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Quasiclassical Approach to Kinetic Equations for Superfluid ^3He : General Theory and Application to the Spin Dynamics

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Using the method of the quasiclassical Green function, we derive a set of kinetic equations which describe general nonequilibrium situations in the quasiclassical regime, i.e., when the external frequency and wave vector, ω and \mathbf{q} are small compared to the atomic scale ($\omega \ll \mu$, $|\mathbf{q}| \ll p_F$). The equations consist of a Boltzmann equation for the quasiparticle distribution function, labeled by the energy and the direction of the momentum (particle representation), coupled to a time dependent Ginzburg–Landau equation for the order parameter. We discuss extensively the properties of these equations, and apply them to orbital and spin dynamics. Solving the Boltzmann equation in a well defined approximation, we are able to derive the expressions for the linewidths for all temperatures, with the correct identification of the phenomenological relaxation times. Furthermore, we discuss the connection between various relaxation times used in non-equilibrium situations, and we give a detailed comparison of the particle representation with the excitation representation which is used frequently in other work on non-equilibrium phenomena in superfluid ^3He and in superconductors.

1. INTRODUCTION

Since their discovery [1], a lot of experimental and theoretical work has been devoted to the new phases (for experimental and theoretical reviews, see Wheatley [2] and Leggett [3], respectively; a more recent theoretical review was also given by Wölfle [4]). It is now believed that the superfluid phases are a realization of a BCS-like state in which the Cooper pairs have a relative angular momentum of $l = 1$, and spin $S = 1$, in contrast to the $l = 0$ —spin singlet pairing of electrons. Furthermore, there is now strong confirmation that the A-phase has to be identified with the ABM-state [5, 6], and the B-phase with the BW-state [7]. The A_1 -phase is believed to show the pairing of one spin species only.

In fact, the existence of more than one superfluid phase, and also the behavior of some thermodynamic quantities, for example, the specific heat jump in the normal to A-phase transition was rather surprising, since the BCS-approximation (for $l = 1$ -pairing) leads to a unique absolute minimum, namely the BW-state. This puzzle could

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be explained by Anderson and Brinkman [6] who showed that the formation of the superfluid state modified the pairing interaction, and that the precise nature of the modification depends on the particular kind of state which is formed. More generally, as shown by Rainer and Serene [8], this feedback mechanism can be understood as a correction to the BCS-approximation (“strong coupling corrections”), which are of order T_c/μ small¹ (T_c is the transition temperature, and μ the chemical potential). Although this quantity is small, $\sim 10^{-3}$, it is still one order of magnitude greater than it is in most superconductors, and only small energies are needed to stabilize one phase against the other if both have the same relative angular momentum. The strong coupling theory is now in good qualitative agreement with experimental results.

Besides the static properties, the new phases show a variety of unexpected time dependent phenomena, for example, the behavior seen in ultrasound experiments, where a strong increase of absorption is found due to the coupling of density variations to internal collective modes of the order parameter, or the nuclear magnetic resonance experiments, which show unusual resonance frequencies due to the crucial role of the nuclear dipole energy (compare the cited reviews [2–4] and references therein). In general, a description of dynamic situations can be done on different levels, depending on the type of questions to be answered: A hydrodynamic description, relying on the conservation laws and the symmetries, and introducing dissipation through various parameters; the use of kinetic equations which are a coupled set for the quasiparticle distribution function and the order parameter; or calculations involving the full Green functions.

Starting from microscopic theory, i.e., from the equation of motion for the full Green function, we derive in this paper a set of kinetic equations for superfluid ^3He . As is generally accepted for time dependent phenomena, we neglect strong coupling corrections of the type discussed above, relying thus on the model of a gas of Landau quasiparticles [9], and introducing an additional p -wave pairing force whose strength is a parameter of the theory. For the derivation, we use the quasiclassical approach [10–14], i.e., the Green function integrated with respect to the variable $\xi_{\mathbf{p}} = \mathbf{p}^2/2m^* - \mu$, where \mathbf{p} is the momentum, and m^* the effective mass. This approach can be applied in the whole quasiclassical regime, i.e., when the external frequency and wave vector, ω and \mathbf{q} , are small compared to the atomic scale ($\omega \ll \mu$, $|\mathbf{q}| \ll p_F$). A great advantage of this method is that the quasiclassical Green function obeys a normalization condition, and this allows the definition of a scalar distribution function (in contrast to the matrix function of Betbeder–Matibet and Nozières [15]); i.e., it will be diagonal in the additional particle–hole index characteristic for superconductivity. Furthermore, the distribution function is labeled by the energy, E , and the direction of the momentum, \mathbf{p}_F , of the quasiparticles, which allows one to include easily strong pair breaking situations such as superconductors with paramagnetic impurities or the gapless case very near T_c .

