Single electron tunneling rates in multijunction circuits

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The rate of electron tunneling through normal metal tunnel junctions is calculated for the case of ultrasmall junction capacitances. The so-called Coulomb blockade of electron tunneling at low temperatures is shown to be strongly affected by the external electrical circuit. Under the common experimental condition of a low impedance environment the Coulomb blockade is suppressed for single tunnel junctions. However, a Coulomb gap structure emerges for junctions embedded in a high impedance environment. For a double junction setup a Coulomb blockade of tunneling arises even for low impedance environments due to the charge quantization on the metallic island between the junctions. An approach using circuit analysis is presented which allows to reduce the calculation of tunneling rates in multijunction circuits to those of a single junction in series with an effective capacitance. The range of validity of the so-called local rule and global rule rates is clarified. It is found that the tunneling rate tends towards the global rule rate as the number of junctions is increased. Some specific results are given for a one-dimensional array of tunnel junctions.

1. Introduction

For ultrasmall tunnel junctions with capacitances in the fF range or below the charging energy $e^2/2C$ of a single electron can easily exceed the energy $k_B T$ of thermal fluctuations at sub Kelvin temperatures. In this situation new phenomena such as the Coulomb blockade of tunneling can arise. In fact, charging effects were observed many years ago on tunnel junctions containing metal grains within the barrier [1] and these effects were studied theoretically quite extensively in the last decade [2]. While a large body of experimental work [3] supports most of the theoretical predictions, it is only now that a clear physical picture is emerging. It is important to note [4–9] that charging effects in tunnel junctions are strongly affected by the electrical circuit loading the junction. In fact, the influence of the electromagnetic environment on single electron tunneling is reminiscent of the role played by the crystal lattice in the Mössbauer effect [7]. When addressing the problem of charge tunneling in an ultrasmall junction it is thus essential to treat the coupled system formed by the tunnel junction and its electromagnetic environment. Moreover, in a circuit composed of several junctions the question arises as to whether the rate of tunneling through one of the junctions depends only on the parameters and on the voltage across this junction or if it depends also on the rest of the circuit. Two simple rate formulas have been proposed in the literature [2, 10]. According to the so-called local rule the energy governing the tunneling rate is the difference of electrostatic energy of the junction capacitor immediately before and after the tunnel event. In contrast, according to the global rule the tunneling rate is governed by the difference in energy of the equilibrium configuration of the whole circuit before and after the tunnel event. In this paper we calculate the tunneling rates of multijunction circuits for a general electrodynamic environment. We also specify the conditions for the validity of simplified rate formulas.

A microscopic approach to electron transfer rates through tunnel barriers in the presence of an electromagnetic field propagating in a specified geometry was put forward by Nazarov [5]. In another approach by Devoret et al. [7] the effect of the external electrical circuit is expressed directly in terms of the environmental impedance function. This allows for a straightforward calculation of measurable quantities under realistic experimental conditions [7, 9]. Here, we present some results of our earlier work [7] in greater detail and extend the theory to the case of multijunction circuits. A short summary of these results was given in [11].

In the following, we first consider the case of a single tunnel junction. Section 2 deals with the effect of the external electrical circuit on the quantum mechanical behavior of the charge on the junction capacitance in the absence of tunneling through the junction barrier. In Sect. 3 we then introduce the tunneling Hamiltonian and calculate the rates for electron transitions between the electrodes of a single tunnel junction. This problem is treated for arbitrary environmental impedance and temperature. Apart from general results, two limiting cases where the electromagnetic environment has a very low or high impedance are presented. It is shown that the Coulomb blockade of tunneling is washed out by quantum fluctuations of the charge except for very high impedance environments. In Sect. 4 we then consider a double junction. There the quantized charge on the metallic island between the junctions introduces new features and causes a Coulomb blockade of tunneling even for low impedance environments. In Sect. 5 we apply network theory to show that the double junction results may be easily explained in terms of a single junction in series with an effective capacitance. This reduction scheme naturally extends to more complicated multi-junction circuits. In Sect. 6 the theory is illustrated by applying it to a one-dimensional array of tunnel junctions. Finally, in Sect. 7 we present our conclusions.

2. The charge on the junction capacitance

Let us first consider a single junction of capacitance C embedded in an electromagnetic environment. We first suppose there is no electron tunneling through the junction barrier. Since we are interested in phenomena occurring at low temperatures and for low voltages, only the properties of the electromagnetic environment well below the plasma frequency will be important. We shall assume that the environment is linear, that is there are no further tunnel junctions in the circuit. The environment can then be modelled by an impedance $Z(\omega)$ in series with a voltage source (cf. Fig. 1). This model is appropriate even for the case of a current-biased junction with a capacitance in the fF range [7]. The leads attached to the junction have capacitances that always exceed the junction capacitance by several orders of magnitude. These parasitic capacitances, which are polarized by the average voltage across the junction, will act as a voltage source. The electromagnetic environment as seen from the junction can then effectively be described by the model depicted in Fig. 1.

From a phenomenological point of view the dynamical behavior of the capacitance C in parallel with the external impedance $Z(\omega)$ can be described in terms of the charge Q on the junction capacitance. Q is the surface charge on the junction electrodes arising from a displacement of the metallic electrons on either side of the junction with respect to the background of opposite charge formed by the metal ions. Thus, Q is a collective or macroscopic variable. Since the junction electrodes are connected to an external circuit, small displacements of the electron clouds may lead to arbitrarily small changes of Q. For instance, $Q=e$ for a 100 nm × 100 nm junction corresponds to a displacement of the order of 1 fermi. Hence, Q has a continuous spectrum.

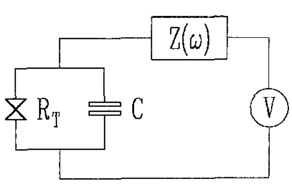

Fig. 1. Model of a tunnel junction with tunneling resistance R_T and capacitance C used in the theory. The junction is coupled to an environment with an impedance $Z(\omega)$ and an ideal voltage source

Assume now that at time $t=0$ the external voltage applied to the junction is suddenly changed from V_0 to V. In the absence of tunneling, the charge on the junction capacitance then relaxes from its initial value $Q_0=CV_0$ towards its new steady state value $Q(\infty)=CV$ according to

$$Q(t)=CV+C(V_0-V)\,R(t). \tag{1}$$

The charge relaxation function $R(t)$ is given in terms of its Fourier transform

$$\tilde{R}(\omega)=\int_0^{\infty} \mathrm{d}t\, e^{-i\omega t}\, R(t)=CZ_t(\omega) \tag{2}$$

where

$$Z_t(\omega)=\frac{1}{i\omega C+Z^{-1}(\omega)} \tag{3}$$

is the total impedance of the capacitance C in parallel with the environmental impedance $Z(\omega)$.

So far the electrical circuit was treated classically. In the absence of electron tunneling through the junction barrier the corresponding quantum mechanical treatment is straightforward. Since the equation of motion for the charge is linear, the quantum mechanical response of the average charge to a voltage perturbation coincides with the corresponding classical response as a consequence of Ehrenfest's theorem. The quantum mechanical correlation function can then readily be determined with the fluctuation-dissipation theorem, provided the electromagnetic environment is in thermal equilibrium. Alternatively, we may write down a Caldeira-Leggett Hamiltonian [12] for the environment with a spectral density determined by the impedance. The variable canonically conjugate to the charge Q is the phase φ which is defined in terms of the voltage U across the junction by [13]

$$\varphi(t)=\frac{e}{\hbar}\int_{-\infty}^{t} U(t')\,\mathrm{d}t' \tag{4}$$

where e is the elementary charge. The variables Q and φ obey the commutation relation

$$[\varphi, Q]=ie. \tag{5}$$

We note that the quantum mechanics of a capacitance attached to an electrical circuit is closely related to the problem of quantum Brownian motion with frequency dependent damping [14]. The charging energy corresponds to the kinetic energy of the Brownian particle and the electromagnetic environment plays the role of the heat bath. It can now easily be shown that the classical mean relaxation law (1) is connected with spontaneous quantum mechanical charge fluctuations on the junction capacitance described by the correlation function

$$\langle \delta Q(t)\, \delta Q(0)\rangle = C^2 \int_0^\infty \frac{d\omega}{\pi}\, \hbar\omega\, \mathrm{Re}[Z_t(\omega)] \cdot [\coth(\tfrac{1}{2}\beta\hbar\omega)\cos(\omega t) - i\sin(\omega t)] \quad (6)$$

where $\delta Q(t) = Q(t) - \langle Q(t)\rangle$ and where $\beta = 1/k_B T$ is the inverse temperature of the environment. For later purposes we also give the result for the correlation function of phase fluctuations in the steady state

$$J(t) = \langle [\varphi(t) - \varphi(0)]\, \varphi(0)\rangle. \quad (7)$$

To avoid clumsy formulae in the sequel the phase fluctuation will also be denoted by $\varphi(t)$. Using $C\dot{\varphi} = (e/\hbar) Q$ we find

$$J(t) = \int_0^\infty \frac{d\omega}{\omega} \frac{\mathrm{Re}[Z_t(\omega)]}{R_Q} (\coth(\tfrac{1}{2}\beta\hbar\omega)[\cos(\omega t) - 1] - i\sin(\omega t)) \quad (8)$$

where

$$R_Q = \frac{h}{2e^2} \quad (9)$$

is the resistance quantum.

