LOCAL ASYMPTOTICS FOR THE AREA UNDER THE RANDOM WALK EXCURSION

ELENA PERFILEV * ** AND
VITALI WACHTEL,* *** *Universität Augsburg*

Abstract

We study the tail behaviour of the distribution of the area under the positive excursion of a random walk which has negative drift and light-tailed increments. We determine the asymptotics for local probabilities for the area and prove a local central limit theorem for the duration of the excursion conditioned on the large values of its area.

Keywords: Random walk; excursion; subexponential distribution

2010 Mathematics Subject Classification: Primary 60G50
Secondary 60G40; 60F17

1. Introduction and statement of results

Let $\{S_n;\ n \geq 1\}$ be a random walk with independent and identically distributed (i.d.d.) increments $\{X_k;\ k \geq 1\}$ and let τ be first time when S_n is nonpositive, i.e.

$$\tau := \min\{n \geq 1 \colon S_n \leq 0\}.$$

Define also the area under the trajectory $\{S_0, S_1, \ldots, S_\tau\}$:

$$A_\tau := \sum_{k=1}^{\tau-1} S_k.$$

If the increments of the random walk have nonpositive mean then the random variables τ and A_τ are finite and we are interested in the tail behaviour of the area A_τ.

In the case of the driftless ($\mathbb{E}X_1 = 0$) random walk with finite variance $\sigma^2 := \mathbb{E}X_1^2 \in (0, \infty)$, we have a universal tail behaviour

$$\lim_{x\to\infty} x^{1/3}\mathbb{P}(A_\tau > x) = 2C_0\sigma^{1/3}\mathbb{E}\biggl[\int_0^1 e(t)\,\mathrm{d}t\biggr]^{1/3}, \tag{1}$$

where $e(t)$ denotes the standard Brownian excursion and the constant C_0 is taken from $\mathbb{P}(\tau = n) \sim C_0 n^{-3/2}$. Proposition 1 of Vysotsky [15] states that (1) holds for some particular classes of random walks. But we can easily see that the proof from [15] remains valid for all random walks with zero mean and finite variance. Later we will provide an alternative proof of (1).

* Postal address: Institut für Mathematik, Universität Augsburg, 86135 Augsburg, Germany.
** Email address: elena.perfilev@math.uni-augsburg.de
*** Email address: vitali.wachtel@math.uni-augsburg.de

If the mean of X_1 is negative then the distribution of A_τ becomes sensitive to the tail behaviour of the increments. Borovkov *et al.* [4] showed that if the tail of X_1 is a regularly varying function then, as $x \to \infty$,

$$\mathbb{P}(A_\tau > x) \sim \mathbb{P}(M_\tau > \sqrt{2|\mathbb{E}X_1|}x^{1/2}) \sim \mathbb{E}\tau\mathbb{P}(X_1 > \sqrt{2|\mathbb{E}X_1|}x^{1/2}), \tag{2}$$

where $M_\tau := \max_{n<\tau} S_n$. Behind this relation there is a simple heuristic explanation. In order to have a large area under the excursion, the random walk has to make a large jump at the very beginning and then the random walk behaves according to the law of large numbers. More precisely, if the jump of size h appears, after which the random walk goes linearly down with the slope $-\mu$, where $\mu := |\mathbb{E}X_1|$, then the duration of the excursion will be of order h/μ. Consequently, the area will be of order $h^2/2\mu$. If we want the area to be of order x then the jump has to be of order $\sqrt{2\mu}x^{1/2}$. The same strategy is optimal for large values of M_τ. As a result, we have both asymptotic equivalences in (2).

This close connection between the maximum M_τ and the area A_τ is not valid for random walks with light tails. Let $\varphi(t)$ be the moment generating function of X_1, i.e.

$$\varphi(t) := \mathbb{E}e^{tX_1}, \qquad t \ge 0.$$

We will consider random walks satisfying the Cramer condition

$$\varphi(\lambda) = 1 \quad \text{for some } \lambda > 0. \tag{3}$$

Moreover, we assume that

$$\varphi'(\lambda) < \infty \quad \text{and} \quad \varphi''(\lambda) < \infty. \tag{4}$$

If (3) and (4) hold then the most likely path to a large value of M_τ is piecewise linear. The random walk first goes up with the slope $\varphi'(\lambda)/\varphi(\lambda)$; see Lemma 4.1 of [4]. After arrival at the desired level h, it goes down with the slope $-\mu$. This follows from the law of large numbers. If this path is optimal for the area then

$$\mathbb{P}(A_\tau > x) \approx \mathbb{P}\left(M_\tau > \sqrt{\frac{2\mu\varphi'(\lambda)}{\varphi'(\lambda)+\mu\varphi(\lambda)}}x^{1/2}\right).$$

Since $\mathbb{P}(M_\tau > y) \sim Ce^{-\lambda y}$, we arrive at the contradiction to the known results for random walks with two-sided exponentially distributed increments; see [10] and [11].

Duffy and Meyn [8] determined the asymptotic behaviour of $\log \mathbb{P}(A_\tau > x)$ for random walks with not necessarily i.i.d. increments. Specialising their result to our setting, we conclude that the optimal path to a large area is a rescaling of the function

$$\psi(u) := -\frac{1}{\lambda}\log\varphi(\lambda(1-u)), \qquad u \in [0,1].$$

They also showed that

$$\lim_{x\to\infty} \frac{1}{x^{1/2}} \log \mathbb{P}(A_\tau > x) = -\theta,$$

where

$$\theta := 2\lambda\sqrt{I} \quad \text{and} \quad I := \int_0^1 \psi(u)\,du.$$

Our purpose is to derive precise, without logarithmic scaling, asymptotics for A_τ.

Theorem 1. *Assume that X_1 is integer valued and aperiodic. Assume also that (3) and (4) hold. Then there exists a positive constant κ such that*

$$\mathbb{P}(A_\tau = x) \sim \kappa x^{-3/4} \mathrm{e}^{-\theta\sqrt{x}}, \qquad x \to \infty. \tag{5}$$

Using a simple summation, we see that (5) yields

$$\mathbb{P}(A_\tau > x) \sim \frac{2\kappa}{\theta} x^{-1/4} \mathrm{e}^{-\theta\sqrt{x}}. \tag{6}$$

An analogue of this relation was obtained by Guillemin and Pinchon [10] for an M/M/1 queue and by Kearney [11] for a geo/geo/1 queue.

Equation (6) confirms the conjecture of Kulik and Palmowski [12] for all integer-valued random walks. Unfortunately, the authors do not know how to derive a version of (5) for nonlattice random walks. Moreover, the authors do not know how to derive (6) without local asymptotics. One can derive an upper bound for $\mathbb{P}(A_\tau > x)$ via the exponential Chebyshev inequality. This leads to

$$\mathbb{P}(A_\tau > x) \le C x^{1/4} \mathrm{e}^{-\theta\sqrt{x}}. \tag{7}$$

For the proof of this estimate, see Subsection 2.2. Comparing (6) and (7), we see that the Chebyshev inequality provides the correct logarithmic rate of divergence and that the error in (7) is of order $\sqrt{x}$. Such an error is quite standard for the exponential Chebyshev inequality. In the most classical situation of sums of i.i.d. random variables, we have an error of order $\sqrt{n}$. In order to avoid this error and to obtain (5) we apply an appropriate exponential change of measure and analyse, under transformed measure, the asymptotic behaviour of the local probabilities for S_n and $A_n := \sum_{k=1}^n S_k$ conditioned on the event $\{\tau = n + 1\}$. This approach allows us to obtain the following conditional limit for the duration of the excursion.

Theorem 2. *Under the assumptions of Theorem 1, there exists $\Delta^2 > 0$ such that*

$$\sup_k \left| x^{1/4} \mathbb{P}(\tau = k \mid A_\tau = x) - \frac{1}{\sqrt{2\pi\Delta^2}} \exp\left\{ -\frac{(k - m x^{1/2})^2}{2\Delta^2 x^{1/2}} \right\} \right| \to 0, \qquad x \to \infty,$$

where $m = (\int_0^1 \psi(t)\,\mathrm{d}t)^{-1/2}$.

2. Nonhomogeneous exponential change of measure

Our approach to the derivation of the tail asymptotics for A_τ is based on a careful analysis of large deviation probabilities for the vector (A_n, S_n) conditioned on $\{\tau = n + 1\}$. For every fixed n, we perform the following nonhomogeneous change of measure. Consider a new probability measure $\widehat{\mathbb{P}}$ such that, for every $j \le n$,

$$\widehat{\mathbb{P}}(X_j \in \mathrm{d}y) = \frac{\mathrm{e}^{u_{n,j} y}}{\varphi(u_{n,j})} \mathbb{P}(X_j \in \mathrm{d}y), \tag{8}$$

where

$$u_{n,j} = \lambda \frac{(n - j + 1)}{n}.$$

The nonhomogeneous choice of transformation parameters $u_{n,j}$ can be easily explained by the fact that it corresponds to the exponential change of the distribution of A_n with parameter λ/n.

