FIRST-PASSAGE TIME ASYMPTOTICS OVER MOVING BOUNDARIES FOR RANDOM WALK BRIDGES

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Abstract

We study the asymptotic tail behavior of the first-passage time over a moving boundary for a random walk conditioned to return to zero, where the increments of the random walk have finite variance. Typically, the asymptotic tail behavior may be described through a regularly varying function with exponent $-\frac{1}{2}$, where the impact of the boundary is captured by the slowly varying function. Yet, the moving boundary may have a stronger effect when the tail is considered at a time close to the return point of the random walk bridge, leading to a possible phase transition depending on the order of the distance between zero and the moving boundary.

Keywords: Random walk; first-passage time; moving boundary; bridge

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1. Introduction

The asymptotic behavior of random walks has long been an extremely popular topic in probability. In 1951 Donsker [9] showed that a suitably rescaled random walk converges to a Brownian motion. Many extensions have been studied over the years, such as generalizations to random walks in the domain of attraction of a stable law [18] and additional conditioning properties. For example, an invariance principle was shown for random walk bridges in [14], whereas the authors of [2], [7], and [13] developed invariance principles for random walks conditioned to stay positive. Recently, these two types of conditioning have been combined in [3] to an invariance principle for random walk bridges conditioned to stay positive over the entire interval.

A natural question that arises is whether and how these results extend to moving boundaries. That is, how does the random walk behave asymptotically, conditioned to stays above a boundary sequence that is not necessarily zero or even constant? This topic, as well as the closely related first-passage asymptotics, are addressed in, for example, [1], [4], [5], [11], [12], [15], and [16].

In this paper we include a moving boundary in a particular random walk bridge setting. More precisely, we consider a random walk S_i, $i \geq 0$, conditioned to return to zero at time n with increments that have zero mean and finite variance. The purpose is to derive the asymptotic

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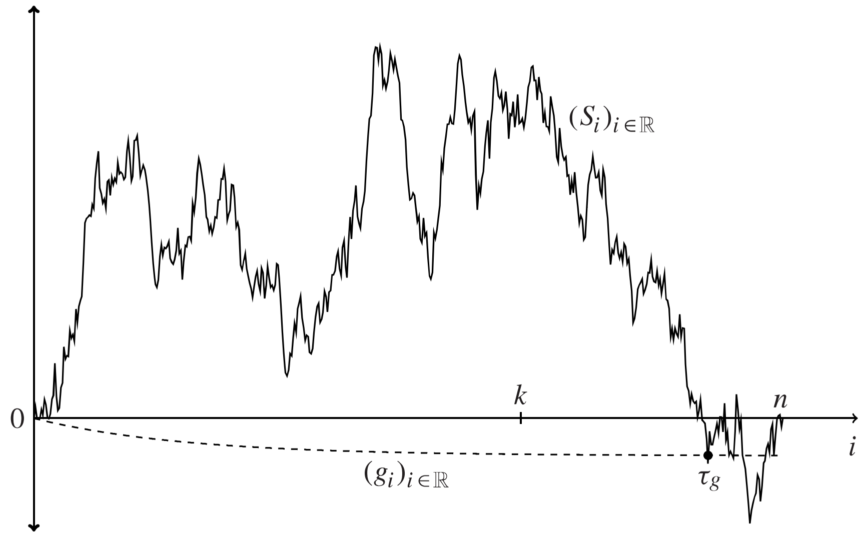

FIGURE 1: Illustration of the random walk bridge.

tail of the first-passage time τ_g over a moving boundary $(g_i)_{i\in\mathbb{R}}$ for this random walk bridge. We stress that we are considering only the tail of τ_g for all times $k := k_n$ that are well before the random walk bridge returns to zero. In other words, we extend a random walk bridge in the Brownian setting to stay above a moving boundary over part of its interval; see Figure 1.

Besides this problem being of intrinsic interest, our inspiration comes from a seemingly unrelated area: cascading failure models. These models are used to describe systems of interconnected components where failures possibly trigger subsequent failures of other components. A typical reliability measure in such problems is the probability that the number of failures exceeds a certain value. Analytic results are obtained for particular settings (see [6] and [19]) but allow for limited generalizations. It turns out that this problem has an equivalent random-walk bridge representation, where the reliability measure translates to the probability that the first-passage time of a random walk bridge over a moving boundary exceeds a certain time. In Section 3 we demonstrate this relation in detail.

The main goal in this paper is to derive the asymptotic behavior of $\mathbb{P}(\tau_g > k \mid S_n = 0)$. Clearly, the asymptotic behavior of this first-passage time depends on the boundary sequence. We consider all boundary sequences that are within square-root order from zero and, hence, relatively not too far from zero with respect to time. Our results distinguish between two regimes. The first concerns values of k that are significantly smaller than the time that the random walk returns to zero, whereas the second considers k close to the return point. In the first case, the asymptotic tail of the first-passage time can be described through a regularly varying function with exponent $-\frac{1}{2}$. When the boundary sequence satisfies certain additional conditions, as explained in, for example, [5] and in Section 2.2 of this paper, the probability of $\{\tau_g > k \mid S_n = 0\}$ to occur has a power-law decay with a preconstant that can be interpreted in a probabilistic way. However, for the second regime, a phase transition possibly occurs when k is close to the return point of the random walk bridge, depending on how close the boundary remains to zero. This intriguing phenomenon reflects the strong dependence on the boundary: the effect may not solely be captured in a slowly varying function, but can affect the behavior much more drastically.

In our approach, we consider all likely ways for the random walk to stay above the boundary until time k, and then return back to zero in $n - k$ time. In particular, when $k = n - o(n)$ (that is, k is relatively close to the return point n), we observe that $n - k$ is of smaller order than k.

This observation will prove to be crucial in understanding why a phase transition possibly occurs in this case. That is, for the random walk to return to zero in $n-k$ time, it is likely that the random walk at time k is within order $\sqrt{n-k}$ from zero due to the central limit theorem. It turns out that we need to be more careful in evaluating how likely the random walk can reach such values at time k while staying above the moving boundary, which depends heavily on the boundary at time k.

The paper is organized as follows. In Section 2 we state our assumptions and some known results that are used throughout the paper. Our main results are presented in Section 3. In Section 4 we prove the result in the case that k is sufficiently far from the return point, while the proof in the other case can be found in Section 5. In particular, the proof in the latter case requires a result on the uniform convergence of an unconditioned random walk while staying above a moving boundary, which we state and prove in Section 5.1.

2. Preliminaries

Before presenting our main results, we first introduce some notation, state our assumptions, and point out the consequences.

2.1. Notation

Let $X_i,\ i \geq 1$, be independent and identically distributed (i.i.d.) random variables with $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 = 1$ for all $i \geq 1$. Define the random walk

$$S_m := \sum_{i=1}^{m} X_i, \qquad m \geq 1.$$

We refer to $\{g_i\}_{i\in\mathbb{N}}$ as the boundary sequence. Define the stopping time

$$\tau_g := \min\{i \geq 1 \colon S_i \leq g_i\},$$

that is, the first-passage time of the random walk below the (moving) boundary. In the case that $g_i = x$ for all $i \geq 1$, we write $T_x := \tau_g$ for the stopping time to emphasize the fact that the boundary is constant.

Finally, we present the notation used throughout the paper. For sequences $a_n, b_n \in \mathbb{R}$, we write $a_n = o(b_n)$ if $\limsup_{n\to\infty} a_n/b_n = 0$ and $a_n = O(b_n)$ if $\limsup_{n\to\infty} |a_n/b_n| < \infty$. We write $a_n = \omega(b_n)$ if $\lim_{n\to\infty} b_n/a_n = 0$ and $a_n = \Omega(b_n)$ if $\limsup_{n\to\infty} |b_n/a_n| < \infty$. Finally, we write $a_n = \Theta(b_n)$ if both $a_n = O(b_n)$ and $a_n = \Omega(b_n)$, and denote $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$.

2.2. Assumptions and properties

First, we make some assumptions on the increments.

Assumption 2.1. *The increments of the random walk are i.i.d. with mean zero and variance 1. Additionally, we assume that the law of the increments has a density $f(\cdot)$ (almost everywhere) and that there exists an $n_0 \in \mathbb{N}$ such that $f_{n_0}(\cdot)$, the density corresponding to S_{n_0}, is bounded (almost everywhere).*

We point out that the boundedness requirement on the density function of the random walk for some n_0 in Assumption 2.1 is a necessary and sufficient condition for uniform convergence between the scaled density of the position of the random walk towards the standard normal density [17, p. 198]. Specifically, let $\phi(\cdot)$ and $\Phi(\cdot)$ denote the density function and the distribution function of a standard normal random variable, respectively. Then

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(\frac{S_n}{\sqrt{n}}\le x\right)-\Phi(x)\right|=0, \qquad \lim_{n\to\infty}\sup_{x\in\mathbb{R}}|\sqrt{n}f_n(\sqrt{n}x)-\phi(x)|=0. \tag{2.1}$$

Next we assume that the boundary sequence $\{g_i\}_{i\in\mathbb{N}}$ does not move too far from zero.

Assumption 2.2. *The boundary sequence $\{g_i\}_{i\in\mathbb{N}}$ satisfies $|g_i|=o(\sqrt{i})$, and $\mathbb{P}(\tau_g>n)>0$ for all $n\ge 1$.*

Under Assumption 2.2, it is known that the position of the rescaled random walk, conditioned to stay above the boundary, converges to a Rayleigh distribution; see [5]. To be precise, as $n\to\infty$,

$$\mathbb{P}(S_n>g_n+v\sqrt{n}\mid\tau_g>n)\sim \mathrm{e}^{-v^2/2}\quad\text{for all } v\ge 0. \tag{2.2}$$

The first-passage time itself has a regularly varying tail,

$$\mathbb{P}(\tau_g>n)\sim\sqrt{\frac{2}{\pi}}\frac{L_g(n)}{\sqrt{n}}, \tag{2.3}$$

where $L_g(\cdot)$ is a positive, slowly varying function.

The slowly varying function in (2.3) has a probabilistic interpretation:

$$L_g(n)=\mathbb{E}(S_n-g_n;\ \tau_g>n)\sim\mathbb{E}(-S_{\tau_g};\ \tau_g\le n)\in(0,\infty).$$

The literature offers many discussions and results for which this slowly varying function converges to a finite constant $L_g(\infty):=\lim_{n\to\infty}L_g(n)$. We note that in the case that $L_g(\infty)<\infty$ exists, the slowly varying term can be replaced by the constant $\mathbb{E}(-S_{\tau_g})$.

To the best of the authors' knowledge, [5] provides the weakest conditions that allow for the existence of a finite $L_g(\infty)$. These conditions are somewhat cumbersome and, as an alternative, we mention two easily checked cases here that are known from the literature.

In [11], it was shown that if the boundary sequence g_n, $n\ge 1$, is nonincreasing and concave, then $L_g(\infty)$ exists and

$$\sum_{n=1}^{\infty}\frac{-g_n}{n^{3/2}}<\infty \quad\Longleftrightarrow\quad L_g(\infty)=\mathbb{E}(-S_{\tau_g})\in(0,\infty).$$

In particular, this holds for all finite constant boundaries. In [20, Theorem 5], the concavity condition is relaxed, but a stronger summability condition is required. Specifically, it was shown that if g_n, $n\ge 1$, is nonincreasing,

$$\sum_{n=1}^{\infty}\frac{\log^{1/2}n}{n^{3/2}}(-g_n)<\infty \quad\Longrightarrow\quad L_g(\infty)=\mathbb{E}(-S_{\tau_g})\in(0,\infty).$$

In this paper we consider a random walk that is conditioned to return to zero at time n. The objective is to derive the asymptotic behavior of the probability that this random walk stay above a moving boundary over part of its interval. That is, given that the random walk returns to zero at time n, what is the asymptotic probability of the random walk staying above the moving boundary up to time $k:=k_n$? We assume that this time k is at least $\omega(1)$ distance from both zero and n.

Assumption 2.3. *Time $k:=k_n$ satisfies both $k\to\infty$ and $n-k\to\infty$ as $n\to\infty$.*

3. Main results

We distinguish between two cases: one where k is not too close to the point of return of the random walk bridge, and one where it is.

Theorem 3.1. *Suppose that* $\limsup_{n\to\infty} k/n < 1$. *Then, uniformly in* k/n,

$$\mathbb{P}(\tau_g > k \mid S_n = 0) \sim \sqrt{\frac{2}{\pi}} L_g(k) \sqrt{\frac{n-k}{n}} k^{-1/2} \quad \text{as } n \to \infty.$$

Note that when k is not too close to the boundary, the impact of the boundary is completely captured by the slowly varying function. When k moves closer to n, the behavior of the boundary becomes more relevant and possibly results in a change in the asymptotics.