3. The tunneling Hamiltonian and electron tunneling rates

A. The rate formula for single junctions

We now take into account the tunneling of electrons across the junction which has been neglected so far. The Hamiltonian of the entire system formed by a tunnel junction and the external electrical circuit may be written as

$$H = H_0 + H_T. \quad (10)$$

Here,

$$H_0 = H_{qp} + H_{em} \quad (11)$$

is the Hamiltonian in the absence of tunneling of electrons through the junction barrier. It consists of the usual quasiparticle Hamiltonian H_{qp} for the two junction electrodes and the electromagnetic Hamiltonian H_{em}. This latter part may explicitly been written as a Caldeira-Leggett Hamiltonian [12] with Q as the macroscopic variable. It is important to note that although the charge operator and quasiparticle creation and annihilation operators can eventually be expressed in terms of true electron operators, they commute in the limit where the electron density is large or, equivalently, when Q is small enough that quasiparticle states are not affected by the electron mean displacements. In the sequel such microscopic considerations will not be persued and we shall no longer distinguish between quasiparticles and electrons. The second part in (10)

$$H_T = \sum_{\sigma k_1 k_2} T_{k_2 k_1} c^\dagger_{k_2\sigma} c_{k_1\sigma} \exp(-i\varphi) + \text{H.c.} \quad (12)$$

is the tunneling Hamiltonian [7, 15, 16]. Here, $T_{k_2 k_1} c^\dagger_{k_2\sigma} c_{k_1\sigma}$ is the usual tunneling term [17] which annihilates an electron with wave vector k_1 on side "1" of the barrier and creates an electron with wave vector k_2 on side "2". σ denotes the spin quantum number. The term $\exp(-i\varphi)$ describes the change of the charge Q by one elementary charge e as a consequence of the tunneling event. In fact, we have

$$\exp(i\varphi)\, Q \exp(-i\varphi) = Q - e \quad (13)$$

which follows from the commutation relation (5). The charge shift operators in (12) couple the tunneling transitions to the electromagnetic environment. From a more microscopic point of view, the factor $\exp(-i\varphi)$ can be explained as a modification of the transition amplitude $T_{k_2 k_1}$ due to the electric field in the barrier region caused by the electromagnetic environment.

The tunneling Hamiltonian transfers charges between the electrodes "1" and "2" of the junction. The corresponding tunneling rates will be denoted by $\vec{\Gamma}$ and $\overleftarrow{\Gamma}$, respectively. To calculate these rates we shall treat the tunneling Hamiltonian as a small perturbation. This approach will turn out to be sufficient if the tunneling resistance R_T introduced below is large compared with the resistance quantum R_Q. Before the tunneling transition occurs the system is assumed to have reached the equilibrium state of H_0 at a given voltage V and environmental temperature T. The Fermi level of electrode "2" is then shifted with respect to the Fermi level of electrode "1" by $-eV$. We choose the signs of electrical variables such that a positive voltage will favor transitions from "1" to "2" described by $\vec{\Gamma}$. These transitions are caused by the terms in H_T that are explicitly written out in (12). The operators $c_{k_1\sigma}$ and $c^\dagger_{k_2\sigma}$ transfer an electron and thereby disturb the equilibrium state of the junction electrodes while the term $\exp(-i\varphi)$ disturbs the equilibrium between the charge distribution on the junction capacitance and the external circuit. Now, proceeding along the usual lines, we obtain to second order in H_T the golden rule result

$$\vec{\Gamma}(V) = \frac{1}{\pi\hbar} \frac{R_Q}{R_T} \int_{-\infty}^{+\infty} dE \int_{-\infty}^{+\infty} dE' f(E)[1 - f(E')] \cdot P(E + eV - E'). \quad (14)$$

Here, $1/R_T$ is the usual tunneling conductance [17] which is proportional to $|T|^2$ and to the densities of electronic states on either side of the junction. $f(E) = [1 + \exp(\beta E)]^{-1}$ is the Fermi function. In formula (14) the initial energy E of the tunneling electron in electrode "1" and its final energy E' in electrode "2" are measured relative to the Fermi energies. Of course, E' would equal $E+eV$ if the equilibrium between the charge distribution on the capacitance and the external circuit were not disturbed. However, the charge shift operators in H_T affect the coupled system formed by the capacitance and the external circuit. This leads to the appearance of the function [7]

$$P(E)=\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} \mathrm{d}t \exp(\mathrm{i}Et/\hbar) \cdot\langle \exp[\mathrm{i}\varphi(t)] \exp[-\mathrm{i}\varphi(0)]\rangle \tag{15}$$

which can be interpreted as the probability that a tunneling electron looses the energy E to the electromagnetic environment.

Since the fluctuations of φ are Gaussian, we have

$$\langle \exp[\mathrm{i}\varphi(t)] \exp[-\mathrm{i}\varphi(0)]\rangle = \exp[J(t)] \tag{16}$$

where $J(t)$ is the phase correlation function introduced previously. We thus obtain

$$P(E)=\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} \mathrm{d}t \exp[J(t)+\mathrm{i}Et/\hbar]. \tag{17}$$

In connection with (3) and (8), this relation determines $P(E)$ for arbitrary environmental impedance and temperature. Using properties of the Fermi function, the result (14) may also be written as

$$\vec{\Gamma}(V)=\frac{1}{\pi\hbar}\frac{R_Q}{R_T}\int_{-\infty}^{+\infty} \mathrm{d}E \frac{E}{1-\exp(-\beta E)} P(eV-E). \tag{18}$$

The rate for transitions in the opposite direction is given by

$$\overleftarrow{\Gamma}(V)=\vec{\Gamma}(-V). \tag{19}$$

Since $P(E)$ satisfies the detailed balance symmetry $P(E)=\exp(\beta E)P(-E)$, we find that the forward and backward rates are also related by

$$\overleftarrow{\Gamma}(V)=\exp(-\beta eV)\vec{\Gamma}(V). \tag{20}$$

In general, these rates have to be calculated numerically. Here, we shall first consider two limiting cases.

B. High- and low-impedance environments

The phase correlation function $J(t)$ introduced in (8) depends on the impedance ratio $\mathrm{Re}[Z_t(\omega)]/R_Q$. Hence, for a low-impedance environment we find approximately $P(E)=\delta(E)$. In this limit a tunneling electron cannot exchange energy with the environment and (18) reduces to

$$\vec{\Gamma}(V)=\frac{1}{\pi\hbar}\frac{R_Q}{R_T}\frac{eV}{1-\exp(-\beta eV)}, \tag{21}$$

which is the familiar result obtained in the conventional treatment where the coupling to the electromagnetic environment is neglected [17]. Hence, a very low-impedance environment influences the dynamics of the tunnel junction like a voltage bias.