Indeed,

$$\mathbb{E}\left[\exp\left\{\frac{\lambda}{n}A_n\right\}\right] = \mathbb{E}\left[\exp\left\{\frac{\lambda}{n}\sum_1^n (n-j+1)X_j\right\}\right] = \prod_{j=1}^n \varphi\left(\frac{n-j+1}{n}\lambda\right).$$

We have also the following relations between probabilities $\widehat{\mathbb{P}}$ and $\mathbb{P}$:

$$\mathbb{P}(A_n \in \mathrm{d}x,\ S_n \in \mathrm{d}y) = \mathrm{e}^{-\lambda x/n}\prod_{j=1}^n \varphi(u_{n,j})\widehat{\mathbb{P}}(A_n \in \mathrm{d}x,\ S_n \in \mathrm{d}y)$$

and

$$\mathbb{P}(A_n \in \mathrm{d}x,\ S_n \in \mathrm{d}y,\ \tau > n) = \mathrm{e}^{-\lambda x/n}\prod_{j=1}^n \varphi(u_{n,j})\widehat{\mathbb{P}}(A_n \in \mathrm{d}x,\ S_n \in \mathrm{d}y,\ \tau > n). \quad (9)$$

2.1. Simple properties of the change of measure

We now collect some elementary properties of the measure change from (8). We first note that, by the definition of $\widehat{\mathbb{P}}$,

$$\widehat{\mathbb{E}}X_j = \frac{\varphi'(u_{n,j})}{\varphi(u_{n,j})}, \qquad j = 1, 2, \ldots, n.$$

This implies that if $j/n \to t \in [0, 1]$ then

$$\widehat{\mathbb{E}}X_j \to \frac{\varphi'(\lambda(1-t))}{\varphi(\lambda(1-t))} \quad \text{and} \quad \frac{1}{n}\widehat{\mathbb{E}}S_j \to \int_0^t \frac{\varphi'(\lambda(1-u))}{\varphi(\lambda(1-u))}\,\mathrm{d}u = \psi(t).$$

More precisely, there exists a constant C such that, for all $j = 1, 2, \ldots, n$,

$$\left|\widehat{\mathbb{E}}S_j - n\psi\left(\frac{j}{n}\right)\right| \le C.$$

This statement is a standard error estimate for the Riemann sum approximation of integrals of a function with bounded derivative. Furthermore,

$$\widehat{\mathrm{var}}X_j = \frac{\varphi''(u_{n,j})}{\varphi(u_{n,j})} - \left(\frac{\varphi'(u_{n,j})}{\varphi(u_{n,j})}\right)^2$$

and, consequently,

$$\frac{1}{n}\widehat{\mathrm{var}}S_j \to \int_0^t \left(\frac{\varphi''(\lambda(1-u))}{\varphi(\lambda(1-u))} - \left(\frac{\varphi'(\lambda(1-u))}{\varphi(\lambda(1-u))}\right)^2\right)\mathrm{d}u.$$

From these asymptotics for the first two moments and from the Kolmogorov inequality, we infer that

$$\sup_{t\in[0,1]}\left|\frac{S_{[nt]}}{n} - \psi(t)\right| \to 0 \quad \text{in } \widehat{\mathbb{P}} \text{ probability.}$$

Fix some $\varepsilon > 0$. For every $j = 1, 2, \ldots, n$, we have

$$\widehat{\mathbb{E}}[X_j^2;\ |X_j| > \varepsilon\sqrt{n}] = \frac{1}{\varphi(u_{n,j})}\mathbb{E}[X_j^2 \mathrm{e}^{u_{n,j}X_j};\ |X_j| > \varepsilon\sqrt{n}]$$
$$\leq \frac{1}{\min_{u\in[0,\lambda]}\varphi(u)}(\mathbb{E}[X_1^2\mathrm{e}^{\lambda X_1};\ X_1 > \varepsilon\sqrt{n}] + \mathbb{E}[X_1^2;\ X_1 < -\varepsilon\sqrt{n}]).$$

Since $\mathbb{E}[X_1^2]$ and $\mathbb{E}[X_1^2\mathrm{e}^{\lambda X_1}]$ are finite, we infer that

$$\frac{1}{n}\sum_{j=1}^{n}\widehat{\mathbb{E}}[X_j^2;\ |X_j| > \varepsilon\sqrt{n}] \to 0 \quad \text{for every } \varepsilon > 0.$$

In other words, the sequence $\{X_j\}_{j=1}^n$ satisfies the Lindeberg condition. Therefore, we have the following version of the functional central limit theorem: the sequence of linear interpolations

$$s_n(t) = n^{-1/2}(S_k + X_k(tn - k - 1) - n\psi(t)), \qquad t \in \left[\frac{k}{n}, \frac{k+1}{n}\right],\ k = 0, 1, \ldots, n-1,$$

converges weakly on $C[0, 1]$ towards a centred Gaussian process $\{\xi(t);\ t \in [0, 1]\}$ with independent increments and second moments

$$\mathbb{E}[\xi(t)]^2 = \int_0^t \sigma^2(u)\,\mathrm{d}u,$$

where

$$\sigma^2(u) := \frac{\varphi''(\lambda(1-u))}{\varphi(\lambda(1-u))} - \left(\frac{\varphi'(\lambda(1-u))}{\varphi(\lambda(1-u))}\right)^2.$$

(This statement can be proven using the standard approach of [3, Section 8]: the convergence of finite-dimensional distributions follows from the classical Lindeberg–Feller theorem, and the tightness is a simple consequence of the Kolmogorov inequality for the maximum of partial sums of independent random variables.)

Convergence on $C[0, 1]$ implies that

$$\left(\frac{S_{[nt]} - n\psi(t)}{\sqrt{n}}, \frac{A_{[nt]} - n^2\int_0^t \psi(s)\,\mathrm{d}s}{n^{3/2}}\right) \xrightarrow{\mathrm{w}} \left(\xi(t), \int_0^t \xi(s)\,\mathrm{d}s\right), \qquad t \in [0, 1].$$

The limiting vector has a normal distribution with zero mean. We now compute the covariance of $\xi(t)$ and $\int_0^t \xi(s)\,\mathrm{d}s$. Using the independence of the increments, we can easily obtain

$$\begin{aligned}
\operatorname{cov}\left(\xi(t), \int_0^t \xi(s)\,\mathrm{d}s\right) &= \int_0^t \operatorname{cov}(\xi(t), \xi(s))\,\mathrm{d}s \\
&= \int_0^t \operatorname{cov}(\xi(s) + \xi(t) - \xi(s), \xi(s))\,\mathrm{d}s \\
&= \int_0^t \operatorname{cov}(\xi(s), \xi(s))\,\mathrm{d}s \\
&= \int_0^t \int_0^s \sigma^2(u)\,\mathrm{d}u\,\mathrm{d}s \\
&= \int_0^t \sigma^2(u)(t-u)\,\mathrm{d}u.
\end{aligned}$$

Moreover,

$$\begin{aligned}\operatorname{cov}\biggl(\int_0^t \xi(s)\,\mathrm{d}s, \int_0^t \xi(s)\,\mathrm{d}s\biggr) &= \int_0^t \int_0^t \operatorname{cov}(\xi(s_1), \xi(s_2))\,\mathrm{d}s_1\,\mathrm{d}s_2 \\ &= 2\int_0^t \mathrm{d}s_1 \int_0^{s_1} \operatorname{cov}(\xi(s_1), \xi(s_2))\,\mathrm{d}s_2 \\ &= 2\int_0^t \mathrm{d}s_1 \int_0^{s_1} \biggl(\int_0^{s_2} \sigma^2(u)\,\mathrm{d}u\biggr)\,\mathrm{d}s_2 \\ &= 2\int_0^t \int_0^{s_1} \sigma^2(u)(s_1-u)\,\mathrm{d}s_1\,\mathrm{d}u \\ &= \int_0^t \sigma^2(u)(t-u)^2\,\mathrm{d}u.\end{aligned}$$

Therefore, the density of $(\xi(t), \int_0^t \xi(s)\,\mathrm{d}s)$ is given by

$$f_t(x,y) := \frac{1}{2\pi\sqrt{\det \Sigma_t}} \exp\biggl(-\frac{1}{2}(x,y)\Sigma_t^{-1}(x,y)^\top\biggr) \tag{10}$$

with the covariance matrix

$$\Sigma_t = \begin{pmatrix} \int_0^t \sigma^2(u)\,\mathrm{d}u & \int_0^t \sigma^2(u)(t-u)\,\mathrm{d}u \\ \int_0^t \sigma^2(u)(t-u)\,\mathrm{d}u & \int_0^t \sigma^2(u)(t-u)^2\,\mathrm{d}u \end{pmatrix}. \tag{11}$$

2.2. Proof of the Chebyshev-type estimate (7)

Lemma 1. *As $n \to \infty$,*

$$\prod_{j=1}^n \varphi(u_{n,j}) = \exp\{-\lambda I n\}(1 + O(n^{-1})). \tag{12}$$

Proof. It is obvious that (12) is equivalent to

$$\sum_{j=1}^n \log\varphi(u_{n,j}) = -\lambda I n + O(n^{-1}). \tag{13}$$

The sum on the left-hand side of (13) can be written as

$$\begin{aligned}\sum_{j=1}^n \log\varphi\biggl(\lambda\frac{n-j+1}{n}\biggr) &= \sum_{j=1}^n \log\varphi\biggl(\lambda\biggl(1-\frac{j-1}{n}\biggr)\biggr) \\ &= -\lambda\sum_{j=1}^n \biggl(-\frac{1}{\lambda}\log\varphi\biggl(\lambda\biggl(1-\frac{j-1}{n}\biggr)\biggr)\biggr) \\ &= -\lambda\sum_{j=1}^n \psi\biggl(\frac{j-1}{n}\biggr) \\ &= -\lambda\sum_{j=0}^{n-1} \psi_n(j), \end{aligned}\tag{14}$$

where $\psi_n(z) := \psi(z/n)$.

