_n, R_n, P_n)$, also denoted by $(\bar{u}_n, R_n, P_n)$, and pseudoholomorphic half cylinders $\bar{u}^\pm$ defined on $(-\infty, 0] \times S^1$ and $[0,\infty) \times S^1$, respectively such that for every sequence $h_n \in \mathbb{R}_+$ and every sequence of diffeomorphisms $\theta_n : [-R_n, R_n] \times S^1 \to [-1,1] \times S^1$ satisfying the assumptions of Remark 44, the following convergence results hold:*

$C^\infty_{loc}-$convergence:

1. *For any sequence $s_n \in [-R_n + h_n, R_n - h_n]$ there exists a constant $\tau_{\{s_n\}} \in [-\tau, \tau]$ (depending on the sequence $\{s_n\}$) such that after passing to a subsequence, the shifted maps $\bar{u}_n(s + s_n, t)$, defined on $[-R_n + h_n - s_n, R_n - h_n - s_n] \times S^1$, converge in C^∞_{loc} to w.*
2. *The left shifts $\bar{u}_n^-(s,t) := \bar{u}_n(s - R_n, t)$, defined on $[0, h_n) \times S^1$, possess a subsequence that converges in C^∞_{loc} to a pseudoholomorphic half cylinder $\bar{u}^- = (\bar{a}^-, \bar{f}^-)$, defined on $[0, +\infty) \times S^1$. The curve $\bar{u}^-$ is asymptotic to $w = (w_a, w_f)$. The maps $\bar{v}_n^- : [-1, -1/2] \times S^1 \to \mathbb{R} \times M$ converge in C^∞_{loc} to $\bar{v}^- : [-1, -1/2) \times S^1 \to \mathbb{R} \times M$ such that $\bar{v}^-$ is asymptotic to w as $s \to -1/2$.*
3. *The right shifts $\bar{u}_n^+(s,t) := \bar{u}_n(s + R_n, t)$, defined on $(-h_n, 0] \times S^1$, possess a subsequence that converges in C^∞_{loc} to a pseudoholomorphic half cylinder $\bar{u}^+ = (\bar{a}^+, \bar{f}^+)$, defined on $(-\infty, 0] \times S^1$. The curve $\bar{u}^+$ is asymptotic to $w = (w_a, w_f)$. The maps $\bar{v}_n^+ : [1/2, 1] \times S^1 \to \mathbb{R} \times M$ converge in C^∞_{loc} to $\bar{v} : (1/2, 1] \times S^1 \to \mathbb{R} \times M$ such that $\bar{v}^+$ is asymptotic to w as $s \to 1/2$.*

C^0-convergence:

1. *The maps $\bar{v}_n : [-1/2, 1/2] \times S^1 \to \mathbb{R} \times M$ converge in C^0 to w.*
2. *The maps $\bar{v}_n^- : [-1, -1/2] \times S^1 \to \mathbb{R} \times M$ converge in C^0 to a map $\bar{v}^- : [-1, -1/2] \times S^1 \to \mathbb{R} \times M$ such that $v^-(-1/2, t) = w$.*
3. *The maps $\bar{v}_n^+ : [1/2, 1] \times S^1 \to \mathbb{R} \times M$ converge in C^0 to a map $\bar{v}^+ : [1/2, 1] \times S^1 \to \mathbb{R} \times M$ such that $v(1/2, t) = w$.*

Proof. We prove only the first and second statements of the C^∞_{loc}- and C^0- convergences because the proofs of the third statements are exactly the same with those of the second statements. For the sequence $h_n \in \mathbb{R}_+$ with the property $h_n, R_n/h_n \to \infty$ as $n \to \infty$, consider the sequence of diffeomorphisms $\theta_n : [-R_n, R_n] \to [-1, 1]$ fulfilling the assumptions of Remark 44. For any sequence $s_n \in [-R_n + h_n, R_n - h_n]$, the shifted maps $\overline{u}_n(\cdot + s_n, \cdot)$, defined on $[-R_n + h_n - s_n, R_n - h_n - s_n] \times S^1$, converge, due to Corollary 79 and Lemma 73, in C^∞_{loc} to w. To prove the second statement of the C^∞_{loc}–convergence we consider the shifted maps $\overline{u}^-_n : [0, h_n] \times S^1 \to \mathbb{R} \times M$, defined by $\overline{u}^-_n(s, t) = \overline{u}_n(s - R_n, t)$. By Lemma 73, these maps have bounded gradients, and hence, after going over to some subsequence, they converge in $C^\infty_{loc}([0, \infty) \times S^1)$ to a usual pseudoholomorphic curve $\overline{u}^- : [0, +\infty) \times S^1 \to \mathbb{R} \times M$ with respect to the standard complex structure i on $[0, +\infty) \times S^1$ and the almost complex structure $J_{-\tau}$ on the domain; here, τ is the limit of $P_n R_n$ as $n \to \infty$. Let us show that $\overline{u}^-$ is asymptotic to $w \in \mathbb{R} \times M$, i.e. let us show that $\lim_{r\to\infty} \overline{u}^-(r, t) = w$. We prove by contradiction. Assume that there exists a sequence $(s_k, t_k) \in [0, \infty) \times S^1$ with $s_k \to \infty$ as $k \to \infty$ such that $\lim_{k\to\infty} \overline{u}^-(s_k, t_k) = w' \in \mathbb{R} \times M$ with $w' \neq w$. Let $\epsilon := \text{dist}_{\overline{g}_0}(w, w') > 0$. For any $k \in \mathbb{N}$ there exists $N_k \in \mathbb{N}$ such that for any $n \geqslant N_k$, $(s_k, t_k) \in [0, h_n]$. Thus for arbitrary k and n such that $n \geqslant N_k$ we have

$$\begin{aligned}\text{dist}_{\overline{g}_0}(w, w') \leqslant\; &\text{dist}_{\overline{g}_0}(w, \overline{u}^-_n(s_k, t_k)) + \text{dist}_{\overline{g}_0}(\overline{u}^-_n(s_k, t_k), \overline{u}^-(s_k, t_k)) \\ &+ \text{dist}_{\overline{g}_0}(\overline{u}^-(s_k, t_k), w').\end{aligned}$$

By Theorem 78, there exists $h > 0$ such that $\text{dist}_{\overline{g}_0}(\overline{u}^-_n(s, t), w) < \epsilon/10$ for all $(s, t) \in [h, h_n] \times S^1$. Choose now k and $n \geqslant N_k$ sufficiently large such that $(s_k, t_k) \in [h, h_n] \times S^1$. Hence, $\text{dist}_{\overline{g}_0}(\overline{u}^-_n(s_k, t_k), w) < \epsilon/10$. Making k and $n \geqslant N_k$ larger we may also assume that $\text{dist}_{\overline{g}_0}(\overline{u}^-(s_k, t_k), w') < \epsilon/10$. After fixing k and making $n \geqslant N_k$ sufficiently large we get $\text{dist}_{\overline{g}_0}(\overline{u}^-_n(s_k, t_k), \overline{u}^-(s_k, t_k)) < \epsilon/10$. As a result, we find $\text{dist}_{\overline{g}_0}(w, w') \leqslant 3\epsilon/10$, which is a contradiction to $\text{dist}_{\overline{g}_0}(w, w') = \epsilon$. The maps $\overline{v}^-_n(s, t) = \overline{u}^-_n((\theta^-_n)^{-1}(s), t)$ converge in C^∞_{loc} to the map $\overline{v}^-(s, t) = \overline{u}^-((\theta^-)^{-1}(s), t)$. This follows from the fact that $(\theta^-_n)^{-1} : [-1, -1/2] \to [0, h_n]$ converge in C^∞_{loc} to the diffeomorphism $(\theta^-)^{-1} : [-1, -1/2) \to [0, +\infty)$. By the asymptotics of $\overline{u}^-$, $\overline{v}^-$ can be continously extended to the whole interval $[-1, -1/2]$ by setting $\overline{v}^-(-1/2, t) = w$. This finishes the proof of the second statement, and so, of the C^∞_{loc}–convergence.
We consider now the first statement of the C^0–convergence. From Corollary 79 it follows that $\text{dist}_{\overline{g}_0}(\overline{v}_n(s, t), w) \to 0$ as $n \to \infty$ for all $(s, t) \in [-1/2, 1/2] \times S^1$, and the proof of the first statement is complete. The proof of the second statement of the C^0–convergence is exactly the same as the proof of Lemma 4.16 in [7] and is omitted here.

□

We are now in the position to prove Theorem 63.

Proof. (of Theorem 63) As before, we focus only on the proofs of the first and second statements of the C^∞_{loc}– and C^0–convergences, because the proofs of the third statements are similar to those of the second statements. For the sequence $h_n \in \mathbb{R}_+$ with the property $h_n, R_n/h_n \to \infty$ as $n \to \infty$, consider the sequence of diffeomorphisms $\theta_n : [-R_n, R_n] \to [-1, 1]$ fulfilling the assumptions of Remark 44. By the construction described in Section B.1, we have

$$\overline{f}_n(s, t) = \phi^\alpha_{P_n s}(f_n(s, t)) \text{ and } d\overline{a}_n = d\Gamma_n + da_n,$$

where $(s, t) \in [-R_n + h_n, R_n - h_n] \times S^1$ and $\Gamma_n : [-R_n, R_n] \times S^1 \to \mathbb{R}$ is a sequence of harmonic functions such that $d\Gamma_n$ has a uniformly bounded L^2–norm. Then we obtain

$$f_n(s, t) = \phi^\alpha_{-P_n s}(\overline{f}_n(s, t)) \text{ and } a_n(s, t) = \overline{a}_n(s, t) - \Gamma_n(s, t). \tag{B.2.6}$$

For the sequence of harmonic functions $\Gamma_n(s, t)$, the L^2–norms of $d\Gamma_n$ are uniformly bounded, while by Remark 68, the functions Γ_n can be chosen to have vanishing average. By Theorems 78 and 105, $\overline{u}_n(0, \cdot), u_n(0, \cdot) \to w = (w_a, w_f) \in \mathbb{R} \times M$ as $n \to \infty$. Hence $\overline{a}_n(0, \cdot), a_n(0, \cdot) \to w_a$. Recall that $P_n R_n \to \tau \in \mathbb{R}_+ \cup \{0\}$. By Theorems 80 and 105, for any sequence $s_n \in [-R_n+h_n, R_n-h_n]$ there exists a subsequence of shifted maps $u_n(\cdot+s_n,\cdot)+S_ns_n$, defined on $[-R_n+h_n-s_n, R_n-h_n-s_n]\times S^1$, that converges in C^∞_{loc} to the constant $(w_a, \phi^\alpha_{-\tau_{\{s_n\}}}(w_f))$, where $\tau_{\{s_n\}}$ is the limit point of P_ns_n. The shifted harmonic 1−form defined on $[-R_n+h_n-s_n, R_n-h_n-s_n]\times S^1$ takes the form $\gamma_n(s+s_n,t) = d\Gamma_n(s+s_n,t)+P_ndt$. Thus by Theorem 105, we have $\gamma_n(s+s_n,t)\to 0$ in C^∞_{loc} as $n\to\infty$, and this finishes the proof of the first statement. To prove the second statement of the $C^\infty_{loc}-$convergence we transfer the convergence results for the shifted maps $\overline{u}_n^- : [0,h_n]\times S^1\to\mathbb{R}\times M$, $\overline{u}_n^-(s,t) = \overline{u}_n(s-R_n,t)$ of Theorem 80 to the maps u_n^-, and use the convergence results of the harmonic functions established in Theorem 105 of Appendix E. The shifted maps $u_n^- = (a_n^-, f_n^-) : [0,h_n]\times S^1\to\mathbb{R}\times M$, defined by $u_n^-(s,t) = u_n(s-R_n,t)$, together with the maps $\overline{u}_n^-$ and the harmonic functions Γ_n^- satisfy

$$\overline{f}_n^-(s,t) = \phi^\alpha_{P_n(s-R_n)}(f_n^-(s,t)) \text{ and } \overline{a}_n^-(s,t) = a_n^-(s,t)+\Gamma_n^-(s,t), \tag{B.2.7}$$

where $\Gamma_n^- : [0,h_n]\times S^1\to\mathbb{R}$ is the left shifted harmonic function, defined by $\Gamma_n^-(s,t) = \Gamma_n(s-R_n,t)$. Hence we obtain

$$f_n^-(s,t) = \phi^\alpha_{-P_n(s-R_n)}(\overline{f}_n^-(s,t)) \text{ and } a_n^-(s,t) = \overline{a}_n^-(s,t)-\Gamma_n^-(s,t). \tag{B.2.8}$$

Thus, by Theorems 80 and 105, $u_n^- - S_nR_n$ converge in C^∞_{loc} to a curve $u^-(s,t) = (a^-(s,t), f^-(s,t)) = (\overline{a}^-(s,t) - \tilde{\Gamma}^-(s,t), \phi^\alpha_\tau(\overline{f}^-(s,t)))$, defined on $[0,\infty)\times S^1$. The map u^- is asymptotic to $(w_a, \phi^\alpha_\tau(w_f))$, and can be regarded as a $\mathcal{H}-$holomorphic map with harmonic perturbation $d\Gamma^-$. This finishes the proof of the second statement. For the third statement, we proceed analogously; the only difference is that the asymptotic of the map u^+ is $(w_a, \phi^\alpha_{-\tau}(w_f))$. To prove the first statement of the C^0-convergence, we consider the maps v_n and recall that

$$\overline{f}_n(s,t) = \phi^\alpha_{P_ns}(f_n(s,t)), \quad \overline{a}_n(s,t) = a_n(s,t)+\Gamma_n(s,t),$$

and

$$v_n(s,t) = \left(\overline{a}_n((\theta_n^{-1})(s),t) - \Gamma_n((\theta_n^{-1})(s),t), \phi^\alpha_{-P_n(\theta_n^{-1})(s)}(\overline{f}_n((\theta_n^{-1})(s),t))\right)$$

for $s\in[-1/2,1/2]$. If $S_nR_n\to\sigma$ as $n\to\infty$ we have, using Theorem 106, that

$$|\overline{a}_n((\theta_n)^{-1}(s),t) - \Gamma_n((\theta_n)^{-1}(s),t) - w_a + 2\sigma s| \to 0$$

for all $s\in[-1/2,1/2]$ as $n\to\infty$. Moreover, there exists a constant $c>0$ such that for all $(s,t)\in[-1/2,1/2]$, there holds

$$\text{dist}_{\overline{g}_0}(\overline{f}_n((\theta_n)^{-1}(s),t), w_f) \geqslant c\,\text{dist}_{\overline{g}_0}(f_n((\theta_n)^{-1}(s),t), \phi^\alpha_{-P_n(\theta_n)^{-1}(s)}(w_f)).$$

Noting that

$$P_n(\theta_n)^{-1}(s) = 2(P_nR_n - P_nh_n)s \tag{B.2.9}$$

for $s\in[-1/2,1/2]$, and that $P_nR_n\to\tau$ and $P_nh_n\to 0$ as $n\to\infty$, it follows that $P_n(\theta_n)^{-1}(s)\to 2\tau s$ in $C^0([-1/2,1/2])$. Hence, for $(s,t)\in[-1/2,1/2]\times S^1$ we have

$$c^{-1}\text{dist}_{\overline{g}_0}(\overline{f}_n((\theta_n)^{-1}(s),t), w_f) + \text{dist}_{\overline{g}_0}(\phi^\alpha_{-P_n(\theta_n)^{-1}(s)}(w_f), \phi^\alpha_{-2\tau s}(w_f)) \geqslant \text{dist}_{\overline{g}_0}(f_n((\theta_n)^{-1}(s),t), \phi^\alpha_{-2\tau s}(w_f)),$$

and $\text{dist}_{\overline{g}_0}(\phi^\alpha_{-P_n(\theta_n)^{-1}(s)}(w_f), \phi^\alpha_{-2\tau s}(w_f))$, $\text{dist}_{\overline{g}_0}(\overline{f}_n((\theta_n)^{-1}(s),t), w_f)\to 0$ as $n\to\infty$. Thus v_n converge in $C^0([-1/2,1/2])$ to $(w_a-2\sigma s, \phi^\alpha_{-2\tau s}(w_f))$ which is a segment of a Reeb trajectory. The proof of the first statement is complete. To prove the second statement we consider the maps v_n^-, for which we have

$$v_n^-(s,t) = (\overline{a}_n^-((\theta_n^-)^{-1}(s),t) - \Gamma_n^-((\theta_n^-)^{-1}(s),t), \phi^\alpha_{-P_n(\theta_n^-)^{-1}(s)+P_nR_n}(\overline{f}_n^-((\theta_n^-)^{-1}(s),t))).$$

If $S_nR_n \to \sigma$ as $n \to +\infty$, Theorem 106 shows that $\overline{a}_n^-((\theta_n^-)^{-1}(s),t) - \Gamma_n^-((\theta_n^-)^{-1}(s),t) - S_nR_n$ converge in C^0 to a function $\overline{a}^-((\theta^-)^{-1}(s),t) - \Gamma^-((\theta^-)^{-1}(s),t)$ on $[-1,-1/2]$. From

$$P_n(\theta_n^-)^{-1}(s) \in [0, P_n h_n] \tag{B.2.10}$$

for all $(s,t) \in [-1,-1/2] \times S^1$, and $P_n(\theta_n^-)^{-1}(s) \to 0$ in $C^0([-1,-1/2])$, it follows that $v_n^-(s,t) - c_nR_n$ converge in $C^0([-1,-1/2])$ to the map

$$(\overline{a}^-((\theta^-)^{-1}(s),t) - \Gamma^-((\theta^-)^{-1}(s),t), \phi_\tau^\alpha(\overline{f}^-((\theta^-)^{-1}(s),t))).$$

This finishes the proof of the second statement of the C^0−convergence, and so, of Theorem 63.

□

B.3 Positive center action

In this section we consider the case when there is no subsequence of $\overline{u}_n$ with vanishing center action. Note that in this case, due to Remark 75, the center action of $\overline{u}_n$ is bounded from below by the constant $\hbar_0 > 0$ defined in Assumption P4. As in the previous section we first characterize the asymptotic behavior of the $\overline{J}_{P_S}$−holomorphic curves with positive center action (Theorem 87). We then prove a convergence result for the transformed psudoholomorphic curves $\overline{u}_n$ and induce a convergence result on the $\mathcal{H}$−holomorphic curves u_n with harmonic perturbations γ_n by undoing the transformation. Theorem 91 establishes the convergence of the transformed pseudoholomorphic curves $\overline{u}_n$.

B.3.1 Behavior of $\overline{J}_{P_S}$−holomorphic curves with positive center action

Via the natural action of S^1 on $C^\infty(S^1, M)$, defined by $(e^{2\pi i\vartheta} \star y)(t) := y(t+\vartheta)$ for $e^{2\pi i\vartheta} \in S^1$, we choose an S^1−invariant neighborhood $\mathcal{W}$ in the loop space $C^\infty(S^1, M)$ of the finitely many loops $t \mapsto x(Tt), 0 \leqslant t \leqslant 1$, defined by the periodic solutions $x(t)$ of X_α with periods $T \leqslant \tilde{E}_0$. Moreover, as the contact form is assumed to be non-degenerate, we choose the neighborhood $\mathcal{W}$ so small that it separates these distinguished loops from each other. The following result, which is similar to Lemma 3.1 of [14], ensures that "long" $\overline{J}_{P_S}$−holomorphic curves $(\overline{u}, R, P)$ with small $d\alpha$−energies and positive center action are close to some periodic orbit of the Reeb vector field.

Lemma 81. *Given any S^1−invariant neighborhood $\mathcal{W} \subset C^\infty(S^1, M)$ in the loop space of the loops defined by the periodic solutions of X_α with periods $T \leqslant \tilde{E}_0$, there exists $h > h_0$ (the constant h_0 is guaranteed by Theorem 72) such that the following hold: For any $R > h$ and any $\overline{J}_{P_S}$−holomorphic curve $(\overline{u}, R, P)$ such that $A(\overline{u}) > 0$ the loops $t \mapsto \overline{f}(s,t)$ satisfy $\overline{f}(s,\cdot) \in \mathcal{W}$ for all $s \in [-R+h, R-h]$. Moreover, with $T = A(\overline{u})$ being the center action, the loops $\overline{f}(s)$ will be in the S^1−invariant neighborhood of a loop $t \mapsto x(Tt)$ corresponding to a T−periodic orbit $x(t)$ of the Reeb vector field.*

According to [14], $\mathcal{W}$ separates the loops of the periodic orbits with periods $T \leqslant \tilde{E}_0$, and so, all these loops $\overline{f}(s,\cdot)$ for $s \in [-R+h, R-h]$ are in the neighborhood component of $\mathcal{W}$ containing precisely one of the distinguished loops defined by a periodic orbit (x, T) with period $T \leqslant \tilde{E}_0$. From $\alpha(X_\alpha) = 1$ we find

$$T = \int_{S^1} x(T\cdot)^*\alpha,$$

and so, given $\epsilon > 0$ we can choose $\mathcal{W}$ so small that

$$\left| \int_{S^1} \bar{f}(s,\cdot)^*\alpha - T \right| \leqslant \epsilon \tag{B.3.1}$$

for all $s \in [-R+h, R-h]$.