In the opposite limit energy can very easily be exchanged between the tunneling particle and the electromagnetic environment. This is the case when the function $\mathrm{Re}[Z_t(\omega)]$, which determines the spectral density of spontaneous fluctuations of Q and φ, is concentrated at low frequencies. In the limit of an high-impedance environment the spectral density is in fact sharply peaked at $\omega=0$ and the short time expansion of (8)

$$J(t)=-\frac{\pi}{2CR_Q}\left(\mathrm{i}t+\frac{1}{\hbar\beta}t^2\right) \tag{22}$$

is valid for all times. Now, (22) combines with (17) to give

$$P(E)=\frac{1}{\sqrt{4\pi E_c k_B T}} \exp[-(E-E_c)^2/4E_c k_B T] \tag{23}$$

where

$$E_c=e^2/2C \tag{24}$$

is the single electron charging energy. Hence, in the high-impedance limit the tunneling rate reads

$$\vec{\Gamma}(V)=\frac{1}{\pi\hbar}\frac{R_Q}{R_T}\int_{-\infty}^{+\infty} \mathrm{d}E \frac{E}{1-\exp(-\beta E)} \cdot\frac{\exp[-(eV-E-E_c)^2/4E_c k_B T]}{\sqrt{4\pi E_c k_B T}} \tag{25}$$

which is seen to reduce at zero temperature to

$$\vec{\Gamma}(V)=\frac{1}{\pi\hbar}\frac{R_Q}{R_T}(eV-E_c)\,\Theta(eV-E_c) \quad \text{for } T=0 \tag{26}$$

where $\Theta(x)$ is the unit step function. The result (26) describes the so-called Coulomb blockade of tunneling, since $\vec{\Gamma}(V)$ vanishes at $T=0$ even for finite positive voltages $V<e/2C$.

C. The I – V characteristic

For small currents the $I-V$ characteristic is related to the tunneling rates by

$$I(V)=e[\vec{\Gamma}(V)-\overleftarrow{\Gamma}(V)] =\frac{1}{\pi\hbar}\frac{R_Q}{R_T}\int_{-\infty}^{+\infty} \mathrm{d}E\cdot E\,\frac{1-\exp(-\beta eV)}{1-\exp(-\beta E)} P(eV-E). \tag{27}$$

This result is only valid provided the system can relax to the steady state between subsequent tunneling transitions. Hence, we should have $I \ll e/\tau_c$ where τ_c is the relaxation time. The full rate (18) will lie between the limiting results (21) and (25). Quite generally, the function $P(E)$ satisfies the sum rules [8, 9]

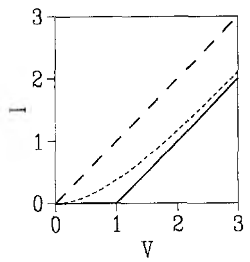

Fig. 2. The current-voltage characteristic of a single tunnel junction at zero temperature. The long-dashed line shows the Ohmic $I-V$ curve for a very low impedance environment and the solid line shows the Coulomb gap structure for a high impedance environment. The short-dashed line depicts an intermediate case with environmental impedance $Z(\omega)=R_Q$. The voltage V is given in units of $e/2C$ and the current I in units of $e/2CR_T$

$$\int_{-\infty}^{+\infty} \mathrm{d}E\, P(E)=1, \qquad \int_{-\infty}^{+\infty} \mathrm{d}E\, E P(E)=E_c. \tag{28}$$

Using these properties one finds that the rate (18) always approaches (26) for large voltages. However, in the interesting region of voltages of the order E_c/e the detailed low frequency behavior of the environmental impedance is relevant. Figure 2 shows the $I-V$ characteristic at $T=0$ for the limiting cases of a high- and low-impedance environment, respectively. Also an intermediate case with an Ohmic environmental impedance $Z(\omega)=R_Q$ is shown. In this latter case a somewhat smeared Coulomb gap structure is found. The form of the $I-V$ curve for various models of the environmental impedance and a range of temperatures was discussed elsewhere [7, 9, 18]. It was noted that the effect of the environment becomes particularly apparent in derivatives of the $I-V$ curve since the form of the second derivative approaches the shape of $P(E)$ for low temperatures [9]. This can easily be seen from (27) for $T\to 0$.

4. The double junction

A. The island charge and electron tunneling rates

We now consider the simplest multijunction circuit, that is two tunnel junctions in series which are connected to an electrical circuit characterized by the impedance $Z(\omega)$ (cf. Fig. 3). As in the case of a single junction, we first study the dynamics in the absence of electron tunneling. We then are left with two capacitances C_1 and C_2 in series with the external impedance. The dynamical behavior of this circuit can be described in terms of the charges Q_1 and Q_2 on the capacitors. Of course, in the absence of tunneling the charge

$$q=Q_1-Q_2 \tag{29}$$

on the metallic island between the junctions will be conserved. Further, the two capacitances in series couple to the external electrical circuit like one capacitance

$$C=\frac{C_1 C_2}{C_1+C_2} \tag{30}$$

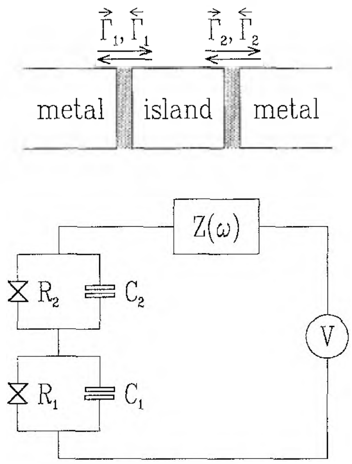

Fig. 3. A double junction system consisting of two tunnel junctions with tunneling resistances R_1 and R_2 and capacitances C_1 and C_2. The system is coupled to an external circuit with impedance $Z(\omega)$ and an ideal voltage source. The tunneling rates calculated in the text are shown in the upper part of the figure

carrying the charge

$$Q=\frac{C_2 Q_1+C_1 Q_2}{C_1+C_2}. \tag{31}$$

In the absence of tunneling the charge Q will obey the mean relaxation law (1) with the charge relaxation function (2). The total impedance $Z_t(\omega)$ is again of the form (3) with the capacitance C given by (30).

To treat the circuit quantum mechanically we introduce the phases φ_1 and φ_2 that are canonically conjugate to the charges Q_1 and Q_2. These phases are related to the voltages U_1 and U_2 across the junctions by relations of the form (4). It is of course advantageous to use the variables q and Q that are uncoupled in the absence of tunneling. From (29) and (31) we obtain for the phase variables canonically conjugate to q and Q

$$\psi=\frac{C_1 \varphi_1-C_2 \varphi_2}{C_1+C_2} \tag{32}$$

and

$$\varphi=\varphi_1+\varphi_2. \tag{33}$$

These variables satisfy the commutation relations

$$[\psi, q]=\mathrm{i}e, \qquad [\varphi, Q]=\mathrm{i}e \tag{34}$$

with all remaining commutators among the four variables vanishing. Now, Q and φ have the same properties as the corresponding variables of a single junction. In particular, the phase correlation function $J(t)=\langle[\varphi(t)-\varphi(0)]\,\varphi(0)\rangle$ is given by (8). On the other hand, the charge q on the island is conserved and quantized in units of e. Hence, in the absence of tunneling, the coupled system has steady states characterized by the voltage V and the charge $q=ne$ on the island where n is an integer.

The Hamiltonian of the entire system formed by the two tunnel junctions and the external circuit may be written as

$$H=H_0+H_1+H_2 \tag{35}$$

where H_0 is again the Hamiltonian in the absence of tunneling. This part contains the Hamiltonian of the electromagnetic environment and its interaction with the charge Q, the energy of the electrons in the junction electrodes, and also the charging energy

$$H_c=\frac{Q_1^2}{2C_1}+\frac{Q_2^2}{2C_2}=\frac{q^2}{2(C_1+C_2)}+\frac{Q^2}{2C} \tag{36}$$

of the two capacitors. H_1 and H_2 describe tunneling through the junction barriers. These operators are of the form (12). Denoting the outer electrode of the first junction by "1", the island by "2", and the other outer electrode by "3" we have, for instance,

$$H_1=\sum_{\sigma k_1 k_2} T^1_{k_2 k_1} c^\dagger_{k_2\sigma} c_{k_1\sigma} \exp(-\mathrm{i}\kappa_1\varphi-\mathrm{i}\psi)+\text{H.c.} \tag{37}$$

where we introduced

$$\kappa_i=\frac{C}{C_i} \quad (i=1, 2). \tag{38}$$

In (37) we have made use of $\varphi_1=\kappa_1\varphi+\psi$, which follows from (32) and (33), to express the charge shift operator $\exp(-\mathrm{i}\varphi_1)$ in terms of φ and ψ.