Applying the Euler–Maclaurin summation formula (see [9, p. 281, Equation (66)] or [1, p. 806, Equation (23.1.30)]), we obtain

$$\sum_{j=0}^{n-1} \psi_n(j) = \int_0^n \psi_n(t)\,\mathrm{d}t + B_1(\psi_n(n) - \psi_n(0)) \\ - \frac{1}{2}\int_0^1 (B_2(t) - B_2) \sum_{j=0}^{n-1} \psi_n''(j+1-t)\,\mathrm{d}t, \tag{15}$$

where B_k and $B_k(t)$ are Bernoulli numbers and Bernoulli polynomials, respectively.

Noting that $\psi_n(n) = \psi(1) = 0 = \psi(0) = \psi_n(0)$, we conclude that the first correction term in (15) disappears. Furthermore, by the definition of ψ_n,

$$\int_0^n \psi_n(t)\,\mathrm{d}t = \int_0^n \psi\left(\frac{t}{n}\right)\mathrm{d}t = n\int_0^1 \psi(t)\,\mathrm{d}t = nI.$$

Consequently, (15) reduces to

$$\sum_{j=0}^{n-1} \psi_n(j) = nI - \frac{1}{2}\int_0^1 (B_2(t) - B_2) \sum_{j=0}^{n-1} \psi_n''(j+1-t)\,\mathrm{d}t. \tag{16}$$

Since $\varphi(z)$, $\varphi'(z)$, and $\varphi''(z)$ are bounded on the interval $[0, \lambda]$, we obtain

$$\sup_{z\in[0,n]} |\psi_n''(z)| = \frac{1}{n^2} \sup_{z\in[0,1]} |\psi''(z)| = \frac{\lambda}{n^2} \sup_{z\in[0,\lambda]} \left| \frac{\varphi''(z)\varphi(z) - (\varphi'(z))^2}{\varphi^2(z)} \right| = \frac{c}{n^2}.$$

Therefore,

$$\left| \int_0^1 (B_2(t) - B_2) \sum_{i=0}^{n-1} \psi_n''(j+1-t)\,\mathrm{d}t \right| \le \frac{c}{n}\int_0^1 |B_2(t) - B_2|\,\mathrm{d}t = O\left(\frac{1}{n}\right).$$

Combining this estimate with (16), we obtain

$$\sum_{j=0}^{n-1} \psi_n(j) = nI + O\left(\frac{1}{n}\right).$$

Taking into account (14), we conclude that (13) is valid. Thus, the proof of the lemma is complete. □

Using (12), we can derive the upper bound (7) for $\mathbb{P}(A_\tau > x)$. Obviously,

$$\mathbb{P}(A_\tau \ge x) = \sum_{n=0}^{\infty} \mathbb{P}(A_n \ge x,\ \tau = n+1).$$

Using the exponential Chebyschev inequality and recalling that

$$A_n = \sum_{i=1}^{n} (n - j + 1) X_j,$$

we obtain

$$\begin{aligned}\mathbb{P}(A_n \ge x,\ \tau = n+1) &\le \mathbb{P}(A_n \ge x) \\ &\le \exp\Big\{-\frac{\lambda}{n}x\Big\}\mathbb{E}\Big[\exp\Big\{\frac{\lambda}{n}A_n\Big\}\Big] \\ &= \exp\Big\{-\frac{\lambda}{n}x\Big\}\prod_{i=1}^{n}\mathbb{E}\Big[\exp\Big\{\lambda\frac{n-j+1}{n}X_j\Big\}\Big] \\ &= \exp\Big\{-\frac{\lambda}{n}x\Big\}\prod_{i=1}^{n}\varphi\Big(\lambda\frac{n-j+1}{n}\Big).\end{aligned}$$

Applying Lemma 1, we have

$$\mathbb{P}(A_n \ge x,\ \tau = n+1) \le \exp\Big\{-\lambda\frac{x}{n} - \lambda I n + O(n^{-1})\Big\} \le C\exp\Big\{-\lambda\frac{x}{n} - \lambda I n\Big\}. \tag{17}$$

Consequently,

$$\mathbb{P}(A_\tau \ge x) = \sum_{n=1}^{\infty}\mathbb{P}(A_n \ge x,\ \tau = n+1) \le C\sum_{n=1}^{\infty}\exp\Big\{-\lambda\frac{x}{n} - \lambda I n\Big\}. \tag{18}$$

Using the result of Bateman [2, p. 146, Equation (25)], we have

$$\sum_{n=1}^{\infty}\exp\Big\{-\lambda\frac{x}{n} - \lambda I n\Big\} \le \int_0^{\infty}\exp\Big\{-\lambda\frac{x}{y} - \lambda I(y+1)\Big\}\,\mathrm{d}y = \exp\{-\lambda I\}\sqrt{\frac{4x}{I}}K_1(2\lambda\sqrt{Ix}).$$

Now using the asymptotics for the modified Bessel function

$$K_1(z) \sim \sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z} \quad \text{as } z \to \infty,$$

we obtain

$$\sum_{n=1}^{\infty}\exp\Big\{-\lambda\frac{x}{n} - \lambda I n\Big\} \le Cx^{1/4}\exp\{-2\lambda\sqrt{Ix}\}.$$

From this bound and (18), we obtain (7).

3. Local limit theorems

We start by proving a standard Gnedenko local limit theorem for the two-dimensional vector $(S_{[nt]}, A_{[nt]})$ under the measure $\widehat{\mathbb{P}}$. The following statement is a one-dimensional case of [6, Theorem 4.2] and we provide its proof for completeness.

Proposition 1. *Assume that the conditions of Theorem 1 are valid. Then, for every $t \in (0, 1]$,*

$$\sup_{x,y}\bigg|n^2\widehat{\mathbb{P}}(S_{[nt]} = x,\ A_{[nt]} = y) - f_t\bigg(\frac{x - n\psi(t)}{\sqrt{n}}, \frac{y - n^2\int_0^t\psi(s)\,\mathrm{d}s}{n^{3/2}}\bigg)\bigg| \to 0,$$

where f_t is defined in (10) and (11).

Proof. Consider centred random variables

$$X_j^0 := X_j - \widehat{\mathbb{E}} X_j$$

and their characteristic functions

$$\varphi_j(v) := \widehat{\mathbb{E}}[\mathrm{e}^{\mathrm{i}vX_j^0}], \qquad 1 \le j \le n.$$

By the inversion formula,

$$\begin{aligned}
&\widehat{\mathbb{P}}(S_{[nt]} = x,\ A_{[nt]} = y) \\
&\quad = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}v_1 x - \mathrm{i}v_2 y} \widehat{\mathbb{E}}[\mathrm{e}^{\mathrm{i}v_1 S_{[nt]} + \mathrm{i}v_2 A_{[nt]}}]\,\mathrm{d}v_1\,\mathrm{d}v_2 \\
&\quad = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}v_1 n^{1/2} x_0 - \mathrm{i}v_2 n^{3/2} y_0} \prod_{j=1}^{n} \varphi_j(v_1 + (n-j+1)v_2)\,\mathrm{d}v_1\,\mathrm{d}v_2,
\end{aligned}$$

where

$$x_0 := \frac{x - \widehat{\mathbb{E}} S_{[nt]}}{n^{1/2}} \quad \text{and} \quad y_0 := \frac{y - \widehat{\mathbb{E}} A_{[nt]}}{n^{3/2}}.$$

Using the change of variables $v_1 \to \sqrt{n} v_1$ and $v_2 \to n^{3/2} v_2$, we obtain

$$\begin{aligned}
&n^2 \widehat{\mathbb{P}}(S_{[nt]} = x,\ A_{[nt]} = y) \\
&\quad = \int_{-\pi n^{1/2}}^{\pi n^{1/2}} \int_{-\pi n^{3/2}}^{\pi n^{3/2}} \mathrm{e}^{-\mathrm{i}v_1 x_0 - \mathrm{i}v_2 y_0} \prod_{j=1}^{n} \varphi_j\biggl(\frac{v_1}{n^{1/2}} + \frac{(n-j+1)v_2}{n^{3/2}}\biggr)\,\mathrm{d}v_1\,\mathrm{d}v_2. \qquad (19)
\end{aligned}$$

By the inversion formula for Fourier transforms,

$$f_t\biggl(\frac{x - \widehat{\mathbb{E}} S_{[nt]}}{n^{1/2}}, \frac{x - \widehat{\mathbb{E}} A_{[nt]}}{n^{3/2}}\biggr) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}v_1 x_0 - \mathrm{i}v_2 y_0} \mathrm{e}^{-(v_1, v_2)\Sigma_t (v_1, v_2)^\top}\,\mathrm{d}v_1\,\mathrm{d}v_2. \qquad (20)$$