Proof. (of Lemma 81) The proof is almost the same as that of Lemma 3.1 from [14]. For completeness reasons we outline the parts which are different. Arguing indirectly, we find a constant $0 < \psi < \hbar_0/2$, a sequence R_n with $R_n \geqslant n + h_0$, and a sequence of $\bar{J}_{P_n s}-$holomorphic curves $(\bar{u}_n, R_n, P_n)$ having positive center actions and satisfying $\bar{f}_n(s_n,\cdot) \notin \mathcal{W}$ for some sequence $s_n \in [-R_n+n, R_n-n]$. By assumption, the center actions are positive. Hence $A(\bar{u}_n) = T_n \geqslant \hbar_0$, and by an earlier inequality, we find that

$$\int_{S^1} \bar{f}_n(s)^*\alpha \geqslant \frac{\hbar_0}{2} - \psi =: \epsilon_0 > 0$$

for all n and all $s \in [-R_n, R_n]$.
We define the new curves $\bar{v}_n = (\bar{b}_n, \bar{g}_n) : [-R_n - s_n, R_n - s_n] \times S^1 \to \mathbb{R} \times M$ by

$$\bar{v}_n(s,t) = (\bar{b}_n(s,t), \bar{g}_n(s,t)) = (\bar{a}_n(s+s_n,t), \bar{f}_n(s+s_n,t)).$$

These curves have bounded total energies, small $d\alpha-$energies, and satisfy

$$\begin{aligned} \pi_\alpha d\bar{g}_n(s,t) \circ i &= \bar{J}_{P_n(s_n+s)}(\bar{g}_n(s,t)) \circ \pi_\alpha d\bar{g}_n(s,t), \\ (\bar{g}_n^*\alpha) \circ i &= d\bar{b}_n \end{aligned}$$

and $\bar{g}_n(0,\cdot) \notin \mathcal{W}$ for all n. The left and right ends of the interval $[-R_n - s_n, R_n - s_n]$ converge to $-\infty$ and $+\infty$, respectively. Define now the sequence of maps $\tilde{v}_n = (b_n, v_n) : [-R_n - s_n, R_n - s_n] \times S^1 \to \mathbb{R} \times M$ by setting $\tilde{v}_n(s,t) = (\bar{b}_n(s,t) - \bar{b}_n(0,0), \bar{g}_n(s,t))$. The maps $\tilde{v}_n$ solve

$$\begin{aligned} \pi_\alpha dv_n(s,t) \circ i &= \bar{J}_{P_n(s_n+s)}(v_n(s,t)) \circ \pi_\alpha dv_n(s,t), \\ (v_n^*\alpha) \circ i &= db_n. \end{aligned}$$

As in the proof of Theorem 72, the gradients of $\tilde{v}_n$ are uniformly bounded. Hence, by Arzelà-Ascoli's theorem, a subsequence of $\tilde{v}_n$ converges in C^∞_{loc}, i.e.

$$\tilde{v}_n \to \tilde{v} \text{ in } C^\infty_{loc}(\mathbb{R} \times S^1, \mathbb{R} \times M),$$

where $\tilde{v} = (b, v) : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ is an usual $\bar{J}_\tau-$holomorhic curve for some $\tau \in [-C, C]$ satisfying

$$\begin{aligned} E_\alpha(\tilde{v}; \mathbb{R} \times S^1) + E_{d\alpha}(\tilde{v}; \mathbb{R} \times S^1) &\leqslant \tilde{E}_0, \\ E_{d\alpha}(\tilde{v}; \mathbb{R} \times S^1) &\leqslant \frac{\hbar_0}{2}, \\ \int_{S^1} v(s,\cdot)^*\alpha &\geqslant \epsilon_0, \text{ for all } s \in \mathbb{R}. \end{aligned}$$

The rest of the proof follows as in Lemma 3.1 of [14].

□

In view of Lemma 81 we fix a non-degenerate periodic solution $x(t)$ of period $T \leqslant \tilde{E}_0$ and analyze the curves $(\bar{u} = (\bar{a}, \bar{f}), R, P)$ with $\bar{f}([-R, R] \times S^1) \subset \mathcal{U}$, where $\mathcal{U}$ is a small tubular neighborhood of $x(\mathbb{R})$.

To study long curves with positive center action we need some special coordinates. Denote by α_0 the standard contact form $\alpha_0 = d\vartheta + x dy$ on $S^1 \times \mathbb{R}^2$ with coordinates (ϑ, x, y). The next lemma introduces the "standard coordinates" near a periodic orbit of the Reeb vector field. For a proof we refer to [13].

Lemma 82. *Let (M, α) be a 3−dimensional manifold equipped with a contact form, and let $x(t)$ be the T−periodic solution of the corresponding Reeb vector field $\dot{x} = X_\alpha(x)$ on M. Let τ_0 be the minimal period such that $T = k\tau_0$ for some positive integer k. Then there exist an open neighborhood $U \subset S^1 \times \mathbb{R}^2$ of $S^1 \times \{0\}$, an open neighborhood $V \subset M$ of $P = x(\mathbb{R})$, and a diffeomorphism $\varphi : U \to V$ mapping $S^1 \times \{0\}$ onto P such that*

$$\varphi^* \alpha = f \cdot \alpha_0$$

for a positive smooth function $f : U \to \mathbb{R}$ satisfying

$$f \equiv \tau_0 \text{ and } df \equiv 0 \tag{B.3.2}$$

on $S^1 \times \{0\}$.

The following description is borrowed from [14]. As $S^1 = \mathbb{R}/\mathbb{Z}$ we work in the covering space and denote by (ϑ, x, y) the coordinates, where ϑ is mod 1. In these coordinates, the contact form α is $\alpha = f \cdot \alpha_0$ for a smooth function $f : \mathbb{R}^3 \to (0, \infty)$ defined near $S^1 \times \{0\}$, being periodic in ϑ, i.e. $f(\vartheta + 1, x, y) = f(\vartheta, x, y)$, and satisfying (B.3.2). The Reeb orbit $X_\alpha = (X_0, X_1, X_2)$ has the components

$$X_0 = \frac{1}{f^2}(f + x\partial_x f), \;\; X_1 = \frac{1}{f^2}(\partial_y f - x\partial_\vartheta f), \;\; X_2 = -\frac{1}{f^2}\partial_x f.$$

The vector field X_α is periodic in ϑ of period 1 and constant along the periodic orbit $x(\mathbb{R})$, i.e. $X_\alpha(\vartheta, 0, 0) = (\tau_0^{-1}, 0, 0)$. The periodic solution is represented as $x(Tt) = (kt, 0, 0)$, where $T = k\tau_0$ is the period, τ_0 the minimal period, and k the covering number of the periodic solution. The subsequent lemma is rather technical and describes the behavior of a long $\bar{J}_{P s}$−holomorphic curve $(\bar{u}, R, P)$ in the coordinates introduced by Lemma 82.

Lemma 83. *For any $N \in \mathbb{N}$, $\delta > 0$, there exists $h > 0$ such that for any $R > h$ and any $\bar{J}_{P s}$−holomorphic curve $(\bar{u}, R, P)$ as in Lemma 81, the representation*

$$\bar{u}(s,t) = (\bar{a}(s,t), \vartheta(s,t), z(s,t) = (x(s,t), y(s,t)))$$

of the cylinder in the above local coordinates satisfies the following: For all $(s,t) \in [-R+h, R-h] \times S^1$ we have

$$|\partial^\alpha(\bar{a}(s,t) - Ts)| \leqslant \delta \text{ and } |\partial^\alpha(\vartheta(s,t) - kt)| \leqslant \delta$$

for $1 \leqslant |\alpha| \leqslant N$, and

$$|\partial^\alpha z(s,t)| \leqslant \delta$$

for all $0 \leqslant |\alpha| \leqslant N$. Here, T is the period and k the covering number of the distinguished periodic solution lying in the center of the tubular neighborhood.

Proof. The proof is more or less the same as that of Lemma 3.3 in [14]. We argue by contradiction. There exist $N \in \mathbb{N}$, $\delta_0 > 0$ such that for any $h_n = 2n$ we find $R_n > 2n$ and the $\bar{J}_{P_n s}$−holomorphic curve $(\bar{u}_n, R_n, P_n)$ satisfying the following. Representing the maps $\bar{u}_n$ in local coordinates by

$$\bar{u}_n(s,t) = (\bar{a}_n(s,t), \vartheta_n(s,t).z_n(s,t)),$$

we assume the existence of a sequence $(s_n, t_n) \in [-R_n - n, R_n - n] \times S^1$ and a multiindex α with $1 \leqslant |\alpha| \leqslant N$ such that

$$|\partial^\alpha \left[(\bar{a}_n - Ts, \vartheta_n - kt)\right](s_n, t_n)| \geqslant \delta_0. \tag{B.3.3}$$

We define the translated sequence $\tilde{v}_n : [-n, n] \times S^1 \to \mathbb{R} \times M$ by

$$\tilde{v}_n(s,t) = (b_n(s,t), v_n(s,t)) = (\overline{a}_n(s+s_n,t) - \overline{a}_n(s_n,t_n), \overline{f}_n(s+s_n,t)).$$

By the $\mathbb{R}$–invariance of $\overline{J}_\tau$ for all τ, these maps satisfy the same assumptions on the energy as the $\overline{u}_n$, and solves

$$\begin{aligned} \pi_\alpha dv_n(s,t) \circ i &= \overline{J}_{\tilde{S}_n(s+s_n)} \circ \pi_\alpha dv_n(s,t), \\ (v_n^*\alpha) \circ i &= db_n. \end{aligned}$$

The rest of the proof is exactly as in [14]. We conclude as in Lemma 81 that the sequence $\tilde{v}_n$ has uniformly bounded gradients on $[-n+1, n-1] \times S^1$, and so, it possesses a C^∞_{loc} converging subsequence. Its limit $\tilde{v} = (b, v) : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ is a $\overline{J}_\tau$–holomorphic cylinder for some $\tau \in [-C, C]$ with the energy bounds

$$\begin{aligned} E_{d\alpha}(\tilde{v}; \mathbb{R} \times S^1) + E_\alpha(\tilde{v}; \mathbb{R} \times S^1) &\leqslant \tilde{E}_0, \\ E_{d\alpha}(\tilde{v}; \mathbb{R} \times S^1) &\leqslant \frac{\hbar_0}{2}. \end{aligned}$$

In addition, due to (B.3.3), the map $\tilde{v}$ is non-constant. Therefore $\tilde{v}$ is a pseudoholomorphic cylinder over a periodic orbit $z(t)$ of period $T' \leqslant \tilde{E}_0$, and so, of the form $\tilde{v}(s,t) = (T's + a_0, z(T't))$. By (B.3.1), the period T' is close to the period T of the distinguished periodic orbit $x(t)$. As this periodic orbit is non-degenerate, there exists a tubular neighborhood U of $x(\mathbb{R})$ which does not contain any other periodic orbit with a period close to T. Hence, choosing the tubular neighborhood sufficiently small, we conclude that $T' = T$ and $z(Tt) = x(Tt)$, so that in local coordinates we have $\tilde{v}(s,t) = (Ts + a_0, kt + \vartheta_0, 0)$ for two constants a_0 and ϑ_0. Using $\tilde{v}_n \to \tilde{v}$ in C^∞_{loc} and setting $s = 0$, it follows that

$$|\partial^\alpha [(\overline{a}_n - Ts, \vartheta_n - kt)] (s_n, t_n)| \to (0, 0)$$

for $|\alpha| \geqslant 1$. This gives a contradiction. Similarly, the last estimate in Lemma 83 is proved by assuming that $|\partial^\alpha z_n(s_n, t_n)| \geqslant \delta_0$ for some α with $0 \leqslant |\alpha| \leqslant N$ and some $\delta_0 > 0$. As the limit map $\tilde{v}$ has its z–component equal to zero, we employ the same arguments to obtain $|\partial^\alpha z_n(s_n, t_n)| \to 0$. This gives again a contradiction and the proof is now complete.

□

As in [14], an immediate consequence is the next corollary showing that the quantity

$$\int_{S^1} \overline{f}^*(s)\alpha$$

gets arbitrary close to the center action $A(\overline{u}) = T$.

Corollary 84. *If the $\overline{J}_{P_S}$–holomorphic curve $(\overline{u}, R, P)$ satisfies the assumption of Lemma 83, then*

$$\int_{S^1} \overline{f}(s)^*\alpha = T + \mathcal{O}(\delta)$$

for all $s \in [-R+h, R-h]$.

For a proof we refer to Corollary 3.4 of [14].
We compute the Cauchy-Riemann equations for the representation

$$\begin{aligned} \overline{u}(s,t) &= (\overline{a}(s,t), \overline{f}(s,t)) = (\overline{a}(s,t), \vartheta(s,t), z(s,t)) \\ &= (\overline{a}(s,t), \vartheta(s,t), x(s,t), y(s,t)) \end{aligned}$$

of a $\bar{J}_{Ps}-$holomorphic curve $(\bar{u}, R, P)$ in the local coordinates $\mathbb{R}\times\mathbb{R}^3$ of the tubular neighborhood given in Lemma 82. In the following, we adopt the same constructions as in [14]. On $\mathbb{R}^3$ we have the contact form $\alpha = f\alpha_0$. At point $m = (t, x, y) \in \mathbb{R}^3$, the contact structure $\xi_m = \ker(\alpha_m)$ is spanned by the vectors

$$E_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } E_2 = \begin{pmatrix} -x \\ 0 \\ 1 \end{pmatrix}.$$

We denote by $\bar{J}_\rho(m)$ the 2×2 matrix representing the compatible almost complex structure on the plane ξ_m in the basis $\{E_1, E_2\}$ for all $\rho \in [-C, C]$. In the basis $\{E_1, E_2\}$, the symplectic structure $d\alpha|_{\xi_m}$ is given by the skew symmetric matrix function $f(m)J_0$, where

$$J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, in view of the compatibility requirement, the complex multiplication $\bar{J}_\rho(m)$ has the properties $\bar{J}_\rho(m)^2 = -\mathrm{id}$, $\bar{J}_\rho(m)^T J_0 \bar{J}_\rho(m) = J_0$ and $-J_0\bar{J}_\rho(m) > 0$. In particular, $J_0\bar{J}_\rho(m)$ is a symmetric matrix for all $\rho \in [-C, C]$, and it follows that

$$\langle x, y\rangle_\rho := \left\langle x, -J_0\bar{J}_\rho(m)y\right\rangle$$

is an inner product on $\mathbb{R}^2$ which is left invariant under $\bar{J}_\rho(m)$, i.e. $\left\langle \bar{J}_\rho(m)x, \bar{J}_\rho(m)y\right\rangle_\rho = \langle x, y\rangle_\rho$ for all $\rho \in [-C, C]$. The Reeb vector field X_α can be written as $X_\alpha = (X_0, X_1, X_2) \in \mathbb{R}\times\mathbb{R}^2$. Setting $z = (x, y) \in \mathbb{R}^2$ we define $Y(t, z) = (X_1(t, z), X_2(t, z)) \in \mathbb{R}^2$. Since $X(t, 0) = (1/\tau_0, 0)$ we have $Y(t, z) = D(t, z)z$, where

$$D(t, z) = \int_0^1 dY(t, \rho z)d\rho,$$

and d is the derivative with respect to the $z-$variable. In particular, if $z = 0$ we obtain

$$D(t, 0) = dY(t, 0) = \frac{1}{\tau_0^2}\begin{pmatrix} \partial_{xy} f & \partial_{yy} f \\ -\partial_{xx} f & -\partial_{xy} f \end{pmatrix}.$$

We introduce the 2×2 matrices depending on $\bar{u}(s, t)$ and Ps by

$$\begin{aligned} J(s, t) &= \bar{J}_{Ps}(\bar{f}(s, t)) = \bar{J}_{Ps}(\vartheta(s, t).z(s, t)),\\ S(s, t) &= [\partial_t\bar{a} - \partial_s\bar{a}\cdot J(s, t)]\, D(\bar{f}(s, t)). \end{aligned}$$

In the basis $\{E_1, E_2\}$ of the contact plane ξ_m at $m = \bar{f}(s, t)$ and for the representation $\bar{u}(s, t) = (\bar{a}(s, t), \vartheta(s, t), z(s, t)) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^2$, we write

$$\pi_\alpha\partial_s\bar{f}(s, t) + \bar{J}_{Ps}(\bar{f}(s, t))\pi_\alpha\partial_t\bar{f}(s, t) = 0.$$

We find

$$z_s + J(s, t)z_t + S(s, t)z = 0$$

and further on, with $z(s, t) = (x(s, t), y(s, t))$,

$$\bar{a}_s = (\vartheta_t + xy_t)f(\bar{f}) \quad\text{and}\quad \bar{a}_t = -(\vartheta_s + xy_s)f(\bar{f}).$$

It is convenient to decompose the matrix $S(s, t)$ into its symmetric and anti-symmetric parts with respect to the inner product $\langle\cdot, -J_0 J(s, t)\cdot\rangle = \left\langle\cdot, -J_0\bar{J}_{Ps}(\bar{f}(s, t))\cdot\right\rangle$ on $\mathbb{R}^2$ by introducing

$$B(s, t) = \frac{1}{2}\left[S(s, t) + S^*(s, t)\right] \quad\text{and}\quad C(s, t) = \frac{1}{2}\left[S(s, t) - S^*(s, t)\right],$$

where S^* is the transpose of S with respect to the inner product $\langle\cdot, -J_0 J(s,t)\cdot\rangle$. Explicitly we have $S^* = JJ_0 S^T J_0 J$, where S^T is the transpose matrix of S with respect to the Euclidean inner product $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^2$. In terms of B and C, the above equation becomes

$$z_s + J(s,t)z_t + B(s,t)z + C(s,t)z = 0.$$

The operator $A(s) : W^{1,2}(S^1, \mathbb{R}^2) \subset L^2(S^1, \mathbb{R}^2) \to L^2(S^1, \mathbb{R}^2)$, given by

$$A(s) = -J(s,t)\frac{d}{dt} - B(s,t),$$

is self-adjoint with respect to the inner product $\langle\cdot,\cdot\rangle_s$ in L^2, defined for $x, y \in L^2(S^1, \mathbb{R}^2)$ by

$$\langle x, y\rangle_s := \int_0^1 \langle x(t), -J_0 J(s,t) y(t)\rangle \, dt.$$

The norms $\|x\|_s^2 := \langle x, x\rangle_s$ are equivalent to the standard $L^2(S^1, \mathbb{R}^2)-$norms (denoted by $\|\cdot\|$) in the following sense:

Lemma 85. *There exist the constants* $h, c > 0$ *such that for all* $R > h$ *and all* $\bar{J}_{Ps}-$*holomorphic curves* $(\overline{u}, R, P)$ *satisfying the assumptions of Lemma 83, all* $x \in L^2(S^1, \mathbb{R}^2)$*, and all* $s \in [-R, R]$*, we have*

$$\frac{1}{c}\|x\|_s \leqslant \|x\| \leqslant c\,\|x\|_s\,.$$

Proof. The first inequality follows from the result according to which for $\rho \in [-C, C]$ and $p \in M$, the domain-dependent complex structure $\bar{J}_\rho(p)$ varies continously in a compact subset of the set of complex structures. For the second inequality, we additionally use the fact that $-J_0 J(s,t)$ is uniformly positive definite.

□

Lemma 86. *There exists a constant* $h > 0$ *such that for every* $R > h$ *and every* $\bar{J}_{Ps}-$*holomorphic curve* $(\overline{u}, R, P)$ *satisfying the assumptions of Lemma 83, the following holds true. If* $\overline{u} = (\overline{a}, \bar{f})$ *is the reparametrization in local coordinates and* $A(s)$ *the associated operator, then there exists a constant* $\eta > 0$ *such that*

$$\|A(s)\xi\|_s \geqslant \eta\,\|\xi\|_s$$

for all $s \in [-R+h, R-h]$ *and all* $\xi \in W^{1,2}(S^1, \mathbb{R}^2)$.