Let us assume that the system has reached the equilibrium state of H_0 at given voltage V, environmental temperature T, and island charge q. The four rates $\vec{\Gamma}_1(V, q)$, $\overleftarrow{\Gamma}_1(V, q)$, $\vec{\Gamma}_2(V, q)$, and $\overleftarrow{\Gamma}_2(V, q)$ for electron tunneling between adjacent electrodes will again be calculated by treating the tunneling Hamiltonians as small perturbations. By convention, the forward rates, $\vec{\Gamma}_i(V, q)$, describe transitions favored by a positive applied voltage V (cf. Fig. 3). For instance, $\vec{\Gamma}_1(V, q)$ is related to the terms explicitly written out in (37). The operators $c_{k_1\sigma}$ and $c^\dagger_{k_2\sigma}$ transfer an electron from the outer electrode to the island. Thereby the island charge is changed from q to $q-e$ as described by the charge shift operator $\exp(-\mathrm{i}\psi)$. Further, the equilibrium between the Q-charge distribution and the external circuit is disturbed by the operator $\exp(-\mathrm{i}\kappa_1\varphi)$. The golden rule rate now takes the form

$$\vec{\Gamma}_1(V, q)=\frac{1}{\pi\hbar}\frac{R_Q}{R_1}\int_{-\infty}^{+\infty} \mathrm{d}E \int_{-\infty}^{+\infty} \mathrm{d}E' f(E)[1-f(E')] \cdot P(\kappa_1, E+E_1(V, q)-E'). \tag{39}$$

Here, R_1 is the tunneling resistance of the first junction and the other quantities are defined as earlier. If the equilibrium between the Q-charge distribution and the electromagnetic environment were not disturbed, energy conservation would give $E'=E+E_1(V, q)$, where

$$E_i(V, q)=\kappa_i e V+\frac{q^2}{2(C_1+C_2)}-\frac{(q-e)^2}{2(C_1+C_2)} =\kappa_i e V+\frac{e\left(q-\frac{e}{2}\right)}{C_1+C_2} \quad (i=1, 2). \tag{40}$$

Note that now only a fraction of an elementary charge, namely $\kappa_1 e$, has to be transferred through the external circuit in order to maintain equilibrium with the environment. In reality, the equilibrium between the Q-charge distribution and the electromagnetic environment is disturbed by the operator $\exp(-\mathrm{i}\kappa_1\varphi)$ which leads in (39) to the appearance of the function

$$P(\kappa_1, E)=\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} \mathrm{d}t \exp[\kappa_1^2 J(t)+\mathrm{i}Et/\hbar]. \tag{41}$$

This function gives the probability that the tunneling electron looses the energy E to the environment. Compared with the corresponding quantity for a single junction, Eq. (17), the effective impedance $Z_t(\omega)$ is now weakened by the factor κ_1^2. The result (39) may also be written as

$$\vec{\Gamma}_1(V, q)=\frac{1}{\pi\hbar}\frac{R_Q}{R_1}\int_{-\infty}^{+\infty} \mathrm{d}E \frac{E}{1-\exp(-\beta E)} \cdot P(\kappa_1, E_1(V, q)-E). \tag{42}$$

The backward tunneling rate $\overleftarrow{\Gamma}_1(V, q)$ is related to $\vec{\Gamma}_1(V, q)$ by

$$\overleftarrow{\Gamma}_1(V, q)=\vec{\Gamma}_1(-V, -q) \tag{43}$$

and by the detailed balance symmetry

$$\overleftarrow{\Gamma}_1(V, q-e)=\exp[-\beta E_1(V, q)]\,\vec{\Gamma}_1(V, q). \tag{44}$$

The rate $\overleftarrow{\Gamma}_2(V, q)$ is obtained from (42) when we exchange the indices 1 and 2 and replace V by $-V$. We find

$$\overleftarrow{\Gamma}_2(V, q)=\frac{1}{\pi\hbar}\frac{R_Q}{R_2}\int_{-\infty}^{+\infty} \mathrm{d}E \frac{E}{1-\exp(-\beta E)} \cdot P(\kappa_2, E_2(-V, q)-E). \tag{45}$$

which is related to the forward rate by

$$\vec{\Gamma}_2(V, q)=\overleftarrow{\Gamma}_2(-V, -q) \tag{46}$$

and the detailed balance symmetry

$$\vec{\Gamma}_2(V, q-e)=\exp[-\beta E_2(-V, q)]\,\overleftarrow{\Gamma}_2(V, q). \tag{47}$$

We mention that for the case of two identical junctions with $C_1=C_2$ and $R_1=R_2$ the four tunneling rates are related by $\vec{\Gamma}_1(V, q)=\overleftarrow{\Gamma}_1(-V, -q)=\overleftarrow{\Gamma}_2(V, -q)=\vec{\Gamma}_2(-V, q)$.

B. High- and low-impedance environments

Let us again study the limiting cases for the electromagnetic environment treated in Sect. 3B. In the low-impedance limit we have $P(\kappa, E)=\delta(E)$ and (42) reduces to

$$\vec{\Gamma}_1(V, q)=\frac{1}{\pi\hbar}\frac{R_Q}{R_1}\frac{E_1(V, q)}{1-\exp[-\beta E_1(V, q)]} \tag{48}$$

which is seen to give at zero temperature

$$\vec{\Gamma}_1(V, q)=\frac{1}{\pi\hbar}\frac{R_Q}{R_1}E_1(V, q)\,\Theta(E_1(V, q)) \quad \text{for } T=0. \tag{49}$$

In view of (38) and (40) we thus find $\vec{\Gamma}_1(V, q)=0$ for $V<(e-2q)/2C_2$. The rate formula (48) corresponds to the so-called global rule rate of the conventional theory of Coulomb charging effects [2, 10]. There, one considers the difference in energy $E_i(V, q)$ of equilibrium configurations of the whole circuit before and after tunneling through the i-th junction. This energy difference (40) takes into account the change in charging energy and the work done by the voltage source to reestablish charge equilibrium. We note that the energy difference (40) may also be written as

$$E_i(V, q)=\frac{e}{C_i}(Q_i-Q_c) \tag{50}$$

where we introduced the critical charge

$$Q_c=\frac{e}{2}(1-\kappa_i). \tag{51}$$

Hence, $E_i(V, q)$ depends only on the charge Q_i on the capacitor through which the electron is tunneling. However, this should not be confused with the local rule which is discussed below.

For a high-impedance environment we have from (22) and (41)

$$P(\kappa, E)=\frac{1}{\sqrt{4\pi\kappa^2 E_c k_B T}}\cdot\exp[-(E-\kappa^2 E_c)^2/4\kappa^2 E_c k_B T]. \tag{52}$$

This yields for the zero temperature rate

$$\vec{\Gamma}_1(V, q)=\frac{1}{\pi\hbar}\frac{R_Q}{R_1}[E_1(V, q)-\kappa_1^2 E_c]\cdot\Theta(E_1(V, q)-\kappa_1^2 E_c) \quad \text{for } T=0. \tag{53}$$

Now, the zero temperature rate vanishes for $V<(e-2q)/2C_2+e/2C_1$. The expression (53) corresponds to the local rule rate of the conventional theory [2, 10] since

$$E_1(V, q)-\kappa_1^2 E_c=\frac{Q_1^2}{2C_1}-\frac{(Q_1-e)^2}{2C_1} \tag{54}$$

gives the change in charging energy on the junction through which the electron is tunneling. As in the case of a low impedance environment one may define a critical charge. In contrast to (51) one finds here $Q_c=e/2$. We emphasize that even for a high impedance environment the local rule rate is not valid at finite temperatures due to the Gaussian nature of $P(\kappa, E)$ in (52).