Define

$$\begin{aligned}
R_2 &= \{(v_1, v_2)\colon v_1 \in [-\varepsilon n^{1/2}, \varepsilon n^{1/2}],\ v_2 \in [-\varepsilon n^{3/2}, \varepsilon n^{3/2}]\}, \\
R_3 &= \{(v_1, v_2)\colon v_1 \in [-\pi n^{1/2}, \pi n^{1/2}],\ v_2 \in [-\pi n^{3/2}, \pi n^{3/2}]\}.
\end{aligned}$$

Combining (19) and (20), we conclude that

$$\sup_{x,y} \biggl| n^2 \widehat{\mathbb{P}}(S_{[nt]} = x,\ A_{[nt]} = y) - f_t\biggl(\frac{x - \mathbb{E} S_{[nt]}}{\sqrt{n}}, \frac{y - \mathbb{E} A_{[nt]}}{n^{3/2}}\biggr)\biggr| \le I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \frac{1}{(2\pi)^2} \int_{-A}^{A} \int_{-B}^{B} \biggl| \prod_{j=1}^{n} \varphi_j\biggl(\frac{v_1}{n^{1/2}} + \frac{(n-j+1)v_2}{n^{3/2}}\biggr) - \exp\{(v_1, v_2)\Sigma_t (v_1, v_2)^\top\}\biggr|\,\mathrm{d}v_1\,\mathrm{d}v_2,$$

$$I_2 = \frac{1}{(2\pi)^2} \iint_{R_2 \setminus [-A,A]\times[-B,B]} \prod_{j=1}^{n} \biggl| \varphi_j\biggl(\frac{v_1}{n^{1/2}} + \frac{(n-j+1)v_2}{n^{3/2}}\biggr)\biggr|\,\mathrm{d}v_1\,\mathrm{d}v_2,$$

$$I_3 = \frac{1}{(2\pi)^2} \iint_{R_3 \setminus R_2} \prod_{j=1}^{n} \left| \varphi_j\left(\frac{v_1}{n^{1/2}} + \frac{(n-j+1)v_2}{n^{3/2}}\right)\right| \mathrm{d}v_1\, \mathrm{d}v_2,$$

$$I_4 = \frac{1}{(2\pi)^2} \int_{|v_1|>A} \int_{|v_2|>B} \exp\{-(v_1, v_2)\Sigma_t (v_1, v_2)^\top\}\, \mathrm{d}v_1\, \mathrm{d}v_2.$$

Choosing large enough A and B, we can make the integral I_4 as small as we please. Furthermore, the weak convergence

$$\left(\frac{S_{[nt]} - \widehat{\mathbb{E}} S_{[nt]}}{\sqrt{n}}, \frac{A_{[nt]} - \widehat{\mathbb{E}} A_{[nt]}}{n^{3/2}}\right) \xrightarrow{\mathrm{w}} \left(\xi(t), \int_0^t \xi(s)\,\mathrm{d}s\right)$$

implies that, uniformly on every compact $[-A, A] \times [-B, B]$,

$$\left| \prod_{j=1}^{n} \varphi_j\left(\frac{v_1}{n^{1/2}} + \frac{(n-j+1)v_2}{n^{3/2}}\right) - \mathrm{e}^{(v_1,v_2)\Sigma_t (v_1,v_2)^\top} \right| \to 0.$$

Consequently, I_1 converges to 0.

It is clear that the random variables X_j^2 are uniformly integrable with respect to the measure $\widehat{\mathbb{P}}$. Therefore, for every small enough ε,

$$|\varphi_j(v)| \le \mathrm{e}^{-\sigma^2(u_{n,j})v^2/4}, \qquad |v| \le 2\varepsilon, \quad 1 \le j \le n.$$

Consequently, there exist constants $c > 0$ and C such that

$$\begin{aligned} \prod_{j=1}^{n} \left| \varphi_j\left(\frac{v_1}{n^{1/2}} + \frac{v_2}{n^{3/2}}\right)\right| &\le \exp\left\{-\sum_{j=1}^{n} \frac{\sigma^2(u_{n,j})}{4}\left(\frac{v_1}{\sqrt{n}} + \frac{(n-j+1)v_2}{n^{3/2}}\right)^2\right\} \\ &\le C \exp\{-c(v_1, v_2)\Sigma_t (v_1, v_2)^\top\} \end{aligned}$$

on the set $|v_1| \le \varepsilon n^{1/2}$, $|v_2| \le \varepsilon n^{1/2}$. Therefore, I_2 can be made as small as we please by choosing A and B large enough.

It remains to bound I_3. Since the distributions of random variables X_j are aperiodic, $|\widehat{\mathbb{E}}[\mathrm{e}^{\mathrm{i}vX_j}]| = 1$ if and only if $v = 2\pi m$. Furthermore, recalling that the distributions of X_j are obtained via the exponential change of measure of the same distribution and that the parameters of these changes are taken from the bounded interval, we conclude that, for every $\delta > 0$, there exists $c_\delta > 0$ such that

$$\max_{1 \le j \le n} |\varphi_j(v)| \le \mathrm{e}^{-c_\delta} \quad \text{for all } v \text{ such that } |v - 2\pi m| > \delta \text{ for all } m \in \mathbb{Z}. \tag{21}$$

For all v_1 and v_2 from the integration region in I_3, we have the following property. At least $n/2$ elements of the sequence $\{v_1/\sqrt{n} + (n-j+1)v_2/n^{3/2}\}_{j=1}^{n}$ are separated from the set $\{2\pi m, m \in \mathbb{Z}\}$. From this fact and (21), we infer that there exists δ_0 such that

$$\prod_{j=1}^{n} \left| \varphi_j\left(\frac{v_1}{n^{1/2}} + \frac{v_2}{n^{3/2}}\right)\right| \le \mathrm{e}^{-c_{\delta_0} n/2}.$$

Consequently, I_3 converges to 0 as $n \to \infty$. Thus, the proof is complete. □

Proposition 2. *Assume that the conditions of Theorem 1 are valid. Then there exists a positive, increasing function $q(a)$ such that, for every $t \in (0,1)$ and every $a \geq 0$,*

$$\sup_{x,y}\left|n^2\widehat{\mathbb{P}}\Big(S_{[nt]}=x,\, A_{[nt]}=y,\, \min_{k\leq[nt]} S_k > -a\Big) - q(a) f_t\left(\frac{x-n\psi(t)}{\sqrt{n}}, \frac{y-n^2\int_0^t \psi(s)ds}{n^{3/2}}\right)\right| \to 0.$$

Proof. Set $m=[\log^2 n]$. Then, by the Markov property at time m,

$$\begin{aligned}
&\widehat{\mathbb{P}}\Big(S_{[nt]}=x,\ A_{[nt]}=y,\ \min_{k\leq n} S_k > -a\Big)\\
&\quad=\sum_{x',y'>0}\widehat{\mathbb{P}}\Big(S_m=x',\ A_m=y',\ \min_{k\leq m} S_k > -a\Big) Q(x'y';\ x,y),
\end{aligned}$$

where

$$\begin{aligned}
&Q(x'y';\ x,y)\\
&\quad=\widehat{\mathbb{P}}\Big(S^{(m)}_{[nt]-m}=x-x',\ A^{(m)}_{[nt]-m}=y-y'-([nt]-m)x',\ \min_{k\leq[nt]-m} S_k^{(m)} > -x'-a\Big)
\end{aligned}$$

and

$$S_k^{(m)}=X_{m+1}+\cdots+X_{m+k}, \qquad A_k^{(m)}=S_1^{(m)}+S_2^{(m)}+\cdots+S_k^{(m)}.$$

By Proposition 1,

$$Q(x'y';\ x,y)\leq\widehat{\mathbb{P}}(S^{(m)}_{[nt]-m}=x-x',\ A^{(m)}_{[nt]-m}=y-y'-([nt]-m)x')\leq\frac{c_t}{n^2}. \tag{22}$$

It follows from the definition of $\widehat{\mathbb{P}}$ that the second moments of X_j are uniformly bounded. Applying the Chebyshev inequality, we then obtain

$$\widehat{\mathbb{P}}(|S_m-\widehat{\mathbb{E}}S_m|\geq\log^{3/2} n)=o(1) \quad\text{and}\quad \widehat{\mathbb{P}}(|A_m-\widehat{\mathbb{E}}A_m|\geq\log^{5/2} n)=o(1). \tag{23}$$

Define

$$D:=\{(x',y'):\ |x'-\widehat{\mathbb{E}}S_m|\leq\log^{3/2} n,\ |y'-\widehat{\mathbb{E}}A_m|\leq\log^{5/2} n\}.$$

Combining (22) and (23), we conclude that, uniformly in $x, y > 0$,

$$\lim_{n\to\infty} n^2\sum_{D^c}\widehat{\mathbb{P}}\Big(S_m=x',\ A_m=y',\ \min_{k\leq m} S_k > -a\Big) Q(x'y';\ x,y)=0. \tag{24}$$