The plan of the paper is as follows: In Chapter 2, we rewrite the microscopic equations of motion into a form convenient for the quasiclassical approach, using the real

¹ We use units such that $\hbar = k_B = 1$.

time technique as developed by Keldysh [16] and applied to superconductors by Larkin and Ovchinnikov [13]. After describing in detail the model appropriate for superfluid ^3He , we discuss the validity of the quasiclassical approximation, and define the distribution function. The kinetic equations near T_c are then considered and applied to the orbital dynamics. Rotations are briefly discussed. Chapter 3 is devoted to the spin dynamics of the A- and B-phase. We study in detail the longitudinal and transverse resonance of the A-phase, and the longitudinal resonance of the B-phase, with emphasis on the damping of the collective modes for all temperatures. The equations (and their solutions) are easily extended to higher frequencies (arbitrary $\omega\tau_E$), where τ_E is a typical inelastic scattering time). The non-linear ringing equations are derived [17, 18], with the correct identification of the phenomenological relaxation times. In Chapter 4, we discuss different relaxation mechanisms and times (compare also [19, 20]) and connect our approach to kinetic equations with the more common one, namely, the concept of Bogoliubov quasiparticles which we call excitation representation (a very complete presentation of this concept was given by Aronov and Gurevich [21]). Final conclusions are found in Chapter 5.

2. FROM MICROSCOPIC THEORY TO KINETIC EQUATIONS

(A) Equation of Motion

The well known perturbation theory is based on a diagrammatic expansion of the Green function with respect to the interaction. At finite temperatures, one has to deal with a Green function which is defined for “imaginary time” or, after Fourier expansion, for a set of discrete points in the complex plane, and physical quantities are obtained through an analytic continuation. If we apply a time dependent external perturbation, the Green function depends on two frequencies with the result that also so called “anomalous” contributions appear [11], besides the usual retarded and advanced quantities. This procedure has been successfully applied; however, it seems generally preferable to avoid this process and start directly with the “physical” quantities defined for real times or real energies. At the cost of introducing a slightly more complicated path formulation of the time development (compared to the zero temperature formalism), or, equivalently, by introducing matrix functions [16], one may describe the theory in an elegant way.

The Green function appropriate for a description of the superfluid phases is a 4×4 -matrix function. We define:

$$\begin{aligned}\mathcal{G}_{\alpha\beta}^>(1, 2) &= -i(\hat{\rho}_3)_{\alpha\gamma}\langle\psi_\gamma(1) \psi^\beta(2)\rangle; \\ \mathcal{G}_{\alpha\beta}^<(1, 2) &= i(\hat{\rho}_3)_{\alpha\gamma}\langle\psi^\beta(2) \psi_\gamma(1)\rangle,\end{aligned}\tag{2.1}$$

where the fermion field operators are in the Heisenberg picture; their arguments given by $1 \equiv (\mathbf{r}_1, t_1)$, $2 \equiv (\mathbf{r}_2, t_2)$; and $\langle \rangle$ denotes the average with the free density

matrix. Furthermore, $\alpha, \beta, \gamma = 1, \dots, 4$ (implicit summation over γ), and the field operators are connected with the usual ones by

$$\psi_1 = \psi_\uparrow, \quad \psi_2 = \psi_\downarrow, \quad \psi_3 = \psi_\uparrow^+, \quad \psi_4 = \psi_\downarrow^+,$$

and

$$\psi^\alpha = (\hat{\rho}_1)_{\alpha\beta} \psi_\beta.$$

We also use the direct product notation between Pauli matrices in particle-hole ($\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3$) and spin ($\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3$) space. For example, $\hat{\rho}_1 \equiv \hat{\rho}_1 \cdot \hat{1}$ is the direct product of the first Pauli matrix and the unit matrix in particle-hole and spin space, respectively. Compared to the usual definition, Eq. (2.1) is “rotated” in Nambu space which is essential for the quasiclassical approach. From the Green function, one may obtain physical quantities in the well known way. For example, the particle number density and the current density are given by

$$\begin{aligned} \rho(1) &= -i\frac{1}{2} \text{Tr}_4 \hat{\mathcal{G}}^<(1, 1); \\ \mathbf{j}(1) &= -(1/4m) \text{Tr}_4 \hat{\rho}_3 [(\nabla_{r_1} - \nabla_{r_2}) \hat{\mathcal{G}}^<(1, 2)]_{1=2}, \end{aligned} \quad (2.2)$$

where m is the mass of a ^3He atom, and Tr_4 denotes the trace in the direct product space. In the rest of this chapter, we consider only situations in which the spin is not important; applied to spin independent functions, $\text{Tr}_4 = 2\text{Tr}_2$, where Tr_2 denotes the trace in particle-hole space (which we write Tr for simplicity).