We see that in both limits the rate shows a Coulomb blockade of tunneling. For instance, for an electrically neutral island, $q=0$, we find that all four tunneling rates vanish at $T=0$ for $|V|<\min(e/2C_1, e/2C_2)$ in the low-impedance limit, and for $|V|<e/2C_1+e/2C_2$ in the high-impedance limit. Note that this latter limit is now much harder to achieve, since the effective environmental impedance is reduced by a factor κ_1^2 or κ_2^2, respectively. For arbitrary impedance the tunneling rates have to be calculated from (42)–(47) using numerical results for $P(\kappa_i, E)$ that may be obtained by the same method as for single junctions. For the double junction the Coulomb blockade arises from the charge quantization on the island. In order to transfer a charge, say, from electrode "1" to electrode "3", one has to occupy the island "2" in an intermediate step, and there is an energy barrier to be overcome. Higher order terms of the perturbation theory for the rate will lead to a finite, yet small current in the presence of a voltage [16]. This problem will not be addressed here.

C. *The I – V characteristic*

For a single junction the current-voltage characteristic can be calculated from the forward and backward tunneling rates by $I(V)=e[\vec{\Gamma}(V)-\overleftarrow{\Gamma}(V)]$. For a double junction, and more generally for multijunction systems, the calculation of the current from the rates is more complicated. In general, one has to solve a master equation. The state of a double junction may be characterized by the voltage V and the island charge q. Since we assume that between two tunneling processes there is enough time to restore equilibrium with the environment, the voltage V is constant during a sequence of tunneling transitions while the island charge q changes. At finite temperatures the time sequence of q is arbitrary since all tunneling rates differ from zero although some tunneling processes are more likely than others. On the other hand, at zero temperature certain tunneling transitions are forbidden depending on V and q as is apparent from (49) and (53).

The current-voltage characteristic of a double junction may be calculated from

$$I=e\sum_{n=-\infty}^{+\infty}p_n(\vec{\Gamma}_1(n)-\overleftarrow{\Gamma}_1(n))=e\sum_{n=-\infty}^{+\infty}p_n(\vec{\Gamma}_2(n)-\overleftarrow{\Gamma}_2(n)) \tag{55}$$

where p_n is the probability that the charge q on the island equals ne. This probability together with the difference of rates $\vec{\Gamma}_1(n)-\overleftarrow{\Gamma}_1(n)$ describes the net current flowing through the first junction. In view of charge conservation the current may also be expressed in terms of the rates at the second junction. Since the tunneling rates have already been discussed in the previous sections, we now have to determine the steady state probabilities p_n. The island charge q may change by tunneling of electrons from or to the island as described by the master equation

$$\dot{p}_n=\sum_{n=-\infty}^{+\infty}[\Gamma_{n,n+1}p_{n+1}+\Gamma_{n,n-1}p_{n-1}-(\Gamma_{n+1,n}+\Gamma_{n-1,n})p_n]. \tag{56}$$

Here, $\Gamma_{n\pm1,n}$ is the rate for the transition from $q=ne$ to $q=(n\pm1)e$. Since we have to consider a change in q by tunneling through the first as well as through the second junction, we get for these rates

$$\Gamma_{n+1,n}=\vec{\Gamma}_1(n)+\overleftarrow{\Gamma}_2(n) \tag{57}$$

$$\Gamma_{n-1,n}=\overleftarrow{\Gamma}_1(n)+\vec{\Gamma}_2(n). \tag{58}$$

The stationary solution of (56) satisfies the detailed balance condition

$$\Gamma_{n,n+1}\,p_{n+1}=\Gamma_{n+1,n}\,p_n: \tag{59}$$

which may be used to express the probability p_n of finding the charge ne on the island in terms of the probability p_0 for the island to be neutral as

$$p_n=p_0\prod_{m=0}^{n-1}\frac{\Gamma_{m+1,m}}{\Gamma_{m,m+1}} \tag{60}$$

and

$$p_{-n}=p_0\prod_{m=-n+1}^{0}\frac{\Gamma_{m-1,m}}{\Gamma_{m,m-1}} \tag{61}$$

where in both formulas $n>0$. p_0 is determined through the normalization condition

$$\sum_{n=-\infty}^{+\infty}p_n=1. \tag{62}$$

The formulae (55) and (60)–(62) together with the rate expression (42) allow us to calculate the current-voltage characteristics for arbitrary temperature and environment. This can easily be done numerically. An example of a current-voltage characteristic is shown in Fig. 4.

Some basic features of a double junction can best be seen from further analytical results available at zero temperature. While for finite temperatures in principle any integer charge can sit on the island, we find from (60) and (61) that for zero temperature the possible charges are restricted due to vanishing rates. Let us first consider two junctions with equal capacitances $C_1=C_2=2C$ but arbitrary tunnel resistances R_1 and R_2. For voltages $V_m=(e/2C)(m+1/2)\,(m=0, 1, 2, \ldots)$ in the low impedance case and $V_m=(e/2C)(m+1)\,(m=0, 1, 2, \ldots)$ in the high impedance case it is possible to evaluate the products and sums appearing in the expressions for the probabilities yielding

$$p_n=\frac{(2m)!}{(m-|n|)!(m+|n|)!}\,\frac{(R_1/R_2)^{n+m}}{(1+(R_1/R_2))^{2m}}. \tag{63}$$

For these voltages we then find the current

$$I=\frac{1}{R_1+R_2}\,\frac{e}{2C}\,m. \tag{64}$$

Hence, these points lie on a straight line in the current-voltage diagram. The slope of this straight line characteristic is given by the total tunneling resistance R_1+R_2 and it shows a Coulomb gap of $e/4C$ and $e/2C$ for the low and high impedance cases, respectively. This property of the $I-V$ characteristic is also apparent from Fig. 4 which in addition shows that for other voltages the current lies above the straight line. The periodic structure of the $I-V$ curve becomes more pronounced as the ratio between the two tunneling resistances is increased. If one of the resistances is very large compared to the other, the tunneling current is restricted by the high resistance junction. In the voltage regime $V_{m-1}<V<V_m$ the island is then charged through the low resistance junction up to the maximal charge $q=\pm me$, the sign of q depending on which of the two junction has high resistance. Since then $p_{\pm m}=1$, the current is determined by the rate $\Gamma(\pm m)$ through the high resistance junction. The corresponding $I-V$ characteristic has the form of a staircase with current jumps at the voltages V_m and segments of constant slope between these voltage values. This so-called Coulomb staircase has been discussed in the literature [19]. In these earlier treatments the coupling to the external circuit was neglected and the results where mostly based on numerical simulations of the master equation. In various experiments on double junction systems [20] the steplike structure has been observed in the current-voltage characteristic as well as in the differential $\mathrm{d}I/\mathrm{d}V-V$ curve.

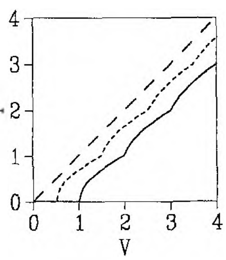

Fig. 4. The current-voltage characteristic of a double junction with tunneling resistances $R_2=10R_1$ and capacitances $C_1=C_2=2C$ at zero temperature. The long-dashed line $I=V/(R_1+R_2)$ shows the Ohmic characteristic of two resistors in series. The short-dashed line and the solid line are for low and high impedance environments, respectively. The voltage V is given in units of $e/2C$

In order to further illustrate the calculation of the current from the rates $\Gamma_{n,n+1}$ let us consider a double junction with capacitances $C_1<C_2$ coupled to a low impedance environment and biased with a positive voltage V. From (49) we find that the rates are different from zero in the following voltage regimes

$$\vec{\Gamma}_1(V,q)\neq0 \quad \text{for } V>(e/2-q)/C_2 \tag{65}$$

$$\overleftarrow{\Gamma}_1(V,q)\neq0 \quad \text{for } V<-(e/2+q)/C_2 \tag{66}$$

$$\vec{\Gamma}_2(V,q)\neq0 \quad \text{for } V>(e/2+q)/C_1 \tag{67}$$

$$\overleftarrow{\Gamma}_2(V,q)\neq0 \quad \text{for } V<-(e/2-q)/C_1. \tag{68}$$

As already discussed above, the current through the double junction vanishes for $|V|<V_1=e/2C_2$ giving rise to a Coulomb gap in the current-voltage characteristic (see

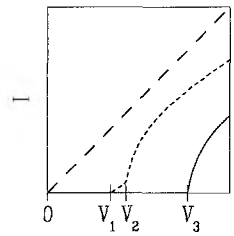

Fig. 5. The current-voltage characteristic of a double junction with tunneling resistances $R_1 = 10R_2$ and capacitances $C_1 = 1.8\,C$ and $C_2 = 2.25\,C$ at zero temperature. The long-dashed line $I = V/(R_1 + R_2)$ shows the Ohmic characteristic of two resistors in series. For a low impedance environment (short-dashed line) the gap extends up to $V_1 = e/2C_2$ and there is a cusp at $V_2 = e/2C_1$ where the sequence of electron tunneling to and from the island becomes statistical. For a high impedance environment (solid line) the Coulomb gap extends up to $V_3 = e/2C$

Fig. 5). In this voltage regime $q=0$ is stable since all rates for transitions starting from $q=0$ vanish. For $q=ne\,(n>0)$ only the rates which decrease q are nonzero and the opposite is true for $q=-ne$. Therefore, even if initially the double junction has non-vanishing probabilities p_n for $n \neq 0$, the stationary solution $p_0 = 1$, $p_n = 0\,(n \neq 0)$ will be approached.

