Long-time tails in quantum Brownian motion

Roland Jung, Gert-Ludwig Ingold, Hermann Grabert

Angaben zur Veröffentlichung / Publication details:

Jung, Roland, Gert-Ludwig Ingold, and Hermann Grabert. 1985. "Long-time tails in quantum Brownian motion." *Physical Review A* 32 (4): 2510–12. https://doi.org/10.1103/physreva.32.2510.

PHYSICAL REVIEW A VOLUME 32, NUMBER 4 OCTOBER 1985

Long-time tails in quantum Brownian motion

Roland Jung and Gert-Ludwig Ingold
Institut für Theoretische Physik, Universität Stuttgart, D-7000 Stuttgart 80, Germany

Hermann Grabert
Institut für Festkörperforschung der Kernforschungsanlage Jülich, D-5170 Jülich, Germany

(Received 7 May 1985)

The authors address the problem of quantum Brownian motion at low temperatures and with arbitrarily strong damping. For a harmonically bound particle the zero-temperature correlation functions are shown to display long-time tails. At finite temperatures a power-law decay at intermediate times is followed by an exponential decay with time constant $\hbar/2\pi k_B T$. The case of free Brownian motion is treated, and some general conclusions for nonlinear systems are drawn.

It has become clear recently[1–7] that even a linear system exhibits interesting quantum effects when coupled to a low-temperature heat bath. The simplest microscopic model to study these phenomena is a particle of mass M moving in a harmonic potential $V(q)=\frac{1}{2}M\omega_0^2 q^2$ while coupled to a heat bath environment consisting of harmonic oscillators. Several authors have examined this model and have shown how the irreversible motion of the Brownian particle arises.[2,5] In a previous work[3] we have derived various exact results for a quantum harmonic oscillator with Ohmic dissipation. In particular, a power-law decay of zero-temperature correlation functions was noted. In this work we present a more detailed study of the long-time behavior of correlation functions at low temperatures for a harmonically bound particle. Furthermore, the limit of free Brownian motion is examined and some general conclusions for nonlinear systems are drawn. As a characteristic feature of dissipative quantum systems, a power-law decay of zero-temperature correlation functions is found.

Dealing first with a strictly linear system,[2] the dynamical susceptibility $\chi(\omega)$ characterizing the linear response of $\langle q(t)\rangle$ to an external force takes the classical form[3,7]

$$\chi(\omega)=M^{-1}[\omega_0^2-\omega^2-i\omega\gamma(\omega)]^{-1}=\chi'(\omega)+i\chi''(\omega)\ , \qquad (1)$$

where $\gamma(\omega)$ is the frequency-dependent damping coefficient. By virtue of the fluctuation-dissipation theorem[8] the imaginary part $\chi''(\omega)$ of the dynamic susceptibility is related to the spectral density

$$J(\omega)=\int_{-\infty}^{\infty} dt\, e^{i\omega t}\langle q(t)q\rangle \qquad (2)$$

of displacement fluctuations in thermal equilibrium according to

$$J(\omega)=2\hbar[1-\exp(-\beta\hbar\omega)]^{-1}\chi''(\omega)\ , \qquad (3)$$

where $\beta=1/k_BT$. Because of $\dot q=p/M$, where p is the momentum, all other pair correlations are obtained from $J(t)$ as time derivatives, while higher-order correlations can be factorized into pair correlations due to the Gaussian property of the process.[3,5,9]

At zero temperature we obtain from (3)

$$J(t)=\frac{\hbar}{\pi}\int_0^{\infty} d\omega\, e^{-i\omega t}\chi''(\omega)\ . \qquad (4)$$

Using (1), the asymptotic expansion of $J(t)$ for large t is found as

$$J(t)=-(\hbar/\pi M\omega_0)\times[2\alpha(\omega_0 t)^{-2}+3\mu(\omega_0 t)^{-4}+O((\omega_0 t)^{-6})]\ , \qquad (5)$$

where

$$\alpha=\gamma/2\omega_0\ , \qquad (6)$$

$$\mu=16\alpha^3-8(1-\gamma')\alpha-\omega_0\gamma''\ , \qquad (7)$$

in which $\gamma=\gamma(\omega=0)$, $\gamma'=-i(\partial\gamma/\partial\omega)_{\omega=0}$, and $\gamma''=(\partial^2\gamma/\partial\omega^2)_{\omega=0}$. Hence, for a system with an Ohmic dissipative mechanism, i.e., $\gamma\neq 0$, the zero-temperature displacement correlation function has a long-time tail[3] $\propto t^{-2}$. On the other hand, if the frictional influence of the environment has no Ohmic part, $J(t)$ fades out more rapidly. For instance, a dissipative mechanism characterized in the classically accessible region by a frictional force $\eta\dddot q$ leads to the long-time behavior $J(t)=-(6\hbar\eta/\pi M)(\omega_0 t)^{-4}$.

