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Relaxation Processes in Superconductors

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By means of a set of kinetic equations, we describe nonequilibrium electrons and phonons in a superconductor. We assign expressions for the distribution functions and the relaxation times of the coupled electron–phonon system in various situations. In order to obtain transparent results, we replace the collision operator by a reduced operator which still satisfies the conservation laws. We also investigate systems at low temperatures.

1. INTRODUCTION

A superconductor can be driven out of thermal equilibrium by various techniques, for example, by electron injection, electromagnetic radiation, or simply heating of the metal. An essential quantity in analyzing the experimental results is the particular relaxation time corresponding to the considered experiment. These are, for example, the quasiparticle recombination time first measured in the experiments of Ginsberg[1] or the "branch mixing" time investigated in the experiment of Clarke.[2] A further interesting topic in nonequilibrium superconductivity is the gap enhancement caused by microwave irradiation of superconductors[3] near T_c.

The nonequilibrium properties of superconductors can be understood in terms of electronic quasiparticles and the condensate of pairs, characterized by a complex order parameter. Furthermore, when the energy of the system varies, nonequilibrium phonons can be excited. Characteristic times for relaxation processes in the coupled electron–phonon system are the inelastic scattering time τ_0 of electrons at the Fermi surface and the

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lifetime τ^{ph} of phonons with energy of order T. In an experiment the thermal contact results in an energy transfer between the probe and the surrounding material. This process, which is essentially effected by phonons escaping from the probe, is conveniently described by a phonon escape time τ_{es}, depending on the geometry of the probe and the velocity of sound. In the limit of strong escape ($\tau_{\mathrm{es}} \ll \tau^{\mathrm{ph}}$), which is often considered, the phonons remain in thermal equilibrium.

The basic equations describing nonequilibrium properties of superconductors are a set of Boltzmann equations for the distribution functions of electrons and phonons, coupled to a time-dependent Ginzburg–Landau equation for the order parameter.[4] In a linear theory one finds two independent modes: one involving changes in the energy of the system and in the magnitude of the order parameter ("energy mode"), the other mode involving changes in the particle number and in the phase of the order parameter ("particle mode").* Consequently, in the energy mode, nonequilibrium phonon distributions are also excited. Due to the electron–phonon collision processes, a complicated integral operator appears in the Boltzmann equation. In this paper, we will replace this collision integral by a reduced operator which satisfies the same conservation laws as the exact operator. This approximation allows a straightforward solution of the coupled kinetic equations, thus providing transparent results for various problems. Some details of the collision processes that are important at low temperatures are not described correctly by the reduced operator; therefore, we will develop an alternative method which applies to this limit.

Concentrating first on the energy mode, we present in Section 2 the appropriate Boltzmann equation for the electron and phonon distribution functions and discuss the reduced operator. We apply this approximation in Section 3 to determine the relaxation time of the energy mode near T_c and consider stationary nonequilibrium situations. In Section 4 we compare the relaxation times corresponding to the energy and the particle modes. In Section 5 we extend the investigation of the particle mode to low temperatures, where, as mentioned above, a different method has to be used to calculate the solution of the kinetic equations. Finally, in Section 6, we discuss the validity of our approximations.

2. THE KINETIC EQUATIONS IN THE ENERGY MODE AND THE REDUCED OPERATOR

Starting from microscopic theory, a set of linear kinetic equations for dirty superconductors was derived by Schmid and Schön.[4] These equations consist of a Boltzmann equation for the electron distribution

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*These modes were called the longitudinal and transverse modes, respectively, in Ref. 4.

function, coupled to a time-dependent Ginzburg–Landau equation (TDGL) for the order parameter. The phonons were assumed to remain in thermal equilibrium. The form of the kinetic equations is different in the two modes, the energy mode and the particle mode. In general, for a finite escape time of the phonons, a nonequilibrium phonon distribution is connected with the energy mode as well. In analogy to Ref. 4, one can derive the Boltzmann equation for the coupled electron–phonon system from microscopic theory, with the following result:

$$\frac{\partial}{\partial t}\begin{Bmatrix} N_1\,\delta f_E \\ (F_E/2N_0)\,\delta N_E \end{Bmatrix} = K\left(\begin{Bmatrix} N_1\,\delta f \\ (F_E/2N_0)\,\delta N_E \end{Bmatrix}\right) - \frac{1}{\tau_{es}}\begin{Bmatrix} 0 \\ (F_E/2N_0)\,\delta N_E \end{Bmatrix} + \frac{\partial}{\partial t}\begin{Bmatrix} h_E \\ 0 \end{Bmatrix} + I_E \tag{1}$$

We restrict ourselves in the following to homogeneous systems, and the notations and definitions are the same as in Ref. 4. Furthermore, δN_E is the deviation of the phonon distribution function from the equilibrium form, and F_E is the phonon density of states. The collision integral K couples electrons and phonons and is given by

$$\begin{aligned} N_1\,\dot{\delta f}_E\big|_{\text{coll}} = {} & -2\pi \int dE'\, M^{(L)}(E,E') \\ & \times \frac{\mu(E-E')}{\operatorname{sh}[(E-E')/2T]\operatorname{ch}(E/2T)\operatorname{ch}(E'/2T)} \\ & \times \left[\operatorname{ch}^2\frac{E}{2T}\,\delta f_E - \operatorname{ch}^2\frac{E'}{2T}\,\delta f_{E'} - \operatorname{sh}^2\frac{E-E'}{2T}\,\delta N_{E-E'}\right] \end{aligned} \tag{2}$$

$$\begin{aligned} \frac{F_E}{2N_0}\,\dot{\delta N}_E\big|_{\text{coll}} = {} & -2\pi \sum_{+,-} \int dE'\, dE''\, M^{(L)}(E',E'') \\ & \times \frac{\mu(E'-E'')}{\operatorname{sh}[(E'-E'')/2T]\operatorname{ch}(E'/2T)\operatorname{ch}(E''/2T)} \\ & \times (\mp)\frac{1}{4}\,\delta(E'-E''\mp E)\left[\operatorname{ch}^2\frac{E'}{2T}\,\delta f_{E'} - \operatorname{ch}^2\frac{E''}{2T}\,\delta f_{E''} \right. \\ & \left. -\operatorname{sh}^2\frac{E'-E''}{2T}\,\delta N_{E'-E''}\right] \end{aligned} \tag{2'}$$