In the following we assume $1 < C_2/C_1 < 3$ for the junction capacitances. Let us first consider the voltage regime $V_1 < V < V_2 = e/2C_1$. When $q=0$ the only non-vanishing rate is $\vec{\Gamma}_1(V, 0)$ bringing the double junction into the state $q=-e$. There, the only possible transition is back to $q=0$ described by $\vec{\Gamma}_2(V, -e)$. The two tunneling events lead to a current through the double junction. In the steady state the non-vanishing probabilities are

$$p_0 = \frac{\vec{\Gamma}_2(V, -e)}{\vec{\Gamma}_1(V, 0) + \vec{\Gamma}_2(V, -e)} \tag{69}$$

$$p_{-1} = \frac{\vec{\Gamma}_1(V, 0)}{\vec{\Gamma}_1(V, 0) + \vec{\Gamma}_2(V, -e)} \tag{70}$$

which yield for the current

$$I = e\Gamma(V) \tag{71}$$

where

$$\frac{1}{\Gamma(V)} = \frac{1}{\vec{\Gamma}_1(V, 0)} + \frac{1}{\vec{\Gamma}_2(V, -e)}\,. \tag{72}$$

For $V_2 < V < 3\,e/2C_2$ the situation becomes more complicated. Starting from $q=0$ there are now two channels $q=0 \to -e \to 0$ and $q=0 \to e \to 0$ both leading to a current through the junctions in the same direction. However, the time order of the tunneling is different. This means that in this voltage regime we can have successive tunneling of two electrons through the same junction. The corresponding probabilities p_{-1}, p_0, and p_1 are given by Eqs. (60)–(62) yielding the $I-V$ characteristic for $V > V_2$ shown in Fig. 5. It is straightforward to calculate the current for larger voltages analytically or numerically and to extend these considerations to other capacitance ratios C_2/C_1.

5. Application of network analysis

As discussed in the previous section the influence of an electromagnetic environment on the behavior of a system of two tunnel junctions may basically be formulated in a way very analogous to the case of a single tunnel junction coupled to an external circuit. In both cases the effect of the environment is characterized through a function $P(E)$ containing an effective external impedance. In this section we present rules from network theory by means of which tunneling rates in a multijunction system can be reduced to rates of a single junction system in series with an effective capacitance.

To introduce the basic ideas let us first consider a voltage-biased single tunnel junction with capacitance C coupled to an environment of impedance $Z(\omega)$ as depicted in Fig. 1. The junction itself is described by a capacitor in parallel with a tunneling element. The tunneling element is a device which transfers electrons with a certain rate. We now look at the circuit shown in Fig. 1 from the viewpoint of the tunneling element. This one-port circuit may be transformed by changing between so-called Thevenin and Norton configurations [21] which are shown in Figs. 6a and b, respectively. If the Thevenin configuration contains a voltage source $V(\omega)$ in series with an impedance $Z(\omega)$, the equivalent Norton configuration consists of this impedance in parallel with a current source $I(\omega) = V(\omega)/Z(\omega)$. We may use this rule to transform the right section of Fig. 7a into its Norton

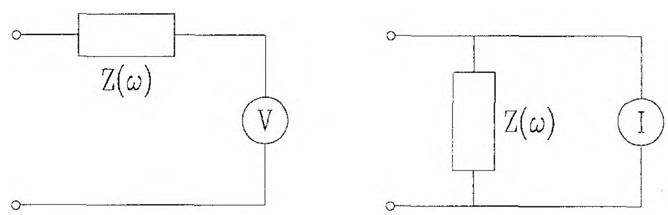

Fig. 6a, b. The Thevenin configuration **a** is equivalent to the Norton configuration **b** if $I = V/Z(\omega)$

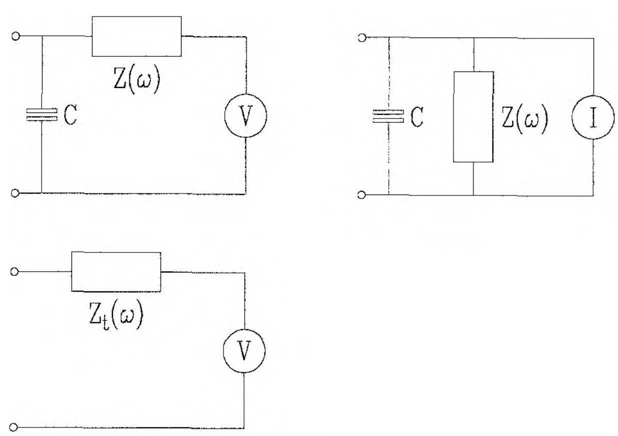

Fig. 7a–c. In three steps the single junction circuit is transformed by network theoretical rules. **a** shows the original circuit as seen from the tunneling element. In the Norton equivalent **b** we may combine the capacitance C and the impedance $Z(\omega)$ leading to the Thevenin-type circuit **c**

equivalent (Fig. 7b). It is advantageous to perform the transformations with an *ac* voltage and to consider the limit $\omega \to 0$ only at the end. In the Norton form we sum up the two parallel impedances $Z(\omega)$ and $1/i\omega C$ yielding the total impedance $Z_t(\omega) = 1/(i\omega C + Z(\omega)^{-1})$. Returning to the Thevenin equivalent we find the one-port circuit shown in Fig. 7c containing the impedance $Z_t(\omega)$ in series with a voltage source $V(\omega)/(1 + i\omega C Z(\omega))$. Since the original voltage source was a *dc* source this again leads to a *dc* voltage source V. Thus the following picture emerges. We have an ideal tunneling element passing an electron through the effective circuit in Fig. 7c. The transfer through the voltage source V leads to a change in the energy by eV. The electromagnetic environment is described by the total impedance $Z_t(\omega)$ introduced in (3) which eventually determines the tunneling rate via $P(E)$. The reduction of a Coulomb gap in the case of a low impedance environment manifests itself in the fact that the effective circuit in the limit $Z(\omega) \to 0$ no longer contains a capacitance.

We now apply this reasoning to the double junction system shown in Fig. 3. For the discussion of tunneling through the first junction we may replace the second junction by the capacitance C_2 thereby disregarding the possibility of simultaneous charge transfer through both junctions. This is correct within lowest order perturbation theory in the tunneling conductances $1/R_1$ and $1/R_2$. Here, we now have to keep track of the charges sitting on the two capacitors. If a capacitor in a Thevenin configuration, where it is in series with a voltage source, carries a charge Q, it will carry the charge $Q - CV$ in the corresponding Norton configuration. Applying this to the circuit of Fig. 8a we obtain the capacitor C_1 with charge Q_1 in parallel with a series connection consisting of the impedance $Z(\omega)$ and the capacitor C_2 with charge $Q_2 - C_2 V$ (Fig. 8b). It is straightforward to calculate the impedance corresponding to the branches in parallel. If we split off a capacitance pole $1/i\omega(C_1 + C_2)$, we are left with the impedance $\kappa_1^2 Z_t(\omega)$ where the capacitance is the total capacitance given by (30). After the transformation back to the Thevenin form the capacitance pole corresponds to the capacitances C_1 and C_2 in parallel carrying a total charge of $Q_1 - Q_2$ which according to (29) is just the island charge q. In addition, the new circuit contains a voltage source with reduced voltage $\kappa_1 V$. We can now use this effective single junction circuit depicted in Fig. 8c to interpret the double junction results presented in the previous section. The energy change (40) related to a tunneling event contains contributions from the transport of the electron through the voltage source $\kappa_1 V$ and from the change of the charging energy on the capacitor with capacitance $C_1 + C_2$ due to a change in the island charge q. Furthermore, the coupling to the impedance $Z_t(\omega)$ which contains the capacitance C of the original circuit in parallel with the original external impedance $Z(\omega)$ is reduced by a factor of κ_1^2.