At finite temperatures (3) gives

$$J(t)=\frac{\hbar}{\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\chi''(\omega)[1-\exp(-\beta\hbar\omega)]^{-1}\ . \qquad (8)$$

In the lower half-plane, $\chi''(\omega)$ has poles at $\omega=-i\lambda_j$. Now, classically $J(t)$ decays as $\exp(-\Omega t)$, where $\Omega=\min_j\{\mathrm{Re}\lambda_j\}$. For instance, for a system with Ohmic dissipation one has $\Omega=\gamma/2$ for $\alpha\leq 1$ and $\Omega=\omega_0[\alpha-(\alpha^2-1)^{1/2}]$ for $\alpha>1$. In the quantum regime, $k_BT \ll \hbar\Omega$, the long-time behavior of $J(t)$ arises from the poles of $[1-\exp(-\beta\hbar\omega)]^{-1}$ at $\omega=-in\nu$, where $\nu=2\pi k_BT/\hbar$. For large times $t\gg\Omega^{-1}$, we find

$$J(t)=-i(\hbar\nu/\pi)\sum_{n=1}^{N}\chi''(-in\nu)\exp(-n\nu t)\ , \qquad (9)$$

where N is of order Ω/ν and where terms of order $\exp(-\Omega t)$ have been disregarded. Expanding $\chi''(-in\nu)$ about the origin, we obtain by virtue of (6) and (7)

$$J(t)=-(\hbar/\pi M\omega_0)\times[2\alpha\psi_1(t)+3\mu\psi_2(t)+O(\theta^6 e^{-\nu t}/(1-e^{-\nu t})^6)]\ , \qquad (10)$$

where $\theta = 2\pi k_B T/\hbar\omega_0$ and where

$$\psi_1(t) = -\theta^2(\partial/\partial\tau)[\exp(\tau)-1]^{-1}\,,$$
$$\psi_2(t) = -\tfrac{1}{6}\theta^4(\partial^3/\partial\tau^3)[\exp(\tau)-1]^{-1}\,, \tag{11}$$

in which $\tau = \nu t$. For $T=0$ one has $\psi_1(t) = (\omega_0 t)^{-2}$, $\psi_2(t) = (\omega_0 t)^{-4}$, and we recover our previous result (5). On the contrary, for finite temperatures $\psi_1(t)$ and $\psi_2(t)$ decay asymptotically as

$$\psi_1(t) \simeq \theta^2 \exp(-\nu t)\,,$$
$$\psi_2(t) \simeq \tfrac{1}{6}\theta^4 \exp(-\nu t)\,, \tag{12}$$

leading to an exponential decay of $J(t)$ for $t >> \nu^{-1}$. On the other hand, for intermediate times $\Omega^{-1} << t << \nu^{-1}$ we obtain from (11)

$$\psi_1(t) \simeq (\omega_0 t)^{-2}, \quad \psi_2(t) \simeq (\omega_0 t)^{-4}\,. \tag{13}$$

Now, for $k_B T << \hbar\omega_0$ there is a finite time interval, where $\omega_0 t >> 1$, but $\nu t << 1$. Hence, for sufficiently low temperatures $k_B T << \hbar\Omega, \hbar\omega_0$, $J(t)$ displays an intermediate power-law decay before it crosses over near $t = \nu^{-1}$ to the asymptotic exponential decay. This behavior is illustrated in Fig. 1 for a system with Ohmic dissipation.

For a free Brownian particle it is convenient to consider the momentum correlation function $J_p(t) = \langle p(t)p\rangle$. By virtue of $J_p(t) = -M^2(\partial^2/\partial t^2)J(t)$ we obtain from (8)

$$J_p(t) = \frac{\hbar}{\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\chi_p''(\omega)[1-\exp(-\beta\hbar\omega)]^{-1}\,, \tag{14}$$

where $\chi_p''(\omega) = M^2\omega^2\chi''(\omega)$, which for a free Brownian particle takes the form

$$\chi_p''(\omega) = M\omega\,\mathrm{Re}(\gamma(\omega))[\omega^2 + |\gamma(\omega)|^2 - 2\omega\,\mathrm{Im}(\gamma(\omega))]^{-1}\,. \tag{15}$$