Theorem 3.2. *Suppose that* $k = n - o(n)$. *In addition to Assumption 2.2, suppose there exists an* $\varepsilon \in (0, 1)$ *such that*

$$\sup_{j\in[(1-\varepsilon)k,k]} |g_j - g_k| \le \alpha(\varepsilon)|g_k| \tag{3.1}$$

for every large enough k *with* $\alpha(\varepsilon) \to 0$ *as* $\varepsilon \downarrow 0$. *Then, as* $n \to \infty$,

$$\mathbb{P}(\tau_g > k \mid S_n = 0) \sim \begin{cases} \sqrt{\frac{2}{\pi}} L_g(k) \frac{\sqrt{n-k}}{k} & \text{if } |g_k| = o(\sqrt{n-k}), \\ \sqrt{\frac{2}{\pi}} L_g(k) \gamma\left(\frac{|g_k|}{\sqrt{n-k}}\right) \frac{\sqrt{n-k}}{k} & \text{if } |g_k| = \Theta(\sqrt{n-k}), \\ 2L_g(k) \frac{|g_k|}{k} & \text{if } |g_k| = \omega(\sqrt{n-k}),\ g_k < 0, \end{cases} \tag{3.2}$$

where

$$\gamma(y) := e^{-y^2/2} - y \int_{x=y}^{\infty} e^{-x^2/2}\, dx.$$

A typical example that is covered by this framework is when $g_i = -i^\alpha$, $i \in \mathbb{N}$, with $\alpha < \frac{1}{2}$. The additional assumption (3.1) is merely technical: it ensures that the boundary does not fluctuate too much as it moves closer to k. That is, for every $\varepsilon > 0$, there is a value $\alpha(\varepsilon) < \infty$ such that the boundary does not fluctuate more than $2\alpha(\varepsilon)g_k$ in the interval $[(1-\varepsilon)k, k]$ for large enough n. In particular, this implies that $\varepsilon \in (0, 1)$ can be chosen small enough such that $\alpha(\varepsilon) < 1$ and, hence, the boundary sequence at time $[(1-\varepsilon)k, k]$ has the same sign (either positive, negative, or zero). Cases where the boundary sequence strongly oscillates close to time k are thus excluded from our framework.

The phase transition that appears in Theorem 3.2 reflects the strong influence of the boundary sequence in this case. It might not be captured solely by the slowly varying function, but can have a much stronger effect. Furthermore, this effect is influenced only by the behavior of the boundary sequence close to time k. This observation is best explained by our approach. We track the position of a random walk at time k, conditioned that it stays above the moving boundary till that point. Then we evaluate how likely a reversed random walk moving back from time n can reach that point. Due to a local limit theorem, the random walk is likely to stay within $\sqrt{n-k} = o(\sqrt{k})$ distance from zero. When $g_k < 0$, those values are thus likely to be of order $\max\{\sqrt{n-k}, |g_k|\}$ distance from the boundary. The phase transition is then a consequence of how likely the random walk staying above the boundary sequence can move to such values.

Example 3.1. As pointed out in the introduction, Theorems 3.1 and 3.2 can be applied to a seemingly unrelated problem in cascading failure models. In this example, we will describe a particular cascading failure model as in [19], and translate it to the random-walk bridge setting we consider in this paper.

Consider a system consisting of n (indistinguishable) components. Each component has a limited capacity for the amount of load it can carry before it fails. The network is initially stable, in the sense that every component has sufficient capacity that exceeds the initial load. We assume that the difference between the initial loads and capacities, which we refer to as the *surplus capacity*, are stochastic random variables that are i.i.d. with continuous distribution function $F(\cdot)$. In order to trigger a possible cascading failure effect, we include an initial disturbance that causes all components to be additionally loaded with $l_n(1)$. When the capacity of a component is exceeded by its load demands, that component fails. Every component failure causes (equal) additional loading on the remaining components, possibly triggering knock-on effects. We write $l_n(i)$ for the total load surge per component when $i-1$ components have failed and assume this is a deterministic nondecreasing function. The cascading failure process continues until the capacities on the remaining components are sufficient to deal with the load increases.

A measure of system reliability is the number of component failures at the end of the cascading failure process, written by A_n. Since $F(\cdot)$ is continuous, it satisfies the identity (see [19])

$$\mathbb{P}(A_n \geq k) = \mathbb{P}(U^n_{(i)} \leq F(l_n(i)),\ i = 1, \ldots, k),$$

where $U^n_{(i)}$ denotes the ith order statistic of n uniformly distributed random variables with support $[0, 1]$. In [19], the goal was to determine which choices of $F(\cdot)$ and $l_n(\cdot)$ asymptotically exhibit power-law behavior for large values of k as in Assumption 2.3. In particular, the authors considered a setting where

$$F(l_n(i)) = \frac{\theta + i - 1}{n}. \tag{3.3}$$

Next we show how this problem can be related to our random-walk bridge framework. Consider the random walk $S_n = n - \sum_{i=1}^n E_i$ where $(E_i)_{i\in\mathbb{N}}$ are independent identically exponentially distribution random variables with mean 1. It is well known that

$$(U^n_{(1)}, U^n_{(2)}, \ldots, U^n_{(n)}) \stackrel{\mathrm{D}}{=} \left(\frac{E_1}{n}, \frac{\sum_{i=1}^2 E_i}{n}, \ldots, \frac{\sum_{i=1}^n E_i}{n} \;\middle|\; \sum_{i=1}^{n+1} E_i = n\right),$$

where '$\stackrel{\mathrm{D}}{=}$' denotes equality in distribution. Then the probability that the number of component failures exceeds k can be written as

$$\begin{aligned}
\mathbb{P}(A_n \geq k) &= \mathbb{P}\left(U^n_{(i)} \leq \frac{\theta + i - 1}{n},\ i = 1, \ldots, k\right) \\
&= \mathbb{P}(S_i \geq 1 - \theta,\ i = 1, \ldots, k \mid S_{n+1} = 1) \\
&\sim \mathbb{P}(S_i \geq 1 - \theta,\ i = 1, \ldots, k \mid S_n = 0) \\
&= \mathbb{P}(T_{1-\theta} > k \mid S_n = 0).
\end{aligned}$$

Theorems 3.1 and 3.2 yield the result immediately. That is, as $n \to \infty$, we obtain

$$\mathbb{P}(A_n \geq k) \sim \sqrt{\frac{2}{\pi}} L_{1-\theta}(k) \sqrt{\frac{n-k}{kn}},$$

and since the boundary is constant,

$$L_{1-\theta}(k) \sim \mathbb{E}(-S_{T_{1-\theta}}) = -(1-\theta) + 1 = \theta,$$

where the equality is due to the memoryless property of exponentials.

Yet, (3.3) is a very specific case. Sloothaak *et al.* [19] explored for which perturbations the power-law behavior prevails. That is, if

$$F(l_n(i)) = \frac{\theta + i - 1 - g_i}{n},$$

which perturbations of $(g_i)_{i\in\mathbb{N}}$ yield power-law behavior? The analytic approach used in [19] allows for relatively limited generalizations. More specifically, they allow only boundaries for which $g_i = 0$ for every $i = \omega(1)$. Theorems 3.1 and 3.2 provide the answer to a much broader range of possible perturbations, and quantify their effect on the prefactor in a probabilistic way.

4. Proof of Theorem 3.1

We first consider the case where $\limsup_{n\to\infty} k/n < 1$ as in Theorem 3.1. Define the reversed random walk as

$$\tilde{S}_m = \sum_{i=1}^{m} \tilde{X}_m, \qquad 1 \le m \le n, \tag{4.1}$$

where $\tilde{X}_m = -X_{n+1-m}$, $1 \le m \le n$. Therefore, $\tilde{S}_m$ obeys the same law as $-S_m$ for all $1 \le m \le n$. If $\limsup_{n\to\infty} k/n < 1$, we can use a direct approach to derive the asymptotic behavior.

Proof of Theorem 3.1. Note that

$$\begin{aligned}
\mathbb{P}(\tau_g > k \mid S_n = 0) &= \int_{u=g_k}^{\infty} \mathbb{P}(\tau_g > k;\ S_k \in \mathrm{d}u \mid S_n = 0) \\
&= \frac{1}{f_n(0)} \int_{u=g_k}^{\infty} \mathbb{P}(S_k \in \mathrm{d}u;\ \tau_g > k) \tilde{f}_{n-k}(u) \\
&= \frac{\mathbb{P}(\tau_g > k)}{f_n(0)} \int_{u=g_k}^{\infty} \mathbb{P}(S_k \in \mathrm{d}u \mid \tau_g > k) \tilde{f}_{n-k}(u),
\end{aligned}$$

where $\tilde{f}_{n-k}(\cdot)$ is the density of the reversed random walk at time $n-k$. Since the reversed random walk has i.i.d. increments with zero mean and finite variance, it also satisfies (2.1) and, hence, there is a uniform convergence to the normal density. Note that since $\lim_{n\to\infty} k/n < 1$, it holds that $\lim_{n\to\infty} k/(n-k) < \infty$. Therefore,

$$\begin{aligned}
&\mathbb{P}(\tau_g > k \mid S_n = 0) \\
&\quad = (1+o(1)) \frac{\mathbb{P}(\tau_g > k)}{1/\sqrt{2\pi n}} \int_{u=g_k}^{\infty} \mathbb{P}(S_k \in \mathrm{d}u \mid \tau_g > k) \frac{\exp(-u^2/2(n-k))}{\sqrt{2\pi(n-k)}} \\
&\quad = (1+o(1)) \mathbb{P}(\tau_g > k) \sqrt{\frac{n}{n-k}} \mathbb{E}\biggl(\exp\biggl(-\frac{S_k^2}{2(n-k)}\biggr) \biggm| \tau_g > k\biggr).
\end{aligned}$$

It follows from (2.2) that

$$\mathbb{E}\biggl(\exp\biggl(-\frac{S_k^2}{2(n-k)}\biggr) \biggm| \tau_g > k\biggr) \sim \int_0^{\infty} \exp\biggl(-\frac{v^2}{2}\frac{k}{n-k}\biggr) v \exp\biggl(-\frac{v^2}{2}\biggr) \mathrm{d}v = \frac{n-k}{n}.$$

Using (2.3), we conclude that

$$\mathbb{P}(\tau_g > k \mid S_n = 0) = (1 + o(1))\sqrt{\frac{n-k}{n}}\mathbb{P}(\tau_g > k) = (1 + o(1))\sqrt{\frac{2}{\pi}}L_g(k)\sqrt{\frac{n-k}{kn}}. \qquad \square$$

5. The $k = n - o(n)$ case

Unfortunately, the analysis in the previous section does not follow through when $k = n - o(n)$. In particular, in our analysis we consider all ways for the random walk to stay above the moving boundary until time k, and in time $n - k$ to get back to its return point. In view of (2.2), at time k the random walk conditioned on $\{\tau_g > k\}$ is most likely to be at a position of order $\Theta(\sqrt{k})$. However, for $u = \Theta(\sqrt{k})$, we can no longer replace $\tilde{f}_{n-k}(u)$ by an appropriately scaled normal density if $n - k = o(k)$. We therefore need to refine our approach, which we elaborate on in this section.

5.1. Density of random walk at time k

For the evaluation of the first-passage time of the random walk bridge, it is sensible to consider the position of a random walk at time k itself. A uniform convergence result can be found in [8, Proposition 18] in the case of constant boundaries. As this result is crucial in our analysis, we pose it here for our setting.