The network analysis presented here may easily be extended to other multijunction circuits. A simple example is presented in the next section and further interesting multijunction circuits are treated elsewhere [22].

6. Electron tunneling in a 1D-array

A. Charging energy and tunneling rates

We shall consider here an array of N tunnel junctions with capacitances C_i and tunneling resistances R_i ($i = 1, \ldots, N$). The array is connected to an electrical circuit consisting of a voltage source in series with the environmental impedance $Z(\omega)$ (see Fig. 9a). The charge on the i-th junction will be denoted by Q_i. The charging energy of the whole array may then be written

$$H_c = \sum_{i=1}^{N} \frac{Q_i^2}{2C_i}. \tag{73}$$

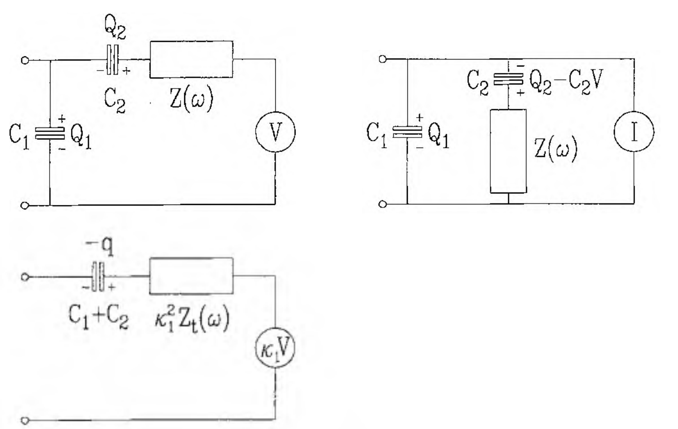

Fig. 8a–c. A double junction circuit is transformed into an effective single junction circuit. The right part of the original circuit **a** may be cast in the Norton form **b**. If the signs of the charges on C_1 and C_2 are properly taken into account one finally gets the circuit **c** with a reduced voltage $\kappa_1 V$ and reduced impedance $\kappa_1^2 Z_t(\omega)$

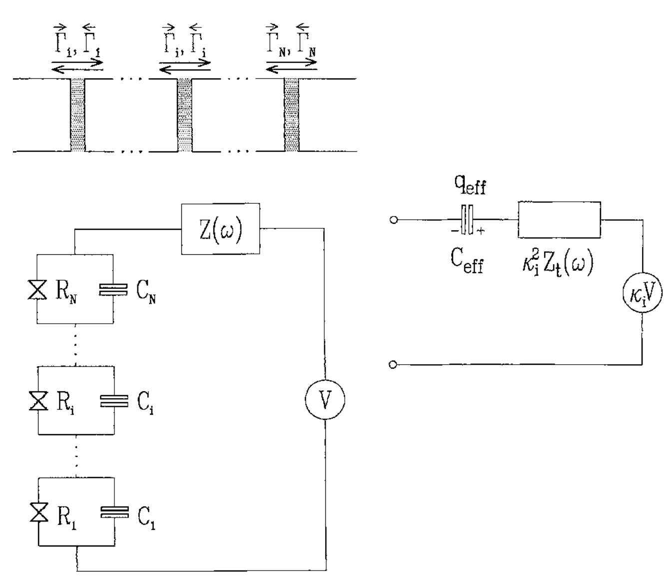

Fig. 9. a A schematic drawing of an 1D-array. The electron tunneling rates for the first, the i-th, and the last junction are indicated. **b** Reduced effective circuit which may be used to calculate the rates $\vec{\Gamma}_i$ and $\overleftarrow{\Gamma}_i$

The array of capacitances presents itself to the environment as one capacitance C with

$$\frac{1}{C}=\sum_{i=1}^{N}\frac{1}{C_i} \tag{74}$$

which carries the charge

$$Q=C\sum_{i=1}^{N}\frac{Q_i}{C_i}. \tag{75}$$

The charge Q is again a variable with a continuous spectrum. On the other hand the charges

$$q_k=Q_k-Q_{k+1} \qquad (k=1,\ldots,N-1) \tag{76}$$

on the $N-1$ metallic islands between the junctions are quantized in units of e. Of course, it is advantageous to work with the variables Q, q_k in terms of which the charging energy (73) reads

$$H_c=\frac{Q^2}{2C}+\sum_{k,l=1}^{N-1}\frac{1}{2}(C^{-1})_{kl}\,q_k\,q_l. \tag{77}$$

Here, we introduced the capacitance matrix

$$C_{kl}=\begin{cases}C_k+C_{k+1} & \text{for } l=k\\ -C_{k+1} & \text{for } l=k+1\\ -C_k & \text{for } l=k-1\\ 0 & \text{otherwise.}\end{cases} \tag{78}$$

The self capacitance of the k-th island equals the sum of the capacitances of the two junctions to the left and to the right of the island. In addition, there is a capacitive coupling between adjacent islands given by the capacitance of the junction separating the islands. The inverse of the capacitance matrix is found to be

$$(C^{-1})_{kl}=C\sum_{m=1}^{\min(k,l)}\frac{1}{C_m}\sum_{n=\max(k,l)+1}^{N}\frac{1}{C_n}. \tag{79}$$

We note that usually one also has to account for the capacitive coupling of the metallic islands to ground. This gives rise to a modification of the diagonal elements of the capacitance matrix (78). Here we are mainly interested in the effect of the external circuit on electron tunneling rates. A modification of the self capacitances can always be accounted for later. Of course, in order to describe charge solitons in 1D-arrays of tunnel junctions [23] one has to take into account the capacitive coupling to ground since it determines the size of the soliton.

Before we apply network analysis to this problem, let us briefly outline the direct calculation of tunneling rates from the tunneling Hamiltonian. To this purpose we introduce the phases φ and ψ_k that are canonically conjugate to the charges Q and q_k. Hence, these variables satisfy the commutation relations

$$[\varphi, Q]=\mathrm{i}\,e, \qquad [\psi_k, q_k]=\mathrm{i}\,e \tag{80}$$

with all remaining commutators among these variables vanishing. The phase φ_i canonically conjugate to the charge Q_i on the i-th junction reads in terms of the new phase variables

$$\begin{aligned}&\varphi_1=\psi_1+\kappa_1\varphi\\ &\varphi_i=\psi_i-\psi_{i-1}+\kappa_i\varphi \qquad (i=2,\ldots,N-1)\\ &\varphi_N=-\psi_{N-1}+\kappa_N\varphi\end{aligned} \tag{81}$$

where we introduced the capacitance ratio

$$\kappa_i=C/C_i \qquad (i=1,\ldots,N). \tag{82}$$

The Hamiltonian of the entire system formed by the 1D-array of junctions and the electrical circuit loading the array splits into

$$H=H_0+\sum_{i=1}^{N}H_i \tag{83}$$

where H_0 is the Hamiltonian in the absence of tunneling which includes H_c, and H_i describes tunneling through the i-th junction. The steady states of H_0 are characterized by the voltage V, the temperature T, and the island charges q_k that are conserved quantities in the absence of tunneling. A straightforward extension of the reasoning in the preceding sections now yields for the golden rule rates through the i-th junction (cf. Fig. 9a)

$$\vec{\Gamma}_i(V,q_k)=\frac{1}{\pi\hbar}\frac{R_Q}{R_i}\int_{-\infty}^{+\infty}\mathrm{d}E\,\frac{E}{1-\exp(-\beta E)}\cdot P(\kappa_i, E_i(V,q_k)-E) \tag{84}$$

where

$$\begin{aligned}&E_1(V,q_k)=\kappa_1 eV+\varepsilon(q_k)-\varepsilon(q_1-e, q_2,\ldots,q_{N-1})\\ &E_i(V,q_k)=\kappa_i eV+\varepsilon(q_k)-\varepsilon(q_1,\ldots,q_{i-2},q_{i-1}+e,q_i-e,\\ &\qquad q_{i+1},\ldots,q_{N-1}) \qquad (i=2,\ldots,N-1)\\ &E_N(V,q_k)=\kappa_N eV+\varepsilon(q_k)-\varepsilon(q_1,\ldots,q_{N-2},q_{N-1}+e).\end{aligned} \tag{85}$$

Here,

$$\varepsilon(q_k)=\sum_{k,l=1}^{N-1}\tfrac{1}{2}(C^{-1})_{kl}\,q_k\,q_l \tag{86}$$

is the internal charging energy of the array.