In the lower half-plane $\chi_p''(\omega)$ has poles at $\omega = -i\tilde{\lambda}_j$. Introducing $\tilde{\Omega} = \min_j\{\mathrm{Re}\tilde{\lambda}_j\}$, which is given by $\tilde{\Omega} = \gamma$ for Ohmic dissipation, we find in the quantum regime, $k_B T << \hbar\tilde{\Omega}$, for $\tilde{\Omega} t >> 1$,

$$J_p(t) = -(M\hbar\gamma/\pi) \times [\phi_1(t) + 3\mu_p\phi_2(t) + O(\kappa^6 e^{-\nu t}/(1-e^{-\nu t})^6)\,, \tag{16}$$

where $\kappa = 2\pi k_B T/\hbar\gamma$, $\mu_p = 2 + \gamma\gamma'' - 4\gamma' + 2\gamma'^2$, and where γ, γ', and γ'' have been introduced previously. The functions $\phi_1(t)$ and $\phi_2(t)$ take the same form as $\psi_1(t)$ and $\psi_2(t)$, except that θ is replaced by κ. In the time interval $\tilde{\Omega}^{-1} << t << \nu^{-1}$ we have $\phi_1(t) = (\gamma t)^{-2}$ and $\phi_2(t) = (\gamma t)^{-4}$. This gives rise to an intermediate power-law decay of $J_p(t)$ which for $T=0$ merges into a long-time tail.

It is worthwhile noting that the zero-temperature momentum correlation function of a free Brownian particle with Ohmic dissipation reads asymptotically for large times $J_p(t) \simeq -(\hbar M/\pi\gamma)t^{-2}$, while for a bound particle we have $J_p(t) \simeq (6\hbar M\gamma/\pi\omega_0^4)t^{-4}$. On the other hand, a strongly overdamped bound particle, i.e., $\gamma >> \omega_0$, behaves for $t << \omega_0^{-1}$ effectively like a free particle, and its momentum correlation function can easily be shown to take the form $J_p(t) \simeq -(\hbar M/\pi\gamma)t^{-2}$ in the time interval $\gamma^{-1} << t << \omega_0^{-1}$ before it crosses over to the t^{-4} behavior for larger times $t >> \omega_0^{-1}$. Accordingly, the displacement correlation function $J(t)$ displays an intermediate logarithmic decay $J(t) \simeq -(\hbar/\pi M\gamma)(C - \ln(2\alpha) + \ln(\omega_0 t))$, where C is the Euler constant which is followed by the asymptotic power-law decay $J(t) \simeq -(\hbar\gamma/\pi M\omega_0^2)(\omega_0 t)^{-2}$. This behavior is illustrated in Fig. 2. In the strongly overdamped case at fin-

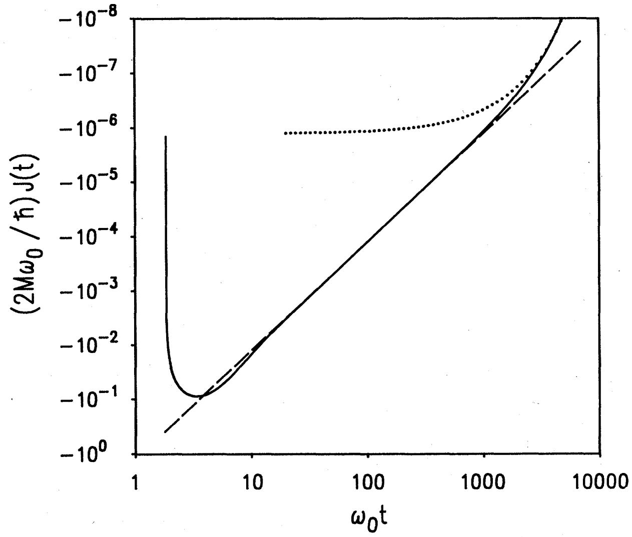

FIG. 1. The real part of the dimensionless displacement correlation function $(2M\omega_0/\hbar)J(t)$ is shown as a function of $\omega_0 t$ for a harmonic oscillator with Ohmic damping $\gamma = 2\omega_0$ (i.e., $\alpha = 1$) at temperature $T = 10^{-3}\hbar\omega_0/2\pi k_B$ (i.e., $\theta = 10^{-3}$). The intermediate power-law decay $(2M\omega_0/\hbar)J(t) \simeq -(4\alpha/\pi)(\omega_0 t)^{-2}$ is shown as a dashed line, and the asymptotic decay $(2M\omega_0/\hbar)J(t) \simeq -(4\alpha/\pi)\theta^2\exp(-\nu t)$ as a dotted line. Note that the real part of $J(t)$ is negative for large $\omega_0 t$ so that $J(t)$ approaches 0 from below as $t \to \infty$.