Proposition 5.1. (See Doney [8, Proposition 18].) *Let $x := x_n \geq 0$ denote the starting point of a random walk (depending on n) and let $y := y_n$ be a sequence of nonnegative numbers. Let $U(\cdot)$ denote the renewal function in the (strict) increasing ladder height process, and $V(\cdot)$ the renewal function corresponding to the decreasing ladder height process. Let $\mathbb{E}(-S_{T_0})$ be the expected position of a random walk at stopping time T_0, and $\mathbb{E}(-\tilde{S}_{T_0})$ the expected position of a random walk with increments $-X_i,\ i \geq 0$, at stopping time T_0. Then the following results hold uniformly for every $\Delta \in (0, \infty)$ as $n \to \infty$.*

(i) *For* $\max\{x/\sqrt{n}, y/\sqrt{n}\} \to 0$,

$$\mathbb{P}(S_n \in [y, y + \Delta), T_0 > n \mid S_0 = x) \sim \frac{V(x)\int_y^{y+\Delta} U(w)\,\mathrm{d}w}{\sqrt{2\pi}n^{3/2}}. \tag{5.1}$$

(ii) *For any (fixed) $D > 1$ with $x/\sqrt{n} \to 0$ and $y/\sqrt{n} \in [D^{-1}, D]$,*

$$\mathbb{P}(S_n \in [y, y + \Delta), T_0 > n \mid S_0 = x) \sim \sqrt{\frac{2}{\pi}}\frac{\mathbb{E}(-S_{T_0})V(x)\Delta}{\sqrt{n}}\frac{y}{n}\mathrm{e}^{-y^2/2n}, \tag{5.2}$$

and uniformly for $y/\sqrt{n} \to 0$ and $x/\sqrt{n} \in [D^{-1}, D]$,

$$\mathbb{P}(S_n \in [y, y + \Delta), T_0 > n \mid S_0 = x) \sim \sqrt{\frac{2}{\pi}}\frac{\mathbb{E}(-\tilde{S}_{T_0})U(y)\Delta}{\sqrt{n}}\frac{x}{n}\mathrm{e}^{-x^2/2n}. \tag{5.3}$$

(iii) *For any (fixed) $D > 1$ with $x/\sqrt{n} \in [D^{-1}, D]$ and $y/\sqrt{n} \in [D^{-1}, D]$,*

$$\mathbb{P}(S_n \in [y, y + \Delta), T_0 > n \mid S_0 = x) \sim \frac{\Delta q(x/\sqrt{n}, y/\sqrt{n})}{\sqrt{n}}, \tag{5.4}$$

where $q(x,y)$ is the density of $\mathbb{P}(W(1)\in \mathrm{d}y, \inf_{0\le t\le 1} W(t)>0 \mid W(0)=x)$ with $\{W(t),\ t\ge 0\}$ the standard Wiener process. This has the explicit form (see [10]):

$$q(u,v)=\frac{1}{\sqrt{2\pi}}\left(\exp\left(\frac{-(u-v)^2}{2}\right)-\exp\left(\frac{-(u+v)^2}{2}\right)\right)\quad \textit{for every } u,v>0.$$

The asymptotic behaviors of $V(\cdot)$ and $U(\cdot)$ are quite well understood: the functions are both nondecreasing functions and regularly varying with exponent 1. In particular, as $t\to\infty$,

$$U(t)\sim\frac{t}{\mathbb{E}(-\tilde{S}_{T_0})},\qquad V(t)\sim\frac{t}{\mathbb{E}(-S_{T_0})}. \tag{5.5}$$

Moreover, for all random walks with finite variance $\sigma^2=1$, it holds that

$$\mathbb{E}(-\tilde{S}_{T_0})\mathbb{E}(-S_{T_0})=\tfrac{1}{2}\sigma^2=\tfrac{1}{2}. \tag{5.6}$$

The goal is to exploit Proposition 5.1 to derive the asymptotic behavior of the random walk at time k, while staying above the moving boundary. Intuitively, we derive this by looking at the position of the random walk at time $(1-\varepsilon)k$, where $\varepsilon\in(0,1)$ satisfies (3.1). Due to the additional assumption (3.1), one can replace the boundary between $(1-\varepsilon)k$ and k by a constant boundary with value (approximately) g_k. The density is then derived using (2.2), (2.3), and the result of Doney [8] with constant boundaries. This strategy yields the following result.

Proposition 5.2. *Let $t\ge g_k$ with $t-g_k=\Theta(|g_k|)$ and $(t-g_k)\to\infty$ as $k\to\infty$. Then, uniformly as $k\to\infty$,*

$$\frac{\mathbb{P}(S_k\in \mathrm{d}t;\,\tau_g>k)}{\mathrm{d}t}\sim\sqrt{\frac{2}{\pi}}\frac{L_g(k)}{k^{3/2}}\begin{cases}\mathbb{E}(-\tilde{S}_{T_0})U(t-g_k) & \textit{if } t=o(\sqrt{k}),\\ t\mathrm{e}^{-t^2/2k} & \textit{if } t=\Theta(\sqrt{k}).\end{cases}$$

For the proof of Proposition 5.2, we separate three cases depending on the position of the random walk at time $(1-\varepsilon)k$: a position close to the boundary (Lemmas 5.1 and 5.2), a position very far from the boundary (Lemmas 5.3 and 5.4), or in a typical distance from the boundary (proof of Proposition 5.2). We will show that the first two cases are very unlikely to occur with respect to the final case.

In the following two lemmas we show that it is unlikely for the random walk to be close to its boundary at time $(1-\varepsilon)k$.

Lemma 5.1. *Suppose that $t=o(\sqrt{k})$ such that $t-g_k=\Omega(|g_k|)$ and $(t-g_k)\to\infty$ as $k\to\infty$. Let $\varepsilon\in(0,1)$ be such that (3.1) is satisfied, and choose $x_\varepsilon>0$ small enough such that*

$$1-\exp\left(-\frac{-{x_\varepsilon}^2}{2(1-\varepsilon)}\right)<\varepsilon^{3/2} \tag{5.7}$$

holds. Let $v_{\varepsilon,k}=g_{(1-\varepsilon)k}+x_\varepsilon\sqrt{k}$. There exists a constant $C_1\in(0,\infty)$ such that, for all $\varepsilon\in(0,1)$,

$$\limsup_{k\to\infty}\frac{k^{3/2}}{L_g(k)U(t-g_k)}\frac{\mathbb{P}(S_k\in \mathrm{d}t;\ S_{(1-\varepsilon)k}<v_{\varepsilon,k};\ \tau_g>k)}{\mathrm{d}t}\le C_1\frac{x_\varepsilon}{\sqrt{(1-\varepsilon)}}.$$

Proof. Define

$$g^+_{k,\varepsilon} = g_k - \alpha(\varepsilon)|g_k|, \tag{5.8}$$

and note that

$$\begin{aligned}&\mathbb{P}(S_k \in \mathrm{d}t;\ S_{(1-\varepsilon)k} < v_{\varepsilon,k};\ \tau_g > k)\\ &\quad\le \mathbb{P}(S_{(1-\varepsilon)k} \le v_{\varepsilon,k};\ \tau_g > (1-\varepsilon)k) \sup_{v\in[g_{(1-\varepsilon)k},\, v_{\varepsilon,k}]} \mathbb{P}(S_{\varepsilon k} \in \mathrm{d}t;\ T_{g^+_{k,\varepsilon}} > \varepsilon k \mid S_0 = v).\end{aligned}$$

For $v = o(\sqrt{k})$, (5.1) yields

$$\frac{\mathbb{P}(S_{\varepsilon k} \in \mathrm{d}t;\ T_{g^+_{k,\varepsilon}} > \varepsilon k \mid S_0 = v)}{\mathrm{d}t} \sim \frac{U(t - g^+_{k,\varepsilon})}{\sqrt{2\pi}\,(\varepsilon k)^{3/2}} V(v - g^+_{k,\varepsilon})$$

uniformly in $v = o(\sqrt{k})$ and $t = o(\sqrt{k})$. On the other hand, if $v = \Theta(\sqrt{k})$ then (5.3) yields

$$\frac{\mathbb{P}(S_{\varepsilon k} \in \mathrm{d}t;\ T_{g^+_{k,\varepsilon}} > \varepsilon k \mid S_0 = v)}{\mathrm{d}t} \sim \sqrt{\frac{2}{\pi}}\mathbb{E}(-\tilde{S}_{T_0})\frac{U(t - g^+_{k,\varepsilon})}{(\varepsilon k)^{3/2}}(v - g^+_{k,\varepsilon})\exp\left(-\frac{(v - g^+_{k,\varepsilon})^2}{2\varepsilon k}\right)$$

uniformly in $v = \Theta(\sqrt{k})$ and $t = o(\sqrt{k})$. We observe that $\mathrm{e}^{-x} \le 1$ for all $x \ge 0$, and $\mathbb{E}(-\tilde{S}_{T_0}) \in (0, \infty)$ since the increments of the random walk have finite variance. Moreover, due to (5.5) and (5.6), there exists a constant $c_1 \in (0, \infty)$ such that

$$\sup_{v\in[g_{(1-\varepsilon)k},\, v_{\varepsilon,k}]} \mathbb{P}(S_{\varepsilon k} \in \mathrm{d}t;\ T_{g^+_{k,\varepsilon}} > \varepsilon k \mid S_0 = v) \le c_1(v_{\varepsilon,k} - g^+_{k,\varepsilon})\frac{U(t - g^+_{k,\varepsilon})}{(\varepsilon k)^{3/2}}\,\mathrm{d}t.$$

Due to assumption (3.1), we have $(g_{(1-\varepsilon)k} - g^+_{k,\varepsilon}) < 2\alpha(\varepsilon)|g_k| = o(\sqrt{k})$. Also, as $U(\cdot)$ is nondecreasing and (5.5) holds, there exists a constant $c_2 \in (0, 1)$ such that

$$U(t - g^+_{k,\varepsilon}) \le c_2(1 + \alpha(\varepsilon))U(t - g_k).$$

Consequently, there exists a $c_3 \in (0, \infty)$ such that

$$\sup_{v\in[g_{(1-\varepsilon)k},\, v_{\varepsilon,k}]} \frac{\mathbb{P}(S_{\varepsilon k} \in \mathrm{d}t;\ T_{g^+_{k,\varepsilon}} > \varepsilon k \mid S_0 = v)}{\mathrm{d}t} \le c_3 x_\varepsilon \frac{U(t - g_k)}{\varepsilon^{3/2}k}.$$

Finally, since (2.2) and (2.3) hold with $L_g(\cdot)$ a slowly varying function,

$$\begin{aligned}\mathbb{P}(S_{(1-\varepsilon)k} \le v_{\varepsilon,k};\ \tau_g > (1-\varepsilon)k) &= \mathbb{P}(S_{(1-\varepsilon)k} \le v_{\varepsilon,k} \mid \tau_g > (1-\varepsilon)k)\mathbb{P}(\tau_g > (1-\varepsilon)k)\\ &\sim \left(1 - \exp\left(-\frac{-x_\varepsilon{}^2}{2(1-\varepsilon)}\right)\right)\sqrt{\frac{2}{\pi}}\frac{L_g(k)}{\sqrt{(1-\varepsilon)k}}\\ &< \sqrt{\frac{2}{\pi}}\frac{\varepsilon^{3/2}}{\sqrt{1-\varepsilon}}\frac{L_g(k)}{\sqrt{k}}.\end{aligned}$$

Multiplying the final two expressions yields the result. □

Next we prove a similar result as Lemma 5.1, but where $t = \Theta(\sqrt{k})$.

Lemma 5.2. *Suppose that $t = \Theta(\sqrt{k})$ such that $t \geq g_k$. Let $\varepsilon \in (0,1)$ be such that (3.1) holds, and choose x_ε small enough such that (5.7) is satisfied. Define $g^+_{k,\varepsilon}$ as in (5.8) and let $v_{\varepsilon,k} = g_{(1-\varepsilon)k} + x_\varepsilon\sqrt{k}$. There exists a constant $C_2 \in (0,\infty)$ such that, for all $\varepsilon \in (0,1)$,*

$$\limsup_{k\to\infty} \frac{k^{3/2}}{L_g(k)t\mathrm{e}^{-t^2/(2k)}} \frac{\mathbb{P}(S_k \in \mathrm{d}t;\ S_{(1-\varepsilon)k} < v_{\varepsilon,k};\ \tau_g > k)}{\mathrm{d}t} \leq C_2 \frac{x_\varepsilon}{\sqrt{(1-\varepsilon)}}.$$

Proof. The proof is similar to the proof of Lemma 5.1, but in this case we have to consider the asymptotics for $t = \Theta(\sqrt{k})$. Note that

$$\begin{aligned}
&\mathbb{P}(S_k \in \mathrm{d}t;\ S_{(1-\varepsilon)k} < v_{\varepsilon,k};\ \tau_g > k)\\
&\quad\leq \mathbb{P}(S_{(1-\varepsilon)k} \leq v_{\varepsilon,k};\ \tau_g > (1-\varepsilon)k) \sup_{v\in[g_{(1-\varepsilon)k},\, v_{\varepsilon,k}]} \mathbb{P}(S_{\varepsilon k} \in \mathrm{d}t;\ T_{g^+_{k,\varepsilon}} > \varepsilon k \mid S_0 = v).
\end{aligned}$$