The second term on the right-hand side of Eq. (1) describes the phonon escape, which is assumed not to depend on the phonon energy. In the limit of zero level broadening, the spectral function N_1 reduces to the normalized BCS density of state,[5] and $R_2 = (\Delta/E)N_1$. In this case, the collision operator, Eq. (2), has been derived by Bardeen *et al.*[6] using golden rule arguments.

In order to simplify the above equations, we introduce a short-hand notation by

$$|F\rangle = \begin{Bmatrix} \delta f \\ \delta N \end{Bmatrix}$$

Then the Boltzmann equation becomes

$$\frac{\partial}{\partial t} M|F\rangle = K|F\rangle - m_{\mathrm{es}}|F\rangle + \frac{\partial}{\partial t}|h\rangle + |I\rangle \tag{1'}$$

where the definitions of the various quantities follow by comparing (1) and (1′). Furthermore, we define a scalar product by

$$\langle F_1|F_2\rangle = \int dE \left[\delta f_1 \operatorname{ch}^2 \frac{E}{2T} \delta f_2 + \delta N_1 \operatorname{sh}^2 \frac{E}{2T} \delta N_2 \right] \tag{3}$$

The collision operator K has the following properties:

(a) K is Hermitian.

(b) All eigenvalues of K are smaller than or equal to zero. The eigenfunctions corresponding to the eigenvalue 0 are thermal distributions corresponding to a shifted temperature. This means $K|F_0\rangle = 0$, where

$$|F_0\rangle = \begin{Bmatrix} E/\operatorname{ch}^2(E/2T) \\ E/\operatorname{sh}^2(E/2T) \end{Bmatrix}$$

(c) K conserves the energy. This can be expressed in our notation by $\dot{E}|_{\mathrm{coll}} = 2N_0\langle F_0|K|F\rangle = 0$ for arbitrary $|F\rangle$.

Furthermore, one can split the collision operator into two contributions, which we call "scattering out" and "scattering in" terms. The scattering out term is given by

$$K_{\mathrm{out}}|F\rangle = -m|F\rangle = \begin{Bmatrix} (1/\tau_E)N_1\,\delta f \\ (1/\tau_E^{\mathrm{ph}})(F_E/2N_0)\,\delta N \end{Bmatrix} \tag{4}$$

where $1/\tau_E$ and $1/\tau_E^{\mathrm{ph}}$ are the temperature- and energy-dependent scattering rates for electrons and phonons, respectively. The scattering in term involves the remaining integrals of Eq. (2). In a normal metal, the energy dependence of the scattering rates is given by $1/\tau_E \simeq (1/\tau_0)(1 + v|E/T|^3)$ and $1/\tau_E^{\mathrm{ph}} = (1/\tau^{\mathrm{ph}})|E/T|$, where the former result is an approximation and $v = 2/21\zeta(3)$. Furthermore, we can express the ratio of the characteristic times τ_0 and τ^{ph} by the ratio of the specific heats of the normal electron and the phonon system, c_N^{el} and c^{ph}, respectively.* The relation for arbitrary temperature is as follows:

$$\frac{\tau^{\mathrm{ph}}}{\tau_0}=\frac{35\zeta(3)}{4\pi^2}\frac{c^{\mathrm{ph}}}{c_N^{\mathrm{el}}} \tag{5}$$

In a superconductor, numerical solution of an integral equation is necessary to calculate the energy dependence of $1/\tau_E$, since this rate in turn enters into the spectral quantities N_1 and R_2. An iterative procedure, however, converges rapidly.

Numerical methods have been used by Chang and Scalapino[7,8] to solve the Boltzmann equation. However, great accuracy is required in order to calculate correctly the effect of the superconducting state, because of the peak in the density of states at the gap edge. Thus we propose to employ another method, namely to replace the exact operator K by a reduced operator K_r, which has the same scattering out term as K and is given by

$$K_r=-m+\frac{m|F_0\rangle\langle F_0|m}{\langle F_0|m|F_0\rangle} \tag{6}$$

This collision operator is simple enough to allow an explicit solution of the Boltzmann equation, but on the other hand, still preserves the properties (a)–(c) of the exact operator. While both the exact operator K and the reduced K_r have the eigenfunction $|F_0\rangle$ with eigenvalue zero, the higher eigenfunctions of K are described by K_r only in a relaxation approximation. Notice that the choice of K_r is not unique.

3. THE ENERGY MODE

3.1. Relaxation Rate near T_c

When determining the free relaxation rate of a nonequilibrium distribution in the energy mode, we have to consider a situation without external perturbations, i.e., we set $|I\rangle$ equal to zero in Eq. (1). The term $|\dot{h}\rangle$, which is proportional to $\delta\dot{\Delta}$, can be eliminated by using the time-dependent Ginzburg–Landau equation. As long as $\Delta\tau_E \gg 1$, which excludes the case of a gapless superconductor, we obtain

$$|h\rangle=\gamma|r\rangle\langle r|F\rangle \tag{7}$$

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*c^{ph} is the specific heat of the phonons that couple to the electrons: in our model these are the longitudinal phonons.