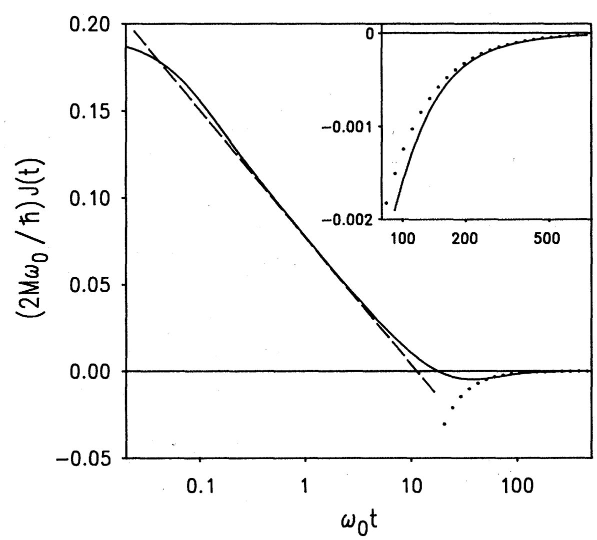

FIG. 2. Same as Fig. 1 for an overdamped harmonic oscillator with $\gamma = 20\omega_0$ (i.e., $\alpha = 10$) at zero temperature. The intermediate logarithmic decay $(2M\omega_0/\hbar)J(t) \simeq -(1/\pi\alpha)(C - \ln(2\alpha) + \ln(\omega_0 t))$ is shown as a dashed line, and the asymptotic power-law decay $(2M\omega_0/\hbar)J(t) \simeq -(4\alpha/\pi)(\omega_0 t)^{-2}$ as a dotted line. Note that the correlation function is shown on a linear scale as distinguished from Fig. 1.

ite but low temperatures one finds a region where $J(t)$ decays logarithmically, followed by a region where the decay is algebraic $\propto t^{-2}$ before the asymptotic exponential decay $\propto \exp(-\nu t)$ sets in.

So far we have considered linear systems only. However, the long-time tails of zero-temperature correlation functions are a rather general feature of dissipative quantum systems for the following reason. As a consequence of the fluctuation-dissipation theorem, spectral densities of correlation functions have poles at the Matsubara frequencies $\omega = n\nu$ which come closer together as T is lowered and give rise to a cut contribution at zero temperature. This cut determines the long-time behavior of the correlation function and leads to power-law decay. For instance, the result (5) likewise applies to a particle moving in a stable nonlinear potential provided that $\gamma(\omega)$ and ω_0 are replaced by the corresponding renormalized quantities.[10] The discussion of the behavior at finite but low temperatures can be extended to the nonlinear case accordingly. However, it must be noted that the determination of the low-frequency behavior of the response functions, which determines the amplitudes of the asymptotic decay laws derived here, may itself be a complicated problem for a nonlinear system near $T=0$.

We would like to thank U. Weiss and P. Schramm for valuable discussions and acknowledge partial support by the Deutsche Forschungsgemeinschaft.

[1]R. H. Koch, D. L. van Harlingen, and J. Clarke, Phys. Rev. Lett. **45**, 2132 (1980); A. Schmid, J. Low Temp. Phys. **49**, 609 (1982).

[2]A. O. Caldeira and A. J. Leggett, Ann. Phys. (N.Y.) **149**, 374 (1983); Physica A **121**, 587 (1983).

[3]H. Grabert, U. Weiss, and P. Talkner, Z. Phys. B **55**, 87 (1984).

[4]P. Schramm, R. Jung, and H. Grabert, Phys. Lett. **107A**, 385 (1985).

[5]G. W. Ford, M. Kac, and P. Mazur, J. Math. Phys. **6**, 504 (1965); P. Ullersma, Physica **32**, 27 (1966); **32**, 56 (1966); **32**, 74 (1966); **32**, 90 (1966).

[6]K. Lindenberg and B. J. West, Phys. Rev. A **30**, 568 (1984).

[7]P. S. Riseborough, P. Hänggi, and U. Weiss, Phys. Rev. A **31**, 471 (1985); F. Haake and R. Reibold, Acta Phys. Austriaca **56**, 37 (1985).

[8]H. B. Callen and T. A. Welton, Phys. Rev. **83**, 34 (1951).

[9]P. Talkner, Z. Phys. B **41**, 365 (1981).

[10]K. S. J. Nordholm and R. Zwanzig, J. Stat. Phys. **11**, 143 (1974); H. Grabert, in *Projection Operator Techniques in Nonequilibrium Statistical Mechanics,* edited by G. Höhler, Springer Tracts in Modern Physics, Vol. 95 (Springer-Verlag, Berlin, 1982).