

Long-time tails in quantum Brownian motion

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The authors address the problem of quantum Brownian motion at low temperatures and with arbitrarily strong damping. For a harmonically bound particle the zero-temperature correlation functions are shown to display long-time tails. At finite temperatures a power-law decay at intermediate times is followed by an exponential decay with time constant $\hbar/2\pi k_B T$. The case of free Brownian motion is treated, and some general conclusions for nonlinear systems are drawn.

It has become clear recently¹⁻⁷ that even a linear system exhibits interesting quantum effects when coupled to a low-temperature heat bath. The simplest microscopic model to study these phenomena is a particle of mass M moving in a harmonic potential $V(q) = \frac{1}{2}M\omega_0^2 q^2$ while coupled to a heat bath environment consisting of harmonic oscillators. Several authors have examined this model and have shown how the irreversible motion of the Brownian particle arises.^{2,5} In a previous work³ we have derived various exact results for a quantum harmonic oscillator with Ohmic dissipation. In particular, a power-law decay of zero-temperature correlation functions was noted. In this work we present a more detailed study of the long-time behavior of correlation functions at low temperatures for a harmonically bound particle. Furthermore, the limit of free Brownian motion is examined and some general conclusions for nonlinear systems are drawn. As a characteristic feature of dissipative quantum systems, a power-law decay of zero-temperature correlation functions is found.

Dealing first with a strictly linear system,² the dynamical susceptibility $\chi(\omega)$ characterizing the linear response of $\langle q(t) \rangle$ to an external force takes the classical form^{3,7}

$$\chi(\omega) = M^{-1}[\omega_0^2 - \omega^2 - i\omega\gamma(\omega)]^{-1} = \chi'(\omega) + i\chi''(\omega), \quad (1)$$

where $\gamma(\omega)$ is the frequency-dependent damping coefficient. By virtue of the fluctuation-dissipation theorem⁸ the imaginary part $\chi''(\omega)$ of the dynamic susceptibility is related to the spectral density

$$J(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \langle q(t)q \rangle \quad (2)$$

of displacement fluctuations in thermal equilibrium according to

$$J(\omega) = 2\hbar [1 - \exp(-\beta\hbar\omega)]^{-1} \chi''(\omega), \quad (3)$$

where $\beta = 1/k_B T$. Because of $\dot{q} = p/M$, where p is the momentum, all other pair correlations are obtained from $J(t)$ as time derivatives, while higher-order correlations can be factorized into pair correlations due to the Gaussian property of the process.^{3,5,9}

At zero temperature we obtain from (3)

$$J(t) = \frac{\hbar}{\pi} \int_0^{\infty} d\omega e^{-i\omega t} \chi''(\omega). \quad (4)$$

Using (1), the asymptotic expansion of $J(t)$ for large t is found as

$$J(t) = -(\hbar/\pi M\omega_0) \times [2\alpha(\omega_0 t)^{-2} + 3\mu(\omega_0 t)^{-4} + O((\omega_0 t)^{-6})], \quad (5)$$

where

$$\alpha = \gamma/2\omega_0, \quad (6)$$

$$\mu = 16\alpha^3 - 8(1 - \gamma')\alpha - \omega_0\gamma'', \quad (7)$$

in which $\gamma = \gamma(\omega = 0)$, $\gamma' = -i(\partial\gamma/\partial\omega)_{\omega=0}$, and $\gamma'' = (\partial^2\gamma/\partial\omega^2)_{\omega=0}$. Hence, for a system with an Ohmic dissipative mechanism, i.e., $\gamma \neq 0$, the zero-temperature displacement correlation function has a long-time tail³ $\propto t^{-2}$. On the other hand, if the frictional influence of the environment has no Ohmic part, $J(t)$ fades out more rapidly. For instance, a dissipative mechanism characterized in the classically accessible region by a frictional force $\eta\dot{q}$ leads to the long-time behavior $J(t) = -(6\hbar\eta/\pi M)(\omega_0 t)^{-4}$.