Proof. We prove by contradiction by adapting the proof given in [14] to our setting. Assume that the inequality does not hold. Then for any $h_n = 2n$ there exist $R_n \in \mathbb{R}_+$ with $R_n > 2n$ and a sequence of $\bar{J}_{P_n s}-$holomorphic curves $(\overline{u}_n, R_n, P_n)$ satisfying the assumptions of Lemma 83 and

$$\int_{S^1} \bar{f}_n^* \alpha \geqslant \epsilon_0$$

for all $s \in [-R_n, R_n]$. Here $\epsilon_0 > 0$ is the constant defined by Theorem 72. Representing $\overline{u}_n$ in local coordinates as $\overline{u}_n(s,t) = (\overline{a}_n(s,t), \vartheta_n(s,t), z_n(s,t))$, consider the associated operator

$$A_n(s) = -J_n(s,t)\frac{d}{dt} - B_n(s,t),$$

where $S_n(s,t)$ and $B_n(s,t)$ are defined as above, and $J_n(s,t) = \bar{J}_{P_n s}(\bar{f}_n(s,t))$. Further on, assume that there exist the sequences $s_n \in [-R_n - n, R_n + n]$ and $\xi_n \in W^{1,2}(S^1, \mathbb{R}^2)$ such that

$$\|\xi_n\|_{s_n} = 1 \text{ and } \|A_n(s_n)\xi_n\|_{s_n} \to 0, \tag{B.3.4}$$

and consider the translated maps

$$\tilde{v}_n(s,t) = (b_n(s,t), v_n(s,t)) = (\overline{a}_n(s+s_s,t) - \overline{a}_n(s_n,0), \overline{f}_n(s+s_n,t))$$

for all n and $(s,t) \in [-n,n] \times S^1$. Arguing as before we find $\tilde{v}_n \to \tilde{v}$ in $C^\infty_{loc}(\mathbb{R} \times S^1, \mathbb{R} \times M)$, where $\tilde{v}$ is a cylinder over a distinguished periodic orbit $x(t)$ lying in the center of the tubular neighborhood. Hence, in local coordinates, we can write $\tilde{v}(s,t) = (Ts + a_0, kt + \vartheta_0, 0)$ with two constants a_0 and ϑ_0. Setting $s = 0$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\overline{a}_n(s_n,t) &\to 0, \\
\frac{\partial}{\partial s}\overline{a}_n(s_n,t) &\to T, \\
\vartheta_n(s_n,t) &\to kt + \vartheta_0, \\
z_n(s_n,t) &\to 0
\end{aligned}$$

as $n \to \infty$, uniformly in t. Consequently,

$$B_n(s_n,t) \to T\overline{J}_{\tau_{\{s_n\}}}(kt+\vartheta_0,0) \cdot dY(kt+\vartheta_0,0), \tag{B.3.5}$$

$$J_n(s_n,t) \to \overline{J}_{\tau_{\{s_n\}}}(kt+\vartheta_0,0) \tag{B.3.6}$$

as $n \to \infty$, uniformly in t and for some $\tau_{\{s_n\}}$ given by $P_n s_n \to \tau_{\{s_n\}}$. In using $\|J_n(s,\cdot)\xi\|_s = \|\xi\|_s$ for every $\xi \in L^2(S^1,\mathbb{R}^2)$ and Lemma 85, we find that there exists a constant $c > 0$ such that for all $n \in \mathbb{N}$ and $\xi \in W^{1,2}(S^1,\mathbb{R}^2)$,

$$\left\|\dot{\xi}\right\| \leqslant c\left(\|A_n(s_n)\xi\| + \|B_n(s_n,\cdot)\xi\|\right). \tag{B.3.7}$$

Consequently, the sequence ξ_n given by (B.3.4) is bounded in $W^{1,2}$. Since $W^{1,2}$ is compactly embedded in L^2, a subsequence of ξ_n converges in L^2. Therefore, by assumption (B.3.4), the limits (B.3.5) and (B.3.6), and the estimate (B.3.7) we have that after going over to a subsequence, ξ_n is a Cauchy sequence in $W^{1,2}(S^1,\mathbb{R}^2)$; thus,

$$\xi_n \to \xi \text{ in } W^{1,2}(S^1,\mathbb{R}^2).$$

From

$$A_n(s_n)\xi_n = -J_n(s_n,t)\dot{\xi}_n - B_n(s_n,t)\xi_n \to 0 \text{ in } L^2(S^1,\mathbb{R}^2)$$

together with (B.3.5) and (B.3.6) we conclude that ξ solves the equation

$$\frac{d}{dt}\xi(t) = TdY(kt+\vartheta_0,0)\xi(t).$$

This is a contradiction to the fact that the periodic orbits $x(t) = (kt+\vartheta_0, 0)$ was assumed to be non-degenerate.

□

The next theorem is similar to Theorem 1.3 of [14]; the only difference is that it is formulated for pseudoholomorphic curves with respect to a domain-dependent almost complex structure on the target space $\mathbb{R} \times M$.

Theorem 87. *Let $h_0 > 0$ be the constant appearing in Theorem 72 and being associated with $0 < \psi < \hbar_0/2$. Then there exist the positive constants δ_0, μ, and $\nu < \min\{4\pi, 2\mu\}$ such that the following hold: Given $0 < \delta \leqslant \delta_0$, there exists $h \geqslant h_0$ such that for any $R > h$ and any $\overline{J}_{Ps}-$holomorphic curve $(\overline{u}, R, P)$ such that $A(\overline{u}) > 0$, there exists a unique (up to a phase shift) periodic orbit $x(t)$ of the Reeb vector field X_α with period*

$T = A(\overline{u}) \leqslant \tilde{E}_0$ *satisfying*

$$\left|\int_{S^1} \overline{f}(0)^*\alpha - T\right| < \frac{\psi}{2} \text{ and } \left|\int_{S^1} \overline{f}(s)^*\alpha - T\right| < \hbar_0, \text{ for all } s \in [-R, R].$$

In addition, there exists a tubular neighborhood $U \cong S^1 \times \mathbb{R}^{2n}$ *around the periodic orbit* $x(\mathbb{R}) \cong S^1 \times \{0\}$ *such that* $\overline{f}(s,t) \in U$ *for all* $(s,t) \in [-R+h, R-h] \times S^1$. *Using the covering* $\mathbb{R}$ *of* $S^1 = \mathbb{R}/\mathbb{Z}$, *the map* $\overline{u}$ *is represented in local coordinates* $\mathbb{R} \times U$ *as*

$$\begin{aligned}\overline{u}(s,t) &= (\overline{a}(s,t), \vartheta(s,t), z(s,t))\\ &= (Ts + a_0 + \tilde{a}(s,t), kt + \vartheta_0 + \tilde{\vartheta}(s,t), z(s,t)),\end{aligned}$$

where $(a_0, \vartheta_0) \in \mathbb{R}^2$ *are constants. The functions* $\tilde{a}$, $\tilde{\vartheta}$, *and* z *are* $1-$*periodic in* t, *and the positive integer* k *is the covering number of the* $T-$*periodic orbit represented by* $x(Tt) = (kt, 0, 0)$. *For all multiindices* α *there exists a constant* C_α *such that for all* $(s,t) \in [-R+h, R-h] \times S^1$ *the following estimates hold:*

$$|\partial^\alpha z(s,t)|^2 \leqslant C_\alpha \delta^2 \frac{\cosh(\mu s)}{\cosh(\mu(R-h))}$$

and

$$|\partial^\alpha \tilde{a}(s,t)|^2, |\partial^\alpha \tilde{\vartheta}(s,t)|^2 \leqslant C_\alpha \delta^2 \frac{\cosh(\nu s)}{\cosh(\nu(R-h))}.$$

For the proof of the theorem we need the following

Remark 88. By Lemma 83, which is similar to Lemma 3.3 from [14], we have $|\partial_s^\alpha \overline{f}(s,t)| \leqslant \delta$ for all $\alpha \geqslant 1$ and all $(s,t) \in [-R+h, R-h] \times S^1$. As a result, the derivatives with respect to the s coordinate of $J(s,t)$ and $B(s,t)$ contain factors estimated by δ. This can be seen as follows. Recalling that $J(s,t) = \overline{J}_{Ps}(\vartheta(s,t), z(s,t))$ we find

$$\partial_s J(s,t) = P\partial_\rho \overline{J}_{Ps}(\overline{f}(s,t)) + \partial_\vartheta \overline{J}_{Ps}(\overline{f}(s,t))\partial_s\vartheta + \partial_z \overline{J}_{Ps}(\overline{f}(s,t))\partial_s z.$$

For R sufficiently large, the assumption on the universal bound of the conformal co-period gives $|P| \leqslant \delta$; consequently, $|\partial_s J(s,t)| \in \mathcal{O}(\delta)$. In a similar way it can be shown that $|\partial_s^2 J(s,t)|, |\partial_s B(s,t)| \in \mathcal{O}(\delta)$.

The proof of Theorem 87 which is omitted here, proceeds as in [14] by using Lemma 86 and Remark 88.

B.3.2 Proof of Theorem 65

Applying Theorem 87 to the sequence of $\overline{J}_{P_n s}-$holomorphic curves $(\overline{u}_n, R_n, P_n)$ we find the following.

Corollary 89. *For every* $\epsilon > 0$ *there exist* $h > 0$ *and* $N_{\epsilon,h} \in \mathbb{N}$ *such that for every* $n \geqslant N_{\epsilon,h}$, *we have* $R_n > h$ *and*

$$d(\overline{f}_n(s,t), x(Tt)) < \epsilon \text{ and } |\overline{a}_n(s,t) - Ts - a_0| < \epsilon \tag{B.3.8}$$

for all $(s,t) \in [-R_n + h, R_n - h] \times S^1$ *uniformly in* $t \in S^1$ *and some* $a_0 \in \mathbb{R}$.

For $h > 0$ sufficiently small and in regard of Condition P2 we continue to denote the cylinder $[-R_n + h, R_n - h] \times S^1$ by $[-R_n, R_n] \times S^1$. In view of (B.3.8) we assume that the quantities

$$\overline{r}_n^- := \inf_{t \in S^1} \overline{a}_n(-R_n, t) \text{ and } \overline{r}_n^+ := \sup_{t \in S^1} \overline{a}_n(R_n, t)$$

satisfy $\overline{r}_n^+ - \overline{r}_n^- \to \infty$ as $n \to \infty$.

Recalling that P_n, S_n and $1/R_n$ are zero sequences we reformulate the above findings as in Corollary 79.

Corollary 90. *For every sequence $h_n \in \mathbb{R}_+$ satisfying $h_n < R_n$ and $h_n, R_n/h_n \to \infty$ and every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$dist_{\overline{g}_0}(\overline{f}_n(s,t), x(Tt)) < \epsilon \text{ and } |\overline{a}_n(s,t) - Ts - a_0| < \epsilon$$

for all $n \geqslant N$ and some $a_0 \in \mathbb{R}$. Moreover, for the period P_n and co-period S_n we obtain that $h_nP_n, h_nS_n \to 0$ as $n \to \infty$.

Proof. The proof of the first and second statement follows as in Corollary 79.

□

The next theorem which states a $C^\infty_{\mathrm{loc}}-$ and a C^0-convergence result for the maps $\overline{u}_n$ with positive center action is the analog of Theorem 63.

Theorem 91. *There exists a subsequence of the sequence of $\overline{J}_{P_n s}-$holomorphic curves $(\overline{u}_n, R_n, P_n)$, also denoted by $(\overline{u}_n, R_n, P_n)$, and the pseudoholomorphic half cylinders $u^\pm$ defined on $(-\infty, 0] \times S^1$ and $[0, \infty) \times S^1$, respectively such that for every sequence $h_n \in \mathbb{R}_+$ and every sequence of diffeomorphisms $\theta_n : [-R_n, R_n] \times S^1 \to [-1, 1] \times S^1$ satisfying the assumptions of Remark 44, the following convergence results hold:*

$C^\infty_{loc}-$convergence:

1. *For any sequence $s_n \in [-R_n + h_n, R_n - h_n]$ there exists a subsequence of the shifted maps $\overline{u}_n(s + s_n, t) - Ts_n$, defined on $[-R_n + h_n - s_n, R_n - h_n - s_n] \times S^1$, that converges in C^∞_{loc} to $(Ts + a_0, x(Tt))$.*
2. *The left shifts $\overline{u}^-_n(s,t) := \overline{u}_n(s - R_n, t) - \overline{r}^-_n$, defined on $[0, h_n) \times S^1$, possess a subsequence that converges in C^∞_{loc} to a pseudoholomorphic half cylinder $\overline{u}^- = (\overline{a}^-, \overline{f}^-)$, defined on $[0, +\infty) \times S^1$. The curve $\overline{u}^-$ is asymptotic to $(Ts + a_0, x(Tt))$. The maps $\overline{v}^-_n$ converge in C^∞_{loc} on $[-1, -1/2) \times S^1$ to $\overline{v}^-$, where $\overline{f}^-((\theta^-)^{-1}(-1/2), t) = x(Tt)$ for all $t \in S^1$.*
3. *The right shifts $\overline{u}^+_n(s,t) := \overline{u}_n(s + R_n, t) - \overline{r}^+_n$, defined on $(-h_n, 0] \times S^1$, possess a subsequence that converges in C^∞_{loc} to a $\mathcal{H}-$holomorphic half cylinder $\overline{u}^+ = (\overline{a}^+, \overline{f}^+)$, defined on $(-\infty, 0] \times S^1$. The curve $\overline{u}^+$ is asymptotic to $(Ts + a_0, x(Tt))$. The maps $\overline{v}^+_n$ converge in C^∞_{loc} on $(1/2, 1] \times S^1$ to $\overline{v}^+$, where $\overline{f}^+((\theta^+)^{-1}(1/2), t) = x(Tt)$ for all $t \in S^1$.*

C^0-convergence:

1. *The maps $\overline{f}_n \circ \theta_n^{-1} : [-1/2, 1/2] \times S^1 \to M$ converge in C^0 to $x(Tt)$.*
2. *The maps $\overline{f}^-_n \circ (\theta^-_n)^{-1} : [-1, -1/2] \times S^1 \to M$ converge in C^0 to a map $\overline{f}^- \circ (\theta^-)^{-1} : [-1, -1/2] \times S^1 \to M$ such that $\overline{f}^-((\theta^-)^{-1}(-1/2), t) = x(Tt)$.*
3. *The maps $\overline{f}^+_n \circ (\theta^+_n)^{-1} : [1/2, 1] \times S^1 \to M$ converge in C^0 to a map $\overline{f}^+ \circ (\theta^+)^{-1} : [1/2, 1] \times S^1 \to M$ such that $\overline{f}^+((\theta^+)^{-1}(1/2), t) = x(Tt)$.*
4. *For any $R > 0$, there exist $\rho > 0$ and $N \in \mathbb{N}$ such that $\overline{a}_n \circ \theta_n^{-1}(s,t) \in [\overline{r}^-_n + R, \overline{r}^+_n - R]$ for all $n \geqslant N$ and all $(s,t) \in [-\rho, \rho] \times S^1$.*

Proof. As in Theorem 80 we prove only the first and second statements of the $C^\infty_{loc}-$convergence. Let $h_n \in \mathbb{R}_+$ be a sequence satisfying $h_n < R_n$ and $h_n, R_n/h_n \to \infty$ as $n \to \infty$ and let the sequence of diffeomorphisms $\theta_n : [-R_n, R_n] \to [-1, 1]$ fulfill the assumptions of Remark 44. To prove the first statement we consider the shifted maps $\overline{u}_n(\cdot + s_n, \cdot)$, defined on $[-R_n + h_n - s_n, R_n - h_n - s_n] \times S^1$, for any sequence $s_n \in [-R_n + h_n, R_n - h_n]$. By Corollary 90, there exists a subsequence of $\overline{u}_n(s + s_n, t) - Ts_n$ that converges in C^∞_{loc} to a trivial cylinder $(Ts + a_0, x(Tt))$ over the Reeb orbit $x(Tt)$. To prove the second statement, we consider the shifted maps $\overline{u}^-_n : [0, h_n] \times S^1 \to \mathbb{R} \times M$, defined by $\overline{u}^-_n(s, t) = \overline{u}_n(s - R_n, t) - \overline{r}^-_n$, where $\overline{r}^-_n := \inf_{t \in S^1} \overline{a}_n(-R_n, t)$. By Lemma 73, $\overline{u}^-_n$ converge in $C^\infty_{loc}([0, \infty) \times S^1)$ to a usual pseudoholomorphic curve $\overline{u}^- = (\overline{a}^-, \overline{f}^-) : [0, +\infty) \times S^1 \to \mathbb{R} \times M$ with respect to the standard complex structure i on $[0, +\infty) \times S^1$ and the almost complex structure $\overline{J}_{-\tau}$ on the domain, where$\tau = \lim_{n\to\infty} P_n R_n$. We show that $\overline{u}^-$ is asymptotic to a trivial cylinder over the Reeb orbit x, i.e. $(Ts + a_0, x(Tt))$. In fact, for proving

$$\lim_{r\to\infty} \left(\overline{a}^-(r, t) - Tr - a_0, \overline{f}^-(r, t)\right) = (0, x(Tt)) \tag{B.3.9}$$

we argue by contradiction. Assume that there exists a sequence $(s_k, t_k) \in [0, \infty) \times S^1$ with $s_k \to \infty$ as $k \to \infty$, and since S^1 is compact, also assume that $t_k \to t^*$ as $k \to \infty$ such that $\lim_{k\to\infty} \overline{f}^-(s_k, t_k) = x'(T't^*) \in M$, where x' is some Reeb orbit with $w' := x'(T't^*) \neq w := x(Tt^*)$. Letting $\epsilon := \text{dist}_{\overline{g}_0}(w, w') > 0$, using Corollary 90 and employing the same arguments as in Theorem 80 we are led to the contradiction $\text{dist}_{\overline{g}_0}(w, w') \leqslant 3\epsilon/10$. Consider now the $\mathbb{R}-$coordinate $\overline{a}_n$. To prove (B.3.9) for the $\mathbb{R}-$coordinate it is sufficient to replace $\overline{f}^-$ by the function $\overline{a}^-(r, t) - Tr - a_0$ and to repeat the above arguments. Because, $(\theta^-_n)^{-1} : [-1, -1/2] \to [0, h_n]$ converge in C^∞_{loc} to the diffeomorphism $(\theta^-)^{-1} : [-1, -1/2) \to [0, +\infty)$, the maps $\overline{u}^-_n((\theta^-_n)^{-1}(s), t)$ converge in C^∞_{loc} to the map $\overline{u}^-((\theta^-)^{-1}(s), t)$ on $[-1, -1/2) \times S^1$. This finishes the proof of the $C^\infty_{loc}-$convergence.
To prove the first statement of the C^0-convergence, we use Corollary 79 which yields $\text{dist}_{\overline{g}_0}(\overline{f}_n(\theta_n^{-1}(s), t), x(Tt)) < 1/n$ for all $(s, t) \in [-1/2, 1/2] \times S^1$, so that, we conclude that $\overline{f}_n$ converge in $C^0([-1/2, 1/2] \times S^1)$ to $x(Tt)$ uniformly. For the second statement we take into account that the maps $\overline{f}^-_n((\theta^-_n)^{-1}(s), t)$ converge in C^∞_{loc} to $\overline{f}^-((\theta^-)^{-1}(s), t)$ on $[-1, -1/2) \times S^1$, so that by the asymptotics of $\overline{f}^-$, $\overline{f}^-$ can be continously extended to the whole interval $[-1, -1/2]$ by setting $\overline{v}^-(-1/2, t) = x(Tt)$. Now, the proof of the convergence of $\overline{f}^-_n$ in $C^0([-1, -1/2])$ to $\overline{f}^-$ is exactly the same as the proof of Lemma 4.16 in [7]. For the maps $\overline{v}^+_n$ we proceed analgously, while for the fourth statement we apply Proposition 92. Thus the proof of the C^0-convergence is complete.