We now turn to the asymptotic behaviour of $Q(x'y';\ x,y)$ for (x',y') belonging to the set D. Obviously,

$$\begin{aligned}
Q(x'y';\ x,y)&=\widehat{\mathbb{P}}(S^{(m)}_{[nt]-m}=x-x',\ A^{(m)}_{[nt]-m}=y-y'-([nt]-m)x')\\
&\quad-\widehat{\mathbb{P}}\Big(S^{(m)}_{[nt]-m}=x-x',\ A^{(m)}_{[nt]-m}=y-y'-([nt]-m)x',\\
&\qquad\min_{k\leq[nt]-m} S_k^{(m)}\leq -x'-a\Big).
\end{aligned} \tag{25}$$

We can apply Proposition 1 to the first probability term on the right-hand side of (25). As a result, uniformly in $x, x', y, y' > 0$,

$$\begin{aligned} &n^2\widehat{\mathbb{P}}(S^{(m)}_{[nt]-m} = x - x',\ A^{(m)}_{[nt]-m} = y - y' - ([nt]-m)x') \\ &\quad - f_t\left(\frac{x - x' - n\psi(t)}{\sqrt{n}}, \frac{y - y' - n^2\int_0^t \psi(s)\,\mathrm{d}s}{n^{3/2}}\right) \\ &\quad \to 0. \end{aligned} \tag{26}$$

Furthermore, it follows easily from the definition of the measure $\widehat{\mathbb{P}}$ that $\widehat{\mathbb{E}}X_j \sim \varphi'(\lambda)/\varphi(\lambda)$ for each $j \le m$. Therefore, $\widehat{\mathbb{E}}S_m \sim (\varphi'(\lambda)/\varphi(\lambda))\log^2 n$ and $\widehat{\mathbb{E}}A_m \sim (\varphi'(\lambda)/2\varphi(\lambda))\log^4 n$. From these relations, we infer that

$$f_t\left(\frac{x - x' - n\psi(t)}{\sqrt{n}}, \frac{y - y' - n^2\int_0^t \psi(s)\,\mathrm{d}s}{n^{3/2}}\right) - f_t\left(\frac{x - n\psi(t)}{\sqrt{n}}, \frac{y - n^2\int_0^t \psi(s)\,\mathrm{d}s}{n^{3/2}}\right) \to 0$$

uniformly in $x, y > 0$ and $(x', y') \in D$. Combining this with (26), we conclude that

$$\begin{aligned} &n^2\widehat{\mathbb{P}}(S^{(m)}_{[nt]-m} = x - x',\ A^{(m)}_{[nt]-m} = y - y' - ([nt]-m)x') \\ &\quad - f_t\left(\frac{x - n\psi(t)}{\sqrt{n}}, \frac{y - n^2\int_0^t \psi(s)\,\mathrm{d}s}{n^{3/2}}\right) \\ &\quad \to 0 \end{aligned}$$

uniformly in $x, y > 0$ and $(x', y') \in D$.

Moreover, for every $(x', y') \in D$ and all sufficiently large n, we have

$$\begin{aligned} &\widehat{\mathbb{P}}\Big(S^{(m)}_{[nt]-m} = x - x',\ A^{(m)}_{[nt]-m} = y - y' - ([nt]-m)x',\ \min_{k \le [nt]-m} S^{(m)}_k \le -x'\Big) \\ &\quad \le \widehat{\mathbb{P}}\Big(\min_{k \le [nt]-m} S^{(m)}_k \le -x'\Big) \\ &\quad \le \widehat{\mathbb{P}}\Big(\min_{k \le [nt]-m} S^{(m)}_k \le -\log^{3/2} n\Big). \end{aligned}$$

By the exponential Chebyshev inequality,

$$\widehat{\mathbb{P}}(S_k \le -\log^{3/2} n) = \widehat{\mathbb{P}}(-S_k \ge \log^{3/2} n) \le \mathrm{e}^{-\lambda h \log^{3/2} n}\widehat{\mathbb{E}}[\mathrm{e}^{-\lambda h S_k}]. \tag{27}$$

Furthermore, from the definition of $\widehat{\mathbb{P}}$, it follows that

$$\begin{aligned} \widehat{\mathbb{E}}[\mathrm{e}^{-\lambda h S_k}] &= \prod_{j=1}^k \widehat{\mathbb{E}}[\mathrm{e}^{-\lambda h X_j}] \\ &= \prod_{j=1}^k \frac{\varphi(u_{n,j} - \lambda h)}{\varphi(u_{n,j})} \\ &= \exp\left\{-\lambda \sum_{j=1}^k \psi\left(\frac{j-1}{n} + h\right) + \lambda \sum_{j=1}^k \psi\left(\frac{j-1}{n}\right)\right\}. \end{aligned} \tag{28}$$

Using the Euler–Maclaurin summation formula (15), we infer that

$$\sum_{j=1}^{k}\psi\Big(\frac{j-1}{n}\Big)-\sum_{j=1}^{k}\psi\Big(\frac{j-1}{k}+h\Big)\le c+n\Big(\int_0^{k/n}\psi(u)\,\mathrm{d}u-\int_h^{h+k/n}\psi(u)\,\mathrm{d}u\Big).$$

If $h<1-t$ then the function $s\mapsto\int_0^s\psi(u)\,\mathrm{d}u-\int_h^{h+s}\psi(u)\,\mathrm{d}u$ achieves its maximum either at 0 or at t. Therefore,

$$\max_{s\in[0,t]}\Big(\int_0^s\psi(u)\,\mathrm{d}u-\int_h^{s+h}\psi(u)\,\mathrm{d}u\Big)=\Big(\int_0^t\psi(u)\,\mathrm{d}u-\int_h^{t+h}\psi(u)\,\mathrm{d}u\Big)^+.$$

If h is so small that $\psi(h)<\psi(t+h)$ then

$$\int_0^t\psi(u)\,\mathrm{d}u-\int_h^{t+h}\psi(u)\,\mathrm{d}u<0$$

and, consequently,

$$\max_{k\le nt}\Big(\sum_{j=1}^{k}\psi\Big(\frac{j-1}{n}\Big)-\sum_{j=1}^{k}\psi\Big(\frac{j-1}{n}+h\Big)\Big)\le c.$$

Substituting this into (28), we obtain

$$\max_{k\le nt}\widehat{\mathbb{E}}[\mathrm{e}^{-\lambda hS_k}]\le\mathrm{e}^c.$$

Combining this estimate and (27), we finally obtain

$$\widehat{\mathbb{P}}\Big(\min_{k\le nt}S_k\le-\log^{3/2}n\Big)\le\sum_{j=1}^{nt}\widehat{\mathbb{P}}(S_k<-\log^{3/2}n)\le nt\mathrm{e}^c\mathrm{e}^{-\lambda h\log^{3/2}n}=o\Big(\frac{1}{n^2}\Big).$$

So we have, uniformly in $x,y>0$ and $(x',y')\in D$,

$$n^2Q(x',y';\,x,y)-\frac{1}{n^2}f_t\Big(\frac{x-n\psi(t)}{\sqrt{n}},\frac{y-n^2\int_0^t\psi(s)\,\mathrm{d}s}{n^{3/2}}\Big)\to0. \tag{29}$$

Combining (23) and (29), we conclude that

$$\begin{aligned}n^2&\sum_D\widehat{\mathbb{P}}\Big(S_m=x',\ A_m=y',\ \min_{k\le m}S_k>-a\Big)Q(x'y';\,x,y)\\&=f_t\Big(\frac{x-n\psi(t)}{\sqrt{n}},\frac{y-n^2\int_0^t\psi(s)\,\mathrm{d}s}{n^{3/2}}\Big)\sum_D\widehat{\mathbb{P}}\Big(S_m=x',\ A_m=y',\ \min_{k\le m}S_k>-a\Big)\\&\quad+o(1)\\&=f_t\Big(\frac{x-n\psi(t)}{\sqrt{n}},\frac{y-n^2\int_0^t\psi(s)\,\mathrm{d}s}{n^{3/2}}\Big)\widehat{\mathbb{P}}\Big(\min_{k\le m}S_k>-a\Big)+o(1).\end{aligned} \tag{30}$$

For every fixed $m_0\ge1$ and all $m\ge m_0$, we have

$$\widehat{\mathbb{P}}\Big(\min_{k\le m}S_k>-a\Big)\le\widehat{\mathbb{P}}\Big(\min_{k\le m_0}S_k>-a\Big)$$

and
$$\widehat{\mathbb{P}}\Big(\min_{k\le m} S_k > -a\Big) \ge \widehat{\mathbb{P}}\Big(\min_{k\le m_0} S_k > -a\Big) - \widehat{\mathbb{P}}\Big(\min_{m_0<k\le m} S_k \le -a\Big).$$

For the second probability term on the right-hand side, we have

$$\begin{aligned}\widehat{\mathbb{P}}\Big(\min_{m_0<k\le m} S_k \le -a\Big) &\le \widehat{\mathbb{P}}(S_{m_0} < m_0^{2/3}) + \widehat{\mathbb{P}}\Big(\min_{k\le m-m_0} S_k < -m_0^{2/3}\Big)\\ &\le \widehat{\mathbb{P}}(S_{m_0} < m_0^{2/3}) + \sum_{k=1}^{m-m_0} \widehat{\mathbb{P}}(S_k < -m_0^{2/3}).\end{aligned}$$