At finite temperatures (3) gives

$$J(t) = \frac{\hbar}{\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \chi''(\omega) [1 - \exp(-\beta\hbar\omega)]^{-1}. \quad (8)$$

In the lower half-plane, $\chi''(\omega)$ has poles at $\omega = -i\lambda_j$. Now, classically $J(t)$ decays as $\exp(-\Omega t)$, where $\Omega = \min_j \{\text{Re}\lambda_j\}$. For instance, for a system with Ohmic dissipation one has $\Omega = \gamma/2$ for $\alpha \leq 1$ and $\Omega = \omega_0[\alpha - (\alpha^2 - 1)^{1/2}]$ for $\alpha > 1$. In the quantum regime, $k_B T \ll \hbar\Omega$, the long-time behavior of $J(t)$ arises from the poles of $[1 - \exp(-\beta\hbar\omega)]^{-1}$ at $\omega = -in\nu$, where $\nu = 2\pi k_B T/\hbar$. For large times $t \gg \Omega^{-1}$, we find

$$J(t) = -i(\hbar\nu/\pi) \sum_{n=1}^N \chi''(-in\nu) \exp(-n\nu t), \quad (9)$$

where N is of order Ω/ν and where terms of order $\exp(-\Omega t)$ have been disregarded. Expanding $\chi''(-in\nu)$ about the origin, we obtain by virtue of (6) and (7)

$$J(t) = -(\hbar/\pi M\omega_0) \times [2\alpha\psi_1(t) + 3\mu\psi_2(t) + O(\theta^6 e^{-\nu t}/(1 - e^{-\nu t})^6)], \quad (10)$$

where $\theta = 2\pi k_B T / \hbar \omega_0$ and where

$$\psi_1(t) = -\theta^2 (\partial/\partial\tau) [\exp(\tau) - 1]^{-1}, \quad (11)$$

$$\psi_2(t) = -\frac{1}{6} \theta^4 (\partial^3/\partial\tau^3) [\exp(\tau) - 1]^{-1},$$

in which $\tau = \nu t$. For $T=0$ one has $\psi_1(t) = (\omega_0 t)^{-2}$, $\psi_2(t) = (\omega_0 t)^{-4}$, and we recover our previous result (5). On the contrary, for finite temperatures $\psi_1(t)$ and $\psi_2(t)$ decay asymptotically as

$$\psi_1(t) \simeq \theta^2 \exp(-\nu t), \quad (12)$$

$$\psi_2(t) \simeq \frac{1}{6} \theta^4 \exp(-\nu t),$$

leading to an exponential decay of $J(t)$ for $t \gg \nu^{-1}$. On the other hand, for intermediate times $\Omega^{-1} \ll t \ll \nu^{-1}$ we obtain from (11)

$$\psi_1(t) \simeq (\omega_0 t)^{-2}, \quad \psi_2(t) \simeq (\omega_0 t)^{-4}. \quad (13)$$

Now, for $k_B T \ll \hbar \omega_0$ there is a finite time interval, where $\omega_0 t \gg 1$, but $\nu t \ll 1$. Hence, for sufficiently low temperatures $k_B T \ll \hbar \Omega, \hbar \omega_0$, $J(t)$ displays an intermediate power-law decay before it crosses over near $t = \nu^{-1}$ to the asymptotic exponential decay. This behavior is illustrated in Fig. 1 for a system with Ohmic dissipation.