□

Proposition 92. *For any* $R > 0$*, there exist* $\rho > 0$ *and* $N \in \mathbb{N}$ *such that* $\overline{a}_n \circ \theta_n^{-1}(s, t) \in [\overline{r}^-_n + R, \overline{r}^+_n - R]$ *for all* $n \geqslant N$ *and all* $(s, t) \in [-\rho, \rho] \times S^1$.

Proof. The proof follows exactly the steps from Lemma 4.10, Lemma 4.13, and Lemma 4.17 of [7].

□

We give now the proof of Theorem 65, which closely follows the proof of Theorem 63.

Proof. (of Theorem 65) We start by proving the first statement of the $C^\infty_{loc}-$convergence. Let $h_n \in \mathbb{R}_+$ be a sequence satisfying $h_n < R_n$ and $h_n, R_n/h_n \to \infty$ as $n \to \infty$ and let the sequence of diffeomorphisms $\theta_n : [-R_n, R_n] \to [-1, 1]$ fulfill the assumptions of Remark 44. As in the proof of Theorem 63 we consider for $(s, t) \in [-R_n + h_n, R_n - h_n] \times S^1$ the maps (cf. (B.2.6))

$$f_n(s, t) = \phi^{\alpha}_{-P_n s}(\overline{f}_n(s, t)) \text{ and } a_n(s, t) = \overline{a}_n(s, t) - \Gamma_n(s, t), \tag{B.3.10}$$

and that by Remark 68, the functions Γ_n can be chosen to have vanishing average. By Theorem 78, $\overline{u}_n(0, \cdot), u_n(0, \cdot) \to (a_0, x(Tt)) \in \mathbb{R} \times M$ as $n \to \infty$. Hence $\overline{a}_n(0, \cdot), a_n(0, \cdot) \to a_0$. By Theorems 91 and 104, for any sequence $s_n \in [-R_n + h_n, R_n - h_n]$ there exists a subsequence of shifted maps $u_n(\cdot + s_n, \cdot) - Ts_n + S_n s_n$, defined on $[-R_n + h_n -$

$s_n, R_n - h_n - s_n] \times S^1$, that converges in C^∞_{loc} to the twisted trivial cylinder $(Ts + a_0, \phi^\alpha_{-\tau_{\{s_n\}}}(x(Tt)) = x(Tt - \tau_{\{s_n\}}))$ over the Reeb orbit $x(Tt)$, where $\tau_{\{s_n\}} = \lim_{n\to\infty} P_n s_n$. To prove the second statement of the $C^\infty_{loc}-$convergence, we consider the shifted maps $\overline{u}^-_n : [0, h_n] \times S^1 \to \mathbb{R} \times M$ which are defined by $\overline{u}^-_n(s,t) = \overline{u}_n(s - R_n, t) - \overline{r}^-_n$, where $\overline{r}^-_n := \inf_{t\in S^1} \overline{a}_n(-R_n, t)$. The shifted maps $u^-_n = (a^-_n, f^-_n) : [0, h_n] \times S^1 \to \mathbb{R} \times M$, defined by $u^-_n(s,t) = u_n(s - R_n, t) - \overline{r}^-_n$, where $\overline{r}^-_n := \inf_{t\in S^1} a_n(-R_n, t)$, together with the maps $\overline{u}^-_n$ satisfy (B.2.7) giving (B.2.8); whence by Theorems 91 and 105, $u^-_n - S_n R_n$ converge in C^∞_{loc} to a curve $u^-(s,t) = (a^-(s,t), f^-(s,t)) = (\overline{a}^-(s,t) - \Gamma^-(s,t), \phi^\alpha_\tau(\overline{f}^-(s,t)))$, defined on $[0,\infty) \times S^1$. The map u^- is the twisted trivial cylinder $(Ts + a_0, \phi^\alpha_\tau(x(Tt)) = x(Tt + \tau))$, and can be regarded as a $\mathcal{H}-$holomorphic map with harmonic perturbation $d\Gamma^-$. As in Theorem 63, the statement concerning the harmonic perturbations γ_n follows from Corollary 107, while the proof of the third statement proceeds analogously; the only difference is that the asymptotic of the map u^+ is $(Ts + a_0, \phi^\alpha_{-\tau}(x(Tt)))$.

To prove the first statement of the C^0-convergence, we consider the maps f_n satisfying $\overline{f}_n(s,t) = \phi^\alpha_{P_n s}(f_n(s,t))$ and

$$f_n(s,t) = f_n(\theta_n^{-1}(s), t) = \phi^\alpha_{-P_n\theta_n^{-1}(s)}(\overline{f}_n((\theta_n^{-1}(s), t))$$

for $s \in [-1/2, 1/2]$. There exists a constant $c > 0$ such that for all $(s,t) \in [-1/2, 1/2]$ we have

$$\mathrm{dist}_{\overline{g}_0}(\overline{f}_n(\theta_n^{-1}(s), t), x(Tt)) \geqslant c\,\mathrm{dist}_{\overline{g}_0}(f_n(\theta_n^{-1}(s), t), \phi^\alpha_{-P_n\theta_n^{-1}(s)}(x(Tt)))$$

Accounting of (B.2.9) we deduce that for $(s,t) \in [-1/2, 1/2] \times S^1$ we have

$$c^{-1}\mathrm{dist}_{\overline{g}_0}(\overline{f}_n(\theta_n^{-1}(s), t), x(Tt)) + \mathrm{dist}_{\overline{g}_0}(\phi^\alpha_{-P_n\theta_n^{-1}(s)}(x(Tt)), \phi^\alpha_{-2\tau s}(x(Tt))) \geqslant \mathrm{dist}_{\overline{g}_0}(f_n(\theta_n^{-1}(s), t), \phi^\alpha_{-2\tau s}(x(Tt)))$$

and $\mathrm{dist}_{\overline{g}_0}(\overline{f}_n(\theta_n^{-1}(s), t), x(Tt))$,$\mathrm{dist}_{\overline{g}_0}(\phi^\alpha_{-P_n\theta_n^{-1}(s)}(x(Tt)), \phi^\alpha_{-2\tau s}(x(Tt))) \to 0$ as $n \to \infty$. Thus f_n converge in $C^0([-1/2, 1/2])$ to $\phi^\alpha_{-2\tau s}(x(Tt))$ which is a segment of a Reeb trajectory. To prove the second statement we consider the maps v^-_n satisfying

$$f^-_n((\theta^-_n)^{-1}(s), t) = \phi^\alpha_{-P_n(\theta^-_n)^{-1}(s) + P_n R_n}(\overline{f}^-_n((\theta^-_n)^{-1}(s), t)),$$

and use (B.2.10) to conclude that $v^-_n(s,t)$ converge in $C^0([-1, -1/2])$ to the map $\phi^\alpha_\tau(\overline{f}^-((\theta^-_n)^{-1}(s), t))$. The third statement is proved in a similar manner, while the last statement follows from Proposition 92 and the fact that the harmonic functions Γ_n are uniformly bounded in C^0. More precisely, with $\overline{\Gamma}_n$ as defined in Appendix E, we can write

$$a_n(\theta_n^{-1}(s), t) = \overline{a}_n(\theta_n^{-1}(s), t) - S_n\theta_n^{-1}(s) - \overline{\Gamma}_n(s,t)$$

for $(s,t) \in [-1, 1] \times S^1$. Hence by Theorem 104 there exists a constant $C_0 > 0$ such that $\overline{\Gamma}_n$ is uniformly bounded in $C^0([-1, 1] \times S^1)$ by $C_0 > 0$. Since we have assumed that the sequence S_n is positive we get

$$-S_n R_n - C_0 \leqslant S_n\theta_n^{-1}(s) + \overline{\Gamma}_n(s,t) \leqslant S_n R_n + C_0$$

for all $s \in [-1, 1]$. On the other hand, from Proposition 92 we have that for every $R > 0$ there exist $\rho > 0$ and $N \in \mathbb{N}$ such that $\overline{a}_n(\theta_n^{-1}(s), t) \in [\overline{r}^-_n + R, \overline{r}^+_n - R]$ for all $n \geqslant N$ and all $(s,t) \in [-\rho, \rho] \times S^1$. Thus we obtain

$$a_n(\theta_n^{-1}(s), t) \in [\overline{r}^-_n - S_n R_n - C_0 + R, \overline{r}^+_n + S_n R_n + C_0 - R]$$

for all $n \geqslant N$ and all $(s,t) \in [-\rho, \rho] \times S^1$. If we assume that $S_n R_n \to \sigma$ as $n \to \infty$ we find

$$a_n(\theta_n^{-1}(s), t) \in [\overline{r}^-_n - \sigma - 1 - C_0 + R, \overline{r}^+_n + \sigma + 1 + C_0 - R]$$

for all $n \geqslant N$ and all $(s,t) \in [-\rho, \rho] \times S^1$. For $C := \sigma + 1 + C_0$ the last statement readily follows. The proof of Theorem 65 is finished.

□

Appendix C

Half cylinders with small energy

This appendix is devoted to the description of the convergence of a sequence of pseudoholomorphic half cylinders $u_n = (a_n, f_n) : [0,\infty) \times S^1 \to \mathbb{R} \times M$ with uniformly bounded $\alpha-$ and $d\alpha-$energies. More precisely, we assume that there exists a constant $\tilde{E}_0 > 0$ such that $E(u_n; [0,\infty) \times S^1) \leqslant \tilde{E}_0$ and

$$E_{d\alpha}(u_n; [0,\infty) \times S^1) \leqslant \frac{\hbar_0}{2}, \tag{C.0.1}$$

where $\hbar_0 > 0$ and $\tilde{E}_0$ is defined as in Section 3.2.1 Step 3. Since the $d\alpha-$energy is smaller than $\hbar_0/2$ it follows, from the usual bubbling-off analysis, that the gradients of u_n are uniformly bounded with respect to the standard Euclidean metric on the cylinder $[0,\infty) \times S^1$ and the induced cylindrical metric on $\mathbb{R} \times M$. To analyze the convergence of such a sequence we use the results of Appendix A and Appendix B. As before we split the analysis of the convergence in two parts, namely the $C^\infty_{\text{loc}}-$ and the C^0-convergence. Before stating the convergence results we need some auxiliary results similar to those from Appendix B. We begin with a remark on the asymptotic of a pseudoholomorphic half cylinder.

Remark 93. Let $u = (a, f) : [0,\infty) \times S^1 \to \mathbb{R} \times M$ be a pseudoholomorphic half cylinder with $E(u; [0,\infty) \times S^1) \leqslant \tilde{E}_0$ and $E_{d\alpha}(u; [0,\infty) \times S^1) \leqslant \hbar_0/2$. To describe the behavior of u as $s \to \infty$, we first assume that u has a bounded image in $\mathbb{R} \times M$. Consider the conformal transformation $h : [0,\infty) \times S^1 \to D\backslash\{0\}$, $(s,t) \mapsto e^{-2\pi(s+it)}$. Then $u \circ h^{-1} = (a \circ h^{-1}, f \circ h^{-1})$ is a pseudoholomorphic punctured disk satisfying the same assumption as u does. By the removal of singularity, $u \circ h^{-1}$ can be defined on the whole disk D. In this case we use the results from Appendix A to describe the convergence. If u has an unbounded image in $\mathbb{R} \times M$, then due to Proposition 5.6 from [6], there exists $T \neq 0$ and a periodic orbit x of X_α such that x is of period $|T|$ and

$$\lim_{s\to\infty} f(s,t) = x(Tt) \ \text{ and } \ \lim_{s\to\infty} \frac{a(s,t)}{s} = T \ \text{ in } C^\infty(S^1).$$

To analyze the convergence of the sequence of pseudoholomorphic half cylinders $u_n = (a_n, f_n) : [0,\infty) \times S^1 \to \mathbb{R} \times M$ we distinguish two cases.

In the first case each element of a subsequence of u_n, still denoted by u_n, has a bounded image in the symplectization $\mathbb{R} \times M$. By Remark 93 we consider the sequence of pseudoholomorphic disks $u_n \circ h^{-1} : D \to \mathbb{R} \times M$ having uniformly bounded energies and small $d\alpha-$energies. After applying bubbling-off analysis and accounting on the uniform energy bounds as well as on the small $d\alpha-$energies, we obtain a subsequence having uniform gradient bounds with respect to the Euclidean metric on the domains and the induced metric on $\mathbb{R} \times M$. After a specific shift in the $\mathbb{R}-$coordinate, $u_n \circ h^{-1}$ converge in C^∞ to a pseudoholomorphic disk $u : D \to \mathbb{R} \times M$.

In the second case each element of a subsequence of u_n, still denoted by u_n, has an unbounded image in $\mathbb{R} \times M$. In the following we assume that after a specific shift in the $\mathbb{R}-$coordinate, $a_n(0,0) = 0$. Before describing the convergence of u_n, we prove an asymptotic result for punctures which is similar to that given in [6].

Proposition 94. *After going over to a subsequence the pseudoholomorphic half cylinders u_n are asysmptotic to the same Reeb orbit, i.e. there exists a Reeb orbit x of period $|T| \neq 0$ with $|T| \leqslant C$ and a sequence $c_n \in S^1$ such that*

$$\lim_{s\to\infty} f_n(s,t) = x(T(t+c_n)) \quad \textit{and} \quad \lim_{s\to\infty} \frac{a_n(s,t)}{s} = T.$$

Moreover, $u_n \to u$ in C^∞_{loc}, where u is a pseudoholomorphic half cylinder $u : [0,\infty) \times S^1 \to \mathbb{R} \times M$ which is asymptotic to the same Reeb orbit $x(T(t+c^*))$ of period T as above. Here, $c^* \in S^1$ and $c_n \to c^*$ as $n \to \infty$.

Proof. Let the sequence u_n be asymptotic to some Reeb orbit. More precisely, for all $n \in \mathbb{N}$ there exist $T_n \neq 0$ and a periodic orbit x_n of period $|T_n|$ such that

$$\lim_{s\to\infty} f_n(s,t) = x_n(T_n t) \quad \text{and} \quad \lim_{s\to\infty} \frac{a_n(s,t)}{s} = T_n$$

in $C^\infty(S^1)$. For simplicity, choose a subsequence of T_n, also denoted by T_n, which is always positive (positive puncture). Since we are in the non-degenerate case and $T_n \leqslant E_0$, assume, after going to some subsequence, that $T_n = \overline{T} > 0$ and $x_n(\overline{T}t) = \overline{x}(\overline{T}(t+c_n))$, where $c_n \in S^1$ for all n. Thus after going over to some subsequence we may assume that $c_n \to c^* \in S^1$. From the uniform boundedness of the gradients of u_n, the elliptic regularity, and Arzelà-Ascoli theorem, we have $u_n \to u : [0,\infty) \times S^1 \to \mathbb{R} \times M$ in C^∞_{loc}. Here u is a pseuhoholomorphic half cylinder with bounded energy and a small $d\alpha-$energy which is asymptotic to some periodic orbit with period $\underline{T}$ or a point; both being denoted by $\underline{x}$. Choose the sequences $\underline{N}_n, \overline{N}_n \overset{n\to\infty}{\longrightarrow} \infty$ and $\underline{N}_n < \overline{N}_n$ such that after going over to a subsequence we have

$$\lim_{n\to\infty} f_n(\underline{N}_n, t) = \underline{x}(\underline{T}t) \quad \text{and} \quad \lim_{n\to\infty} f_n(\overline{N}_n, t) = \overline{x}(\overline{T}(t+c^*)) \quad \text{in } C^\infty(S^1),$$

and consider the maps

$$v_n = u_n|_{[\underline{N}_n, \overline{N}_n]\times S^1}.$$

which have by construction $d\alpha-$energy tending to 0. Performing the same analysis as in [6] we conclude that $\underline{x} = \overline{x}$ and $\underline{T} = \overline{T}$.

□

To describe the C^0-convergence of u_n we use the results established in Appendix B. In view of Proposition 94, choose a sequence $R_n > 0$ such that $R_n \to \infty$ and $a_n(R_n,t) - TR_n \to 0$ as $n \to \infty$. Consider the shifted maps $\overline{u}_n(s,t) := u_n(s+R_n,t) - TR_n$ for $(s,t) \in [-R_n, R_n] \times S^1$. These are pseudoholomorphic cylinders with uniformly bounded $\alpha-$ and $d\alpha-$energies and a $d\alpha-$energy smaller than $\hbar/2$. Recall that these pseudoholomorphic cylinders are a special case of the $\mathcal{H}-$holomorphic cylinders described in Appendix A. We distinguish two cases corresponding to subsequences with vanishing and non-vanishing center actions. In latter case, the cater action is greater than $\hbar > 0$. By Proposition 94, the first case does not appear and we are left with the case in which $A(\overline{u}_n) \geqslant \hbar$. By Corollary 90, for every $\epsilon > 0$ there exists $h > 0$ such that for all $n \in \mathbb{N}$ and $R_n > h$, $\text{dist}_{\overline{g}_0}(\overline{f}_n(s,t), x(T(t+c_n))) < \epsilon$ and $|\overline{a}_n(s,t) - Ts - a_0| < \epsilon$ for all $(s,t) \in [-R_n+h, R_n-h] \times S^1$. On the other hand, we have the following result: For every $\epsilon > 0$ there exists $h > 0$ such that for all $n \in \mathbb{N}$ and $R_n > h$, $\text{dist}_{\overline{g}_0}(f_n(s,t), x(T(t+c_n))) < \epsilon$ and $|a_n(s,t) - Ts - a_0| < \epsilon$ for all $(s,t) \in [h, 2R_n - h] \times S^1$. As R_n can be chosen arbitrary large the following equivalent statement readily follows:

Corollary 95. *For every $\epsilon > 0$ there exist $h > 0$ and $N \in \mathbb{N}$ such that for all $n \geqslant N$, $dist_{\overline{g}_0}(f_n(s,t), x(T(t+c_n))) < \epsilon$ and $|a_n(s,t) - Ts - a_0| < \epsilon$ for all $(s,t) \in [h,\infty) \times S^1$.*

Consider the diffeomorphism $\theta : [0,\infty) \times S^1 \to \mathbb{R} \times M$ and the maps

$$g_n := f_n \circ \theta^{-1} : [0,1) \times S^1 \to M, \tag{C.0.2}$$

which by Proposition 94 converge in C^∞_{loc} to a map $g := f \circ \theta^{-1} : [0,1) \times S^1 \to M$. By Corollary 95, the maps g_n and g can be continously extended to $[0,1] \times S^1$ by $g_n(1,t) = g(1,t) = x(Tt+c_n)$ for all $n \in \mathbb{N}$ and all $t \in S^1$. Hence due to Corollary 95, g_n converge in C^0 to g. As a consequence, we formulate the following compactness property of the sequence of pseudoholomorphic half cylinders $u_n : [0,\infty) \times S^1 \to \mathbb{R} \times M$ with uniformly bounded energies and $d\alpha-$energies less than $\hbar/2$:

Theorem 96. *Let u_n be a sequence of pseudoholomorphic curves having uniformly bounded energy by E_0 and satisfying condition (C.0.1). Then there exists a subsequence of u_n, still denoted by u_n, such that the following is satisfied.*

1. *u_n is asysmptotic to the same Reeb orbit, i.e. there exists a Reeb orbit x of period $|T| \neq 0$ with $|T| \leqslant C$ and a sequence $c_n \in S^1$ such that*

$$\lim_{s\to\infty} f_n(s,t) = x(T(t+c_n)) \quad \textit{and} \quad \lim_{s\to\infty} \frac{a_n(s,t)}{s} = T.$$

   for all $n \in \mathbb{N}$.

2. *u_n converge in C^∞_{loc} to a pseudoholomorphic half cylinder $u : [0,\infty) \times S^1 \to \mathbb{R} \times M$ having uniformly bounded energy by the constant E_0 and satisfying condition (C.0.1).*

3. *The maps $g_n : [0,1] \times S^1 \to M$ converge in C^0 to a map $g : [0,1] \times S^1 \to M$ and satisfy $g(1,t) = x(T(t+c^*))$, where x is a Reeb orbit of period $|T| \neq 0$.*

Appendix D

Special coordinates

Let S be a compact surface with boundary, and let j_n and j be complex structures on S for all $n \in \mathbb{N}$. Additionally, let h_n and h be the hyperbolic structures on S with respect to j_n and j, respectively. Assume that $j_n \to j$ and $h_n \to h$ in $C^\infty(S)$. In this appendix we construct a sequence of biholomorphic coordinates around some point in S with respect to the complex structure j_n that converges in a certain sense to the biholomorphic coordinates with respect to j. This result is used in Section 3 for proving the convergence on the thick part.