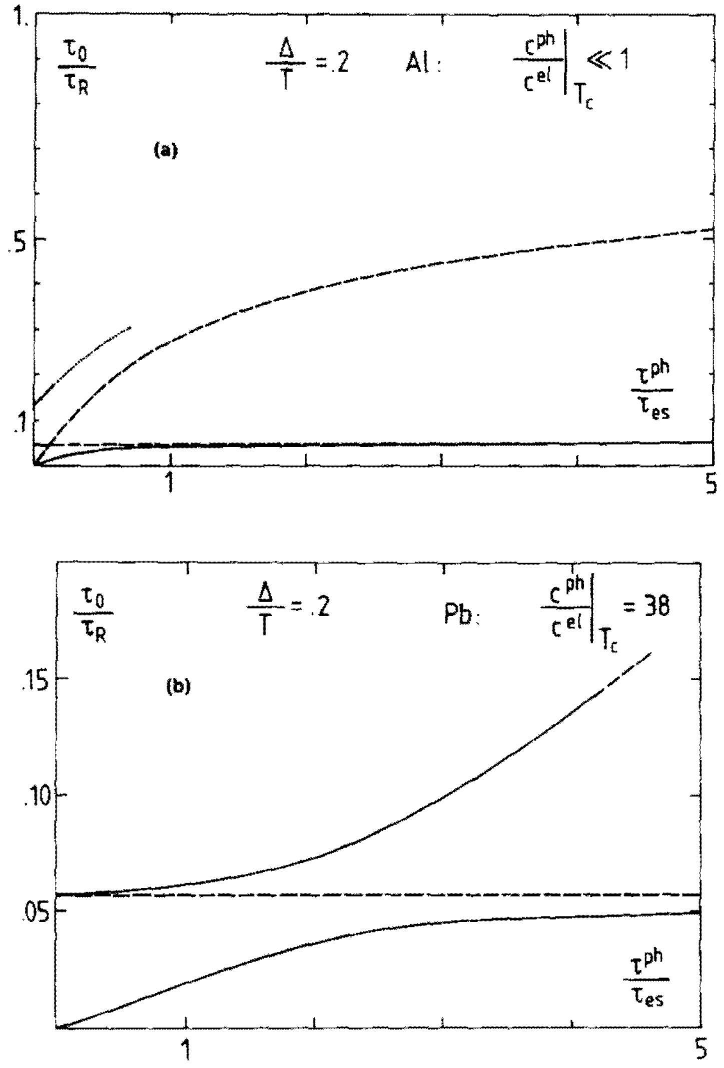

Fig. 1. The energy relaxation rate $1/\tau_R$ near T_c, plotted as a function of τ^{ph}/τ_{es} and compared to the normal state result $1/\tau_R^{(N)}$ (broken line). Δ/T takes the value 0.2. (a) c^{ph}/c^{el} is assumed to be small, appropriate to Al. (b) $c^{ph}/c^{el} = 38$, appropriate to Pb.

where $\gamma = \pi^2 T/[7\zeta(3)\Delta^2]$ and

$$|r\rangle = \left\{\begin{matrix}1\\0\end{matrix}\right\}\frac{R_2(E)}{\mathrm{ch}^2(E/2T)} \tag{8}$$

By solving the Boltzmann equation we find the solution

$$|F\rangle = \left(\frac{M}{\tau_R} - m - m_{es}\right)^{-1}\left\{-m|F_0\rangle + \frac{\langle r|m[(M/\tau_R)-m]^{-1}|F_0\rangle}{(\tau_R/\gamma)+\langle r|[(M/\tau_R)-m]^{-1}|r\rangle}|r\rangle\right\} \tag{9}$$

which is normalized such that

$$\langle F_0|m|F\rangle = \langle F_0|m|F_0\rangle \tag{10}$$

Equation (10) provides an implicit relation for $1/\tau_R$. We are interested only in slow modes, i.e., $1/\tau_R^{\prime}$ small compared to $1/\tau_0$ and $(1/\tau^{\mathrm{ph}}+1/\tau_{\mathrm{es}})$. Thus we expand the matrix elements in $[(m+m_{\mathrm{es}})\tau_R]^{-1}$. Up to the lowest order in Δ/T, we are allowed to neglect the energy dependence of $1/\tau_E$, use the normal state result for $1/\tau_E^{\mathrm{ph}}$, and also set $N_1=1$ and $R_2=\Delta/E$. The result is a quadratic equation in $1/\tau_R$, and the two solutions are plotted in Fig. 1 as a function of the escape rate at a fixed temperature. We distinguish the two cases $c^{\mathrm{ph}}/c^{\mathrm{el}} \ll 1$, which corresponds to materials like aluminum (Fig. 1a), and $c^{\mathrm{ph}}/c^{\mathrm{el}} \gg 1$, corresponding to lead, for example (Fig. 1b). For illustration, we include also the normal state result in Fig. 1a.

3.1.1. Normal State Relaxation

At small escape rates the distribution function is the temperature-shifted thermal function, as can be seen from Eq. (9) using $|r\rangle = 0$. Since the decay of this nonequilibrium distribution involves an energy transfer from the electron–phonon system to the surrounding material, the relaxation becomes arbitrarily slow as $1/\tau_{\mathrm{es}}$ goes to zero. By integration of the Boltzmann equation, we obtain the following equation for the change of the electron and phonon energy:

$$\frac{\partial}{\partial t}(\delta E_{\mathrm{el}}+\delta E_{\mathrm{ph}})+\frac{1}{\tau_{\mathrm{es}}}\delta E_{\mathrm{ph}}=0 \tag{11}$$

where

$$\begin{aligned} \delta E_{\mathrm{el}} &= 2N_0 \int EN_1\, \delta f_E\, dE \\ \delta E_{\mathrm{ph}} &= \int EF(E)\delta N_E\, dE \end{aligned} \tag{12}$$

From Eq. (11) we find the relaxation rate $1/\tau_R$. A particularly simple result is obtained when most of the energy is carried by the phonon system. In this case $1/\tau_R^{(N)} \simeq 1/\tau_{\mathrm{es}}$ and the energy transfer from the electron to the phonon system is of minor importance. More generally, since for *weak escape* the distributions are thermal, we obtain

$$\frac{1}{\tau_R^{(N)}}=\frac{1}{\tau_{\mathrm{es}}}\frac{c^{\mathrm{ph}}}{c^{\mathrm{ph}}+c_N^{\mathrm{el}}} \tag{13}$$

For *large escape rates*, on the other hand, the relaxation is restricted by the energy transfer from the electron to the phonon system and therefore is of order $1/\tau_0$, independent of the ratio of the specific heats.