For a free Brownian particle it is convenient to consider the momentum correlation function $J_p(t) = \langle p(t)p \rangle$. By virtue of $J_p(t) = -M^2 (\partial^2/\partial t^2) J(t)$ we obtain from (8)

$$J_p(t) = \frac{\hbar}{\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \chi_p''(\omega) [1 - \exp(-\beta \hbar \omega)]^{-1}, \quad (14)$$

where $\chi_p''(\omega) = M^2 \omega^2 \chi''(\omega)$, which for a free Brownian par-

ticle takes the form

$$\chi_p''(\omega) = M \omega \operatorname{Re}(\gamma(\omega)) [\omega^2 + |\gamma(\omega)|^2 - 2\omega \operatorname{Im}(\gamma(\omega))]^{-1}. \quad (15)$$

In the lower half-plane $\chi_p''(\omega)$ has poles at $\omega = -i\tilde{\lambda}_j$. Introducing $\tilde{\Omega} = \min_j \{\operatorname{Re}\tilde{\lambda}_j\}$, which is given by $\tilde{\Omega} = \gamma$ for Ohmic dissipation, we find in the quantum regime, $k_B T \ll \hbar \tilde{\Omega}$, for $\tilde{\Omega} t \gg 1$,

$$J_p(t) = - (M \hbar \gamma / \pi) \times [\phi_1(t) + 3\mu_p \phi_2(t) + O(\kappa^6 e^{-\nu t} / (1 - e^{-\nu t})^6)], \quad (16)$$

where $\kappa = 2\pi k_B T / \hbar \gamma$, $\mu_p = 2 + \gamma \gamma'' - 4\gamma' + 2\gamma'^2$, and where γ , γ' , and γ'' have been introduced previously. The functions $\phi_1(t)$ and $\phi_2(t)$ take the same form as $\psi_1(t)$ and $\psi_2(t)$, except that θ is replaced by κ . In the time interval $\tilde{\Omega}^{-1} \ll t \ll \nu^{-1}$ we have $\phi_1(t) = (\gamma t)^{-2}$ and $\phi_2(t) = (\gamma t)^{-4}$. This gives rise to an intermediate power-law decay of $J_p(t)$ which for $T=0$ merges into a long-time tail.

It is worthwhile noting that the zero-temperature momentum correlation function of a free Brownian particle with Ohmic dissipation reads asymptotically for large times $J_p(t) \simeq -(\hbar M / \pi \gamma) t^{-2}$, while for a bound particle we have $J_p(t) \simeq (6\hbar M \gamma / \pi \omega_0^4) t^{-4}$. On the other hand, a strongly overdamped bound particle, i.e., $\gamma \gg \omega_0$, behaves for $t \ll \omega_0^{-1}$ effectively like a free particle, and its momentum correlation function can easily be shown to take the form $J_p(t) \simeq -(\hbar M / \pi \gamma) t^{-2}$ in the time interval $\gamma^{-1} \ll t \ll \omega_0^{-1}$ before it crosses over to the t^{-4} behavior for larger times $t \gg \omega_0^{-1}$. Accordingly, the displacement correlation function $J(t)$ displays an intermediate logarithmic decay $J(t) \simeq -(\hbar / \pi M \gamma) (C - \ln(2\alpha) + \ln(\omega_0 t))$, where C is the Euler constant which is followed by the asymptotic power-law decay $J(t) \simeq -(\hbar \gamma / \pi M \omega_0^2) (\omega_0 t)^{-2}$. This behavior is illustrated in Fig. 2. In the strongly overdamped case at fin-