Lemma 97. *For each $z \in int(S)$ there exist open neighborhoods $U_n(z) = U_n$ and $U(z) = U$ of z and diffeomorphisms*

$$\begin{aligned} \psi_n : D_1(0) &\to U_n, \\ \psi : D_1(0) &\to U \end{aligned}$$

such that

1. *ψ_n are $i - j_n$—biholomorphisms and ψ is a $i - j$—biholomorphism;*
2. *$\psi_n \to \psi$ in $C^\infty_{loc}(D_1(0))$ as $n \to \infty$ with respect to the Euclidean metric on $D_1(0)$ and h on S;*
3. *$\psi_n(0) = z$ for every n and $\psi(0) = z$.*

Proof. Around $z \in \text{int}(S)$, choose the $i - j$—holomorphic coordinates $c : D_2(0) \to U$ such that $U \subset \text{int}(S)$ and $c(0) = z$, and consider the complex structures $j^{(n)} := c^* j_n$. Since $j_n \to j$ as $n \to \infty$ in C^∞, $j^{(n)} \to i$ in $C^\infty_{\text{loc}}(D_2(0))$ as $n \to \infty$. Let $d^{\mathbb{C}}_n$ be the operator defined by $d^{\mathbb{C}}_n f = df \circ j^{(n)}$ and let $d^{\mathbb{C}}$ be the operator defined by $d^{\mathbb{C}} f = df \circ i$. Denote by $p_x : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x$ the projection onto the first coordinate. Consider the problem of finding a smooth function $f : \overline{D_1(0)} \to \mathbb{R}$ such that

$$\begin{aligned} dd^{\mathbb{C}}_n f &= 0 \text{ on } D_1(0), \\ f &= p_x \text{ on } \partial D_1(0) \end{aligned} \tag{D.0.1}$$

for all n and

$$\begin{aligned} dd^{\mathbb{C}} f &= 0 \text{ on } D_1(0), \\ f &= p_x \text{ on } \partial D_1(0). \end{aligned} \tag{D.0.2}$$

As the second problem translates into

$$\begin{aligned} \Delta f &= 0 \text{ on } D_1(0), \\ f &= p_x \text{ on } \partial D_1(0), \end{aligned} \tag{D.0.3}$$

where Δ is the standard Laplace operator in $\mathbb{R}$, the unique solution is $f(x,y)=x$ for all $(x,y)\in\overline{D_1(0)}$. To see the uniqueness observe that the difference of f with any other solution of (D.0.3) solves $\Delta u=0$ with $u|_{\partial D_1(0)}=0$. Thus from the maximum principle for harmonic functions we deduce that $u\equiv 0$, and so, that (D.0.3) has the unique solution f. In coordinates representation, $j^{(n)}$ can be written as

$$j^{(n)}=\begin{pmatrix} j_{11}^{(n)} & j_{12}^{(n)} \\ j_{21}^{(n)} & j_{22}^{(n)} \end{pmatrix}$$

and take notice that $j^{(n)}\to i$ in C^∞ on $D_1(0)$ as $n\to\infty$. The solutions of (D.0.1) are equivalent to the solutions of

$$\begin{aligned} dd_n^{\mathbb{C}}\tilde{f} &= t_n \text{ on } D_1(0), \\ \tilde{f} &= 0 \text{ on } \partial D_1(0), \end{aligned} \tag{D.0.4}$$

where $t_n=-dd_n^{\mathbb{C}}p_x$. Hence $dd_n^{\mathbb{C}}$ is an elliptic and coercive operator, and thus by Proposition 5.10 from [18], the problem (D.0.4) has a uniquely weak solution $\tilde{f}_n\in W^{1,2}(D_1(0))$ for all n. From regularity theorem, the solutions $\tilde{f}_n$ are smooth for all n. Thus $f_n:=\tilde{f}_n+p_x$ is the smooth unique solution of (D.0.1). Let us show that $f_n\to f$ in $C^\infty_{loc}(D_1(0))$ as $n\to\infty$. For $u_n:=f_n-f$ we have

$$\begin{aligned} dd_n^{\mathbb{C}}u_n &= g_n \text{ on } D_1(0), \\ u_n &= 0 \text{ on } \partial D_1(0). \end{aligned}$$

Here, $g_n\in C^\infty(D_1(0))$ is defined by $g_n:=dd_n^{\mathbb{C}}f$, and because of $j^{(n)}\to i$ in $C^\infty(D_1(0))$ as $n\to\infty$, g_n converges to 0 in $C^\infty_{loc}(D_1(0))$ as $n\to\infty$. For every $m\in\mathbb{N}_0$ we consider the bounded operator $dd_n^{\mathbb{C}}:W_\partial^{2+m,2}(D_1(0),\mathbb{R})\to W^{m,2}(D_1(0),\mathbb{R})$, where $W_\partial^{2+m,2}(D_1(0),\mathbb{R})$ consists of maps from $W^{2+m,2}(D_1(0),\mathbb{R})$ that vanish at the boundary. By Proposition 5.10 together with Propositions 5.18 and 5.19 of [18] we deduce that the operator $dd_n^{\mathbb{C}}$ is bounded invertible; hence $u_n=(dd_n^{\mathbb{C}})^{-1}g_n$. Since $dd_n^{\mathbb{C}}\to\Delta$ in operator norm, $(dd_n^{\mathbb{C}})^{-1}$ is a uniformly bounded family, and so, $\|u_n\|_{W^{m+2,2}}\to 0$ as $n\to\infty$. Further on, as $m\in\mathbb{N}_0$ was arbitrary, the Sobolev embedding theorem yields $u_n\to 0$ in $C^\infty_{loc}(D_1(0))$ as $n\to\infty$. Thus we have constructed a unique sequence of solutions $\{f_n:\overline{D_1(0)}\to\mathbb{R}\}_{n\in\mathbb{N}}$ of (D.0.1), and a unique solution $f:\overline{D_1(0)}\to\mathbb{R}$, $(x,y)\mapsto x$ of (D.0.2) satisfying $f_n\to f$ in $C^\infty_{loc}(D_1(0))$ as $n\to\infty$. According to Lemma 6.8.1 of [16], there exists a $j^{(n)}$—i—holomorphic function $F_n:D_1(0)\to\mathbb{C}$ and a i—i—holomorhic function $F:D_1(0)\to\mathbb{C}$ such that $f_n=\Re(F_n)$ and $f=\Re(F)$. Let us investigate the unique extensions of the functions F_n and F. For doing this we set $F_n=f_n+ib$ and $F=f+ib$, where $b_n,b:D_1(0)\to\mathbb{R}$ are harmonic functions. As F_n and F are $j^{(n)}-i$—holomorphic and $i-i$—holomorphic, respectively, they solve the equations

$$dF_n+i\circ dF_n\circ j^{(n)}=0$$

and

$$dF+i\circ dF\circ i=0,$$

respectively, which in turn, are equivalent to

$$db_n=-df_n\circ j^{(n)}$$

and

$$db=-df\circ i,$$

respectively. By the harmonicity of f_n and f, and the application of Poincare lemma on $D_1(0)$, we find the solutions b_n and b which are unique up to addition with some constant. They can be make unique by requiring that $b_n(0)=0$ and $b(0)=0$. In particular, we find $F(x,y)=x+iy$. Then we get $db_n\to db$ in $C^\infty_{loc}(D_1(0))$ as $n\to\infty$, and from $b_n(0)=0$ and $b(0)=0$, we actually get $b_n\to b$ in $C^\infty_{loc}(D_1(0))$ as $n\to\infty$. Hence $F_n\to F=\mathrm{id}$ in $C^\infty_{loc}(D_1(0))$ as $n\to\infty$.

For n large, F_n is bijective onto its image (maybe after shrinking the domain). This follows from the proof of the inverse function theorem. With $\tilde{F}_n = F_n - f_n(0)$, the maps ψ_n and ψ are defined by $\psi_n = c \circ \tilde{F}_n : D_1(0) \to U_n$ and $\psi = c \circ F : D_1(0) \to U$ for sufficiently large n, respectively.

$\square$

Appendix E

Asymptotics of Harmonic Cylinders

In this section we describe the C^∞_{loc}- and C^0- convergence of a sequence of harmonic functions Γ_n on cylinders $[-R_n, R_n] \times S^1$. This result is used in the proof of Theorems 63 and 65. The analysis is performed in the following setting:

R1 $R_n \to \infty$;

R2 Γ_n is a harmonic function on $[-R_n, R_n] \times S^1$ such that $d\Gamma_n$ is a harmonic 1−form with respect to the standard complex structure i on $\mathbb{R}\times S^1$, i.e. if s, t are the coordinates on $\mathbb{R}\times S^1$, $\partial_s\Gamma_n + i\partial_t\Gamma_n : [-R_n, R_n]\times S^1 \to \mathbb{C}$ is holomorphic;

R3 Γ_n has vanishing average over the cylinder $[-R_n, R_n] \times S^1$, i.e. for all $n \in \mathbb{N}$ we have

$$\frac{1}{2R_n}\int_{[-R_n,R_n]\times S^1} \Gamma_n(s,t)dsdt = 0;$$

R4 the L^2−norm of $d\Gamma_n$ is uniformly bounded, i.e. there exists a constant $C > 0$ such that

$$\|d\Gamma_n\|^2_{L^2([-R_n,R_n]\times S^1)} := \int_{[-R_n,R_n]\times S^1} d\Gamma_n \circ i \wedge d\Gamma_n \leqslant C$$

for all $n \in \mathbb{N}$.

The subsequent lemma gives a decomposition of Γ_n in a linear term and a harmonic function satisfying properties R1-R4 and having a uniformly bounded L^2−norm.

Lemma 98. *There exists a sequence $S_n \in \mathbb{R}$ with $|S_n| \leqslant \sqrt{C/2R_n}$ such that the harmonic function $\Gamma_n : [-R_n, R_n]\times S^1 \to \mathbb{R}$ can be decomposed as $\Gamma_n(s,t) = S_n s + \tilde{\Gamma}_n(s,t)$, where $\tilde{\Gamma}_n : [-R_n, R_n]\times S^1 \to \mathbb{R}$ is a harmonic function satisfying properties R1-R4 and additionally*

$$\|\tilde{\Gamma}_n\|^2_{L^2([-R_n,R_n]\times S^1)} \leqslant \|d\tilde{\Gamma}_n\|^2_{L^2([-R_n,R_n]\times S^1)}. \tag{E.0.1}$$

Proof. We consider the Fourier series of the harmonic function Γ_n, i.e.

$$\Gamma_n(s,t) = \sum_{k\in\mathbb{Z}} c^n_k(s)e^{2\pi ikt} = c^n_0(s) + \sum_{k\in\mathbb{Z}\setminus\{0\}} c^n_k(s)e^{2\pi ikt}.$$

Because Γ_n has vanishing mean value, we have

$$0=\int_{[-R_n,R_n]\times S^1}\Gamma_n(s,t)dsdt=\int_{-R_n}^{R_n}\int_0^1\Gamma_n(s,t)dtds=\int_{-R_n}^{R_n}c_0^n(s)ds. \tag{E.0.2}$$

As Γ_n is a harmonic function, the coefficients c_k^n can be readily computed; we find

$$c_k^n(s)=\begin{cases}A_k^n\sinh(2\pi ks)+B_k^n\cosh(2\pi ks), & k\in\mathbb{Z}\backslash\{0\}\\ S_ns+d_n, & k=0\end{cases},$$

where $A_k^n, B_k^n, S_n, d_n\in\mathbb{C}$. By (E.0.2), $d_n=0$, and the Fourier expansion of Γ_n takes the form

$$\Gamma_n(s,t)=S_ns+\sum_{k\in\mathbb{Z}\backslash\{0\}}c_k^n(s)e^{2\pi ikt}=S_ns+\tilde\Gamma_n(s,t),$$

where

$$\tilde\Gamma_n(s,t)=\Gamma_n(s,t)-S_ns=\sum_{k\in\mathbb{Z}\backslash\{0\}}c_k^n(s)e^{2\pi ikt}. \tag{E.0.3}$$

For every $s\in[-R_n,R_n]$ we have

$$S_n=\int_{\{s\}\times S^1}\partial_s\Gamma_n(s,t)dt\in\mathbb{R}$$

and so, $\tilde\Gamma_n$ is a real valued harmonic function. On the other hand by Hölder's inequality we find the estimate

$$|S_n|\leqslant\frac{1}{2R_n}\int_{[-R_n,R_n]\times S^1}|\partial_s\Gamma_n(s,t)|dsdt\leqslant\sqrt{\frac{C}{2R_n}}.$$

We show now that $d\tilde\Gamma_n$ has a uniform L^2—bound. By (E.0.3) and Hölder's inequality we get

$$\begin{aligned}\|d\tilde\Gamma_n\|^2_{L^2([-R_n,R_n]\times S^1)}&=\|d\Gamma_n\|^2_{L^2([-R_n,R_n]\times S^1)}-2S_n\int_{[-R_n,R_n]\times S^1}d\Gamma_n\circ i\wedge ds\\&\quad+2S_n^2R_n\\&\leqslant 4C.\end{aligned}$$

Thus $\tilde\Gamma_n$ satisfies the property R4 from above, and obviously, properties R1-R3. Next we prove estimate (E.0.1). By (E.0.3), the L^2—norm of $\tilde\Gamma_n$ computes as follows

$$\|\tilde\Gamma_n\|^2_{L^2([-R_n,R_n]\times S^1)}=\sum_{k\in\mathbb{Z}\backslash\{0\}}\|c_k^n\|^2_{L^2([-R_n,R_n])}.$$

On the other hand we have

$$\partial_t\tilde\Gamma_n(s,t)=\sum_{k\in\mathbb{Z}\backslash\{0\}}2\pi ikc_k^n(s)e^{2\pi ikt}$$

and

$$\|\partial_t\tilde\Gamma_n\|^2_{L^2([-R_n,R_n]\times S^1)}=\sum_{k\in\mathbb{Z}\backslash\{0\}}4\pi^2k^2\|c_k^n\|^2_{L^2([-R_n,R_n])}\geqslant\|\tilde\Gamma_n\|^2_{L^2([-R_n,R_n]\times S^1)},$$

while from

$$\|\partial_t\tilde\Gamma_n\|^2_{L^2([-R_n,R_n]\times S^1)}\leqslant\|d\tilde\Gamma_n\|^2_{L^2([-R_n,R_n]\times S^1)}$$

we end up with

$$\|\tilde{\Gamma}_n\|^2_{L^2([-R_n,R_n]\times S^1)} \leqslant \|d\tilde{\Gamma}_n\|^2_{L^2([-R_n,R_n]\times S^1)}.$$

□

Remark 99. The quantity S_n can be interpreted as the co-period of the harmonic 1–form $d\Gamma_n$ over the closed curve $\{0\}\times S^1$ with respect to the standard complex structure i on $\mathbb{R}\times S^1$.

In particular, we see that for all n we have

$$\|\tilde{\Gamma}_n\|^2_{L^2([-R_n,R_n]\times S^1)} \leqslant 4C. \tag{E.0.4}$$

The next lemma establishes uniform bounds on the derivatives of $\tilde{\Gamma}_n$.

Lemma 100. *For any $\delta > 0$ and $k \in \mathbb{N}_0$ there exists a constant $\tilde{C} = \tilde{C}(\delta, k, C) > 0$ such that*

$$\|\tilde{\Gamma}_n\|_{C^k([-R_n^\delta,R_n^\delta]\times S^1)} \leqslant \tilde{C}$$

for all $n \in \mathbb{N}$ and $R_n^\delta := R_n - \delta$.

Proof. Set $F_n := \partial_s\tilde{\Gamma}_n + i\partial_t\tilde{\Gamma}_n : [-R_n, R_n]\times S^1 \to \mathbb{C}$, and note that F_n is a holomorphic function with uniformly bounded L^2–norm, i.e.

$$\int_{[-R_n,R_n]\times S^1} |F_k|^2 ds dt \leqslant 4C \tag{E.0.5}$$

for all $n \in \mathbb{N}$. As $\tilde{\Gamma}_n$ is harmonic it is obvious that

$$\Delta|F_n|^2 = 2|\nabla F_n|^2 \geqslant 0.$$

Hence $|F_n|^2$ is subharmonic. By using the mean value property for subharmonic functions we conclude that for any $\delta > 0$ and any $z = (s,t) \in [-R_n^{\delta/2}, R_n^{\delta/2}]\times S^1$,

$$|F_n(z)|^2 \leqslant \frac{32}{\pi\delta^2}\int_{B_{\frac{\delta}{4}}(z)} |F_n(s,t)|^2 ds dt \leqslant \frac{32}{\pi\delta^2}\|F_n\|^2_{L^2([-R_n^{\frac{\delta}{2}},R_n^{\frac{\delta}{2}}]\times S^1)}.$$

Since these estimates hold for all $z \in [-R_n^{\delta/2}, R_n^{\delta/2}]\times S^1$ we obtain

$$\|F_n\|^2_{C^0([-R_n^{\frac{\delta}{2}},R_n^{\frac{\delta}{2}}]\times S^1)} \leqslant \frac{32}{\pi\delta^2}\|F_n\|^2_{L^2([-R_n^{\frac{\delta}{2}},R_n^{\frac{\delta}{2}}]\times S^1)}.$$

In particular, by using (E.0.5), we find

$$\|F_n\|_{C^0([-R_n^{\frac{\delta}{2}},R_n^{\frac{\delta}{2}}]\times S^1)} \leqslant \frac{8\sqrt{2C}}{\delta\sqrt{\pi}} \tag{E.0.6}$$

for all $n \in \mathbb{N}$. By the Cauchy integral formula for holomorphic functions and (E.0.6) we deduce that the derivatives of F_n are uniformly bounded on $[-R_n^\delta, R_n^\delta]\times S^1$. Indeed, for $k \in \mathbb{N}$ we have

$$|F_n^{(k)}(z)| = \frac{k!}{2\pi}\left|\int_{\partial B_{\frac{\delta}{2}}(z)} \frac{F_n(\xi)}{(\xi - z)^{k+1}} d\xi\right| = \frac{k!}{2\pi}\left|\int_0^{2\pi} 2^k i\frac{F_n(z+\delta e^{it})}{\delta^k e^{ikt}} dt\right| \leqslant \frac{2^{k+3}k!\sqrt{2C}}{\delta^{k+1}\sqrt{\pi}}$$

for all $z \in [-R_n^\delta, R_n^\delta] \times S^1$ and $n \in \mathbb{N}$. Since $z \in [-R_n^\delta, R_n^\delta] \times S^1$ was arbitrary, we obtain

$$\left\| F_n^{(k)} \right\|_{C^0([-R_n^\delta, R_n^\delta] \times S^1)} \leqslant \frac{2^{k+3} k! \sqrt{2C}}{\delta^{k+1} \sqrt{\pi}}.$$

Using (E.0.4) and the mean value property and Hölder inequality for $\tilde{\Gamma}_n$ we find that for all $z \in [-R_n^\delta, R_n^\delta] \times S^1$,

$$|\tilde{\Gamma}_n(z)| \leqslant \frac{4}{\pi \delta^2} \int_{B_{\frac{\delta}{2}}(z)} |\tilde{\Gamma}_n(s,t)| ds dt = \frac{4\sqrt{C}}{\delta \sqrt{\pi}}.$$

Hence we get

$$\|\tilde{\Gamma}_k\|_{C^0([-R_n^\delta, R_n^\delta] \times S^1)} \leqslant \frac{4\sqrt{C}}{\delta \sqrt{\pi}}$$

for all $n \in \mathbb{N}$.

$\square$

Remark 101. We note the following.

1. From the proof of Lemma 100 the following result can be established: By the Arzelà-Ascoli theorem, for any sequence $s_n \in [-R_n^\delta, R_n^\delta]$, the sequence of maps $F_n(\cdot + s_n, \cdot)$ defined on $[-R_n^\delta - s_n, R_n^\delta - s_n] \times S^1$, where $F_n = \partial_s \tilde{\Gamma}_n + i \partial_t \tilde{\Gamma}_n$, contains a subsequence, also denoted by $F_n(\cdot + s_n, \cdot)$, that converges in C^∞_{loc} to some holomorphic map F; F depends on the sequence $\{s_n\}$, has bounded L^2- and C^0- norms, and is defined either on a half cylinder or on $\mathbb{R} \times S^1$. In the later case, when $R_n^\delta - s_n$ and $R_n^\delta + s_n$ diverge, F has to be 0. Indeed, by Liouville's theorem, F has to be constant, while from the boundedness of the L^2—norm we conclude that F is 0.