3.1.2. Relaxation Below T_c

In a superconductor an additional branch resulting from the order-parameter relaxation appears, which has a typical relaxation rate of order $\Delta/T\tau_0$ for all escape rates. When the normal metal result $1/\tau_R^{(N)}$ is of the same order, we find the usual splitting and thus two branches in the spectrum. The eigenfunctions of the lower branch for small escape and the upper branch for large escape are approximately the distribution functions discussed above. Furthermore, the relaxation rate of the lower branch for small escape is again given by Eq. (13), with the replacement of c_N^{el} by c_S^{el}, the latter being the electronic specific heat of the superconductor.

The order-parameter relaxation is represented by the upper branch for weak escape and by the lower branch for strong escape. It is connected with the decay of a nonequilibrium distribution created by a change in the magnitude of the order parameter. We obtain the following limiting values for the relaxation rate *for weak escape*:

$$\frac{1}{\tau_R} = \begin{cases} \dfrac{7\zeta(3)}{\pi^3} \dfrac{\Delta}{T\tau_0} \dfrac{c_S^{\text{el}}}{c_N^{\text{el}}} & \text{for} \quad c^{\text{ph}} \ll c^{\text{el}} \qquad (14\text{a}) \\[2ex] \dfrac{7\zeta(3)}{\pi^3} \dfrac{\Delta}{T\tau_0} & \text{for} \quad c^{\text{ph}} \gg c^{\text{el}} \qquad (14\text{b}) \end{cases}$$

For strong escape, we find the same relaxation rate as given in Eq. (14b), independent of the ratio of the specific heats. This case has been studied in Ref. 4. In this limit, the phonons remain in thermal equilibrium ($\delta N_E \simeq 0$), and only relaxation between excitations and the order parameter is important. In this process the temperature of the electrons approaches the bath temperature.

For weak escape, on the other hand, the energy of the electron–phonon system can be considered constant during the order-parameter relaxation. In order to illustrate this case, we start with a situation where the order-parameter is changed while the temperature of the electrons is the bath temperature. Then the order parameter and the electrons relax to an equilibrium distribution with higher temperature. In other words, the thermal background of electrons and the nonthermal distribution approach one another (like the two blades of a pair of scissors), and this leads to an increase of the relaxation rate compared to the limit of strong escape proportional to $c_S^{\text{el}}/c_N^{\text{el}}$. In the case $c^{\text{ph}} \gg c^{\text{el}}$, however, the phonons effectively play the role of a heat bath, and the relaxation rate is the same as for strong escape.

As an illustration, we consider a system where only *particle–particle collision processes* are important. This applies, for example, to the superfluid phases of ^{3}He; for simplicity, we consider only the isotropic B phase. Using the approximation of the reduced collision operator, we find the relaxation rate *for zero escape*:

$$\frac{1}{\tau_R} = \frac{7\zeta(3)}{\pi^3} \frac{\Delta}{T\tau_p} \frac{c_S}{c_N} \tag{15}$$

where $1/\tau_p$ is the particle–particle collision rate on the Fermi surface. This is obviously analogous to the case $c^{\text{ph}} \ll c^{\text{el}}$. Note, however, that because of the direct coupling between particles and the surroundings, the relaxation rate *for strong escape* is given by

$$\frac{1}{\tau_R} = \frac{7\zeta(3)}{\pi^3} \frac{\Delta}{T\tau_{\text{es}}} \tag{16}$$

3.2. Stationary Nonthermal Situations

In this section, we examine systems driven out of thermal equilibrium due to the action of a perturbation. We consider two cases: (a) the irradiation of a superconductor by microwaves, and (b) the injection of electrons from a normal metal through a tunnel junction into a superconductor. The inhomogeneity entering into the Boltzmann equation is in case (a) given by[4]

$$|I_a\rangle = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \frac{W_0}{2N_0\nu^2} \{[N_1(E)N_1(E+) + R_2(E)R_2(E+)](f^0_{E+} - f^0_E) + (+ \leftrightarrow -)\} \tag{17a}$$

where $E\pm = E \pm \nu$; ν is the frequency of the electromagnetic radiation; and W_0 is the power absorbed in the normal state at the same electric field. In case (b), we have[4]

$$|I_b\rangle = \text{const} \times \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} N_1(E)(f^0_{E+} + f^0_{E-} - 2f^0_E) \tag{17b}$$

where $E\pm = E \pm eV$ and eV is the potential difference between normal metal and superconductor. Corresponding to the symmetry in the energy mode, we have written down only the part of $|I_b\rangle$ that is odd in energy.

In the stationary case, we easily find from Eqs. (1) and (6) the solution of the Boltzmann equation, and plots of δf_E and δN_E are given in Figs. 2 and 3 for both cases. We used the same parameters as Chang and Scalapino,[7] to allow comparison of the results. The electron distributions are, of course, dominated by the inhomogeneities, which in our examples act only on the electronic part. Therefore, as far as the electrons are concerned, it is sufficient to treat the influence of the scattering-in term in the global way as we do in our approximation.