The backward tunneling rate $\overleftarrow{\Gamma}_i(V,q_k)$ is related to $\vec{\Gamma}_i(V,q_k)$ by

$$\overleftarrow{\Gamma}_i(V,q_k)=\vec{\Gamma}_i(-V,-q_k) \tag{87}$$

and by the detailed balance symmetry

$$\begin{aligned}&\overleftarrow{\Gamma}_1(V,q_1-e,q_2,\ldots,q_{N-1})\\ &\quad=\exp[-\beta E_1(V,q_k)]\,\vec{\Gamma}_1(V,q_1,\ldots,q_{N-1}),\\ &\overleftarrow{\Gamma}_i(V,q_1,\ldots,q_{i-2},q_{i-1}+e,q_i-e,q_{i+1},\ldots,q_{N-1})\\ &\quad=\exp[-\beta E_i(V,q_k)]\,\vec{\Gamma}_i(V,q),\end{aligned}$$

$$\vec{\Gamma}_N(V, q_1, \ldots, q_{N-2}, q_{N-1}+e) = \exp[-\beta E_N(V, q_k)]\, \vec{\Gamma}_N(V, q_1, \ldots, q_{N-1}). \tag{88}$$

In view of (82) and the factor κ_i^2 in the exponent of the integrand of (41), for an array of junctions with capacitances of the same order of magnitude the effective environmental impedance $Z(\omega)$ is reduced by a factor N^2 as compared to a single junction. Hence, for large arrays the low-impedance limit is mostly appropriate and the rates reduce to the global rule result [2]

$$\vec{\Gamma}_i(V, q_k) = \frac{1}{\pi\hbar}\frac{R_Q}{R_i}\frac{E_i(V, q_k)}{1-\exp[-\beta E_i(V, q_k)]}. \tag{89}$$

At zero temperature this gives

$$\vec{\Gamma}_i(V, q_k) = \frac{1}{\pi\hbar}\frac{R_Q}{R_i} E_i(V, q_k)\, \Theta(E_i(V, q_k)). \tag{90}$$

Now, using (85) and (86), the energy $E_i(V, q_k)$ which must be positive for the rate not to vanish may be written as

$$E_i(V, q_k) = e\kappa_i\left(V - \frac{e}{2C}(1-\kappa_i) - \sum_{k=1}^{i-1}\frac{q_k}{C_k^-} + \sum_{k=i}^{N-1}\frac{q_k}{C_k^+}\right). \tag{91}$$

Here, we introduced the capacitances of the junction arrays to the left and right of the k-th junction

$$C_k^- = \left(\sum_{j=1}^{k}\frac{1}{C_j}\right)^{-1}, \quad C_k^+ = \left(\sum_{j=k+1}^{N}\frac{1}{C_j}\right)^{-1}. \tag{92}$$

In (91) the energy change is expressed in terms of the state variables (V, q_k) of the array. By means of the relation (76) between the island and junction charges, it can be shown that the energy E_i for the i-th junction depends just on the charge Q_i on this junction. We then obtain the simple form

$$E_i(Q_i) = \frac{e}{C_i}(Q_i - Q_c) \tag{93}$$

with the critical charge

$$Q_c = \frac{e}{2}(1-\kappa_i). \tag{94}$$

This is in accordance with the result (50), (51) for the double junction in the low impedance limit.

From (91) we find that the Coulomb gap is determined by the junction with the largest capacitance, i.e. at zero temperature no current will flow for voltages $|V| < \min_i(1 - C/C_i)\, E_c/e$. For the special case of a double junction this reduces to the result given in Sect. 4B. Considering an array consisting of tunnel junctions with the same capacitance, i.e. $C_i = nC$, we find for the Coulomb gap $(1-1/n)\, E_c/e$. Therefore, for a large array, the Coulomb gap at zero temperature is given by the charging energy E_c corresponding to the capacitance C of the whole array.

We now want to apply the rules for network transformations introduced in Sect. 5 to the 1D-array. Let us pick an arbitrary tunnel junction "i" of the array. All other tunnel junctions may be treated like capacitances. In a first step we rearrange the capacitances in the circuit leaving the tunnel junction fixed. All junctions to the left of the i-th junction are shifted through the voltage source and the external impedance. Then they may be combined with the capacitances to the right of the tunnel junction resulting in a total capacitance

$$\tilde{C}_i = \frac{C_i C}{C_i - C} \tag{95}$$

carrying the charge

$$\tilde{Q}_i = \frac{C_i C}{C_i - C}\sum_{j \neq i}\frac{Q_j}{C_j}. \tag{96}$$

This capacitance may then be considered analogous to the capacitance C_2 of the double junction circuit discussed in Sect. 5. Along the same lines of reasoning we find the circuit shown in Fig. 9b. It contains an effective voltage source $\kappa_i V$, a reduced total impedance $\kappa_i^2 Z_t(\omega)$ and a capacitance

$$C_{\text{eff}} = \frac{C_i^2}{C_i - C} \tag{97}$$

carrying the charge

$$q_{\text{eff}} = \frac{C_i}{C_i - C}\left(-Q_i + C\sum_j \frac{Q_j}{C_j}\right). \tag{98}$$

This charge may be expressed entirely in terms of properties of the i-th junction and the voltage as

$$q_{\text{eff}} = \frac{C_i}{C_i - C}(-Q_i + CV). \tag{99}$$

During the tunneling process an electron is transferred through the effective voltage source and an electron charge is added to the charge present on the effective capacitor. Summing up the corresponding energy changes we find the result (93) for $E_i(Q_i)$ which depends only on the charge of the tunnel junction under consideration. The environment of the i-th junction is characterized by the effective impedance $\kappa_i^2 Z_t(\omega)$ and we are thus led directly to the rate formula (84) with E_i given by (93). As we have already mentioned for large arrays this rate expression can mostly be simplified to the global rule rate (89) since $\kappa_1^2 Z_t(\omega) \ll R_Q$ under common experimental conditions.

7. Conclusions

The transport of electrons in circuits with several normal tunnel junctions was examined for the case of ultrasmall junction capacitances where Coulomb blockade of tunneling may arise. The analysis was based on perturbation theory in the tunneling Hamiltonian which is appropriate for junctions with tunneling resistances R_T that are large compared with the resistance quantum R_Q. It was pointed out that the dynamics of these mesoscopic multijunction circuits is usually strongly affected by the external electrical circuit loading the device.

In multijunction circuits the discreteness of the charge transfer through the junctions implies a quantization of island charges which in turn causes a Coulomb blockade of tunneling. This effect is more pronounced for systems embedded in a high impedance environment. In fact, for single junctions the suppression of charge fluctuations by high impedance leads is the only reason for a Coulomb gap structure. In multijunction circuits, on the other hand, the effective environmental impedance decreases roughly as one over the square of the number of junctions. As a consequence, electrons tunneling in large arrays see only a very low impedance environment and the tunneling rates may be calculated using the global rule rate of the conventional theory. This is true for arbitrary temperatures. On the other hand, the local rule rates were shown to become correct in the limit of a very high impedance environment and at zero temperature only. For arbitrary environmental impedance the tunneling rates have to be computed numerically from the general rate formulas derived in this paper.

We have demonstrated that network analysis allows one to reduce the calculation of tunneling rates in multijunction circuits to those of an effective single junction circuit of the form given in Fig. 9b where the tunneling element is in series with an effective capacitance representing the internal charging energy of the circuit, a reduced impedance, and a reduced voltage source. Explicit results were given for a one-dimensional array but the method can directly be used to treat more complicated cases. One of the main challenges for future work is an extension of the results presented here to the case of tunnel junctions with higher tunneling conductances.

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