For $v = o(\sqrt{k})$, (5.2) yields

$$\frac{\mathbb{P}(S_{\varepsilon k} \in \mathrm{d}t;\ T_{g^+_{k,\varepsilon}} > \varepsilon k \mid S_0 = v)}{\mathrm{d}t} \sim \sqrt{\frac{2}{\pi}}\mathbb{E}(-S_{T_0})\frac{V(v - g^+_{k,\varepsilon})}{(\varepsilon k)^{3/2}}(t - g^+_{k,\varepsilon})\exp\biggl(-\frac{(t-g^+_{k,\varepsilon})^2}{2\varepsilon k}\biggr)$$

uniformly in $t = \Theta(\sqrt{k})$ and $v = o(\sqrt{k})$. Note that $\mathrm{e}^{-x} \leq 1$ for all $x \geq 0$ and $g^+_{k,\varepsilon} = o(\sqrt{k})$. Since $V(\cdot)$ is nondecreasing and satisfies (5.5), we find that there exists a $c_1 \in (0,\infty)$ such that

$$\begin{aligned}
&\sup_{v\in[g_{(1-\varepsilon)k},\, v_{\varepsilon,k}]} \sqrt{\frac{2}{\pi}}\mathbb{E}(-S_{T_0})\frac{V(v - g^+_{k,\varepsilon})}{(\varepsilon k)^{3/2}}(t - g^+_{k,\varepsilon})\exp\biggl(-\frac{(t-g^+_{k,\varepsilon})^2}{2\varepsilon k}\biggr)\\
&\quad\leq c_1\frac{x_\varepsilon\sqrt{k}}{(\varepsilon k)^{3/2}} t\exp\biggl(-\frac{t^2}{2k}\biggr)\exp\biggl(-\frac{(1-\varepsilon)t^2}{2\varepsilon k}\biggr)\\
&\quad\leq c_1\frac{x_\varepsilon}{\varepsilon^{3/2}k} t\exp\biggl(-\frac{t^2}{2k}\biggr).
\end{aligned}$$

On the other hand, if $v = \Theta(\sqrt{k})$ then (5.3) yields

$$\begin{aligned}
&\frac{\mathbb{P}(S_{\varepsilon k} \in \mathrm{d}t;\ T_{g^+_{k,\varepsilon}} > \varepsilon k \mid S_0 = v)}{\mathrm{d}t}\\
&\quad\sim \frac{1}{\sqrt{2\pi\varepsilon k}}\biggl(\exp\biggl(-\frac{(v-t)^2}{2\varepsilon k}\biggr) - \exp\biggl(-\frac{(v+t-2g^+_{k,\varepsilon})^2}{2\varepsilon k}\biggr)\biggr)\\
&\quad\sim \frac{1}{\sqrt{2\pi\varepsilon k}}\biggl(\exp\biggl(-\frac{(v-t)^2}{2\varepsilon k}\biggr) - \exp\biggl(-\frac{(v+t)^2}{2\varepsilon k}\biggr)\biggr)
\end{aligned}$$

uniformly in $t = \Theta(\sqrt{k})$ and $v = \Theta(\sqrt{k})$. Using a Taylor expansion, we obtain

$$\begin{aligned}
&\biggl(\exp\biggl(-\frac{(v-t)^2}{2\varepsilon k}\biggr) - \exp\biggl(-\frac{(v+t)^2}{2\varepsilon k}\biggr)\biggr)\\
&\quad= \exp\biggl(-\frac{v^2}{2\varepsilon k} - \frac{t^2}{2\varepsilon k}\biggr)\biggl(\exp\biggl(\frac{vt}{\varepsilon k}\biggr) - \exp\biggl(-\frac{vt}{2\varepsilon k}\biggr)\biggr)\\
&\quad\leq \exp\biggl(-\frac{t^2}{2k}\biggr)\biggl(2\frac{vt}{\varepsilon k} + o\biggl(\frac{vt}{\varepsilon k}\biggr)\biggr).
\end{aligned}$$

Therefore, there exists a $c_2 \in (0, \infty)$ such that

$$\sup_{v\in[g_{(1-\varepsilon)k},\, v_{\varepsilon,k}]} \frac{1}{\sqrt{2\pi\varepsilon k}}\left(\exp\left(-\frac{(v-t)^2}{2\varepsilon k}\right) - \exp\left(-\frac{(v+t)^2}{2\varepsilon k}\right)\right) \le c_2 \frac{x_\varepsilon}{\varepsilon^{3/2}k} t \exp\left(-\frac{t^2}{2k}\right).$$

We can conclude that there must exist a $c_3 \in (0, \infty)$ such that

$$\sup_{v\in[g_{(1-\varepsilon)k},\, v_{\varepsilon,k}]} \frac{\mathbb{P}(S_{\varepsilon k} \in \mathrm{d}t;\ T_{g^+_{k,\varepsilon}} > \varepsilon k \mid S_0 = v)}{\mathrm{d}t} \le c_3 \frac{x_\varepsilon}{\varepsilon^{3/2}k} t \exp\left(-\frac{t^2}{2k}\right).$$

Again, since (2.2) and (2.3) hold with $L_g(\cdot)$ a slowly varying function,

$$\begin{aligned}\mathbb{P}(S_{(1-\varepsilon)k} \le v_{\varepsilon,k};\ \tau_g > (1-\varepsilon)k) &\sim \left(1-\exp\left(-\frac{-x_\varepsilon{}^2}{2(1-\varepsilon)}\right)\right)\sqrt{\frac{2}{\pi}}\frac{L_g(k)}{\sqrt{(1-\varepsilon)k}}\\ &\le \sqrt{\frac{2}{\pi}}\frac{\varepsilon^{3/2}}{\sqrt{1-\varepsilon}}\frac{L_g(k)}{\sqrt{k}}.\end{aligned}$$

Multiplying this with the previous expression then concludes the proof. □

The next two lemmas imply that it is unlikely for the random walk to be very far above the boundary at time $(1-\varepsilon)k$.

Lemma 5.3. *Suppose that $t = o(\sqrt{k})$ such that $t - g_k = \Omega(|g_k|)$ and $(t-g_k) \to \infty$ as $k \to \infty$. Let $\varepsilon \in (0, 1)$ be such that (3.1) is satisfied. Then there exist constants C_3, $C_4 \in (0, \infty)$ such that, for all $\varepsilon \in (0, 1)$,*

$$\begin{aligned}&\limsup_{k\to\infty} \frac{k^{3/2}}{L_g(k)U(t-g_k)}\frac{\mathbb{P}(S_k \in \mathrm{d}t;\ S_{(1-\varepsilon)k} > \sqrt{k/\varepsilon};\ \tau_g > k)}{\mathrm{d}t}\\ &\qquad\le C_3(1 + C_4\alpha(\varepsilon))\frac{1}{\varepsilon\sqrt{1-\varepsilon}}\exp\left(-\frac{1}{\varepsilon(1-\varepsilon)}\right).\end{aligned}$$

Proof. Let $g_{k,\varepsilon}$ be defined as in (5.8). Since $t = o(\sqrt{k})$, it follows from (5.3) that there exists a $c_1 < \infty$ such that, for every $v = \Omega(\sqrt{k})$,

$$\frac{\mathbb{P}(S_{\varepsilon k} \in \mathrm{d}t;\ T_{g^+_{k,\varepsilon}} > \varepsilon k \mid S_0 = v)}{\mathrm{d}t} \le c_1 \frac{U(t - g^+_{k,\varepsilon})}{\sqrt{\varepsilon k}}.$$

Then it follows that

$$\begin{aligned}&\frac{\mathbb{P}(S_k \in \mathrm{d}t;\ S_{(1-\varepsilon)k} > \sqrt{k/\varepsilon};\ \tau_g > k)}{\mathrm{d}t}\\ &\quad\le \int_{v=\sqrt{k/\varepsilon}}^{\infty} \mathbb{P}(S_{(1-\varepsilon)k} \in \mathrm{d}v;\ \tau_g > (1-\varepsilon)k)\frac{\mathbb{P}(S_{\varepsilon k} \in \mathrm{d}t;\ T_{g^+_{k,\varepsilon}} > \varepsilon k \mid S_0 = v)}{\mathrm{d}t}\\ &\quad\le c_1 \frac{U(t-g^+_{k,\varepsilon})}{\varepsilon k}\mathbb{P}\left(S_{(1-\varepsilon)k} > \sqrt{\frac{k}{\varepsilon}};\ \tau_g > (1-\varepsilon)k\right).\end{aligned}$$

In view of (2.2),

$$\begin{aligned}\mathbb{P}\left(S_{(1-\varepsilon)k} > \sqrt{\frac{k}{\varepsilon}};\ \tau_g > (1-\varepsilon)k\right) &\sim \mathbb{P}(\tau_g > (1-\varepsilon)k)\exp\left(-\frac{1}{\varepsilon(1-\varepsilon)}\right)\\ &\sim \sqrt{\frac{2}{\pi}}\frac{L_g((1-\varepsilon)k)}{\sqrt{(1-\varepsilon)k}}\exp\left(-\frac{1}{\varepsilon(1-\varepsilon)}\right), \qquad (5.9)\end{aligned}$$

where $L_g(\cdot)$ is slowly varying and, hence,

$$\limsup_{k\to\infty} \frac{L_g((1-\varepsilon))}{L_g(k)} = 1.$$

Due to (3.1) and (5.5),

$$\limsup_{k\to\infty} \frac{U(t-g^+_{k,\varepsilon})}{U(t-g_k)} = 1 + \alpha(\varepsilon) \limsup_{k\to\infty} \frac{|g_k|}{t-g_k} \le 1 + c_2\alpha(\varepsilon)$$

for some constant $c_2 \in (0,\infty)$, since we assume that $t - g_k = \Omega(|g_k|)$. Therefore,

$$\limsup_{k\to\infty} \frac{k^{3/2}}{L_g(k)U(t-g_k)} \mathbb{P}\biggl(S_k \in \mathrm{d}t;\ S_{(1-\varepsilon)k} > \sqrt{\frac{k}{\varepsilon}};\ \tau_g > k\biggr)$$
$$\le \sqrt{\frac{2}{\pi}} c_1(1 + c_2\alpha(\varepsilon)) \frac{1}{\varepsilon\sqrt{1-\varepsilon}} \exp\biggl(-\frac{1}{\varepsilon(1-\varepsilon)}\biggr). \qquad \square$$

Next we prove a similar result as in Lemma 5.3, but where $t = \Theta(\sqrt{k})$.

Lemma 5.4. *Suppose that $t = \Theta(\sqrt{k})$ such that $t \ge g_k$. Let $\varepsilon \in (0,1)$ be such that (3.1) holds. Then there exists a constant $C_5 \in (0,\infty)$ such that, for all $\varepsilon \in (0,1)$,*

$$\limsup_{k\to\infty} \frac{k^{3/2}}{L_g(k) t \mathrm{e}^{-t^2/(2k)}} \mathbb{P}\biggl(S_k \in \mathrm{d}t;\ S_{(1-\varepsilon)k} > \sqrt{\frac{k}{\varepsilon}};\ \tau_g > k\biggr)$$
$$\le \frac{C_5}{\sqrt{\varepsilon(1-\varepsilon)}} \exp\biggl(-\frac{1}{\varepsilon(1-\varepsilon)}\biggr) \mathrm{d}t.$$

Proof. The proof proceeds along the same lines as the proof of Lemma 5.3. Again, let $g_{k,\varepsilon}$ be defined as in (5.8). Since $t = \Omega(\sqrt{k})$ and (5.4), there exists a constant $c_1 < \infty$ such that, for every $v = \Omega(\sqrt{k})$,

$$\frac{\mathbb{P}(S_{\varepsilon k} \in \mathrm{d}t;\ T_{g^+_{k,\varepsilon}} > \varepsilon k \mid S_0 = v)}{\mathrm{d}t} \le \frac{c_1}{\sqrt{\varepsilon k}}$$

and, hence,

$$\frac{\mathbb{P}(S_k \in \mathrm{d}t;\ S_{(1-\varepsilon)k} > \sqrt{k/\varepsilon};\ \tau_g > k)}{\mathrm{d}t} \le \frac{c_1}{\sqrt{\varepsilon k}} \mathbb{P}\biggl(S_{(1-\varepsilon)k} > \sqrt{\frac{k}{\varepsilon}};\ \tau_g > (1-\varepsilon)k\biggr).$$

In view of (2.2), (5.9) again holds with $L_g(\cdot)$ a slowly varying function. That is, there exists a constant $c_2 \in (0,\infty)$ such that

$$\mathbb{P}\biggl(S_{(1-\varepsilon)k} > \sqrt{\frac{k}{\varepsilon}};\ \tau_g > (1-\varepsilon)k\biggr) \le c_2 \frac{L_g(k)}{\sqrt{(1-\varepsilon)k}} \exp\biggl(-\frac{1}{\varepsilon(1-\varepsilon)}\biggr).$$

The result follows by combining these two expressions and noting that

$$\limsup_{k\to\infty} \frac{k^{1/2}}{t\mathrm{e}^{-t^2/(2k)}} < \infty. \qquad \square$$

Next we will prove Proposition 5.2.