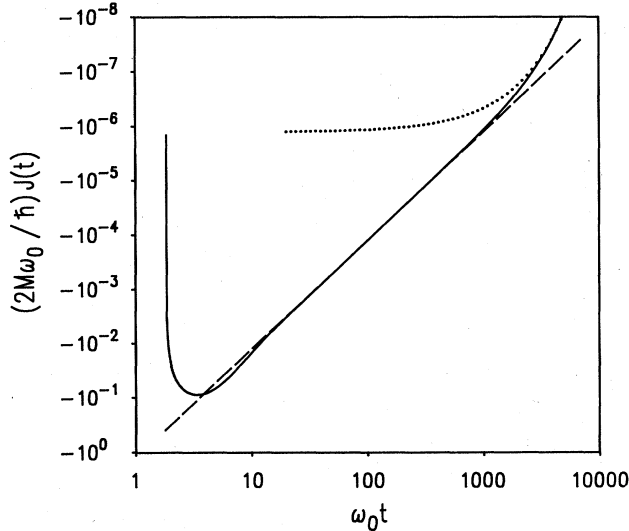


FIG. 1. The real part of the dimensionless displacement correlation function $(2M\omega_0/\hbar)J(t)$ is shown as a function of $\omega_0 t$ for a harmonic oscillator with Ohmic damping $\gamma = 2\omega_0$ (i.e., $\alpha = 1$) at temperature $T = 10^{-3} \hbar \omega_0 / 2\pi k_B$ (i.e., $\theta = 10^{-3}$). The intermediate power-law decay $(2M\omega_0/\hbar)J(t) \simeq -(4\alpha/\pi)(\omega_0 t)^{-2}$ is shown as a dashed line, and the asymptotic decay $(2M\omega_0/\hbar)J(t) \simeq -(4\alpha/\pi)\theta^2 \exp(-\nu t)$ as a dotted line. Note that the real part of $J(t)$ is negative for large $\omega_0 t$ so that $J(t)$ approaches 0 from below as $t \rightarrow \infty$.

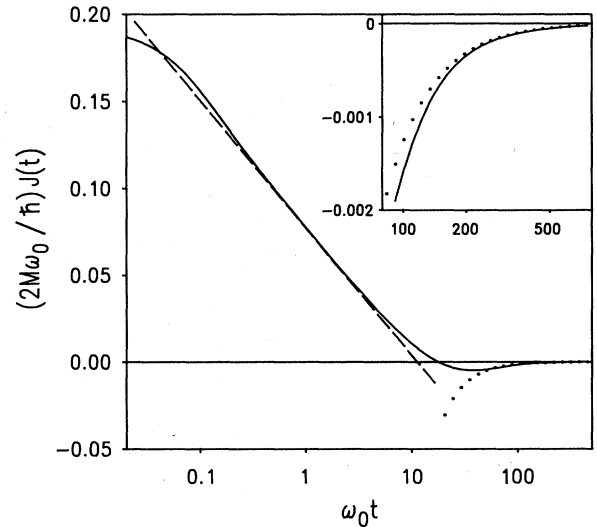


FIG. 2. Same as Fig. 1 for an overdamped harmonic oscillator with $\gamma = 20\omega_0$ (i.e., $\alpha = 10$) at zero temperature. The intermediate logarithmic decay $(2M\omega_0/\hbar)J(t) \simeq -(1/\pi\alpha)(C - \ln(2\alpha) + \ln(\omega_0 t))$ is shown as a dashed line, and the asymptotic power-law decay $(2M\omega_0/\hbar)J(t) \simeq -(4\alpha/\pi)(\omega_0 t)^{-2}$ as a dotted line. Note that the correlation function is shown on a linear scale as distinguished from Fig. 1.

ite but low temperatures one finds a region where $J(t)$ decays logarithmically, followed by a region where the decay is algebraic $\propto t^{-2}$ before the asymptotic exponential decay $\propto \exp(-\nu t)$ sets in.

So far we have considered linear systems only. However, the long-time tails of zero-temperature correlation functions are a rather general feature of dissipative quantum systems for the following reason. As a consequence of the fluctuation-dissipation theorem, spectral densities of correlation functions have poles at the Matsubara frequencies $\omega = n\nu$ which come closer together as T is lowered and give rise to a cut contribution at zero temperature. This cut determines the long-time behavior of the correlation function and leads to power-law decay. For instance, the result

(5) likewise applies to a particle moving in a stable nonlinear potential provided that $\gamma(\omega)$ and ω_0 are replaced by the corresponding renormalized quantities.¹⁰ The discussion of the behavior at finite but low temperatures can be extended to the nonlinear case accordingly. However, it must be noted that the determination of the low-frequency behavior of the response functions, which determines the amplitudes of the asymptotic decay laws derived here, may itself be a complicated problem for a nonlinear system near $T=0$.

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