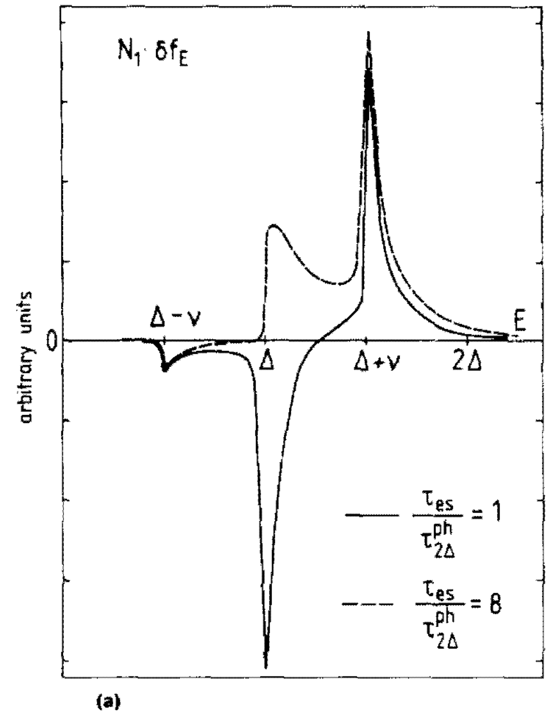

Fig. 2. The electron distribution multiplied by the reduced density of states $N_1\,\delta f_E$, plotted as a function of E. Only the part with $E>0$ is shown; the values for $E<0$ are obtained from $\delta f_E = -\delta f_{-E}$. The temperature is $T = 0.5T_c$; two different escape rates are considered: $\tau_{es}/\tau_{2\Delta}^{ph} = 1$ and 8. These parameters approximately agree with values chosen by Chang and Scalapino. The linewidth corresponds to lead: $1/\tau_0(T=T_c) = T_c/3$. (a) The system is excited by microwave radiation with $\nu = 0.5\Delta$. (b) The system is excited by tunnel injection with $eV = 1.5\Delta$.

A more serious test for our approximation scheme is the investigation of the phonon distribution, since the phonons are only indirectly excited by collisions with nonequilibrium electrons. In our approximation, δN_E does not depend on the form of the perturbation. A remarkable feature is the peak at energies $E \gtrsim 2\Delta$. This is due to the strong increase of the phonon relaxation rate $1/\tau_E^{ph}$ at the same energy, which in turn results from the possibility of scattering electrons across the gap by phonons with $E \gtrsim 2\Delta$. This process is known as phonon trapping. The reduced operator yields correct results for thermally shifted distributions; thus we expect K_r to be a good approximation in the case of electron injection, where δf_E approximately is of this form (compare also Ref. 7). In the case of microwave irradiation, on the other hand, the detailed form of δf_E is important; the shift of electrons from the gap edge to higher energies in this case results in an exact treatment in a negative δN_E for $E \gtrsim 2\Delta$, an effect which cannot be described by the reduced operator. To conclude, we mention another detail: since we are taking into account the linewidth in the density of states, we have, in contrast to the results of Chang and Scalapino,[7] no jumps in the distribution functions.

The effect of microwave radiation on the order parameter in superconductors[3,9,10] follows from the Ginzburg–Landau equation:

$$\left\{\frac{T_c - T}{T_c} - \frac{7\zeta(3)}{8\pi^2}\frac{\Delta^2}{T^2} + \chi\right\}\Delta = 0 \qquad (18)$$

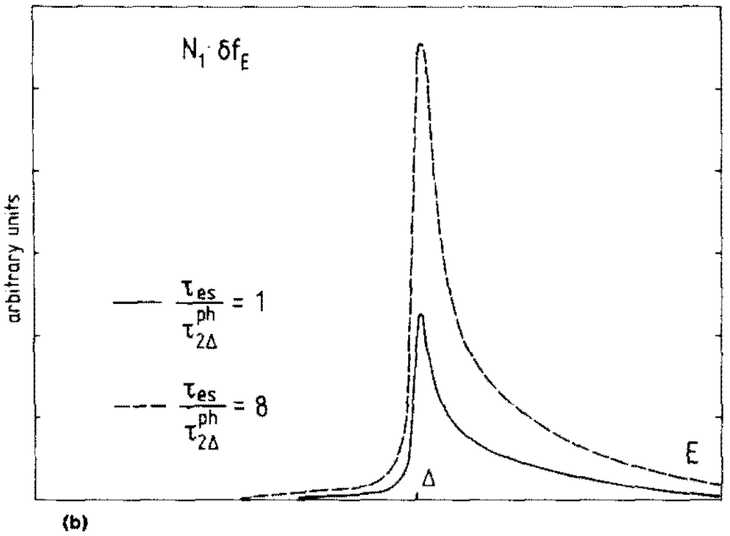

Fig. 2. Continued.

where $\chi = -\langle r|F\rangle/\Delta$. The electronic part of $|F\rangle$ consists of two contributions, one resulting directly from the inhomogeneity [Eq. (17a)], and the second from the scattering-in term. Correspondingly, we split $\chi = \chi^{\text{inh}} + \chi^{\text{coll}}$. Furthermore, the former part can be expanded in ν/T. The first contribution yields* $\chi^{(\text{rad})} = BG(\Delta^2/\nu^2)$, which leads to the gap enhancement and has been discussed by Schmid.[10] The second term, $\chi_{(2)}^{\text{inh}}$, which near T_c does not depend on Δ, can be interpreted together with χ^{coll} as an effective increase of the temperature, as follows:

$$\frac{T_c - T^*}{T_c} = \frac{T_c - T}{T_c} - (\chi_{(2)}^{\text{inh}} + \chi^{\text{coll}}) \tag{19}$$

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*We have $B = \tau_0 W_0/(8N_0 \nu T_c)$.