Proof of Proposition 5.2. We consider the position of the random walk at time $(1-\varepsilon)k$ with $\varepsilon \in (0,1)$. Specifically, fix $\varepsilon \in (0,1)$ such that (3.1) holds and, additionally,

$$\alpha(\varepsilon) < \liminf_{k\to\infty} \frac{t-g_k}{|g_k|} \tag{5.10}$$

is satisfied. Note that the right-hand side is of order $\Omega(1)$ due to our assumptions and, hence, such a $\varepsilon \in (0,1)$ satisfying (5.10) exists.

We will partition the event $\{S_k \in \mathrm{d}t;\ \tau_g > k\}$ into three disjoint events depending on the position of the random walk at time $(1-\varepsilon)k$. Let $v_{\varepsilon,k} = g_{(1-\varepsilon)k} + x_\varepsilon\sqrt{k}$, where $x_\varepsilon > 0$ is chosen small enough such that (5.7) is satisfied. Note that this choice of x_ε implies that $x_\varepsilon/\sqrt{\varepsilon(1-\varepsilon)} \to 0$ as $\varepsilon \downarrow 0$, since

$$\lim_{\varepsilon\downarrow 0} \frac{x_\varepsilon}{\sqrt{\varepsilon(1-\varepsilon)}} < \lim_{\varepsilon\downarrow 0} \sqrt{\frac{-2\log(1-\varepsilon^{3/2})}{\varepsilon}} = 0.$$

Then

$$\begin{aligned}\mathbb{P}(S_k \in \mathrm{d}t;\ \tau_g > k) &= \mathbb{P}(S_k \in \mathrm{d}t;\ S_{(1-\varepsilon)k} < v_{\varepsilon,k};\ \tau_g > k)\\ &\quad + \mathbb{P}\left(S_k \in \mathrm{d}t;\ S_{(1-\varepsilon)k} \in \left[v_{\varepsilon,k}, \sqrt{\frac{k}{\varepsilon}}\right];\ \tau_g > k\right)\\ &\quad + \mathbb{P}\left(S_k \in \mathrm{d}t;\ S_{(1-\varepsilon)k} > \sqrt{\frac{k}{\varepsilon}};\ \tau_g > k\right).\end{aligned} \tag{5.11}$$

We consider an upper and lower limiting bound of $\mathbb{P}(S_k \in \mathrm{d}t;\ \tau_g > k)$ as $k \to \infty$, and show that they coincide as $\varepsilon \downarrow 0$. For readability of the proof, we consider the $t = o(\sqrt{k})$ and $t = \Theta(\sqrt{k})$ cases separately.

First, suppose that $t = o(\sqrt{k})$, and we will derive an upper bound. Lemma 5.1 provides an upper bound for the first term in (5.11), and Lemma 5.3 yields an upper bound for the third term in (5.11). Recalling definition (5.8), we see that the second term in (5.11) can be bounded by

$$\begin{aligned}&\mathbb{P}\left(S_k \in \mathrm{d}t;\ S_{(1-\varepsilon)k} \in \left[v_{\varepsilon,k}, \sqrt{\frac{k}{\varepsilon}}\right];\ \tau_g > k\right)\\ &\quad = \int_{v=v_{\varepsilon,k}}^{\sqrt{k/\varepsilon}} \mathbb{P}(S_{(1-\varepsilon)k} \in \mathrm{d}v;\ \tau_g > (1-\varepsilon)k)\\ &\qquad\qquad \times \mathbb{P}(S_k \in \mathrm{d}t;\ \tau_g > k \mid S_{(1-\varepsilon)k} = v;\ \tau_g > (1-\varepsilon)k)\\ &\quad \le \int_{v=v_{\varepsilon,k}}^{\sqrt{k/\varepsilon}} \mathbb{P}(S_{(1-\varepsilon)k} \in \mathrm{d}v;\ \tau_g > (1-\varepsilon)k)\mathbb{P}(S_{\varepsilon k} \in \mathrm{d}t;\ T_{g^+_{k,\varepsilon}} > \varepsilon k \mid S_0 = v).\end{aligned}$$

Due to Proposition 5.1, it holds that uniformly in $t = o(\sqrt{k})$ and $v = \Theta(\sqrt{k})$ as $k \to \infty$,

$$\begin{aligned}&\frac{\mathbb{P}(S_{\varepsilon k} \in \mathrm{d}t;\ T_{g^+_{k,\varepsilon}} > \varepsilon k \mid S_0 = v)}{\mathrm{d}t}\\ &\quad = (1+o(1))\sqrt{\frac{2}{\pi}}\mathbb{E}(-\tilde{S}_{T_0})\frac{U(t-g^+_{k,\varepsilon})}{\sqrt{\varepsilon k}}\frac{v-g^+_{k,\varepsilon}}{\varepsilon k}\exp\left(-\frac{(v-g^+_{k,\varepsilon})^2}{2\varepsilon k}\right).\end{aligned}$$

This yields

$$\int_{v=v_{\varepsilon,k}}^{\sqrt{k/\varepsilon}} \mathbb{P}(S_{(1-\varepsilon)k} \in \mathrm{d}v;\ \tau_g > (1-\varepsilon)k) \frac{\mathbb{P}(S_{\varepsilon k} \in \mathrm{d}t;\ T_{g^+_{k,\varepsilon}} > \varepsilon k \mid S_0 = v)}{\mathrm{d}t}$$

$$= (1+o(1))\mathbb{P}(\tau_g > (1-\varepsilon)k)\sqrt{\frac{2}{\pi}}\mathbb{E}(-\tilde{S}_{T_0})\frac{U(t-g^+_{k,\varepsilon})}{\sqrt{\varepsilon k}}$$

$$\times \int_{v=v_{\varepsilon,k}}^{\sqrt{k/\varepsilon}} \frac{v-g^+_{k,\varepsilon}}{\varepsilon k} \exp\left(-\frac{(v-g^+_{k,\varepsilon})^2}{2\varepsilon k}\right)\mathbb{P}(S_{(1-\varepsilon)k} \in \mathrm{d}v \mid \tau_g > (1-\varepsilon)k).$$

First, due to (2.3) and the fact that $L_g(\cdot)$ is slowly varying,

$$\mathbb{P}(\tau_g > (1-\varepsilon)k) = (1+o(1))\sqrt{\frac{2}{\pi}}\frac{L_g((1-\varepsilon)k)}{\sqrt{(1-\varepsilon)k}} = (1+o(1))\sqrt{\frac{2}{\pi}}\frac{L_g(k)}{\sqrt{(1-\varepsilon)k}}. \tag{5.12}$$

Second, note that due to (5.5) and (3.1),

$$\limsup_{k\to\infty} \frac{U(t-g^+_{k,\varepsilon})}{U(t-g_k)} = 1+\alpha(\varepsilon)\limsup_{k\to\infty}\frac{|g_k|}{t-g_k} \le 1+c_1\alpha(\varepsilon) \quad \text{for some } c_1 \in (0,\infty)$$

(since $t - g_k = \Omega(|g_k|)$). Finally, invoking (2.2) yields

$$\int_{v=v_{\varepsilon,k}}^{\sqrt{k/\varepsilon}} \frac{v-g^+_{k,\varepsilon}}{\varepsilon k} \exp\left(-\frac{(v-g^+_{k,\varepsilon})^2}{2\varepsilon k}\right)\mathbb{P}(S_{(1-\varepsilon)k} \in \mathrm{d}v \mid \tau_g > (1-\varepsilon)k)$$

$$= (1+o(1))\int_{z=x_\varepsilon/\sqrt{1-\varepsilon}}^{1/\sqrt{\varepsilon(1-\varepsilon)}} \frac{\sqrt{1-\varepsilon}}{\varepsilon\sqrt{k}} z\exp\left(-\frac{z^2}{2}\frac{1-\varepsilon}{\varepsilon}\right) z\exp\left(-\frac{z^2}{2}\right)\mathrm{d}z$$

$$\le (1+o(1))\frac{\sqrt{1-\varepsilon}}{\varepsilon\sqrt{k}}\int_{z=0}^{\infty} z^2\exp\left(-\frac{z^2}{2\varepsilon}\right)\mathrm{d}z$$

$$= \sqrt{\frac{\pi}{2}}\sqrt{\frac{\varepsilon(1-\varepsilon)}{k}}.$$

We conclude that, for every $\varepsilon \in (0,1)$,

$$\limsup_{k\to\infty}\frac{k^{3/2}}{L_g(k)U(t-g_k)}\mathbb{P}\left(S_k \in \mathrm{d}t;\ S_{(1-\varepsilon)k} \in \left[v_{\varepsilon,k}, \sqrt{\frac{k}{\varepsilon}}\right];\ \tau_g > k\right)$$

$$\le \sqrt{\frac{2}{\pi}}\mathbb{E}(-\tilde{S}_{T_0})(1+c_1\alpha(\varepsilon)).$$

Then this expression, together with invoking Lemma 5.1 for the first term in (5.11) and Lemma 5.3 for the third term in (5.11), yields the upper bound

$$\limsup_{k\to\infty}\frac{k^{3/2}}{L_g(k)U(t-g_k)}\mathbb{P}(S_k \in \mathrm{d}t;\ \tau_g > k)$$

$$\le C_1\frac{x_\varepsilon}{\sqrt{(1-\varepsilon)}}\sqrt{\frac{2}{\pi}}\mathbb{E}(-\tilde{S}_{T_0}) + \sqrt{\frac{2}{\pi}}\mathbb{E}(-\tilde{S}_{T_0})(1+c_1\alpha(\varepsilon))$$

$$+ C_3(1+C_4\alpha(\varepsilon))\frac{1}{\varepsilon\sqrt{1-\varepsilon}}\exp\left(-\frac{1}{\varepsilon(1-\varepsilon)}\right) \quad \text{for every } \varepsilon > 0.$$

Letting $\varepsilon \downarrow 0$, we conclude that

$$\limsup_{k\to\infty} \frac{k^{3/2}}{L_g(k)U(t-g_k)} \mathbb{P}(S_k \in \mathrm{d}t;\ \tau_g > k) \le \sqrt{\frac{2}{\pi}} \mathbb{E}(-\tilde{S}_{T_0}). \tag{5.13}$$

The proof of the lower bound follows along similar lines. Define $g^-_{k,\varepsilon} = g_k + \alpha(\varepsilon)|g_k|$. In view of (5.11), we bound it from below by the second term only. That is,

$$\begin{aligned}
&\mathbb{P}(S_k \in \mathrm{d}t;\ \tau_g > k)\\
&\quad \ge \mathbb{P}\Big(S_k \in \mathrm{d}t;\ S_{(1-\varepsilon)k} \in \Big[v_{\varepsilon,k}, \sqrt{\frac{k}{\varepsilon}}\Big];\ \tau_g > k\Big)\\
&\quad \ge \int_{v=v_{\varepsilon,k}}^{\sqrt{k/\varepsilon}} \mathbb{P}(S_{(1-\varepsilon)k} \in \mathrm{d}v;\ \tau_g > (1-\varepsilon)k)\mathbb{P}(S_{\varepsilon k} \in \mathrm{d}t;\ T_{g^-_{k,\varepsilon}} > \varepsilon k \mid S_0 = v)\\
&\quad = (1+o(1))\mathbb{P}(\tau_g > (1-\varepsilon)k)\sqrt{\frac{2}{\pi}}\mathbb{E}(-\tilde{S}_{T_0})\frac{U(t-g^-_{k,\varepsilon})}{\sqrt{\varepsilon k}}\,\mathrm{d}t\\
&\qquad \times \int_{v=v_{\varepsilon,k}}^{\sqrt{k/\varepsilon}} \frac{v-g^-_{k,\varepsilon}}{\varepsilon k}\exp\Big(-\frac{(v-g^-_{k,\varepsilon})^2}{2\varepsilon k}\Big)\mathbb{P}(S_{(1-\varepsilon)k} \in \mathrm{d}v \mid \tau_g > (1-\varepsilon)k).
\end{aligned}$$