2. By Lemma 100, (E.0.4) and the Liouville theorem for harmonic functions, $\tilde{\Gamma}_n(0, \cdot)$ converges to 0. By Lemma 100 and Remark 101, the sequence of harmonic functions $\tilde{\Gamma}_n(\cdot + s_n, \cdot)$ with $s_n \in [R_n^\delta, R_n^\delta]$, contains a subsequence that converges in C^∞_{loc} to some harmonic function defined either on a half cylinder or on $\mathbb{R} \times S^1$. In the later case the limit harmonic function has to be 0 by the same arguments as above.

To simplify notation we drop the index δ. We define the harmonic functions $\tilde{\Gamma}_n^- : [0, 2R_n] \times S^1 \to \mathbb{R}$ and $\tilde{\Gamma}_n^+ : [-2R_n, 0] \times S^1 \to \mathbb{R}$ by $\tilde{\Gamma}_n^-(s,t) := \tilde{\Gamma}_k(s - R_n, t)$ and $\tilde{\Gamma}_n^+(s,t) := \tilde{\Gamma}_n(s + R_n, t)$, respectively. By Lemma 100, there exist harmonic functions $\tilde{\Gamma}^- : [0, +\infty) \times S^1 \to \mathbb{R}$ and $\tilde{\Gamma}^+ : (-\infty, 0] \times S^1 \to \mathbb{R}$ such that $\tilde{\Gamma}_n^- \xrightarrow{C^\infty_{loc}} \tilde{\Gamma}^-$ and $\tilde{\Gamma}_n^+ \xrightarrow{C^\infty_{loc}} \tilde{\Gamma}^+$. The next proposition plays an important role in establishing a C^∞_{loc}- and C^0- convergence of the harmonic functions $\tilde{\Gamma}_n$.

Proposition 102. *For any $\epsilon > 0$ there exists $h > 0$ such that for any $R_n > h$ we have*

$$\|\tilde{\Gamma}_n\|_{C^0([-R_n + h, R_n - h] \times S^1)} < \epsilon.$$

Proof. Assume that this is not the case. Then there exist $\epsilon_0, C_0 > 0$ such that for any $h_k := k$ there exist $R_{n_k} > k$ and a sequence $(s_k, t_k) \in [-R_{n_k} + k, R_{n_k} - k] \times S^1$ such that $|\tilde{\Gamma}_{n_k}(s_k, t_k)| \geqslant \epsilon_0$. From $s_k \in [-R_{n_k} + k, R_{n_k} - k]$ it follows that $|R_{n_k} - s_k| \to \infty$ as $k \to \infty$. Consider the harmonic functions $H_k : [-R_{n_k} - s_k, R_{n_k} - s_k] \times S^1 \to \mathbb{R}$ defined by $H_k(s,t) = \tilde{\Gamma}_{n_k}(s + s_k, t)$. Obviously, we have $H_k(0, t_k) = \tilde{\Gamma}_{n_k}(s_k, t_k)$ and by Remark 101 we conclude that the H_k converge in C^∞_{loc} to some harmonic function $H : \mathbb{R} \times S^1 \to \mathbb{R}$ with bounded L^2 and C^0—norms. By the Liouville theorem for harmonic functions, $H \equiv 0$. This gives a contradiction to $|H_k(0, t_k)| = |\tilde{\Gamma}_{n_k}(s_k, t_k)| \geqslant \epsilon_0$, and the proof is finished.

□

Corollary 103. *For every sequence $h_n \in \mathbb{R}_+$ satisfying $h_n < R_n$ and $h_n, R_n/h_n \to \infty$ and every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$\|\tilde{\Gamma}_n\|_{C^0([-R_n+h_n, R_n-h_n]\times S^1)} < \epsilon$$

for all $n \geqslant N$. Moreover, for the co-period S_n we obtain that $h_n S_n \to 0$ as $n \to \infty$.

Proof. Consider a sequence $h_n \in \mathbb{R}_+$ with $h_n < R_n$ and $h_n, R_n/h_n \to \infty$ as $n \to \infty$ and let $\epsilon > 0$ be given. By Proposition 102, there exist $h_\epsilon > 0$ and $N_\epsilon \in \mathbb{N}$ such that for all $n \geqslant N_\epsilon$ we have $R_n > h_\epsilon$ and $\|\tilde{\Gamma}_n\|_{C^0([-R_n+h_\epsilon, R_n-h_\epsilon]\times S^1)} < \epsilon$. By taking N_ϵ sufficiently large and since $h_n \to \infty$ we assume that for all $n \geqslant N_\epsilon$, we have $R_n > h_n > h_\epsilon$, giving $\|\tilde{\Gamma}_n\|_{C^0([-R_n+h_n, R_n-h_n]\times S^1)} < \epsilon$. It follows from $R_n S_n \to \sigma$ and $R_n/h_n \to \infty$ as $n \to \infty$ that $h_n S_n \to 0$ as $n \to \infty$.

□

In the following the subsequence $\tilde{\Gamma}_k$ will be denoted by $\tilde{\Gamma}_n$. To describe the C^0-convergence of the maps $\tilde{\Gamma}_n$ for a sequence $h_n \in \mathbb{R}_+$with $h_n < R_n$ and $h_n, R_n/h_n \to \infty$ as $n \to \infty$, we consider the sequence of diffeomorphisms θ_n defined in Remark 44. Further on, let us introduce the maps

$$\begin{aligned}
\overline{\Gamma}_n(s,t) &= \tilde{\Gamma}_n(\theta_n^{-1}(s),t),\ \ s \in [-1,1],\\
\overline{\Gamma}_n^-(s,t) &= \tilde{\Gamma}_n^-((\theta_n^-)^{-1}(s),t),\ \ s \in [-1,-1/2],\\
\overline{\Gamma}_n^+(s,t) &= \tilde{\Gamma}_n^+((\theta_n^+)^{-1}(s),t),\ \ s \in [1/2,1],\\
\overline{\Gamma}^-(s,t) &= \tilde{\Gamma}^-((\theta^-)^{-1}(s),t),\ \ s \in [-1,-1/2),\\
\overline{\Gamma}^+(s,t) &= \tilde{\Gamma}^+((\theta^+)^{-1}(s),t),\ \ s \in (1/2,1].
\end{aligned}$$

We prove the following

Theorem 104. *For every sequence $h_n \in \mathbb{R}_+$ satisfying $h_n < R_n$ and $h_n, R_n/h_n \to \infty$ as $n \to \infty$, the following convergence results hold for the maps $\tilde{\Gamma}_n$ and $\overline{\Gamma}_n$ and their left and right shifts $\tilde{\Gamma}_n^\pm$ and $\overline{\Gamma}_n^\pm$, respectively.*

$C^\infty_{loc}-$convergence:

1. *For any sequence $s_n \in [-R_n+h_n, R_n-h_n]$ there exists a subsequence of the sequence of shifted harmonic functions $\tilde{\Gamma}_n(\cdot+s_n,\cdot)$, also denoted by $\tilde{\Gamma}_n(\cdot+s_n,\cdot)$, which is defined on $[-R_n+h_n-s_n, R_n-h_n-s_n]\times S^1$ and converges in C^∞_{loc} to 0.*
2. *The harmonic functions $\tilde{\Gamma}_n^- : [0,h_n]\times S^1 \to \mathbb{R}$ converge in C^∞_{loc} to a harmonic function $\tilde{\Gamma}^- : [0,+\infty)\times S^1 \to \mathbb{R}$ which is asymptotic to 0. Furthermore, $\overline{\Gamma}_n^- : [-1,-1/2]\times S^1 \to \mathbb{R}$ converge in $C^\infty_{loc}([-1,-1/2)\times S^1)$ to the map $\overline{\Gamma}^- : [-1,-1/2)\times S^1 \to \mathbb{R}$ that is asymptotic to 0 at $\{-1/2\}\times S^1$.*
3. *The harmonic functions $\tilde{\Gamma}_n^+ : [-h_n,0]\times S^1 \to \mathbb{R}$ converge in C^∞_{loc} to a harmonic function $\tilde{\Gamma}^+ : (-\infty,0]\times S^1 \to \mathbb{R}$ which is asymptotic to 0. Furthermore, $\overline{\Gamma}_n^+ : [1/2,1]\times S^1 \to \mathbb{R}$ converge in $C^\infty_{loc}((1/2,1]\times S^1)$ to the map $\overline{\Gamma}^+ : (1/2,1]\times S^1 \to \mathbb{R}$ that is asymptotic to 0 at $\{1/2\}\times S^1$.*

C^0-convergence:

1. *The functions $\overline{\Gamma}_n$ converge in $C^0([-1/2,1/2]\times S^1)$ to 0.*

2. *The functions* $\overline{\Gamma}_n^-$ *converge in* $C^0([-1,-1/2]\times S^1)$ *to a function* $\overline{\Gamma}^- : [-1,-1/2]\times S^1 \to \mathbb{R}$ *with* $\overline{\Gamma}^-(-1/2,t) = 0$ *for all* $t \in S^1$.
3. *The functions* $\overline{\Gamma}_n^+$ *converge in* $C^0([1/2,1]\times S^1)$ *to a function* $\overline{\Gamma}^+ : [1/2,1]\times S^1 \to \mathbb{R}$ *with* $\overline{\Gamma}^+(1/2,t) = 0$ *for all* $t \in S^1$.

Proof. First we prove the C^∞_{loc}−convergence of the harmonic functions Γ_n.

1. By Remark 101, for any sequence $s_n \in [-R_n+h_n, R_n-h_n]$ the sequence of shifted harmonic maps $\Gamma_n(\cdot+s_n,\cdot)$ contains a subsequence, also denoted by $\Gamma_n(\cdot+s_n,\cdot)$, which is defined on $[-R_n+h_n-s_n, R_n-h_n-s_n]\times S^1$ and converges in C^∞_{loc} to 0.
2. Consider the shifted maps $\Gamma_n^- : [0,h_n]\times S^1 \to \mathbb{R}\times M$. By Lemma 100, these maps have uniformly bounded derivatives, and hence after going over to some subsequence, they converge in $C^\infty_{\text{loc}}([0,\infty)\times S^1)$ to the harmonic function $\Gamma^- : [0,+\infty)\times S^1 \to \mathbb{R}\times M$. In the following we show that Γ^- is asymptotic to 0, i.e. that $\lim_{s\to\infty}\Gamma^-(s,t) = 0$. We prove by contradiction. Because the Γ_n are uniformly bounded in C^0, we assume that there exists a sequence $(s_k,t_k)\in[0,\infty)\times S^1$ with $s_k\to\infty$ as $k\to\infty$ such that $\lim_{k\to\infty}\Gamma^-(s_k,t_k) = w \in \mathbb{R}\backslash\{0\}$. Putting $\epsilon := |w| > 0$, using Proposition 102, and arguing as in Theorem 63 we are led to the contradiction $\epsilon = |w| \leqslant 3\epsilon/10$. As $(\theta_n^-)^{-1} : [-1,-1/2] \to [0,h_n]$ converge in C^∞_{loc} to the diffeomorphism $(\theta^-)^{-1} : [-1,-1/2) \to [0,+\infty)$, the maps $\Gamma_n^-((\theta_n^-)^{-1}(s),t)$ converge in C^∞_{loc} to the map $\Gamma^-((\theta^-)^{-1}(s),t)$. By the asymptotics of Γ^-, $\overline{\Gamma}^-$ is asymptotic to 0 at $\{-1/2\}\times S^1$.
3. The proof for the maps Γ_n^+ proceeds as in Case 2.

To prove the C^0−convergence of the harmonic functions Γ_n, we prove that the functions $\overline{\Gamma}_n$, $\overline{\Gamma}_n^-$ and $\overline{\Gamma}_n^+$ converge in C^0.

1. From Corollary 103 it follows that $\left\|\overline{\Gamma}_n\right\|_{C^0([-1/2,1/2]\times S^1)} \to 0$ as $n\to\infty$.
2. Consider now the maps $\overline{\Gamma}_n^-(s,t)$. The $\overline{\Gamma}_n^-$ converge in C^∞_{loc} to $\overline{\Gamma}^-$ on $[-1,-1/2)\times S^1$. By the asymptotics of Γ^-, $\overline{\Gamma}^-$ can be continously extended to the whole cylinder $[-1,-1/2]\times S^1$ by setting $\overline{\Gamma}^-(-1/2,t) = 0$. As a matter of fact, the maps $\overline{\Gamma}_n^-$ converge in $C^0([-1,-1/2])$ to $\overline{\Gamma}^-$. The proof of this statement is as in Lemma 4.16 of [7], and for completeness reasons, it is here described. Let $\delta > 0$ be given. By the C^∞_{loc}−convergence of the maps $\overline{\Gamma}_n^-$ to $\overline{\Gamma}^-$ on $[-1,-1/2)\times S^1$ it suffices to find $\sigma > 0$ and $N\in\mathbb{N}$ such that $|\overline{\Gamma}_n^-(s,t)| \leqslant \delta$ for all $(s,t)\in[-(1/2)-\sigma,-1/2]\times S^1$ and $n \geqslant N$. From Proposition 102 there exist $N\in\mathbb{N}$ and $h > 0$ such that for all $n \geqslant N$ and $(s,t)\in[-R_n+h, R_n-h]\times S^1$, we have $|\Gamma_n(s,t)| \leqslant \delta$. Recall that $(\theta^-)^{-1}$ maps $[-1,-1/2)$ diffeomorphically onto $[0,\infty)$. Thus we find $\sigma > 0$ such that $(\theta^-)^{-1}(-\sigma) \geqslant h+1$. By the C^∞_{loc}−convergence, we obtain $\theta_n^{-1}(-\sigma) + R_n = (\theta_n^-)^{-1}(-\sigma) \geqslant h$ for n sufficiently large; hence, $\theta_n^{-1}(-\sigma) \geqslant -R_n + h$. Therefore, by the monotonicity of θ_n, we have $\theta_n^-([-(1/2)-\sigma,-1/2]) \subset [-R_n+h, R_n-h]$ and we end up with $|\overline{\Gamma}_n^-(s,t)| = |\Gamma_n^-((\theta_n^-)^{-1}(s),t)| \leqslant \delta$.
3. For the maps $\overline{\Gamma}_n^+$ we proceed analogously.

□

See Figure E.0.1.

In the following, we establish a convergence result for the harmonic functions Γ_n. For this purpose, we define the harmonic functions $\Gamma_n^- : [0,2R_n]\times S^1 \to \mathbb{R}$ and $\Gamma_n^+ : [-2R_n,0]\times S^1 \to \mathbb{R}$ by $\Gamma_n^-(s,t) := \Gamma_n(s-R_n,t) =$

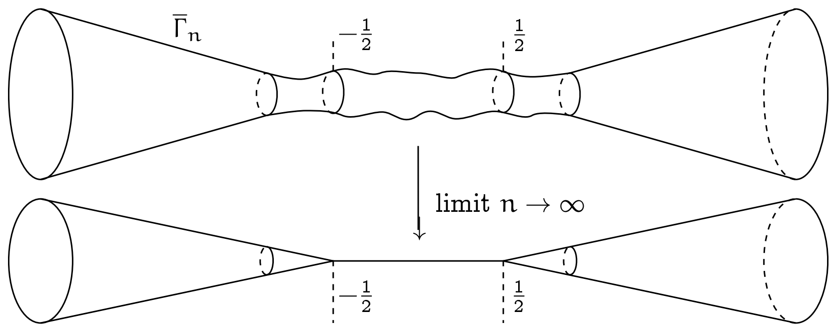

Figure E.0.1: The sequence $\overline{\Gamma}_n$ and the limit object.

$S_n(s-R_n)+\tilde{\Gamma}_n^-(s,t)$ and $\Gamma_n^+(s,t) := \Gamma_n(s+R_n,t) = S_n(s+R_n)+\tilde{\Gamma}_n^+(s,t)$, respectively. Since $\tilde{\Gamma}_n^- \to \tilde{\Gamma}^-$ and $\tilde{\Gamma}_n^+ \to \tilde{\Gamma}^+$ in C^∞_{loc}, $\Gamma_n^- + S_nR_n \to \tilde{\Gamma}^-$ converge in C^∞_{loc} on $[0,+\infty)\times S^1$ and $\Gamma_n^+ - S_nR_n \to \tilde{\Gamma}^+$ converge in C^∞_{loc} on $(-\infty,0]\times S^1$. Moreover, by means of the homeomorphism θ_n, we define the maps

$$\underline{\Gamma}_n(s,t) = \Gamma_n(\theta_n^{-1}(s),t) = S_n\theta_n^{-1}(s) + \overline{\Gamma}_n(s,t),\ \ s\in[-1,1],$$
$$\underline{\Gamma}_n^-(s,t) = \Gamma_n^-((\theta_n^-)^{-1}(s),t) = S_n((\theta_n^-)^{-1}(s)-R_n) + \overline{\Gamma}_n^-(s,t),\ \ s\in[-1,-1/2],$$
$$\underline{\Gamma}_n^+(s,t) = \Gamma_n^+((\theta_n^+)^{-1}(s),t) = S_n((\theta_n^+)^{-1}(s)+R_n) + \overline{\Gamma}_n^+(s,t),\ \ s\in[1/2,1].$$

We are now in the position to derive a convergence result for the sequence of harmonic functions Γ_n.

Theorem 105. *For every sequence of harmonic functions Γ_n satisfying assumptions R1-R5 the following holds. For every sequence $h_n \in \mathbb{R}_+$ satisfying $h_n < R_n$ and $h_n, R_n/h_n \to \infty$ as $n\to\infty$, the following C^∞_{loc}−convergence results hold for the maps Γ_n and $\underline{\Gamma}_n$ and their left and right shifts $\Gamma_n^\pm$ and $\underline{\Gamma}_n^\pm$, respectively:*

1. *For any sequence $s_n \in [-R_n+h_n, R_n-h_n]$ there exists a subsequence of the sequence of shifted harmonic functions $\Gamma_n(\cdot+s_n,\cdot)$, also denoted by $\Gamma_n(\cdot+s_n,\cdot)$ and defined on $[-R_n+h_n-s_n, R_n-h_n-s_n]\times S^1$, such that $\Gamma_n(\cdot+s_n,\cdot)-S_ns_n$ converges in C^∞_{loc} to 0.*

2. *The harmonic functions $\Gamma_n^- + S_nR_n : [0,h_n]\times S^1 \to \mathbb{R}$ converge in C^∞_{loc} to a harmonic function $\tilde{\Gamma}^- : [0,+\infty)\times S^1 \to \mathbb{R}$ which is asymptotic to 0. Furthermore, $\underline{\Gamma}_n^- + S_nR_n : [-1,-1/2]\times S^1 \to \mathbb{R}$ converge in $C^\infty_{loc}([-1,-1/2)\times S^1)$ to the map $\overline{\Gamma}^- : [-1,-1/2)\times S^1 \to \mathbb{R}$ such that*

$$\lim_{s\to-\frac{1}{2}} \overline{\Gamma}^-(s,t) = 0$$

   in $C^\infty(S^1)$.

3. *The harmonic functions $\Gamma_n^+ - S_nR_n : [-h_n,0]\times S^1 \to \mathbb{R}$ converge in C^∞_{loc} to a harmonic function $\tilde{\Gamma}^+ : (-\infty,0]\times S^1 \to \mathbb{R}$ which is asymptotic to 0. Furthermore, $\underline{\Gamma}_n^+ - S_nR_n : [1/2,1]\times S^1 \to \mathbb{R}$ converge in $C^\infty_{loc}((1/2,1]\times S^1)$ to the map $\overline{\Gamma}^+ : (1/2,1]\times S^1 \to \mathbb{R}$ such that*

$$\lim_{s\to\frac{1}{2}} \overline{\Gamma}^+(s,t) = 0$$

   in $C^\infty(S^1)$.

Proof. For $(s,t)\in[-R_n+h_n-s_n, R_n-h_n-s_n]\times S^1$, we have $\Gamma_n(s+s_n,t)-S_ns_n = S_ns+\tilde{\Gamma}_n(s+s_n,t)$. By Theorem 104, the first assertion readily follows. Putting $\Gamma_n^-(s,t)-S_nR_n = S_ns+\tilde{\Gamma}_n^-(s,t)$, using the fact that

$\underline{\Gamma}_n^-(s,t) + S_n R_n = S_n(\theta_n^-)^{-1}(s) + \overline{\Gamma}_n^-(s,t)$ converge in C^∞_{loc} to $\overline{\Gamma}^- : [-1,-1/2) \times S^1 \to \mathbb{R}$ which is asymptotic to 0 as $s \to -1/2$, and applying Theorem 104 finishes the proof of the second assertion. The third assertion is proved in a similar manner.