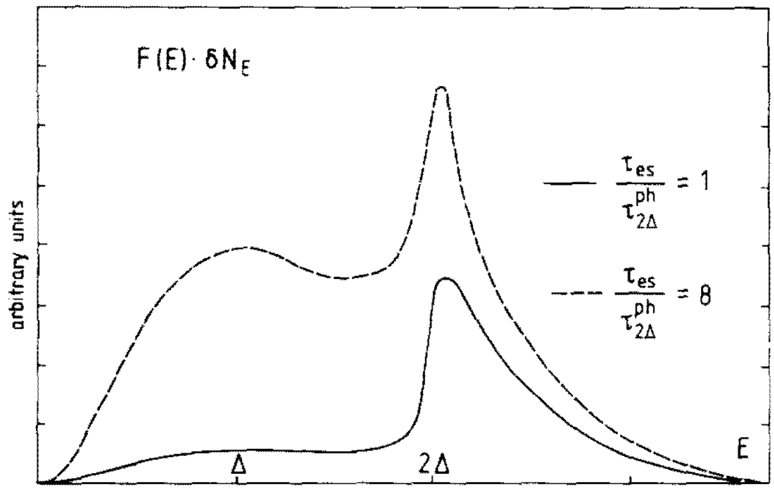

Fig. 3. The phonon distribution function δN_E for the same parameters as in Fig. 2. It does not depend on the special form of the electronic inhomogeneity.

We obtain $\chi_{(2)}^{\text{inh}} = 1.41 B\nu/T_c$ and $\chi^{\text{coll}} = \alpha B\nu/T_c$, where

$$\alpha = \begin{cases} [210\zeta(3)/\pi^4]\tau_{\text{es}}/\tau^{\text{ph}} & \text{for} \quad \tau_{\text{ph}}/\tau_{\text{es}} \ll 1 \\ 7\zeta(3)/15\zeta(5) & \text{for} \quad \tau_{\text{ph}}/\tau_{\text{es}} \gg 1 \end{cases} \tag{20}$$

In the limit of weak escape, this temperature increase is large, since the energy cannot be transferred to the bath. If the escape is strong, a comparison with the result from a variational calculation[11] reveals a weakness of the reduced operator. In this limit, we find $T^* - T = 1.95 B\nu$, whereas $(T^* - T)_{\text{variation}} = 0.7 B\nu$. We will discuss this deviation in Chapter 6.

4. ENERGY RELAXATION VERSUS PARTICLE RELAXATION NEAR T_c

In certain experiments, e.g., in the case of electron injection through a tunnel barrier as in the experiments of Clarke and Paterson,[2] the particle mode is also excited. When deriving the kinetic equations of this mode, it is important to realize that due to the properties of the superconducting ground state, the particle number is not a conserved quantity in collision processes.

For definiteness, we write down the Boltzmann equation in the stationary case. In this mode, the perturbation I_E is an even function of energy; thus the total energy remains constant and nonthermal phonons are not excited. We have

$$-K^{(T)}(\delta f_E) + 2\,\Delta N_2\,\delta f_E = I_E \tag{21}$$

The collision integral is split into a number-conserving and a nonconserving part, which form the left-hand side of Eq. (21). The integral kernel of $K^{(T)}$ differs from the collision operator of the energy mode in that the coherence factor $M^{(L)}(E, E')$ has to be replaced by the quantity $M^{(T)}(E, E') = N_1(E)N_1(E')$. The difference can be neglected near T_c, but is important at low temperatures, and this will be considered in the next section.

In correspondence to the technique of Section 2, we replace the conserving part $K^{(T)}$ by a reduced operator guaranteeing particle number conservation:

$$K^{(T)}(\delta f) \to K_r^{(T)}(\delta f)$$

$$= -\frac{1}{\tau_E^*} N_1\,\delta f + \frac{1}{\tau_E^*}\,\frac{N_1}{\operatorname{ch}^2(E/2T)}\,\frac{\int dE\,(1/\tau_E^*) N_1\,\delta f}{\int dE\,(1/\tau_E^*) N_1/\operatorname{ch}^2(E/2T)} \tag{22}$$

The eigenfunction with eigenvalue zero in this case is the distribution function describing a shift in the chemical potential. It is straightforward to solve for δf_E and calculate the relaxation time, given by the ratio of nonequilibrium electrons to injected electrons, or

$$\tau_R^{(T)} = \frac{\int dE\, N_1\, \delta f}{2\Delta \int dE\, N_2\, \delta f} \tag{23}$$

Furthermore, in the injection case, the perturbation is given by[4]

$$I_E = P_E^{(T)} = \text{const} \times N_1 \{ f_{E-}^0 - f_{E+}^0 \} \tag{24}$$

Near T_c, we may neglect the energy dependence of the inelastic scattering rate. Furthermore, there is no difference between τ_E and τ_E^*, and we reproduce the result of Ref. 4, namely

$$\tau_R^{(T)} = \frac{4T}{\pi\Delta}\left[\frac{\tau_0}{2\Gamma}\left(1 + \frac{\Gamma}{2\tau_0\Delta^2}\right)\right]^{1/2} \tag{25}$$

This result is valid for arbitrary pair-breaking Γ, for example, due to inelastic electron–phonon collisions and paramagnetic impurities: $\Gamma = 1/\tau_s + 1/2\tau_0$.