First, we observe that (5.12) holds for the lower bound. Second, note that due to (5.5) and (5.10),

$$\liminf_{k\to\infty} \frac{U(t-g^-_{k,\varepsilon})}{U(t-g_k)} = 1 - \alpha(\varepsilon)\liminf_{k\to\infty}\frac{|g_k|}{t-g_k} \in (0,\infty).$$

Finally, invoking (2.2) and using partial integration, we obtain

$$\begin{aligned}
&\int_{v=v_{\varepsilon,k}}^{\sqrt{k/\varepsilon}} \frac{v-g^-_{k,\varepsilon}}{\varepsilon k}\exp\Big(-\frac{(v-g^-_{k,\varepsilon})^2}{2\varepsilon k}\Big)\mathbb{P}(S_{(1-\varepsilon)k} \in \mathrm{d}v \mid \tau_g > (1-\varepsilon)k)\\
&\quad = (1+o(1))\frac{\sqrt{1-\varepsilon}}{\varepsilon\sqrt{k}}\int_{z=x_\varepsilon/\sqrt{1-\varepsilon}}^{1/\sqrt{\varepsilon(1-\varepsilon)}} z^2 \mathrm{e}^{-z^2/2\varepsilon}\,\mathrm{d}z\\
&\quad = (1+o(1))\Big(\frac{x_\varepsilon}{\sqrt{k}}\exp\Big(-\frac{x_\varepsilon^2}{2\varepsilon(1-\varepsilon)}\Big) - \sqrt{\frac{\varepsilon}{k}}\exp\Big(-\frac{1}{2\varepsilon^2(1-\varepsilon)}\Big)\\
&\qquad\qquad + \sqrt{\frac{\varepsilon(1-\varepsilon)}{k}}\int_{y=x_\varepsilon/\sqrt{\varepsilon(1-\varepsilon)}}^{1/\varepsilon\sqrt{1-\varepsilon}} \mathrm{e}^{-y^2/2}\,\mathrm{d}y\Big).
\end{aligned}$$

We conclude that, for every $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\liminf_{k\to\infty} \frac{k^{3/2}}{L_g(k)U(t-g_k)}\frac{\mathbb{P}(S_k \in \mathrm{d}t;\ \tau_g > k)}{\mathrm{d}t}\\
&\quad \ge \sqrt{\frac{2}{\pi}}\mathbb{E}(-\tilde{S}_{T_0})\Big(1 - \alpha(\varepsilon)\liminf_{k\to\infty}\frac{|g_k|}{t-g_k}\Big)\\
&\qquad \times \sqrt{\frac{2}{\pi}}\Big(\frac{x_\varepsilon}{\sqrt{\varepsilon(1-\varepsilon)}}\exp\Big(-\frac{x_\varepsilon^2}{2\varepsilon(1-\varepsilon)}\Big) - \frac{1}{\sqrt{1-\varepsilon}}\exp\Big(-\frac{1}{2\varepsilon^2(1-\varepsilon)}\Big)\\
&\qquad\qquad + \int_{y=x_\varepsilon/\sqrt{\varepsilon(1-\varepsilon)}}^{1/\varepsilon\sqrt{1-\varepsilon}} \mathrm{e}^{-y^2/2}\,\mathrm{d}y\Big).
\end{aligned}$$

We note that

$$\lim_{\varepsilon\downarrow 0}\biggl(\frac{x_\varepsilon}{\sqrt{\varepsilon(1-\varepsilon)}}\mathrm{e}^{-x_\varepsilon^2/2\varepsilon(1-\varepsilon)}-\frac{1}{\sqrt{1-\varepsilon}}\mathrm{e}^{-1/2\varepsilon^2(1-\varepsilon)}+\int_{y=x_\varepsilon/\sqrt{\varepsilon(1-\varepsilon)}}^{1/\varepsilon\sqrt{1-\varepsilon}}\mathrm{e}^{-y^2/2}\,\mathrm{d}y\biggr)=\sqrt{\frac{\pi}{2}},$$

and

$$\lim_{\varepsilon\downarrow 0}\biggl(1-\alpha(\varepsilon)\liminf_{k\to\infty}\frac{|g_k|}{t-g_k}\biggr)=1.$$

In conclusion,

$$\liminf_{k\to\infty}\frac{k^{3/2}}{L_g(k)U(t-g_k)}\frac{\mathbb{P}(S_k\in\mathrm{d}t;\ \tau_g>k)}{\mathrm{d}t}\ge\sqrt{\frac{2}{\pi}}\mathbb{E}(-\tilde{S}_{T_0}).$$

Note this coincides with the upper bound (5.13), proving the result in the $t=o(\sqrt{k})$ case.

Next we consider the $t=\Theta(\sqrt{k})$ case. An upper bound for the first and third terms in (5.11) is given by Lemmas 5.2 and 5.4. The second term in (5.11) can again be bounded by

$$\begin{aligned}&\mathbb{P}\biggl(S_k\in\mathrm{d}t;\ S_{(1-\varepsilon)k}\in\biggl[v_{\varepsilon,k},\sqrt{\frac{k}{\varepsilon}}\biggr];\ \tau_g>k\biggr)\\&\qquad\le\int_{v=v_{\varepsilon,k}}^{\sqrt{k/\varepsilon}}\mathbb{P}(S_{(1-\varepsilon)k}\in\mathrm{d}v;\ \tau_g>(1-\varepsilon)k)\mathbb{P}(S_{\varepsilon k}\in\mathrm{d}t;\ T_{g^+_{k,\varepsilon}}>\varepsilon k\mid S_0=v).\end{aligned}$$

Due to Proposition 5.1, it holds that uniformly in $t=\Theta(\sqrt{k})$ and $v=\Theta(\sqrt{k})$ as $k\to\infty$,

$$\begin{aligned}&\frac{\mathbb{P}(S_{\varepsilon k}\in\mathrm{d}t;\ T_{g^+_{k,\varepsilon}}>\varepsilon k\mid S_0=v)}{\mathrm{d}t}\\&\qquad\sim\frac{1}{\sqrt{2\pi\varepsilon k}}\biggl(\exp\biggl(-\frac{(v-t)^2}{2\varepsilon k}\biggr)-\exp\biggl(-\frac{(v+t-2g^+_{k,\varepsilon})^2}{2\varepsilon k}\biggr)\biggr)\end{aligned}$$

and, hence,

$$\begin{aligned}&\frac{\mathbb{P}(S_k\in\mathrm{d}t;\ S_{(1-\varepsilon)k}\in[v_{\varepsilon,k},\sqrt{k/\varepsilon}];\ \tau_g>k)}{\mathrm{d}t}\\&\qquad\le(1+o(1))\frac{\mathbb{P}(\tau_g>(1-\varepsilon)k)}{\sqrt{2\pi\varepsilon k}}\\&\qquad\quad\times\int_{v=v_{\varepsilon,k}}^{\sqrt{k/\varepsilon}}\biggl(\exp\biggl(-\frac{(v-t)^2}{2\varepsilon k}\biggr)-\exp\biggl(-\frac{(v+t-2g^+_{k,\varepsilon})^2}{2\varepsilon k}\biggr)\biggr)\\&\qquad\quad\times\mathbb{P}(S_{(1-\varepsilon)k}\in\mathrm{d}v\mid\tau_g>(1-\varepsilon)k).\end{aligned}$$

Again, we find that (5.12) holds. Moreover, due to (2.2),

$$\begin{aligned}&\int_{v=v_{\varepsilon,k}}^{\sqrt{k/\varepsilon}}\biggl(\exp\biggl(-\frac{(v-t)^2}{2\varepsilon k}\biggr)-\exp\biggl(-\frac{(v+t-2g^+_{k,\varepsilon})^2}{2\varepsilon k}\biggr)\biggr)\mathbb{P}(S_{(1-\varepsilon)k}\in\mathrm{d}v\mid\tau_g>(1-\varepsilon)k)\\&\quad=(1+o(1))\int_{z=x_\varepsilon/\sqrt{1-\varepsilon}}^{1/\sqrt{\varepsilon(1-\varepsilon)}}\biggl(\exp\biggl(-\frac{(z-t/\sqrt{(1-\varepsilon)k})^2}{2\varepsilon/(1-\varepsilon)}\biggr)\\&\qquad\qquad-\exp\biggl(-\frac{(z+t/\sqrt{(1-\varepsilon)k})^2}{2\varepsilon/(1-\varepsilon)}\biggr)\biggr)z\mathrm{e}^{-z^2/2}\,\mathrm{d}z.\end{aligned}$$

Applying Lemma A.1,

$$\int_{v=v_{\varepsilon,k}}^{\sqrt{k/\varepsilon}}\Big(\exp\Big(-\frac{(v-t)^2}{2\varepsilon k}\Big)-\exp\Big(-\frac{(v+t-2g^+_{k,\varepsilon})^2}{2\varepsilon k}\Big)\Big)\mathbb{P}(S_{(1-\varepsilon)k}\in \mathrm{d}v \mid \tau_g>(1-\varepsilon)k)$$
$$\le (1+o(1))\int_{z=0}^{\infty}\Big(\exp\Big(-\frac{(z-t/\sqrt{(1-\varepsilon)k})^2}{2\varepsilon/(1-\varepsilon)}\Big) - \exp\Big(-\frac{(z+t/\sqrt{(1-\varepsilon)k})^2}{2\varepsilon/(1-\varepsilon)}\Big)\Big)z\mathrm{e}^{-z^2/2}\,\mathrm{d}z$$
$$=(1+o(1))\sqrt{2\pi}\sqrt{\varepsilon(1-\varepsilon)}\frac{t}{\sqrt{k}}\mathrm{e}^{-t^2/2k}.$$

We conclude that, for every $\varepsilon\in(0,1)$,

$$\limsup_{k\to\infty}\frac{k^{3/2}}{L_g(k)t\mathrm{e}^{-t^2/(2k)}}\frac{\mathbb{P}(S_k\in\mathrm{d}t;\ S_{(1-\varepsilon)k}\in[v_{\varepsilon,k},\sqrt{k/\varepsilon}];\ \tau_g>k)}{\mathrm{d}t}\le\sqrt{\frac{2}{\pi}}.$$

Recalling (5.11) and invoking Lemmas 5.2 and 5.4, we derive, for every $\varepsilon\in(0,1)$,

$$\limsup_{k\to\infty}\frac{k^{3/2}}{L_g(k)t\mathrm{e}^{-t^2/(2k)}}\frac{\mathbb{P}(S_k\in\mathrm{d}t;\ \tau_g>k)}{\mathrm{d}t}$$
$$\le C_2\frac{x_\varepsilon}{\sqrt{(1-\varepsilon)}}+\sqrt{\frac{2}{\pi}}+\frac{C_5}{\sqrt{\varepsilon(1-\varepsilon)}}\exp\Big(-\frac{1}{\varepsilon(1-\varepsilon)}\Big).$$

Letting $\varepsilon\downarrow 0$, we conclude that

$$\limsup_{k\to\infty}\frac{k^{3/2}}{L_g(k)t\mathrm{e}^{-t^2/(2k)}}\frac{\mathbb{P}(S_k\in\mathrm{d}t;\ \tau_g>k)}{\mathrm{d}t}\le\sqrt{\frac{2}{\pi}}. \tag{5.14}$$

For the lower bound of the probability that $\{S_k\in\mathrm{d}t;\ \tau_g>k\}$ occurs, we observe that

$$\mathbb{P}(S_k\in\mathrm{d}t;\ \tau_g>k)\ge\mathbb{P}\Big(S_k\in\mathrm{d}t;\ S_{(1-\varepsilon)k}\in\Big[v_{\varepsilon,k},\sqrt{\frac{k}{\varepsilon}}\Big];\ \tau_g>k\Big)$$
$$\ge(1+o(1))\frac{\mathbb{P}(\tau_g>(1-\varepsilon)k)}{\sqrt{2\pi\varepsilon k}}$$
$$\times\int_{v=v_{\varepsilon,k}}^{\sqrt{k/\varepsilon}}\Big(\exp\Big(-\frac{(v-t)^2}{2\varepsilon k}\Big)-\exp\Big(-\frac{(v+t-2g^-_{k,\varepsilon})^2}{2\varepsilon k}\Big)\Big)$$
$$\times\,\mathbb{P}(S_{(1-\varepsilon)k}\in\mathrm{d}v\mid\tau_g>(1-\varepsilon)k).$$