□

To derive a notion of C^0 convergence we assume that the sequence $S_n R_n$ converges, i.e. $S_n R_n \to \sigma$ as $n \to \infty$. Note that this assumption is the same as Assumption C9 of Section 3.2.2.

Theorem 106. *For every sequence of harmonic functions Γ_n satisfying assumptions R1-R5 and additionally Assumption C9 of Section 3.2.2 the following holds. For every sequence $h_n \in \mathbb{R}_+$ satisfying $h_n < R_n$ and $h_n, R_n/h_n \to \infty$ as $n \to \infty$, the following C^0-convergence results hold for the maps $\underline{\Gamma}_n$ together with their left and right shift $\underline{\Gamma}_n^\pm$:*

1. *There exists a subsequence of $\underline{\Gamma}_n$ that converges in $C^0([-1/2,1/2] \times S^1)$ to $2\sigma s$.*
2. *There exists a subsequence of $\underline{\Gamma}_n^-$ that converges in $C^0([-1,-1/2] \times S^1)$ to $\overline{\Gamma}^- - \sigma$, where $\overline{\Gamma}^-(-1/2,t) = 0$ for all $t \in S^1$.*
3. *There exists a subsequence of $\underline{\Gamma}_n^+$ that converges in $C^0([1/2,1] \times S^1)$ to $\overline{\Gamma}^+ + \sigma$, where $\overline{\Gamma}^+(+1/2,t) = 0$ for all $t \in S^1$.*

Proof. We consider $\underline{\Gamma}_n(s,t) = S_n\theta_n^{-1}(s) + \overline{\Gamma}_n(s,t)$ for $(s,t) \in [-1/2,1/2] \times S^1$ with

$$S_n\theta_n^{-1}(s) = 2(S_nR_n - S_nh_n)s.$$

Corollary 103 implies that S_nh_ns converges in $C^0([-1/2,1/2] \times S^1)$ to 0, and similarly, that S_nR_ns converges in $C^0([-1/2,1/2] \times S^1)$ to $2\sigma s$. By Theorem 105, $\overline{\Gamma}_n$ converges in $C^0([-1/2,1/2] \times S^1)$ to 0, and so, the first assertion is proved. Setting $\underline{\Gamma}_n^-(s,t) = S_n(\theta_n^-)^{-1}(s) - S_nR_n + \overline{\Gamma}_n^-(s,t)$ for $(s,t) \in [-1,-1/2] \times S^1$, taking into account that $S_n(\theta_n^-)^{-1}(s)$ converges in $C^0([-1,-1/2] \times S^1)$ to 0, and applying Theorem 105 proves the second assertion. The third assertion follows in an analogous manner.

□

See Figure E.0.2.

Finally, we establish a convergence result for the derivative of Γ_n. Due to Lemma 98, we have $d\Gamma_n^- = S_n ds + d\tilde{\Gamma}_n^-$ on $[0,h_n] \times S^1$ and $d\Gamma_n^+ = S_n ds + d\tilde{\Gamma}_n^+$ on $[-h_n,0] \times S^1$. For a sequence $h_n \in \mathbb{R}_+$ satisfying $h_n < R_n$ and $h_n, R_n/h_n \to \infty$ as $n \to \infty$, consider the sequence of diffeomorphisms $\theta_n : [-R_n,R_n] \to [-1,1]$ as in Definition 44. In terms of θ_n we obtain the equations $d\underline{\Gamma}_n^- = S_n[(\theta_n^-)^{-1}]'(s)ds + d\overline{\Gamma}_n^-$ on $[-1,-1/2] \times S^1$ and $d\underline{\Gamma}_n^+ = S_n[(\theta_n^+)^{-1}]'(s)ds + d\overline{\Gamma}_n^+$ on $[1/2,1] \times S^1$. As a consequence of Theorem 105 we have the following

Corollary 107. *After going over to a subsequence, the following $C^\infty_{loc}-$convergence results for the maps $d\Gamma_n^-$, $d\Gamma_n^+$, $\underline{\Gamma}_n^-$ and $\underline{\Gamma}_n^+$ hold:*

1. *The harmonic $1-$forms $d\Gamma_n^-$ converge in $C^\infty_{loc}([0,+\infty) \times S^1)$ to a harmonic $1-$form $d\tilde{\Gamma}^-$ on $[0,+\infty) \times S^1$, which is asymptotic to 0. The $1-$forms $d\underline{\Gamma}_n^-$ converge in $C^\infty_{loc}([0,1/2) \times S^1)$ to a $1-$form $d\overline{\Gamma}^-$ which is asymptotic to a constant for $s \to 1/2$.*

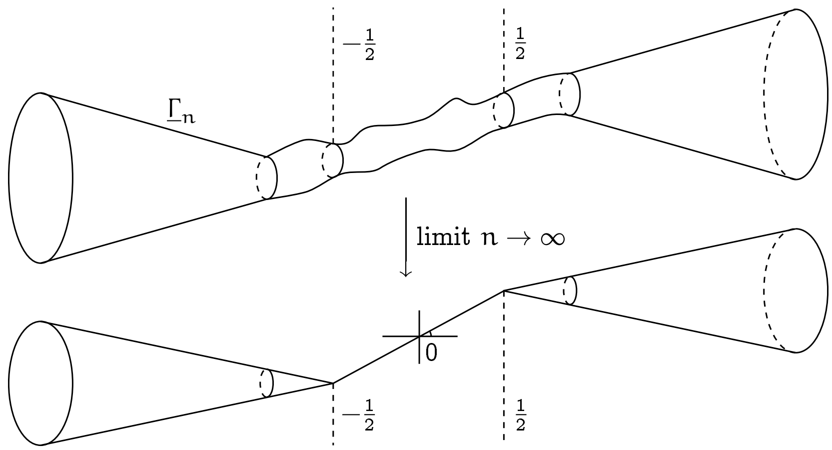

Figure E.0.2: The sequence $\underline{\Gamma}_n$ and the limit object in the case $S_n R_n \to \sigma$ as $n \to \infty$. Between $-1/2$ and $1/2$ the limit object is a linear function of slope σ.

2. *The harmonic* $1-$*forms* $d\Gamma_n^+$ *converge in* $C^\infty_{loc}((-\infty, 0] \times S^1)$ *to a harmonic* $1-$*form* $d\tilde{\Gamma}^+$ *on* $(-\infty, 0] \times S^1$*, which is asymptotic to* 0*. The* $1-$*forms* $d\underline{\Gamma}_n^+$ *converge in* $C^\infty_{loc}((-1/2, 0] \times S^1)$ *to a* $1-$*form* $d\overline{\Gamma}^+$ *which is asymptotic to a constant for* $s \to -1/2$*.*

Appendix F

A version of the Monotonicity Lemma

In this appendix we introduce a notion of monotonicity for the transformed curves $\overline{u}$ as in Definition 71. Before proceeding we recall a version of the isoperimetric inequality. For an arbitrary $a > 0$ let us consider the manifold $W := [-a, a] \times M$ together with defined structure from Section 2.2.

Theorem 108. *Let $p \in W \backslash \partial W$. There exist constants $C_2, \epsilon > 0$ such that for any $\overline{J}_{P_s}$–holomorphic curve $(\overline{u}, R, P)$ and any compact subset $K \subset [-R, R] \times S^1$ with smooth boundary and $\overline{u}(K) \subset B_{\epsilon}^{\overline{g}_0}(p)$, we have*

$$\mathit{area}_{\overline{g}_0}(\overline{u}|_K) \leqslant C_2 \ell^2_{\overline{g}_0}(\overline{u}_{\partial K}).$$

Proof. By Theorem 2.5 from [21] there exists $r_p > 0$ such that $B_{r_p}^{\overline{g}_0}(p) \subset W \backslash \partial W$ and $\exp_p = \exp_p^{\overline{g}_0} : B_{r_p}^{\overline{g}_0}(0) \subset T_p W \to B_{r_p}^{\overline{g}_0}(p)$ defines normal coordinates around p. Consider the standard symplectic form ω_0 on $(T_p W, \overline{J}_{0,p}, \overline{g}_{0,p})$ given by $\omega_0(v, w) := \overline{g}_{0,p}(\overline{J}_{0,p} v, w)$ for $v, w \in T_p W$, where $\overline{J}_{0,p}$ is the domain-dependent almost complex structure $\overline{J}_{P_s}$ evaluated at p for $s = 0$. Pull back ω_0 to $B_{r_p}^{\overline{g}_0}(p) \subset W \backslash \partial W$ with $(\exp_p^{\overline{g}_0})^{-1} = \exp_p^{-1}$ and get an exact symplectic form $\omega := (\exp_p^{-1})^* \omega_0$ on $B_{r_p}^{\overline{g}_0}(p)$, i.e. there exists a 1–form λ such that $\omega = d\lambda$. For any $v \in T_p W$ we have $\omega(v, \overline{J}_{0,p} v) = \|v\|^2_{\overline{g}_0} > 0$. We claim that there exist the constants $c_0, c_1 > 0$ such that for all $v \in T_p W$ and all $\rho \in [-C, C]$ the following inequalities hold:

$$c_1 \|v\|^2_{\overline{g}_0} \geqslant \omega(v, \overline{J}_{\rho,p} v) \geqslant \frac{1}{c_0} \|v\|^2_{\overline{g}_0}.$$

To prove this claim we consider the second inequality and assume that this is not true. Thus, for each constant $c_{0,n} = n$ there exists $v_n \in T_p W$ with $\|v_n\|_{\overline{g}_0} = 1$ and $\rho_n \in [-C, C]$ such that $\omega(v_n, \overline{J}_{\rho_n,p} v_n) < 1/n$. By passing to a subsequence we assume that $v_n \to v$ with $\|v\|_{\overline{g}_0} = 1$ and $\rho_n \to \rho$ as $n \to \infty$. Then we get $\omega(v, \overline{J}_{\rho,p} v) = 0$ and (we work in point p)

$$0 = \omega(v, \overline{J}_{\rho,p} v) = \omega_0(v, \overline{J}_{\rho,p} v).$$

We arrive at $\overline{g}_{\rho,p}(v, v) = 0$, which is a contradiction since the family of metrics $\overline{g}_\rho$ are equivalent. The first inequality is proved in an analogous manner. Now we claim that there exist an open neighborhood $U_p \subset B_{r_p}^{\overline{g}_0}(p)$ of p and the constants $c_0, c_1 > 0$ (making the old ones smaller) such that for all $v \in TU_p$ and all $\rho \in [-C, C]$, the following inequalities hold:

$$c_1 \|v\|^2_{\overline{g}_0} \geqslant \omega(v, \overline{J}_\rho v) \geqslant \frac{1}{c_0} \|v\|^2_{\overline{g}_0}. \tag{F.0.1}$$

The proof of this claim is similar to the previous one (by contradiction). Choose $\epsilon > 0$ to be the largest number such that $B_\epsilon^{\overline{g}_0}(p) \subset U_p$. After eventually making the constants c_0 and c_1 smaller, assume that (F.0.1) holds for all

$v \in TB_\epsilon^{\overline{g}_0}(p)$ and all $\rho \in [-C, C]$.

$$c_1 \|v\|^2_{\overline{g}_0} \geqslant \omega(v, \overline{J}_\rho v) \geqslant \frac{1}{c_0} \|v\|^2_{\overline{g}_0}$$

Now, let $(\overline{u}, R, P)$ be a $\overline{J}_{P_S}$–holomorphic curve and $K \subset [-R, R] \times S^1$ a compact subset with smooth boundary such that $\overline{u}(K) \subset B_\epsilon^{\overline{g}_0}(p)$. Then for $\text{area}_{\overline{g}_0}(\overline{u}|_K)$ defined as in Section II.2 from [15], there exists a constant $\tilde{c} > 0$ (independent of K and ϵ) for which

$$\text{area}_{\overline{g}_0}(\overline{u}|_K) \leqslant \tilde{c} \int_K \|d\overline{u}\|^2_{\overline{g}_0} \text{vol}_{\text{eucl.}} \leqslant \tilde{c}c_0 \int_K \overline{u}^*\omega = \tilde{c}c_0 \int_{\partial K} \overline{u}^*\lambda.$$

By the results of Appendix 1 in [15], there exists a minimal surface $g : K \to B_\epsilon^{\overline{g}_0}(0) \subset T_pW$ with $g|_{\partial K} = \exp_p^{-1} \circ \overline{u}|_{\partial K}$ satisfying the inequality

$$\text{area}_{\overline{g}_{0,p}}(g|_K) \leqslant \frac{1}{4\pi} \ell^2_{\overline{g}_{0,p}}(g|_{\partial K})$$

in $(T_pW, \overline{g}_{0,p})$. Thus

$$\int_{\partial K} \overline{u}^*\lambda = \int_{\partial K} (\exp_p \circ g)^*\lambda = \int_K (\exp_p \circ g)^*\omega = \int_K g^*\omega_0.$$

Wirtinger's inequality for the vector space $(T_pW, \overline{g}_{0,p}, J_{0,p})$ of the 2–form ω_0 states that for all $v, w \in T_pW$, $\omega_0(v, w) \leqslant \|v \wedge w\|_{\overline{g}_{0,p}}$ with respect to $\overline{g}_{0,p}$, where

$$\|v \wedge w\|^2_{\overline{g}_{0,p}} = \det \begin{pmatrix} \overline{g}_{0,p}(v, v) & \overline{g}_{0,p}(v, w) \\ \overline{g}_{0,p}(v, w) & \overline{g}_{0,p}(w, w) \end{pmatrix}.$$

From this we find

$$\int_K g^*\omega_0 \leqslant \text{area}_{\overline{g}_{0,p}}(g|_K),$$

and moreover,

$$\text{area}_{\overline{g}_0}(\overline{u}|_K) \leqslant \tilde{c}c_0 \text{area}_{\overline{g}_{0,p}}(g|_K) \leqslant \frac{\tilde{c}c_0}{4\pi} \ell^2_{\overline{g}_{0,p}}(g|_{\partial K}).$$

Recall that $\exp_p : B_\epsilon^{\overline{g}_0}(0) \to B_\epsilon^{\overline{g}_0}(p)$ is a diffeomorphism with $(d\exp_p)(0) = \text{Id}$. Then there exists a constant $K > 0$ which may depend on p such that

$$\left\|(d_q \exp_p^{-1})(v)\right\|_{\overline{g}_{0,p}} \leqslant K^{\frac{1}{4}} \|v\|_{\overline{g}_0}$$

for all $q \in B_\epsilon^{\overline{g}_0}(p)$ and all $v \in T_qW$. Hence we get

$$\ell^2_{\overline{g}_{0,p}}(g|_{\partial K}) \leqslant K \ell^2_{\overline{g}_0}(\overline{u}|_{\partial K}),$$

while putting all these together we obtain

$$\text{area}_{\overline{g}_0}(\overline{u}|_K) \leqslant \tilde{c}c_0 \frac{1}{4\pi} K \ell^2_{\overline{g}_0}(\overline{u}|_{\partial K}).$$

For the choice $C_2 := c_0 K/(4\pi)$, the assertion then readily follows.

□

Corollary 109. *Let* $(W, \overline{J}_0)$ *be as above, and let* $W^{-\delta} \subset W$ *with* $\delta > 0$ *consist of the points in* W *having distance to* ∂W *(with respect to the metric* $\overline{g}_0$*) at least* δ*. Then there exist constants* $C_3, \epsilon_0 > 0$ *such that for any* $\overline{J}_{P_S}$*–holomorphic curve* $(\overline{u}, R, P)$ *and any compact subset* $K \subset [-R, R] \times S^1$ *with smooth boundary satisfying*

$\overline{u}(K) \subset W^{-\delta}$ *and* $diam_{\overline{g}_0}(\overline{u}(K)) \leqslant \epsilon_0$*, we have*

$$area_{\overline{g}_0}(\overline{u}|_K) \leqslant C_3 \ell^2_{\overline{g}_0}(\overline{u}|_{\partial K}).$$

Proof. Cover $\overline{W^{-\delta}}$ by balls $\bigcup_{p \in W \setminus \partial W} B^{\overline{g}_0}_{\epsilon_p}(p)$, where $\epsilon_p > 0$ is chosen as in Theorem 108. Since $\overline{W^{-\delta}}$ is compact there exists a finite subcover $B^{\overline{g}_0}_{\epsilon_{p_1}}(p_1), ..., B^{\overline{g}_0}_{\epsilon_{p_N}}(p_N)$. For each $B^{\overline{g}_0}_{p_i}(p_i)$ with $i = 1, ..., N$ we obtain from Theorem 108 the constants $\epsilon_{p_i} > 0$ and $C_{p_i} > 0$. Set $\epsilon := \min_{i=1,...,N} \epsilon_{p_i}$ and $C_3 := \max_{i=1,...,N} C_{p_i}$. Let $\lambda > 0$ be the Lebesque number of the covering $B^{\overline{g}_0}_{\epsilon_{p_i}}(p_i)$, for $i = 1, ..., N$. By the choice $\epsilon_0 := \min\{\lambda, \epsilon\}$ the proof is finished.

□

Corollary 110. *For the same setting* $(W, \overline{J}_0)$ *and Hermitian metric* $\overline{g}_0$ *for* $\overline{J}_0$ *and* $\delta > 0$*, there exist constants* $C_3, \epsilon_0 > 0$ *such that for any* $\overline{J}_{P_s}-$*holomorphic curve* $(\overline{u}, R, P)$ *and any compact subset* $K \subset [-R, R] \times S^1$ *with smooth boundary satisfying* $K \subset \overline{u}^{-1}(W^{-\delta}_{\overline{g}_0})$ *and* $diam_{\overline{u}^*\overline{g}_0}(K) \leqslant \epsilon_0$*, we have*

$$area_{\overline{u}^*\overline{g}_0}(K) \leqslant C_3 \ell^2_{\overline{u}^*\overline{g}_0}(\partial K).$$

Remark 111. $\overline{u}^*\overline{g}_0$ is a positive semi-definite Riemann metric, i.e. $\overline{u}^*\overline{g}_0$ vanishes only when the derivative of $\overline{u}$ vanishes. Due to the Carleman similarity principle [22] this occurs only in a finite number of points.

Proof. Let C_3 and ϵ_0 be the constants from Corollary 109, $(\overline{u}, R, P)$ a $\overline{J}_{P_s}-$holomorphic curve, and $K \subset [-R, R] \times S^1$ a compact set with smooth boundary such that $K \subset \overline{u}^{-1}(W^{-\delta})$ and $\mathrm{diam}_{\overline{u}^*\overline{g}_0}(K) \leqslant \epsilon_0$. Noting the inequality

$$\mathrm{diam}_{\overline{u}^*\overline{g}_0}(K) \geqslant \mathrm{diam}_{\overline{g}_0}(\overline{u}(K)), \tag{F.0.2}$$

we obtain $\mathrm{diam}_{\overline{g}_0}(\overline{u}(K)) \leqslant \epsilon_0$, while by means of Corollary 109 we find

$$\mathrm{area}_{\overline{g}_0}(\overline{u}|_K) \leqslant C_3 \ell^2_{\overline{g}_0}(\overline{u}|_{\partial K}).$$

On the other hand, by definition we have

$$\mathrm{area}_{\overline{g}_0}(\overline{u}|_K) = \int_K \mathrm{vol}_{\overline{u}^*\overline{g}_0} = \mathrm{area}_{\overline{u}^*\overline{g}_0}(K),$$

where $\mathrm{vol}_{\overline{u}^*\overline{g}_0}$ is the 2−form defined by

$$\mathrm{vol}_{\overline{u}^*\overline{g}_0}(v, w) = \left[\overline{g}_0(d\overline{u}(v), d\overline{u}(v))\overline{g}_0(d\overline{u}(w), d\overline{u}(w)) - \overline{g}_0(d\overline{u}(v), d\overline{u}(w))^2\right]^{\frac{1}{2}}$$

for $v, w \in T([-R, R] \times S^1)$; by the same reason, we find

$$\ell_{\overline{g}_0}(\overline{u}|_{\partial K}) = \ell_{\overline{u}^*\overline{g}_0}(\partial K),$$

and the proof of Corollary 110 is finished.