Using the exponential Chebyshev inequality once again, we can easily infer that there exists $f(x) \to 0,\ x \to \infty$, such that, for all $n \ge 1$,

$$\widehat{\mathbb{P}}\Big(\min_{m_0<k\le m} S_k < -a\Big) \le f(m_0).$$

Consequently,

$$\widehat{\mathbb{P}}\Big(\min_{k\le m_0} S_k > -a\Big) - f(m_0) \le \widehat{\mathbb{P}}\Big(\min_{k\le m} S_k > -a\Big) \le \widehat{\mathbb{P}}\Big(\min_{k\le m_0} S_k > -a\Big).$$

For every $j \le m_0$, the distribution of X_j converges, as $n \to \infty$, to the distribution of X_1 under $\widehat{\mathbb{P}}$. (Here we note that this distribution does not depend on n.) Let U_k denote a random walk with i.i.d. increments which are distributed according to the limiting distribution of X_j. Then
$$\lim_{n\to\infty} \widehat{\mathbb{P}}\Big(\min_{k\le m_0} S_k > -a\Big) = \mathbb{P}\Big(\min_{k\le m_0} U_k > -a\Big).$$

Now letting $m_0 \to \infty$, we finally obtain

$$\lim_{n\to\infty} \widehat{\mathbb{P}}\Big(\min_{k\le m} S_k > -a\Big) = \mathbb{P}\Big(\min_{k\ge 1} U_k \ge -a\Big) =: q(a).$$

The positivity of the function q follows from the fact that the increments of U_k have positive mean. Applying the previous relation to (30) and taking into account (24), we obtain the desired asymptotics. □

In order to prove local limit theorems for (S_n, A_n) conditioned on $\{\tau > n, S_n = x\}$ with fixed x, we are going to consider the path $\{S_{[n/2]}, S_{[n/2]+1}, \dots, S_n\}$ in reversed time. More precisely, we consider random variables

$$\hat{X}_k = -X_{n-k+1} \quad \text{and} \quad \hat{S}_k = \hat{X}_1 + \hat{X}_2 + \cdots + \hat{X}_k, \qquad k = 1, 2, \dots, n.$$

Proposition 3. *Assume that the conditions of Theorem 1 are valid. Then there exists a positive increasing $\hat{q}$ such that, for every $t \in (0, 1)$,*

$$\begin{aligned}&n^2\widehat{\mathbb{P}}\Big(\hat{S}_{[nt]} = x,\ \hat{A}_{[nt]-1} = y,\ \min_{k\le[nt]} \hat{S}_k > -a\Big)\\ &\quad - \hat{q}(a)\hat{f}_t\left(\frac{x - n\psi(1-t)}{\sqrt{n}}, \frac{y - n^2\int_{1-t}^1 \psi(s)ds}{n^{3/2}}\right)\\ &\quad \to 0\end{aligned}$$

uniformly in x, $y > 0$. The function $\hat{f}_t$ is the density function of the normal distribution with zero mean and the covariance matrix

$$\widehat{\Sigma}_t = \begin{pmatrix} \int_{1-t}^1 \sigma^2(u)\,\mathrm{d}u & \int_{1-t}^1 \sigma^2(u)(t-1+u)\,\mathrm{d}u \\ \int_{1-t}^1 \sigma^2(u)(t-1+u)\,\mathrm{d}u & \int_{1-t}^1 \sigma^2(u)(t-1+u)^2\,\mathrm{d}u \end{pmatrix}.$$

Proof. The proof follows that of Propositions 1 and 2 and we omit it. □

We now state a local limit theorem for a bridge of S_n conditioned to stay positive. This result is the most important ingredient in our approach to the proof of Theorem 1.

Proposition 4. *Assume that the conditions of Theorem 1 are valid. Then, for every fixed x,*

$$n^2\widehat{\mathbb{P}}(A_n = y,\ S_n = x,\ \tau > n) - q(0)\hat{q}(x) f_1\left(0, \frac{y - n^2 I}{n^{3/2}}\right) \to 0.$$

Proof. From the definition of $\hat{S}_k$, it is immediate that $S_k = S_n - \sum_{j=k+1}^n X_j = S_n + \hat{S}_{n-k}$. Therefore, for $\ell(n) = [nt]$ with some fixed $t \in (0, 1)$, we have

$$\begin{aligned}\{A_n = y, S_n = x\} &= \left\{A_{l(n)} + \sum_{k=l(n)+1}^{n} S_k = y, S_{l(n)} - \hat{S}_{n-l(n)} = x\right\} \\ &= \left\{A_{l(n)} + (n - l(n))x + \sum_{l(n)+1}^{n} \hat{S}_{n-k} = y, S_{l(n)} - \hat{S}_{n-l(n)} = x\right\} \\ &= \{A_{l(n)} + \hat{A}_{n-l(n)-1} = y - (n - l(n))x, S_{l(n)} - \hat{S}_{n-l(n)} = x\}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\widehat{\mathbb{P}}(A_n = y,\ S_n = x,\ \tau > n) \\ &\quad= \widehat{\mathbb{P}}(A_{l(n)} + \hat{A}_{n-l(n)-1} = y - (n - l(n))x,\ S_{l(n)} - \hat{S}_{n-l(n)} = x,\ \tau > n) \\ &\quad= \sum_{x',y'} \widehat{\mathbb{P}}(A_{l(n)} = y',\ S_{l(n)} = x',\ \tau > l(n))\hat{Q}(x', y';\ x, y),\end{aligned}$$

where

$$\begin{aligned}&\hat{Q}(x', y';\ x, y) \\ &\quad:= \widehat{\mathbb{P}}\Big(\hat{A}_{n-l(n)-1} = y - y' - (n - l(n))x,\ \hat{S}_{n-l(n)} = x' - x,\ \min_{k \le n-\ell(n)} \hat{S}_k > -x\Big).\end{aligned}$$

Combining Propositions 2 and 3, we conclude that, for every fixed x,

$$n^2\widehat{\mathbb{P}}(A_n = y,\ S_n = x,\ \tau > n) \sim q(0)\hat{q}(x) n^2 \Sigma_n(y),$$

where

$$\begin{aligned}\Sigma_n(y) := \sum_{x',y'} f_t&\left(\frac{x' - n\psi(1/2)}{\sqrt{n}}, \frac{y' - n^2\int_0^{1/2}\psi(s)\,\mathrm{d}s}{n^{3/2}}\right) \\ &\times \hat{f}_{1-t}\left(\frac{x' - n\psi(1/2)}{\sqrt{n}}, \frac{y - y' - n^2\int_{1/2}^1 \psi(s)\,\mathrm{d}s}{n^{3/2}}\right).\end{aligned}$$

From the continuity and boundedness of functions f_t and $\hat{f}_{1-t}$, it is immediate that

$$n^2\Sigma_n(y) \sim \int_{\mathbb{R}^2} f_t(u,v)\hat{f}_{1-t}\left(u, \frac{y-n^2I}{n^{3/2}} - v\right) \mathrm{d}u\,\mathrm{d}v, \qquad n \to \infty,$$

and, consequently,

$$n^2\widehat{\mathbb{P}}(A_n = y,\ S_n = x,\ \tau > n) \sim q(0)\hat{q}(x) \int_{\mathbb{R}^2} f_t(u,v)\hat{f}_{1-t}\left(u, \frac{y-n^2I}{n^{3/2}} - v\right) \mathrm{d}u\,\mathrm{d}v.$$

Since the left-hand side does not depend on t, we infer that the integral on the right-hand side does not depend on t as well. Letting $t \to 1$ and using the continuity of f_t, we infer that

$$\int_{\mathbb{R}^2} f_t(u,v)\hat{f}_{1-t}(u, z-v)\,\mathrm{d}u\,\mathrm{d}v = f_1(0,z). \qquad \square$$

4. Proofs of tail asymptotics

Proof of Theorem 1. Using (9), we obtain

$$\begin{aligned}
\mathbb{P}(A_n = x,\ \tau = n+1) &= \sum_{y=1}^{\infty} \mathbb{P}(A_n = x,\ S_n = y,\ \tau = n+1) \\
&= \sum_{y=1}^{\infty} \mathbb{P}(A_n = x,\ S_n = y,\ \tau > n)\mathbb{P}(X_{n+1} \le -y) \\
&= \mathrm{e}^{-\lambda x/n} \prod_{j=1}^{n} \varphi(u_{n,j}) \sum_{y=1}^{\infty} \widehat{\mathbb{P}}(A_n = x,\ S_n = y,\ \tau > n)\mathbb{P}(X_{n+1} \le -y).
\end{aligned}$$

From Proposition 3, it follows that, for every fixed M,

$$\begin{aligned}
&\sum_{y=1}^{M} \widehat{\mathbb{P}}(A_n = x,\ S_n = y,\ \tau > n)\mathbb{P}(X_{n+1} \le -y) \\
&\qquad = \frac{q(0)}{n^2} h\left(\frac{x-n^2I}{n^{3/2}}\right) \sum_{y=1}^{M} \hat{q}(y)\mathbb{P}(X_1 \le -y) + o\left(\frac{1}{n^2}\right).
\end{aligned} \tag{31}$$