Recall (5.12) and, moreover, due to (2.2),

$$\int_{v=v_{\varepsilon,k}}^{\sqrt{k/\varepsilon}}\Big(\exp\Big(-\frac{(v-t)^2}{2\varepsilon k}\Big)-\exp\Big(-\frac{(v+t-2g^-_{k,\varepsilon})^2}{2\varepsilon k}\Big)\Big)\mathbb{P}(S_{(1-\varepsilon)k}\in \mathrm{d}v \mid \tau_g>(1-\varepsilon)k)$$
$$=(1+o(1))\int_{z=x_\varepsilon/\sqrt{1-\varepsilon}}^{1/\sqrt{\varepsilon(1-\varepsilon)}}\Big(\exp\Big(-\frac{(z-t/\sqrt{(1-\varepsilon)k})^2}{2\varepsilon/(1-\varepsilon)}\Big) - \exp\Big(-\frac{(z+t/\sqrt{(1-\varepsilon)k})^2}{2\varepsilon/(1-\varepsilon)}\Big)\Big)z\mathrm{e}^{-z^2/2}\,\mathrm{d}z.$$

Note that $[x_\varepsilon/\sqrt{1-\varepsilon}, 1/\sqrt{\varepsilon(1-\varepsilon)}] = [0,\infty] \setminus [0, x_\varepsilon/\sqrt{1-\varepsilon}] \setminus [1/\sqrt{\varepsilon(1-\varepsilon)}, \infty]$, and also that it always holds that

$$\begin{aligned}\biggl(\exp\Bigl(-\frac{(z-t/\sqrt{(1-\varepsilon)k})^2}{2\varepsilon/(1-\varepsilon)}\Bigr) - \exp\Bigl(-\frac{(z+t/\sqrt{(1-\varepsilon)k})^2}{2\varepsilon/(1-\varepsilon)}\Bigr)\biggr)&\\ \le \exp\Bigl(-\frac{(z-t/\sqrt{(1-\varepsilon)k})^2}{2\varepsilon/(1-\varepsilon)}\Bigr)&\\ \le 1&\end{aligned}$$

for any $\varepsilon \in (0,1)$. Then invoking Lemma A.1 yields

$$\begin{aligned}&\int_{v=v_{\varepsilon,k}}^{\sqrt{k/\varepsilon}} \biggl(\exp\Bigl(-\frac{(v-t)^2}{2\varepsilon k}\Bigr) - \exp\Bigl(-\frac{(v+t-2g_{k,\varepsilon}^-)^2}{2\varepsilon k}\Bigr)\biggr)\mathbb{P}(S_{(1-\varepsilon)k} \in \mathrm{d}v \mid \tau_g > (1-\varepsilon)k)\\ &\quad \ge (1+o(1))\biggl(\sqrt{2\pi}\sqrt{\varepsilon(1-\varepsilon)}\frac{t}{\sqrt{k}}\mathrm{e}^{-t^2/2k} - \int_{z=0}^{x_\varepsilon/\sqrt{1-\varepsilon}} z\mathrm{e}^{-z^2/2}\,\mathrm{d}z\\ &\qquad\qquad - \int_{z=1/\sqrt{\varepsilon(1-\varepsilon)}}^{\infty} z\mathrm{e}^{-z^2/2}\,\mathrm{d}z\biggr)\\ &\quad = (1+o(1))\biggl(\sqrt{2\pi}\sqrt{\varepsilon(1-\varepsilon)}\frac{t}{\sqrt{k}}\mathrm{e}^{-t^2/2k} - (1-\mathrm{e}^{-x_\varepsilon^2/2(1-\varepsilon)}) - \mathrm{e}^{-1/\varepsilon(1-\varepsilon)}\biggr).\end{aligned}$$

Therefore, for every $\varepsilon \in (0,1)$,

$$\begin{aligned}&\liminf_{k\to\infty} \frac{k^{3/2}}{L_g(k)t\mathrm{e}^{-t^2/(2k)}}\mathbb{P}(S_k \in \mathrm{d}t;\ \tau_g > k)\\ &\quad \ge \liminf_{k\to\infty} \frac{k^{3/2}}{L_g(k)t\mathrm{e}^{-t^2/(2k)}}\mathbb{P}\biggl(S_k \in \mathrm{d}t;\ S_{(1-\varepsilon)k} \in \biggl[v_{\varepsilon,k}, \sqrt{\frac{k}{\varepsilon}}\biggr];\ \tau_g > k\biggr)\\ &\quad \ge \sqrt{\frac{2}{\pi}} - \frac{1}{\pi}\liminf_{k\to\infty}\Bigl(\frac{t}{\sqrt{k}}\mathrm{e}^{-t^2/2k}\Bigr)^{-1}\biggl(\frac{1}{\sqrt{\varepsilon(1-\varepsilon)}}\Bigl(1-\mathrm{e}^{-x_\varepsilon^2/2(1-\varepsilon)}\Bigr)\\ &\qquad\qquad + \frac{1}{\sqrt{\varepsilon(1-\varepsilon)}}\mathrm{e}^{-1/\varepsilon(1-\varepsilon)}\biggr).\end{aligned}$$

We observe that as $\varepsilon \downarrow 0$, this expression tends to $\sqrt{2/\pi}$ due to our choice of x_ε and, hence,

$$\liminf_{k\to\infty} \frac{k^{3/2}}{L_g(k)t\mathrm{e}^{-t^2/(2k)}}\mathbb{P}(S_k \in \mathrm{d}t;\ \tau_g > k) \ge \sqrt{\frac{2}{\pi}}.$$

Since this coincides with the upper bound in (5.14), the proposition follows uniformly for $t = \Theta(\sqrt{k})$. □

5.2. Proof of Theorem 3.2

Recall that $\tilde{S}_m$, $m \ge 1$, denotes the reversed random walk defined in (4.1), and $\tilde{f}_m(\cdot)$ the corresponding density function at time m. Then

$$\begin{aligned}\mathbb{P}(\tau_g > k \mid S_n = 0) &= \int_{u=g_k}^{\infty} \mathbb{P}(\tau_g > k;\ S_k \in \mathrm{d}u \mid S_n = 0)\\ &= \frac{1}{f_n(0)}\int_{u=g_k}^{\infty} \mathbb{P}(S_k \in \mathrm{d}u;\ \tau_g > k)\tilde{f}_{n-k}(u). \qquad (5.15)\end{aligned}$$

It may be clear from the above identity that one may use Proposition 5.2 to derive the main result. Yet, Proposition 5.2 does not provide the asymptotic behavior for the complete interval $[-g_k, \infty]$, and we need to account for the behavior at its extremes, that is, for values within $o(|g_k|)$ distance from the boundary and values that are beyond distance $\Theta(\sqrt{k})$. The next two lemmas turn out to be useful in order to bound the behavior at these values.

Lemma 5.5. *For every stopping time τ_g, irrespective of boundary $(g_i)_{i\in\mathbb{N}}$, there exist a $k_0 \in \mathbb{N}$ and constant $c_1 < \infty$ such that uniformly for all $k \geq k_0$ and for all $x \geq g_k$,*

$$\frac{\mathbb{P}(S_k \in \mathrm{d}x;\ \tau_g > k)}{\mathrm{d}x} \leq c_1 \frac{\mathbb{P}(\tau_g > \lfloor k/2 \rfloor)}{\sqrt{k}}.$$

In particular, if the boundary satisfies $g_i = o(\sqrt{i})$, there exists a constant $c_2 < \infty$ such that

$$\frac{\mathbb{P}(S_k \in \mathrm{d}x;\ \tau_g > k)}{\mathrm{d}x} \leq c_2 \frac{L_g(k)}{k}.$$

Proof. Set $m = \lfloor k/2 \rfloor$. Then

$$\begin{aligned}
\frac{\mathbb{P}(S_k \in \mathrm{d}x;\ \tau_g > k)}{\mathrm{d}x} &\leq \frac{\mathbb{P}(S_k \in \mathrm{d}x;\ \tau_g > m)}{\mathrm{d}x} \\
&= \int_{y=g_m}^{\infty} \mathbb{P}(S_m \in \mathrm{d}y;\ \tau_g > m) \frac{\mathbb{P}(S_{k-m} \in \mathrm{d}x \mid S_0 = y)}{\mathrm{d}x} \\
&= \int_{y=g_m}^{\infty} \mathbb{P}(S_m \in \mathrm{d}y;\ \tau_g > m) f_{k-m}(x-y).
\end{aligned}$$

Assumption 2.1 implies that there exists a constant $c_3 < \infty$ such that

$$\sup_{z\in\mathbb{R}} f_{k-m}(z) \leq \frac{c_3}{\sqrt{k-m}}.$$

Therefore,

$$\frac{\mathbb{P}(S_k \in \mathrm{d}x;\ \tau_g > k)}{\mathrm{d}x} \leq \frac{c_3}{\sqrt{k-m}} \int_{y=g_m}^{\infty} \mathbb{P}(S_m \in \mathrm{d}y;\ \tau_g > m) \leq \sqrt{2}\frac{c_3}{\sqrt{k}} \mathbb{P}(\tau_g > m).$$

For the second assertion, note that due to (2.3) with $L_g(\cdot)$ slowly varying,

$$\mathbb{P}(\tau_g > m) \sim \sqrt{\frac{2}{\pi}} \frac{L_g(m)}{\sqrt{m}} \sim \frac{2}{\sqrt{\pi}} \frac{L_g(k)}{\sqrt{k}} = \sqrt{2}\mathbb{P}(\tau_g > k). \qquad \square$$

In the next lemma we quantify how likely the random walk staying above the moving boundary is to have a position relatively close to the boundary.

Lemma 5.6. *Suppose that $x_k = o(\sqrt{k})$ is such that $x_k = \Omega(|g_k|)$ and $x_k \to \infty$ as $k \to \infty$. There exists a constant $c_1 < \infty$ such that*

$$\mathbb{P}(S_k \leq g_k + x_k;\ \tau_g > k) \leq c_1 x_k^2 \frac{L_g(k)}{k^{3/2}}.$$

Proof. Recall (5.8) for some fixed $\varepsilon \in (0, 1)$, and let $m = \lfloor k/2 \rfloor$. Note that

$$
\begin{aligned}
&\mathbb{P}(S_k \le g_k + x_k;\ \tau_g > k) \\
&\quad = \int_{u=g_m}^{\infty} \mathbb{P}(S_m \in \mathrm{d}u;\ \tau_g > m) \int_{v=g_k}^{g_k+x_k} \mathbb{P}(S_{k-m} \in \mathrm{d}v;\ T_{g^+_{k,\varepsilon}} > k-m \mid S_0 = u) \\
&\quad \le \mathbb{P}(\tau_g > m) x_k \sup_{v\in[g_k, g_k+x_k],\, u\ge g_m} \frac{\mathbb{P}(S_{k-m} \in \mathrm{d}v;\ T_{g^+_{k,\varepsilon}} > k-m \mid S_0 = u)}{\mathrm{d}v}.
\end{aligned}
$$

Applying Lemma 5.5,

$$
\frac{\mathbb{P}(S_{k-m} \in \mathrm{d}v;\ T_{g^+_{k,\varepsilon}} > k-m \mid S_0 = u)}{\mathrm{d}v} \le \frac{c_2}{\sqrt{k-m}} \mathbb{P}\Big(T_{g^+_{k,\varepsilon}-v} > \Big\lfloor \frac{k-m}{2} \Big\rfloor\Big)
$$

for some constant $c_2 < \infty$. Taking its supremum over $v \in [g_k, g_k + x_k]$ yields

$$
\mathbb{P}(S_k \le g_k + x_k;\ \tau_g > k) \le \frac{c_2}{\sqrt{k-m}} \mathbb{P}(\tau_g > m) x_k \mathbb{P}\Big(T_{-(\alpha(\varepsilon)|g_k|+x_k)} > \Big\lfloor \frac{k-m}{2} \Big\rfloor\Big).
$$

In view of (2.3) with $L_g(\cdot)$ slowly varying, it follows that there exists a $c_3 < \infty$ such that

$$
\mathbb{P}(\tau_g > m) \le c_3 \frac{L_g(k)}{\sqrt{k}}.
$$

Moreover, for constant boundaries it holds that, for some constants $c_4 \le c_5 < \infty$,

$$
\mathbb{P}\Big(T_{-(\alpha(\varepsilon)|g_k|+x_k)} > \Big\lfloor \frac{k-m}{2} \Big\rfloor\Big) \le c_4 \frac{U(\alpha(\varepsilon)|g_k| + x_k)}{\sqrt{\lfloor (k-m)/2 \rfloor}} \le c_5 \frac{x_k}{\sqrt{k}},
$$

where the latter inequality follows from (5.5) and since $x_k = \Omega(|g_k|)$. This concludes that there exists a $c_1 < \infty$ such that

$$
\mathbb{P}(S_k \le g_k + x_k;\ \tau_g > k) \le \frac{c_2}{\sqrt{k-m}} c_3 \frac{L_g(k)}{\sqrt{k}} x_k c_5 \frac{x_k}{\sqrt{k}} \le c_1 x_k^2 \frac{L_g(k)}{k^{3/2}}. \qquad \square
$$

Next we prove our main result.