At this stage, a comparison between particle relaxation, Eq. (25), and energy relaxation is of interest, the latter in the limit of infinite escape rate for simplicity. To lowest order in Δ/T, we find, extending the investigation of Section 3 to arbitrary pair-breaking, that

$$\tau_R^{(L)} = \frac{\pi^3}{7\zeta(3)} \frac{T\tau_0}{\Delta}\left[\left(1 + \frac{\Gamma^2}{\Delta^2}\right)^{1/2} - \frac{\Gamma}{\Delta} + \frac{1}{2\Delta\tau_0}\right] \tag{26}$$

If the pair-breaking results only from inelastic phonon scattering, i.e., $\Gamma = 1/2\tau_0$, or in the gapless region $\Delta\tau_0 \ll 1$, we have the simple result

$$\tau_R^{(T)}/\tau_R^{(L)} = 28\zeta(3)/\pi^2 \simeq 3.41 \tag{27}$$

On the other hand, when $\Delta \gg \Gamma \gg 1/2\tau_0$, there is no simple expression for the ratio of the two rates.

5. PARTICLE MODE AT LOW TEMPERATURES

In this section, we will extend the investigation of the particle mode to low temperatures. In this limit, we have to take into account the temperature and energy dependence of the inelastic scattering rate $1/\tau_E^*$ and of the quantity $2\Delta N_2/N_1$. For reasons obvious from the last section, we call the latter the *conversion rate*. The conversion rate can be represented by

$$\frac{2\Delta N_2}{N_1} \simeq \frac{\Delta^2}{E^2 - \Delta^2} \frac{1}{\tau_E} \tag{28}$$

Notice that $1/\tau_E$, not $1/\tau_E^*$, enters into the spectral quantities N_1 and N_2 and appears in Eq. (28), and that this approximate form is valid outside the gapless region and in stationary situations. We recognize that the conversion of electrons becomes favorable near the gap edge, and this leads always to a decrease of the distribution function for $|E| \gtrsim \Delta$.

A straightforward integration (compare also Kaplan *et al.*[12]) yields the following results:

$$\frac{1}{\tau_E^*} = 0.82\frac{1}{\tau_0}\left(\frac{T}{\Delta}\right)^{5/2}\left[1+0.6\left(\frac{|E|-\Delta}{T}\right)^{5/2}\right] \tag{29}$$

$$\frac{1}{\tau_E} = 1.7\frac{1}{\tau_0}\left(\frac{T}{\Delta}\right)^{7/2}\left[1+0.43\left(\frac{|E|-\Delta}{T}\right)^{7/2}\right] \tag{29'}$$

These expressions are valid at low temperatures† $T \ll \Delta$ and not too large energies ($|E| - \Delta \lesssim \Delta$). Concentrating first on the case of thermal injection, i.e., $eV \ll T$, into a system where no paramagnetic impurities are present, we calculate the distribution function, applying the scheme of the reduced operator, and find the following relaxation time:

$$\tau_R^{(T)} \simeq 0.8(\Delta/T)^{5/2}\tau_0 = 1.4(T/\Delta)\tau_{E=\Delta} \tag{30}$$

Except for a numerical factor, this is the same result as near T_c for $\Delta\tau_0 \gg 1$. It is important, however, that $1/\tau_E$ and not the scattering-out rate $1/\tau_E^*$ of this mode appears in the final expression. Due to the increase of the conversion near the gap edge, the electrons at these energies are converted very rapidly. Therefore, the distribution function is $\delta f_E \sim [(|E|-\Delta)/\Delta] \exp(-|E|/T)$ for $\Delta \lesssim |E| \lesssim \Delta + T$. For larger energies, δf_E decreases exponentially.

If we consider injection at a higher voltage, $T \ll eV - \Delta$, a nonequilibrium distribution exists in a broad energy range $\Delta \leq |E| \leq eV$. Furthermore, since the relaxation time at low temperatures [see Eqs. (29) and (30)] is expected to be of the same order as the inelastic scattering rate $1/\tau_E^*$, the scattering processes are not able to build up an approximate thermal distribution. Instead of being scattered, electrons are directly converted. This means, however, that the condition imposed on $K_r^{(T)}$ to be exact for thermal distributions is not sufficient anymore.

For this reason we proceed in a completely different way[13] to study explicitly the influence of the scattering processes in this case. The following calculation relies on the fact that the scattering in term in the Boltzmann equation for $T \ll \Delta$ is approximately independent of temperature.

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† In the low-temperature region, scattering processes involving phonons with energy $E \gtrsim 2\Delta$ do not contribute to $1/\tau_E$ and $1/\tau_E^*$ (they vanish exponentially). This is in contrast to the energy mode, where these processes, often called *recombination processes*, provide the only relaxation mechanism.

Comparing the distribution function for $T=0$ and for $0 \neq T \ll \Delta$, we find that δf_E differs only in the energy range $\Delta \lesssim |E| \lesssim \Delta + T$, which is small compared to the injection interval. Consequently, we have $K_{\text{in}}(T=0) \simeq K_{\text{in}}(0 \neq T \ll \Delta)$, which leads to

$$[\Gamma(E)\,\delta f_E]_{T=0} \simeq [\Gamma(E)\,\delta f_E]_{0 \neq T \ll \Delta} \tag{31}$$

where

$$\Gamma(E) = \frac{1}{\tau_E^*} + \frac{\Delta^2}{E^2 - \Delta^2}\,\frac{1}{\tau_E} \tag{32}$$