Furthermore, applying Proposition 1, we have

$$\begin{aligned}
&\sum_{y=M+1}^{\infty} \widehat{\mathbb{P}}(A_n = x,\ S_n = y,\ \tau > n)\mathbb{P}(X_{n+1} \le -y) \\
&\qquad \le \sum_{y=M+1}^{\infty} \widehat{\mathbb{P}}(A_n = x,\ S_n = y)\mathbb{P}(X_{n+1} \le -y) \\
&\qquad \le \frac{c}{n^2} \sum_{M+1}^{\infty} \mathbb{P}(X_1 \le -y).
\end{aligned}$$

Consequently, uniformly in n,

$$\lim_{M\to\infty} n^2 \sum_{y=M+1}^{\infty} \widehat{\mathbb{P}}(A_n = x,\ S_n = y,\ \tau > n)\mathbb{P}(X_{n+1} \le -y) = 0. \tag{32}$$

Combining (31) and (32), we conclude that

$$\sum_{y=1}^{\infty} \widehat{\mathbb{P}}(A_n = x,\ S_n = y, \tau > n)\mathbb{P}(X_{n+1} \le -y)$$
$$= \frac{1}{n^2} h\left(\frac{x - n^2 I}{n^{3/2}}\right) \sum_{y=1}^{\infty} \hat{q}(y)\mathbb{P}(X_1 \le -y) + o\left(\frac{1}{n^2}\right).$$

According to Lemma 1,

$$\prod_{j=1}^{n} \varphi(u_{n,j}) = \exp\{-\lambda I n\}(1 + O(n^{-1})).$$

Therefore,

$$\mathbb{P}(A_n = x,\ \tau = n+1) = \frac{Q + o(1)}{n^2} \exp\left\{-\frac{\lambda x}{n} - \lambda n I\right\} h\left(\frac{x - n^2 I}{n^{3/2}}\right), \tag{33}$$

where $Q := q(0)\sum_{y=1}^{\infty} \hat{q}(y)\mathbb{P}(X_1 \le -y)$. In particular, there exists a constant C such that

$$\mathbb{P}(A_n,\ \tau = n+1) \le \frac{C}{n^2} \exp\left\{-\frac{\lambda x}{n} - \lambda n I\right\}. \tag{34}$$

Define $n_- : = \max\{n \in \mathbb{N}\colon n \le \sqrt{x/I}\}$ and $n_+ := n_- + 1$. Changing the summation index and splitting the series into two parts, we obtain

$$\begin{aligned}
\sum_{n=n_+}^{\infty} \mathbb{P}(A_n = x,\ \tau = n+1) &= \sum_{k=0}^{\infty} \mathbb{P}(A_{n_+ + k} = x,\ \tau = n_+ + k + 1) \\
&= \sum_{k \le M n_+^{1/2}} \mathbb{P}(A_{n_+ + k} = x,\ \tau = n_+ + k + 1) \\
&\quad + \sum_{k > M n_+^{1/2}} \mathbb{P}(A_{n_+ + k} = x,\ \tau = n_+ + k + 1).
\end{aligned} \tag{35}$$

Applying (33) to the summands in the first sum, we have

$$\sum_{k \le M n_+^{1/2}} \mathbb{P}(A_{n_+ + k} = x,\ \tau = n_+ + k + 1)$$
$$= \frac{Q}{n_+^2} \sum_{k \le M n_+^{1/2}} \exp\left\{-\frac{\lambda x}{n_+ + k} - \lambda I (n_+ + k)\right\} h\left(\frac{x - (n_+ + k)^2 I}{(n_+ + k)^{3/2}}\right) + o(n_+^{-3/4}).$$

Since $x - n_+^2 I + k^2 I = o(n^{3/2})$ uniformly in $k \le M n_+^{1/2}$,

$$h\left(\frac{x-(n_+ + k)^2 I}{(n_+ + k)^{3/2}}\right) \sim h\left(-\frac{2Ik}{n_+^{1/2}}\right). \tag{36}$$

Furthermore,

$$\begin{aligned}\frac{\lambda x}{n_+ + k} + \lambda I(n_+ + k) &= \frac{\lambda x}{n_+}\left(1 - \frac{k}{n_+} + \frac{k^2}{n_+^2} + O\left(\frac{k^3}{n_+^3}\right)\right) + \lambda I n_+ + \lambda I k \\ &= \left(\frac{\lambda x}{n_+} + \lambda I n_+\right) + \lambda I k - \frac{\lambda x}{n_+^2}k + \frac{\lambda x k^2}{n_+^3} + O\left(\frac{\lambda x}{n_+^{5/2}}\right).\end{aligned}$$

Now recalling that $n_+ = \sqrt{x/I} + \varepsilon_x$ with $\varepsilon_x \in (0, 1]$, we have, uniformly in $k \le M n_+^{1/2}$,

$$0 \le \lambda I k - \frac{\lambda x}{n_+^2}k = \left(I - \frac{x}{(x/I)(1+\varepsilon_x\sqrt{I/x})^2}\right)\lambda k \le 2\lambda k \varepsilon_x \sqrt{\frac{I}{x}} = O(x^{-1/4}) = O\left(\frac{1}{n_+^{1/2}}\right).$$

Consequently,

$$\begin{aligned}&\sum_{k \le M n_+^{1/2}} \mathbb{P}(A_{n_+ + k} = x,\ \tau = n_+ + k + 1) \\ &\quad = \frac{Q}{n_+^2}\exp\left\{-\frac{\lambda x}{n_+} - \lambda I n_+\right\}\left[\sum_{k \le M n_+^{1/2}} \exp\left\{-\lambda I \frac{k2}{n_+}\right\} h\left(-\frac{2IK}{n_+^{1/2}}\right) + o(n_+^{3/2})\right] \\ &\quad = \frac{Q}{n_+^{3/2}}\exp\left\{-\frac{\lambda x}{n_+} - \lambda I n_+\right\}\left[\int_0^M \mathrm{e}^{-\lambda I u^2} h(-2Iu)\,\mathrm{d}u + o(n_+^{-3/2})\right] \\ &\quad = \frac{\hat{Q}}{x^{3/4}}\exp\{-2\lambda\sqrt{Ix}\}\left[\int_0^M \mathrm{e}^{-\lambda I u^2} h(-2Iu)\,\mathrm{d}u + o(1)\right].\end{aligned} \tag{37}$$

We split the second sum in (35) into two parts: $k \le n_+$ and $k > n_+$. Using (34), we obtain

$$\begin{aligned}&\sum_{k \in (M n_+^{1/2}, n_+]} \mathbb{P}(A_{n_+ + k} = x,\ \tau = n_+ + k + 1) \\ &\quad \le \frac{C}{n_+^2} \sum_{k \in (M n_+^{1/2}, n_+]} \exp\left\{-\frac{\lambda x}{n_+ + k} - \lambda I(n_+ + k)\right\}.\end{aligned}$$

Noting that $n_+^2 \ge x/I$, we conclude that

$$\begin{aligned}-\frac{\lambda x}{n_+ + k} - \lambda I(n_+ + k) &= -\frac{\lambda x}{n_+} - \lambda I n_+ + \frac{\lambda x k}{n_+^2(1 + k/n_+)} - \lambda I k \\ &\le -\frac{\lambda x}{n_+} - \lambda I n_+ + \lambda I k\left(\frac{1}{1 + k/n_+} - 1\right) \\ &\le -\frac{\lambda x}{n_+} - \lambda I n_+ - \frac{\lambda I k^2}{2n_+}, \qquad k \le n_+.\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{k\in(Mn_+^{1/2},n_+]} \mathbb{P}(A_{n_++k}=x,\ \tau=n_++k+1)\\
&\quad\le \frac{C}{n_+^2}\exp\left\{-\frac{\lambda x}{n_+}-n_+\lambda I\right\}\sum_{k\in(Mn_+^{1/2},n_+]}\exp\left\{-\frac{\lambda I}{2}\frac{k^2}{n_+}\right\}\\
&\quad\le \frac{\hat{C}}{n_+^{3/2}}\exp\left\{-\frac{\lambda x}{n_+}-n_+\lambda I\right\}\int_M^\infty \exp\left\{-\frac{\lambda I u^2}{2}\right\}\mathrm{d}u. \qquad (38)
\end{aligned}$$

For every $k > n_+$, we have

$$\begin{aligned}
-\frac{\lambda x}{n_++k}-\lambda I(n_++k) &= -\frac{\lambda x}{n_+}-\lambda I n_+ + \frac{\lambda x k}{n_+^2(1+k/n_+)}-\lambda I k\\
&\le -\frac{\lambda x}{n_+}-\lambda I n_+ + \lambda I k\left(\frac{1}{1+k/n_+}-1\right)\\
&\le -\frac{\lambda x}{n_+}-\lambda I n_+ - \frac{\lambda I k}{2}.
\end{aligned}$$

Combining this with (17), we obtain

$$\begin{aligned}
\sum_{k>n_+}\mathbb{P}(A_{n_++k}=x,\ \tau=n_++k+1) &\le \sum_{k>n_+}\exp\left\{-\frac{\lambda x}{n_++k}-\lambda I(n_++k)\right\}\\
&\le C\exp\left\{-\frac{\lambda x}{n_+}-\lambda I n_+\right\}\exp\left\{-\frac{n_+\lambda I}{2}\right\}. \qquad (39)
\end{aligned}$$