Proof of Theorem 3.2. As in the proof of Proposition 5.2, we will provide an appropriate upper and lower bound of $\mathbb{P}(\tau_g > k \mid S_n = 0)$, and show that these behave identically in the limit. Fix $\delta \in (0, 1)$ and recall (5.15), that is,

$$
\mathbb{P}(\tau_g > k \mid S_n = 0) = \frac{1}{f_n(0)} \int_{u=g_k}^{\infty} \mathbb{P}(S_k \in \mathrm{d}u;\ \tau_g > k) \tilde{f}_{n-k}(u).
$$

Let $x_k = |g_k|$ if $|g_k| \to \infty$ as $k \to \infty$, and $x_k = (n-k)^{1/4}$ if $g_k = O(1)$. For the upper bound, we will partition the integration area into three intervals, namely $[g_k, g_k+\delta x_k]$, $[g_k+\delta x_k, \delta\sqrt{k}]$, and $[\delta\sqrt{k}, \infty]$.

For values close to the boundary, we observe the following. Assumption 2.1 implies that there exists a constant $c_1 \in [0, \infty)$ such that $\sup_{x\in\mathbb{R}} \tilde{f}_{n-k}(x) \le c_1/\sqrt{n-k}$ and, hence, applying Lemma 5.6,

$$
\begin{aligned}
\int_{u=g_k}^{g_k+\delta x_k} \mathbb{P}(S_k \in \mathrm{d}u;\ \tau_g > k) \tilde{f}_{n-k}(u) &\le \frac{c_1}{\sqrt{n-k}} \mathbb{P}(S_k \in [g_k, g_k + \delta x_k];\ \tau_g > k) \\
&\le \frac{c_2}{\sqrt{n-k}} \frac{\delta^2 x_k^2 L_g(k)}{k^{3/2}} \quad \text{for some } c_2 < \infty.
\end{aligned}
$$

Due to (2.1),

$$\frac{1}{f_n(0)} \sim \sqrt{2\pi n} \sim \sqrt{2\pi k} \tag{5.16}$$

and, hence,

$$\mathbb{P}(\tau_g > k;\ S_k \le g_k + \delta x_k \mid S_n = 0) = \delta^2 O\biggl(\frac{x_k^2 L_g(k)}{k\sqrt{n-k}}\biggr).$$

Note that this is at most of an order stated in (3.2) in all three cases. As $\delta \downarrow 0$, this term is therefore negligible with respect to (3.2).

For values relatively far from the boundary, we observe that, due to Lemma 5.5, there exists a constant $c_3 < \infty$ such that

$$\int_{u=\delta\sqrt{k}}^{\infty} \mathbb{P}(S_k \in \mathrm{d}u;\ \tau_g > k)\tilde{f}_{n-k}(u) \le c_3 \frac{L_g(k)}{k}\mathbb{P}(\tilde{S}_{n-k} > \delta\sqrt{k}).$$

Applying Chebyshev's inequality on $\tilde{S}_{n-k}$ with $\operatorname{var}(\tilde{S}_{n-k}) = n-k$ yields

$$\mathbb{P}(\tilde{S}_{n-k} > \delta\sqrt{k}) \le \frac{1}{\delta^2}\frac{n-k}{k}.$$

Recalling (5.16), we observe that

$$\mathbb{P}(\tau_g > k;\ S_k \ge \delta\sqrt{k} \mid S_n = 0) = \frac{1}{\delta^2} O\biggl(\frac{L_g(k)}{k}\frac{n-k}{\sqrt{k}}\biggr).$$

In all three cases this term is of *strictly* smaller order as $k \to \infty$ than that stated in (3.2) and, hence, is also negligible with respect to (3.2).

For values in $[g_k + \delta x_k, \delta\sqrt{k}]$, we will use Proposition 5.2. Using (5.5) and the fact that $\mathrm{e}^{-x^2} \le 1$ for all $x \in \mathbb{R}$, Proposition 5.2 implies that uniformly for all $u \in [g_k + \delta x_k, \delta\sqrt{k}]$,

$$\frac{\mathbb{P}(S_k \in \mathrm{d}u;\ \tau_g > k)}{\mathrm{d}u} \le (1+o(1))\sqrt{\frac{2}{\pi}}\frac{L_g(k)}{k^{3/2}}(u - g_k).$$

Therefore,

$$\begin{aligned}
&\int_{u=g_k+\delta x_k}^{\delta\sqrt{k}} \mathbb{P}(S_k \in \mathrm{d}u;\ \tau_g > k)\tilde{f}_{n-k}(u) \\
&\quad\le (1+o(1))\sqrt{\frac{2}{\pi}}\frac{L_g(k)}{k^{3/2}} \int_{u=g_k+\delta x_k}^{\delta\sqrt{k}} (u-g_k)\tilde{f}_{n-k}(u) \\
&\quad= (1+o(1))\sqrt{\frac{2}{\pi}}\frac{L_g(k)}{k^{3/2}}\sqrt{n-k}\,\mathbb{E}\biggl(\frac{\tilde{S}_{n-k}-g_k}{\sqrt{n-k}};\ \frac{\tilde{S}_{n-k}}{\sqrt{n-k}} \in \biggl[\frac{g_k+\delta x_k}{\sqrt{n-k}}, \frac{\delta\sqrt{k}}{\sqrt{n-k}}\biggr]\biggr).
\end{aligned}$$

Using the central limit theorem, we observe that as $n \to \infty$,

$$\begin{aligned}
&\mathbb{E}\biggl(\frac{\tilde{S}_{n-k}-g_k}{\sqrt{n-k}};\ \frac{\tilde{S}_{n-k}}{\sqrt{n-k}} \in \biggl[\frac{g_k+\delta x_k}{\sqrt{n-k}}, \frac{\delta\sqrt{k}}{\sqrt{n-k}}\biggr]\biggr) \\
&\quad\sim \begin{cases}
\displaystyle\int_{v=0}^{\infty} \frac{1}{\sqrt{2\pi}} v\mathrm{e}^{-v^2/2}\,\mathrm{d}v = \frac{1}{\sqrt{2\pi}} & \text{if } g_k = o(\sqrt{n-k}), \\
\displaystyle\int_{v=(g_k+\delta|g_k|)/\sqrt{n-k}}^{\infty} \frac{1}{\sqrt{2\pi}}\biggl(v - \frac{g_k}{\sqrt{n-k}}\biggr)\mathrm{e}^{-v^2/2}\,\mathrm{d}v & \text{if } g_k = \Theta(\sqrt{n-k}), \\
\displaystyle\frac{-g_k}{\sqrt{n-k}} & \text{if } g_k = \omega(\sqrt{n-k}),\ g_k < 0.
\end{cases}
\end{aligned}$$

Adding the three terms corresponding to the three disjoint intervals shows that, for every $0 < \delta < 1$, we have the upper bound

$$
\begin{aligned}
&\int_{u=g_k}^{\infty} \mathbb{P}(S_k \in \mathrm{d}u;\ \tau_g > k)\tilde{f}_{n-k}(u) \\
&\quad \leq (1+o(1))\sqrt{\frac{2}{\pi}}\frac{L_g(k)}{k^{3/2}}(n-k) \\
&\quad\quad \times \begin{cases} \dfrac{1}{\sqrt{2\pi}} & \text{if } g_k = o(\sqrt{n-k}), \\ \displaystyle\int_{v=(g_k+\delta|g_k|)/\sqrt{n-k}}^{\infty} \frac{1}{\sqrt{2\pi}}\left(v - \frac{g_k}{\sqrt{n-k}}\right)\mathrm{e}^{-v^2/2}\,\mathrm{d}v & \text{if } g_k = \Theta(\sqrt{n-k}), \\ \dfrac{-g_k}{\sqrt{n-k}} & \text{if } g_k = \omega(\sqrt{n-k}),\ g_k < 0. \end{cases}
\end{aligned}
$$

For the lower bound, Proposition 5.2 and (5.5) imply that uniformly in $u \in [g_k + \delta x_k, \delta\sqrt{k}]$,

$$
\frac{\mathbb{P}(S_k \in \mathrm{d}u;\ \tau_g > k)}{\mathrm{d}u} \geq (1+o(1))\sqrt{\frac{2}{\pi}}\frac{L_g(k)}{k^{3/2}}(u-g_k)\mathrm{e}^{-\delta^2/2}.
$$

Therefore, we obtain the lower bound

$$
\begin{aligned}
&\int_{u=g_k}^{\infty} \mathbb{P}(S_k \in \mathrm{d}u;\ \tau_g > k)\tilde{f}_{n-k}(u) \\
&\quad \geq \int_{u=g_k+\delta x}^{\delta\sqrt{k}} \mathbb{P}(S_k \in \mathrm{d}u;\ \tau_g > k)\tilde{f}_{n-k}(u) \\
&\quad \geq (1-o(1))\sqrt{\frac{2}{\pi}}\frac{L_g(k)}{k^{3/2}}\sqrt{n-k}\,\mathrm{e}^{-\delta^2/2} \\
&\quad\quad \times \begin{cases} \displaystyle\int_{v=0}^{\infty}\frac{1}{\sqrt{2\pi}}v\mathrm{e}^{-v^2/2}\,\mathrm{d}v = \frac{1}{\sqrt{2\pi}} & \text{if } g_k = o(\sqrt{n-k}), \\ \displaystyle\int_{v=(g_k+\delta|g_k|)/\sqrt{n-k}}^{\infty} \frac{1}{\sqrt{2\pi}}\left(v - \frac{g_k}{\sqrt{n-k}}\right)\mathrm{e}^{-v^2/2}\,\mathrm{d}v & \text{if } g_k = \Theta(\sqrt{n-k}), \\ \dfrac{-g_k}{\sqrt{n-k}} & \text{if } g_k = \omega(\sqrt{n-k}), \\ & \quad g_k < 0. \end{cases}
\end{aligned}
$$

We observe that as $\delta \downarrow 0$ the lower and upper bound coincide. Since we have identity (5.15) with (5.16), we conclude that (3.2) holds. $\square$

Appendix A. A useful integral identity

Lemma A.1. *For every $c, d > 0$,*

$$
\begin{aligned}
&\int_{y=0}^{\infty} y\exp\left(-\frac{y^2}{2}\right)\left(\exp\left(-\frac{(y-c)^2}{2d}\right) - \exp\left(-\frac{(y+c)^2}{2d}\right)\right)\mathrm{d}y \\
&\quad = \sqrt{2\pi}c\sqrt{\frac{d}{(1+d)^3}}\exp\left(-\frac{c^2}{2}\frac{1}{1+d}\right).
\end{aligned}
$$

Proof. First note that, for every $a, b \in \mathbb{R}$,

$$\int y \exp\biggl(-\frac{y^2}{2a}+\frac{y}{b}\biggr)\,\mathrm{d}y = -a\exp\biggl(-\frac{y^2}{2a}+\frac{y}{b}\biggr) - \frac{a^{3/2}}{b}\mathrm{e}^{a/2b^2}\int_{s=0}^{(a-by)/\sqrt{a}b}\mathrm{e}^{-s^2/2}\,\mathrm{d}s.$$

Therefore,

$$\begin{aligned}
&\int_{y=0}^{\infty} y\mathrm{e}^{-y^2/2}\biggl(\exp\biggl(-\frac{(y-c)^2}{2d}\biggr)-\exp\biggl(-\frac{(y+c)^2}{2d}\biggr)\biggr)\,\mathrm{d}y \\
&\quad = \mathrm{e}^{-c^2/2d}\int_{y=0}^{\infty} y\exp\biggl(-\frac{y^2}{2}\frac{1+d}{d}\biggr)\bigl(\mathrm{e}^{yc/d}-\mathrm{e}^{-yc/d}\bigr)\,\mathrm{d}y \\
&\quad = \mathrm{e}^{-c^2/2d}\biggl(\frac{d}{1+d}\biggr)^{3/2}\frac{c}{d}\exp\biggl(\frac{1}{2}\frac{d}{1+d}\frac{c^2}{d^2}\biggr)\sqrt{2\pi} \\
&\quad = \sqrt{2\pi}\,c\sqrt{\frac{d}{(1+d)^3}}\exp\biggl(-\frac{c^2}{2}\frac{1}{1+d}\biggr).
\end{aligned}$$

□

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