In the limit $T=0$, on the other hand, the integral equation can be reduced to a differential equation with boundary conditions for $|E| = eV$. At $T=0$ electrons are scattered *into* a given energy level $E>0$ only from higher energies, and thus the integration in the scattering-in part of the collision integral extends from E to eV. By differentiation of the kinetic equation with respect to $|E|$, we reduce it to the following differential equation:

$$z'''(E) + 1.07\,\frac{1}{(|E|-\Delta)^3}\,z(E) = 0 \tag{33}$$

where $z(E) = \Gamma(E)\,\delta f_E$. The boundary conditions are $z'(|E| = eV) = z''(|E| = eV) = 0$, while $z(|E| = eV)$ takes some constant value, depending on the injection current. This differential equation is of the Euler type and can be solved easily. The solution, which is divergent near the gap edge, can be written as follows:

$$[\delta f_E]_{T=0} = \text{const} \times \left(\frac{\Delta}{|E|-\Delta}\right)^{(5/2)+\rho} \tag{34}$$

with $\rho \simeq 0.34$. Thus we obtain, using Eqs. (29) and (31),

$$[\delta f_E]_{0 \neq T \ll \Delta} \simeq [\delta f_E]_{T=0}\left(\frac{|E|-\Delta}{T}\right)^{7/2} \simeq \text{const} \times \left(\frac{\Delta}{T}\right)^{7/2}\left(\frac{|E|-\Delta}{\Delta}\right)^{1-\rho} \tag{35}$$

for $|E| - \Delta \lesssim T$, and

$$[\delta f_E]_{0 \neq T \ll \Delta} \simeq [\delta f_E]_{T=0} = \text{const} \times \left(\frac{\Delta}{|E|-\Delta}\right)^{(5/2)+\rho} \tag{35$'$}$$

for energies $|E| - \Delta \gtrsim T$. Then we find the relaxation time

$$\tau_R^{(T)} \sim \left(\frac{\Delta}{eV-\Delta}\right)^{(1/2)-\rho}\left(\frac{\Delta}{T}\right)^{2+\rho}\tau_0 \tag{36}$$

The proportionality factor in Eq. (36) is of order one, and one has to solve numerically the differential equation (33) to determine it. Compared to the case of thermal injection, the relaxation time is slightly reduced, since conversion over a broad energy interval is more effective.

Comparing with experimental results at low temperatures,[2] we find the agreement unsatisfactory. Only some of the probes show an increase of the relaxation time in this limit. But it should be noted that our results depend critically on the level broadening of the spectral quantities N_1 and N_2, which we assumed to be entirely due to inelastic electron–phonon scattering. Anomalies in the N–S conductivity, on the other hand, indicate that this ideal case is not realized in the samples, and other pair-breaking effects are important.

Finally, we want to mention that the transverse relaxation time is precisely the relevant quantity entering into the determination of the linewidth of the unusual magnetic resonances[14] found in ^{3}He-A and -B. In this case, however, the inelastic particle–particle collision time, which has a completely different temperature dependence,[15] appears in the final result.

6. DISCUSSION AND CONCLUSION

To conclude, we want to add some comments on the approximation scheme that we have adopted in this paper. The method has been used before, among others by Wölfle,[16] in order to treat particle–particle collisions. In this case, the eigenvalues and eigenfunctions of the exact operator for a normal system are known,[17] which allows comparison of the exact and approximate results. Application to collective phenomena such as sound propagation or viscosity yields good agreement, the deviations being of the order of a few percent only.

We wish to stress that in the approximation, the following properties of the exact operator are preserved: (a) The conservation laws retain their validity; (b) both operators have the same eigenvalue zero and the same corresponding eigenfunctions, which are thermal distribution functions with shifted temperature in the energy mode and shifted chemical potential in the particle mode.

Therefore, the use of K_r is justified whenever processes are considered that evolve slowly in time; this means when the process is dominated by the decay of the eigenfunction with the smallest eigenvalue. In a superconductor near T_c, the coupling to the order parameter leads to a considerable slowing down of the relaxation rate; thus our results are fully reliable as long as $1/\tau_R \ll 1/\tau_E^*$. In Fig. 1 this is satisfied everywhere except for the normal state result in the limit of strong escape, which can be considered only an order-of-magnitude estimate.

More generally, we conclude that the use of the reduced operator is justified in the description of collective phenomena where it is essential that the conservation laws are satisfied. Accordingly, the approximation scheme applies to the hydrodynamic modes in superconductors.[18]

On the other hand, when we use a reduced operator to calculate the distribution functions, the results are not always satisfactory. In this case, a criterion can be formulated as follows: assume the eigenfunctions of the exact collision operator to be known and expand the inhomogeneity in the Boltzmann equation in this set of functions. Whenever the eigenfunction with eigenvalue zero is dominant in this expansion, the approximation of K by K_r is valid. This does apply to the problem of electron injection in both modes near T_c and becomes exact for small injection voltages. On the other hand, it does not apply to the particle mode at low temperatures for $eV > \Delta$, since in this case the relaxation time is of the same order as the inelastic scattering rate τ_E^*. Thus, one has carefully to take into account the details of the collision processes, as described in Section 5.

A further shortcoming follows from a special property of the scattering-in part of the collision operator in the energy mode. The term "scattering in" in this case is actually somewhat misleading: in the particle mode or in the case of particle–particle scattering this term always leads to an increase of the distribution function at a specified energy level due to the decay at all other energies. In the energy mode, however, δf_E for small energies is further reduced by the scattering-in term. This is due to the fact that δf_E is an odd function of energy, and that the transition probability has a maximum at energy differences of about $2T$. The reduced scattering-in term, on the other hand, always causes an increase in δf_E. This property, combined with the importance of low-energy electrons in the calculation of the temperature enhancement by microwave irradiation, is responsible for the discrepancy between our result and the result obtained from the variational calculation for $T_c - T^*$.

We thus conclude that a reduced operator cannot exactly describe all details and will lead to errors in certain cases. On the other hand, in the description of collective phenomena which depend essentially on the conservation laws, its application is fully justified. Furthermore, compared to numerical solutions, it allows a more rapid application to different problems characterized by different inhomogeneities in the Boltzmann equation. In addition, this method may reveal more of the underlying physics, since the equations and solutions become transparent.

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