□

For a $\overline{J}_{P_s}-$holomorphic curve $(\overline{u}, R, P)$ we define a positive semi-definite Riemannian metric on $[-R, R] \times S^1$ by

$$h_{\overline{u},(s,t)} = \overline{g}_{P_s}(\overline{u}(s,t))(d\overline{u}(s,t)\cdot, d\overline{u}(s,t)\cdot).$$

Note that this metric is not exactly a pull-back metric since $\overline{g}$ is parameter dependent. By (B.1.10), there exists a constant $C_4 > 0$ such that for all $v \in T([-R, R] \times S^1)$,

$$\frac{1}{C_4} \|v\|_{\overline{u}^*\overline{g}_0} \leqslant \|v\|_h \leqslant C_4 \|v\|_{\overline{u}^*\overline{g}_0} . \tag{F.0.3}$$

From here we have the following

Corollary 112. *For the same setting $(W, \overline{J}_0)$ and Hermitian metric $\overline{g}_0$ for $\overline{J}_0$ and $\delta > 0$ there exist constants $C_5, \epsilon_1 > 0$ such that for any $\overline{J}_{P_S}$−holomorphic curve $(\overline{u}, R, P)$ and any compact subset $K \subset [-R, R] \times S^1$ with smooth boundary satisfying $K \subset \overline{u}^{-1}(W^{-\delta}_{\overline{g}_0})$ and $diam_{h_{\overline{u}}}(K) \leqslant \epsilon_1$, we have*

$$area_{h_{\overline{u}}}(K) \leqslant C_5 \ell^2_{h_{\overline{u}}}(\partial K).$$

Proof. We choose the constants C_3 and ϵ_0 such that Corollary 110 holds. Define $\epsilon_1 := \epsilon_0/C_4$, let $(\overline{u}, R, P)$ be a $\overline{J}_{P_S}$−holomorphic curve and $K \subset [-R, R] \times S^1$ a compact subset such that $K \subset \overline{u}^{-1}(W^{-\delta}_{\overline{g}_0})$ and $\mathrm{diam}_{h_{\overline{u}}}(K) \leqslant \epsilon_1$. For any compact $K \subset [-R, R] \times S^1$, there hold

$$\frac{1}{C_4}\mathrm{diam}_{\overline{u}^*\overline{g}_0}(K) \leqslant \mathrm{diam}_{h_{\overline{u}}}(K) \leqslant C_4\mathrm{diam}_{\overline{u}^*\overline{g}_0}(K).$$

From (F.0.3) it follows (perhaps by enlarging the constant C_4) that

$$\frac{1}{C_4^2}\mathrm{area}_{h_{\overline{u}}}(K) \leqslant \mathrm{area}_{\overline{u}^*\overline{g}_0}(K)$$

and

$$\ell_{\overline{u}^*\overline{g}_0}(\partial K) \leqslant C_4 \ell_{h_{\overline{u}}}(\partial K).$$

Thus $\mathrm{diam}_{\overline{u}^*\overline{g}_0}(K) \leqslant C_4\mathrm{diam}_{h_{\overline{u}}}(K) \leqslant \epsilon_0$. By Corollary 110,

$$\frac{1}{C_4^2}\mathrm{area}_{h_{\overline{u}}}(K) \leqslant \mathrm{area}_{\overline{u}^*\overline{g}_0}(K) \leqslant C_3 \ell^2_{\overline{u}^*\overline{g}_0}(\partial K) \leqslant C_3 C_4^2 \ell^2_{h_{\overline{u}}}(\partial K),$$

and for the choice $C_5 = C_4^4 C_3$ the proof is finished.

□

The next theorem is the key feature in the proof of the monotonicity lemma.

Theorem 113. *Let S be a compact surface with non-empty boundary ∂S and let h be a positive semi-definite Riemannian metric that vanishes only in a finite number of points away from the boundary ∂S. Let $d = d_h$ be the distance function with respect to h. Assume that there exist constants $\tilde{C}, \tilde{\varepsilon} > 0$ such that for all compact subsurfaces $S' \subset S \backslash \partial S$ with $diam_h(S') \leqslant \tilde{\varepsilon}$,*

$$area_h(S') \leqslant \tilde{C}\ell^2_h(\partial S').$$

Then, for all $r \in (0, \tilde{\varepsilon}/2)$ and all $x \in S$ such that $B_r(x) \subset S \backslash \partial S$, we have

$$area_h(B_r(x)) \geqslant \frac{1}{4\tilde{C}} r^2.$$

Proof. Let $P = \{p_1, ..., p_N\} \subset S$ be the points where the metric h vanishes. Let $\tilde{h}$ be an arbitrary Riemann metric on S and consider for $\rho > 0$ the balls $B_\rho(p_i)$ for all $i = 1, ..., N$. After making ρ sufficiently small assume that

$B_\rho(p_i)$ and ∂S are pairwise disjoint for all $i = 1, ..., N$, and for $\rho < r$ that

$$B^h_r(x) \backslash \coprod_{i=1}^{N} B^{\tilde{h}}_\rho(p_i)$$

is a manifold with boundary. Consider the distance function $d_x : S \to \mathbb{R}, y \mapsto d(x, y)$. As this defines a metric on S, d_x is $1-$Lipschitz continous, and by the co-area formula [8], we obtain

$$\begin{aligned}
\text{area}_h(B^h_r(x)) &\geqslant \int_{d_x^{-1}([0,r])\backslash \coprod_{i=1}^{N} B^{\tilde{h}}_\rho(p_i)} \text{vol}_h \\
&\geqslant \int_{d_x^{-1}([0,r])\backslash \coprod_{i=1}^{N} B^{\tilde{h}}_\rho(p_i)} \|\nabla d_x\|_h \,\text{vol}_h \\
&\geqslant \int_0^r \ell_h \left(d_x^{-1}(t) \backslash \coprod_{i=1}^{N} B^{\tilde{h}}_\rho(p_i) \right) dt.
\end{aligned}$$

Hence

$$\text{area}_h(B^h_r(x)) \geqslant \int_0^r \ell_h \left(d_x^{-1}(t) \backslash \coprod_{i=1}^{N} B^{\tilde{h}}_\rho(p_i) \right) dt,$$

while letting $\rho \to 0$ we obtain

$$\text{area}_h(B^h_r(x)) \geqslant A(r) := \int_0^r \ell_h \left(d_x^{-1}(t) \right) dt.$$

From the isoperimetric inequality it follows that

$$\frac{d}{dt}\Big|_{r=r'} A(r) = \ell_h(d_x^{-1}(r')) \geqslant \frac{1}{\sqrt{\tilde{C}}} \sqrt{\text{area}_h(B_{r'}(x))} \geqslant \frac{1}{\sqrt{\tilde{C}}} \sqrt{A(r')}.$$

Separating the variables and integrating with respect to r' over the full measure set of noncritical values of d_x yields

$$2\sqrt{A(r)} \geqslant \frac{1}{\sqrt{\tilde{C}}} r.$$

Hence $\text{area}_h(B_r(x)) \geqslant A(r) \geqslant r^2/(4\tilde{C})$.

□

The next corollaries follow from Theorem 113.

Corollary 114. *Let* $(W, \bar{J}_0)$ *be as above and* $\delta > 0$*. Let* $\bar{g}_0$ *be a Hermitian metric for* $\bar{J}_0$*. Then there exist constants* $C_6, \varepsilon_2 > 0$*, such that for all* $\bar{J}_{P_S}-$*holomorphic curves* $(\bar{u}, R, P)$*, all* $r \in (0, \epsilon_2/2)$*, and all* $x \in [-R, R] \times S^1$ *satisfying* $B^{h_{\bar{u}}}_r(x) \subset \bar{u}^{-1}(W^{-\delta}_{\bar{g}_0}) \cap ([-R, R] \times S^1 \backslash \partial([-R, R] \times S^1))$*, we have*

$$\textit{area}_{h_{\bar{u}}}(B^{h_{\bar{u}}}_r(x)) \geqslant C_6 r^2.$$

Proof. Let $C_5, \varepsilon_1 > 0$ be as in Corollary 112. Pick a $\bar{J}_{P_S}-$holomorphic curve $(\bar{u}, R, P)$. For any compact subset $K \subset [-R, R] \times S^1$ with $K \subset \bar{u}^{-1}(W^{-\delta}_{\bar{g}_0})$ and $\text{diam}_{h_{\bar{u}}}(K) \leqslant \epsilon_1$,

$$\text{area}_{h_{\bar{u}}}(K) \leqslant C_5 \ell^2_{h_{\bar{u}}}(\partial K).$$

Pick $r \in (0, \epsilon_1/2)$ and some $x \in S$ such that $B^{h_{\bar{u}}}_r(x) \subset \bar{u}^{-1}(W^{-\delta}_{\bar{g}_0}) \cap ([-R, R] \times S^1 \backslash \partial([-R, R] \times S^1))$. By Theorem 113 it follows that $\text{area}_{h_{\bar{u}}}(B^{h_{\bar{u}}}_r(x)) \geqslant C_6 r^2$ for some constant $C_6 = 1/(4C_5)$ and $\epsilon_2 = \epsilon_1$.

□

We apply this result to the whole symplectisation $\mathbb{R} \times M$. On $\mathbb{R} \times M$ recall that $\bar{J}_0$ is a cylindrical almost complex structure with the Hermitian metric $\bar{g}_0$, and that $\bar{J}_0$ and $\bar{g}_0$ are $\mathbb{R}-$invariant.

Corollary 115. *There exist constants $C_7, \varepsilon_3 > 0$, such that for all $\bar{J}_{Ps}-$holomorphic curves $(\overline{u}, R, P)$, all $r \in (0, \epsilon_3/2)$ and all $x \in ([-R, R] \times S^1 \backslash \partial([-R, R] \times S^1))$ satisfying $B_r^{h_{\overline{u}}}(x) \subset ([-R, R] \times S^1 \backslash \partial([-R, R] \times S^1))$, we have*

$$\mathit{area}_{h_{\overline{u}}}(B_r^{h_{\overline{u}}}(x)) \geqslant C_7 r^2.$$

Proof. The translations in the $\mathbb{R}-$coordinate are isometries of all metrics $\bar{g}_\rho$. Consider $W_0 := [-2, 2] \times M$ and $W_1 = [-1, 1] \times M$. Let $\delta := \mathrm{dist}_{\bar{g}_0}(\partial W_0, W_1) > 0$ yielding $W_1 \subset W_0^{-\delta}$. For the data $W_0, W_0^{-\delta}, \bar{J}_0$ and $\bar{g}_0$ apply Corollary 114, and obtain the constants $C_6, \epsilon_2 > 0$ such that for all $\bar{J}_{Ps}-$holomorphic curves $(\overline{u}, R, P)$ satisfying $B_r^{h_{\overline{u}}}(x) \subset \overline{u}^{-1}(W_0^{-\delta}) \cap ([-R, R] \times S^1 \backslash \partial([-R, R] \times S^1))$ for all $r \in (0, \epsilon_2/2)$ and all $x \in ([-R, R] \times S^1 \backslash \partial([-R, R] \times S^1))$, $\mathrm{area}_{h_{\overline{u}}}(B_r^{h_{\overline{u}}}(x)) \geqslant C_6 r^2$. Set $\tilde{\epsilon}_2 := \inf_{\tau \in [-C, C]} \mathrm{diam}_{\bar{g}_\rho}(W_1) > 0$ and $\epsilon_3 := \min\{\frac{\epsilon_2}{2}, \frac{\tilde{\epsilon}_2}{2}, \frac{\tilde{\epsilon}_2}{C_6}\}$. Let $(\overline{u}, R, P)$ be a $\bar{J}_{Ps}-$holomorphic curve and pick $r \in (0, \epsilon_3/2)$ and $x \in ([-R, R] \times S^1 \backslash \partial([-R, R] \times S^1))$ such that $B_r^{h_{\overline{u}}}(x) \subset ([-R, R] \times S^1 \backslash \partial([-R, R] \times S^1))$. We get $\mathrm{diam}_{\bar{g}_0}(\overline{u}(B_r^{h_{\overline{u}}}(x))) \leqslant \mathrm{diam}_{\overline{u}^* \bar{g}_0}(B_r^{h_{\overline{u}}}(x)) \leqslant C_6 \mathrm{diam}_{h_{\overline{u}}}(B_r^{h_{\overline{u}}}(x)) \leqslant 2rC_6 \leqslant \epsilon_3 C_6 \leqslant \epsilon_2$. Thus there exists a translation such that after composing it with $\overline{u}$ we obtain $\overline{u}(B_r^{h_{\overline{u}}}(x)) \subset W_1 \subset W_0^{-\delta}$. By Corollary 114, the proof is finished.

□

The same results hold if we replace the parameter-dependent metric $\bar{g}_\rho$ defined by

$$\bar{g}_\rho = dr \otimes dr + \alpha \otimes \alpha + d\alpha(\cdot, \bar{J}_\rho \cdot)$$

by the parameter-dependent metric $\tilde{g}_\rho$ defined by

$$\tilde{g}_\rho = \varphi'(r)\,(dr \otimes dr + \alpha \otimes \alpha) + \varphi(r) d\alpha(\cdot, \bar{J}_\rho \cdot)$$

where $\varphi : \mathbb{R} \to [0, 1]$ satisfies $\varphi(r) > 0$ and $\varphi'(r) > 0$ for all $r \in \mathbb{R}$. By replacing $\bar{g}_\rho$ with $\tilde{g}_\rho$ in the definition of $h_{\overline{u}}$ and by straightforward computation we obtain

$$\mathrm{area}_{h_{\overline{u}}}(B_r^{h_{\overline{u}}}(x)) = \int_{B_r^{h_{\overline{u}}}(x)} \mathrm{vol}_{h_{\overline{u}}} = \int_{B_r^{h_{\overline{u}}}(x)} \overline{u}^* d(\varphi\alpha). \tag{F.0.4}$$

Remark 116. Even though in Corollary 115 the metric $\tilde{g}_\rho$ is not $\mathbb{R}-$invariant, the results established so far are also valid for the family of metrics $\tilde{g}_\rho$ with some fixed function $\varphi : \mathbb{R} \to [0, 1]$ satisfying $\varphi(r) > 0$ and $\varphi'(r) > 0$ for all $r \in \mathbb{R}$.

Using Corollary 115 we can prove the following

Corollary 117. *There exist constants $C_7, \epsilon_3 > 0$ such that for any $\bar{J}_{Ps}-$holomorphic curve $(\overline{u}, R, P)$, any $r \in (0, \epsilon_3/2)$, and any $x \in ([-R, R] \times S^1) \backslash \partial([-R, R] \times S^1)$ satisfying $B_r^{h_{\overline{u}}}(x) \subset ([-R, R] \times S^1 \backslash \partial([-R, R] \times S^1))$, we have*

$$E(\overline{u}|_{B_r^{h_{\overline{u}}}(x)}; B_r^{h_{\overline{u}}}(x)) := \sup_{\varphi \in \mathcal{A}} \int_{B_r^{h_{\overline{u}}}(x)} \overline{u}^* d(\varphi\alpha) \geqslant C_7 r^2.$$

Proof. Fix a function $\varphi \in \mathcal{A}$ such that $\varphi'(r) > 0$ for all $r \in \mathbb{R}$. By (F.0.4) and Corollary 115 there exists constants $C_7, \epsilon_3 > 0$ such that for any $\bar{J}_{Ps}-$holomorphic curves $(\overline{u}, R, P)$ satisfying the hypothesis of Corollary 117 we have

$$E(\overline{u}|_{B_r^{h_{\overline{u}}}(x)}; B_r^{h_{\overline{u}}}(x)) \geqslant \int_{B_r^{h_{\overline{u}}}(x)} \overline{u}^* d(\varphi\alpha) \geqslant C_7 r^2.$$

□

The following version is also valid:

Corollary 118. *There exist constants* $C_8, \epsilon_4 > 0$ *such that for any* $\overline{J}_{P_S}$*—holomorphic curve* $(\overline{u}, R, P)$*, any* $r \in (0, \epsilon_4)$*, and any* $x \in ([-R,R] \times S^1) \backslash \partial([-R,R] \times S^1)$ *satisfying* $B_r^{\overline{g}_0}(\overline{u}(x)) \cap \overline{u}(\partial([-R,R] \times S^1)) = \emptyset$*, we have*

$$E(\overline{u}|_{\overline{u}^{-1}(B_r^{\overline{g}_0}(\overline{u}(x)))}; \overline{u}^{-1}(B_r^{\overline{g}_0}(\overline{u}(x)))) \geqslant C_8 r^2.$$

Proof. First we prove that $\overline{u}(B^{h_{\overline{u}}}_{\frac{r}{C_4}}(x)) \subset B_r^{\overline{g}_0}(\overline{u}(x))$, where C_4 is the constant from (F.0.3). For $y \in B^{h_{\overline{u}}}_{\frac{r}{C_4}}(x)$ we find

$$\mathrm{dist}_{h_{\overline{u}}}(x,y) = \inf_{\gamma, \gamma(0)=x, \gamma(1)=y} \int_0^1 \|\dot{\gamma}(t)\|_{h_{\overline{u}}}\, dt \leqslant \frac{r}{C_4}.$$

and then

$$\begin{aligned}
\mathrm{dist}_{\overline{g}_0}(\overline{u}(x), \overline{u}(y)) &= \inf_{\eta, \eta(0)=\overline{u}(x), \eta(1)=\overline{u}(y)} \int_0^1 \|\dot{\eta}(t)\|_{\overline{g}_0}\, dt \\
&\leqslant \inf_{\overline{u}\circ\gamma, \overline{u}\circ\gamma(0)=\overline{u}(x), \overline{u}\circ\gamma(1)=\overline{u}(y)} \int_0^1 \left\| (\dot{\overline{u} \circ \gamma})(t) \right\|_{\overline{g}_0} dt \\
&= \inf_{\gamma, \gamma(0)=x, \gamma(1)=y} \int_0^1 \|\dot{\gamma}(t)\|_{\overline{u}^* \overline{g}_0}\, dt \\
&\leqslant C_4 \inf_{\gamma, \gamma(0)=x, \gamma(1)=y} \int_0^1 \|\dot{\gamma}(t)\|_{h_{\overline{u}}}\, dt \\
&= C_4 \mathrm{dist}_{h_{\overline{u}}}(x,y) \\
&= r.
\end{aligned}$$

Hence, if $B_r^{\overline{g}_0}(\overline{u}(x)) \cap \overline{u}(\partial([-R,R] \times S^1)) = \emptyset$, we obtain

$$\overline{u}(B^{h_{\overline{u}}}_{\frac{r}{C_4}}(x)) \cap \overline{u}(\partial([-R,R] \times S^1)) = \emptyset,$$

and further on,

$$B^{h_{\overline{u}}}_{\frac{r}{C_4}}(x) \subset ([-R,R] \times S^1) \backslash \partial([-R,R] \times S^1).$$

From Corollary 117 there exist the constants $C_7, \epsilon_3 > 0$ such that for any $\overline{J}_{P_S}$—holomorphic curve $(\overline{u}, R, P)$, any $r \in (0, \epsilon_3/2)$, and any $x \in ([-R,R] \times S^1) \backslash \partial([-R,R] \times S^1)$ satisfying $B_r^{h_{\overline{u}}}(x) \subset ([-R,R] \times S^1 \backslash \partial([-R,R] \times S^1))$, $E(\overline{u}|_{B_r^{h_{\overline{u}}}(x)}) \geqslant C_7 r^2$. Set $\epsilon_4 := C_4 \epsilon_3$, and let $(\overline{u}, R, P)$ be a $\overline{J}_{P_S}$—holomorphic curve, and $r \in (0, \epsilon_4)$ and $x \in ([-R,R] \times S^1) \backslash \partial([-R,R] \times S^1)$ be such that $B_r^{\overline{g}_0}(\overline{u}(x)) \cap \overline{u}(\partial([-R,R] \times S^1)) = \emptyset$. From the above considerations we infer that $B^{h_{\overline{u}}}_{\frac{r}{C_4}}(x) \subset ([-R,R] \times S^1) \backslash \partial([-R,R] \times S^1)$, and we end up with

$$E(\overline{u}|_{\overline{u}^{-1}(B_r^{\overline{g}_0}(\overline{u}(x)))}; \overline{u}^{-1}(B_r^{\overline{g}_0}(\overline{u}(x)))) \geqslant E(\overline{u}|_{B^{h_{\overline{u}}}_{\frac{r}{C_4}}(x)}; B^{h_{\overline{u}}}_{\frac{r}{C_4}}(x)) \geqslant C_7 \frac{r^2}{C_4^2}.$$

□

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