Combining (37)–(39) and letting $M \to \infty$, we conclude that, for some $C_+ > 0$,

$$\sum_{n=n_+}^{\infty}\mathbb{P}(A_n,\ \tau=n+1) \sim \frac{C_+}{x^{3/4}}\exp\{-2\lambda\sqrt{Ix}\}.$$

Similar arguments lead to

$$\sum_{n=1}^{n_-}\mathbb{P}(A_n=x,\ \tau=n+1) \sim \frac{C_-}{x^{3/4}}\exp\{-2\lambda\sqrt{Ix}\}. \qquad \square$$

Proof of Theorem 2. For $k \ge 0$, we have

$$\mathbb{P}(\tau=n_++k+1 \mid A_\tau=x) = \frac{\mathbb{P}(A_{n_++k}=x,\ \tau=n_++k+1)}{\mathbb{P}(A_\tau=x)}.$$

From (33), it follows that

$$\begin{aligned}
&\mathbb{P}(A_{n_++k}=x,\ \tau=n_++k+1)\\
&\quad= \frac{Q}{(n_++k)^2}\exp\left\{-\frac{\lambda x}{n_++k}-\lambda I(n_++k)\right\}\left[f_1\left(0,\frac{x-(n_++k)^2 I}{(n_++k)^{3/2}}\right)+o(1)\right].
\end{aligned}$$

From the definition of h, it is immediate that

$$\varepsilon_M := \max_{u \geq M} f_1(0, u) \to 0 \quad \text{as } M \to \infty.$$

Therefore, for all large enough x and all $k \geq M n_+^{1/2}$,

$$\mathbb{P}(\tau = n_+ + k + 1 \mid A_\tau = x) \leq C\varepsilon_M.$$

For $k < M n_+^{1/2}$, from (36), we have

$$\begin{aligned}&\mathbb{P}(A_{n_+ + k} = x,\ \tau = n_+ + k + 1)\\&\quad \sim \frac{Q}{n_+^2} \exp\left\{-\frac{\lambda x}{n_+} - \lambda I n_+\right\} \exp\left\{-\lambda I \frac{k^2}{n_+}\right\} f_1\left(0, -2I \frac{k}{n_+^{1/2}}\right).\end{aligned}$$

From Theorem 1, it follows that

$$\mathbb{P}(\tau = n_+ + k + 1 \mid A_\tau = x) \sim C x^{1/4} \exp\left\{-\lambda I \frac{k^2}{n_+}\right\} f_1\left(0, -2I \frac{k}{n_+^{1/2}}\right).$$

Recalling that

$$f_1(0, z) = c \exp\left\{-\frac{z^2}{2 \int_0^1 \sigma^2(u)(1-u)^2 \, \mathrm{d}u}\right\},$$

we obtain the desired asymptotics for $k \geq 0$. The $k < 0$ case can be treated in the same manner. This completes the proof. □

Proof of Equation (1). Fix some $\varepsilon > 0$. Then

$$\mathbb{P}(A_\tau > x) = \mathbb{P}(A_\tau > x,\ \tau \leq \varepsilon x^{2/3}) + \sum_{n \geq \varepsilon x^{2/3}} \mathbb{P}(A_\tau > x,\ \tau = n + 1). \tag{40}$$

The relation $A_\tau \leq \tau M_\tau$ implies that $\{A_\tau > x, \tau \leq \varepsilon x^{2/3}\} \subset \{M_\tau > x^{1/3}/\varepsilon\}$. Doney [7] showed that $y\mathbb{P}(M_\tau > y) \to c \in (0, \infty)$. Therefore, there exists a constant C such that

$$x^{1/3} \mathbb{P}(A_\tau > x,\ \tau \leq \varepsilon x^{2/3}) \leq C\varepsilon \quad \text{for all } x > 0. \tag{41}$$

By the functional limit theorem for random walk excursions (see [5] and [13]),

$$\mathbb{P}(A_\tau > x \mid \tau = n + 1) = \bar{G}\left(\frac{x}{\sigma n^{3/2}}\right) + o(1),$$

where $\bar{G}(y) := \mathbb{P}(\int_0^1 e(t) \, \mathrm{d}t > y)$. Furthermore, according to Vatutin and Wachtel [14, Theorem 8],

$$\mathbb{P}(\tau = n + 1) \sim \frac{C_0}{n^{3/2}}.$$

Combining these two relations, we obtain

$$\mathbb{P}(A_\tau > x,\ \tau = n + 1) = \frac{C_0}{n^{3/2}} \bar{G}\left(\frac{x}{\sigma n^{3/2}}\right) + o(n^{-3/2})$$

and, consequently,

$$\sum_{n\geq \varepsilon x^{2/3}} \mathbb{P}(A_\tau > x,\ \tau = n+1) = C_0 \sum_{n\geq \varepsilon x^{2/3}} n^{-3/2}\bar{G}\bigg(\frac{x}{\sigma n^{3/2}}\bigg) + o(x^{-1/3}).$$

Since the sum on the right-hand side can be written as a Riemann sum for the function $y^{-3/2}\bar{G}(y^{-3/2})$, we have

$$\begin{aligned}\sum_{n\geq \varepsilon x^{2/3}} \mathbb{P}(A_\tau > x,\ \tau = n+1) &= \frac{C_0\sigma^{1/3}}{x^{1/3}} \int_{\varepsilon\sigma^{2/3}}^{\infty} y^{-3/2}\bar{G}(y^{-3/2})\,\mathrm{d}y + o(x^{-1/3}) \\ &= \frac{2C_0\sigma^{1/3}}{3x^{1/3}} \int_0^{1/(\varepsilon\sigma)} z^{-2/3}\bar{G}(z)\,\mathrm{d}z + o(x^{-1/3}). \qquad (42)\end{aligned}$$

Combining (40)–(42), we obtain

$$\liminf_{x\to\infty} x^{1/3}\mathbb{P}(A_\tau > x) \geq \frac{2C_0\sigma^{1/3}}{3} \int_0^{1/(\varepsilon\sigma)} z^{-2/3}\bar{G}(z)\,\mathrm{d}z$$

and

$$\limsup_{x\to\infty} x^{1/3}\mathbb{P}(A_\tau > x) \leq \frac{2C_0\sigma^{1/3}}{3} \int_0^{1/(\varepsilon\sigma)} z^{-2/3}\bar{G}(z)\,\mathrm{d}z + C\varepsilon.$$

Now letting $\varepsilon \to 0$, we arrive at the relation

$$\lim_{x\to\infty} x^{1/3}\mathbb{P}(A_\tau > x) = \frac{2C_0\sigma}{3} \int_0^{\infty} z^{-2/3}\bar{G}(z)\,\mathrm{d}z = 2C_0\sigma^{1/3}\mathbb{E}\bigg[\int_0^1 e(t)\,\mathrm{d}t\bigg]^{1/3}. \qquad \square$$

References

[1] Abramowitz, M. and Stegun, I. A. (1964). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* U.S. Government Printing Office, Washington, D.C.

[2] Bateman, H. (1954). *Tables of Integral Transforms*, Vol. 1. McGraw-Hill, New York.

[3] Billingsley, P. (1968). *Convergence of Probability Measures.* John Wiley, New York.

[4] Borovkov, A. A., Boxma, O. J. and Palmowski, Z. (2003). On the integral of the workload process of the single server queue. *J. Appl. Prob.* **40,** 200–225.

[5] Caravenna, F. and Chaumont, L. (2013). An invariance principle for random walk bridges conditioned to stay positive. *Electron. J. Prob.* **18,** 60.

[6] Dobrushin, R. and Hryniv, O. (1996). Fluctuations of shapes of large areas under paths of random walks. *Prob. Theory Relat. Fields* **105,** 423–458.

[7] Doney, R. A. (1983). A note on conditioned random walk. *J. Appl. Prob.* **20,** 409–412.

[8] Duffy, K. R. and Meyn, S. P. (2014). Large deviation asymptotics for busy periods. *Stoch. Systems* **4,** 300–319.

[9] Gel'fond, A. O. (1967). *The Calculus of Finite Differences,* 3rd edn. Nauka, Moscow (in Russian).

[10] Guillemin, F. and Pinchon, D. (1998). On the area swept under the occupation process of an $M/M/1$ queue in a busy period. *Queueing Systems Theory Appl.* **29,** 383–398.

[11] Kearney, M. J. (2004). On a random area variable arising in discrete-time queues and compact directed percolation. *J. Phys. A* **37,** 8421–8431.

[12] Kulik, R. and Palmowski, Z. (2011). Tail behaviour of the area under a random process, with applications to queueing systems, insurance and percolations. *Queueing Systems* **68,** 275–284.

[13] Sohier, J. (2010). A functional limit convergence towards Brownian excursion. Preprint. Available at https://arxiv.org/abs/1012.0118.

[14] Vatutin, V. A. and Wachtel, V. (2009). Local probabilities for random walks conditioned to stay positive. *Prob. Theory Relat. Fields* **143,** 177–217.

[15] Vysotsky, V. (2010). On the probability that integrated random walks stay positive. *Stoch. Process. Appl.* **120,** 1178–1193.