In constructing the equation of motion for the Green functions, we follow the method of Keldysh [16], as it was also used by Larkin and Ovchinnikov [13] for superconductors. In this formalism, one introduces the concept of time ordering along a certain contour C , which runs from $-\infty$ to $+\infty$ and then back to $-\infty$, along the real axis, and defines the corresponding path ordered Green function. This Green function obeys a Dyson equation of usual type, however, with time integrations performed along the path C . For practical calculations, it is more convenient to have all integrations from $-\infty$ to $+\infty$, and this leads to an additional 2×2 -matrix description since one has to indicate whether t_1, t_2 are on the forward or on the backward branch of the contour. It is most convenient to introduce a matrix of the following form

$$\check{\mathcal{G}}(1, 2) = \begin{pmatrix} \hat{\mathcal{G}}^R(1, 2) & \hat{\mathcal{G}}^K(1, 2) \\ 0 & \hat{\mathcal{G}}^A(1, 2) \end{pmatrix}, \quad (2.3)$$

where we defined the retarded, the advanced, and the Keldysh function $\hat{\mathcal{G}}^K$ through the relations:

$$\begin{aligned} \hat{\mathcal{G}}^R &= \Theta(t_1 - t_2)(\hat{\mathcal{G}}^> - \hat{\mathcal{G}}^<); \\ \hat{\mathcal{G}}^A &= -\Theta(t_2 - t_1)(\hat{\mathcal{G}}^> - \hat{\mathcal{G}}^<); \\ \hat{\mathcal{G}}^K &= \hat{\mathcal{G}}^> + \hat{\mathcal{G}}^<, \end{aligned} \quad (2.4)$$

where $\Theta(t)$ is the usual step function.

In deriving the equation of motion for $\check{\mathcal{G}}$ from the Dyson equation for the path ordered Green function, some identities are useful [22]. Consider, for example, expressions like

$$D(t_1, t_2) = \oint_C d\bar{t} A(t_1, \bar{t}) B(\bar{t}, t_2)$$

and define $D^>$ and $D^<$ as usual. Then one finds immediately

$$D^<(t_1, t_2) = \int_{-\infty}^{\infty} d\bar{t} [A^R(t_1, \bar{t}) B^<(\bar{t}, t_2) + A^<(t_1, \bar{t}) B^A(\bar{t}, t_2)], \quad (2.5)$$

and similarly for $D^>$ by interchanging “<” into “>” in the above expressions. The equation of motion for $\check{\mathcal{G}}$ then has the form

$$\begin{aligned} (\hat{Q}_0 - \check{\Sigma}) * \check{\mathcal{G}} &= \check{1} \delta(1 - 2), \\ \check{\mathcal{G}} * (\hat{Q}_0 - \check{\Sigma}) &= \check{1} \delta(1 - 2). \end{aligned} \quad (2.6)$$

In these equations,

$$\hat{Q}_0(1, 3) = \left[i \frac{\partial}{\partial t_1} \hat{p}_3 + \frac{1}{2m} \nabla_{\mathbf{r}_1}^2 + \mu - U_{\text{ext}}(1) \right] \delta(1 - 3) \quad (2.7)$$

is the single particle Hamiltonian with energy measured from the (unperturbed) Fermi level, and $\check{\Sigma}$ denotes the self energy matrix defined analogous to Eq. (2.3). A “transposed” differentiation of the form $\check{\mathcal{G}}(1, 2)(\partial/\partial 2)$ is meant to be equal to $-(\partial/\partial 2)\check{\mathcal{G}}(1, 2)$, and the $*$ -product is defined by:

$$A * B = \int_{-\infty}^{\infty} d3 A(1, 3) B(3, 2). \quad (2.8)$$

Furthermore, the notation \hat{Q}_0 implies that this quantity is a unit matrix in the Keldysh index, as well as the kinetic energy and the external potential U_{ext} are unit matrices also in particle-hole space. From Eq. (2.6), we immediately conclude that the equation of motion for $\check{\mathcal{G}}$ can be written in the form of a commutator, namely,

$$[\hat{Q}_0 - \check{\Sigma}^*, \check{\mathcal{G}}] = 0, \quad (2.9)$$

which is very essential for the following.

Finally, it is more convenient for most applications to consider the Green function as a function of the variables momentum \mathbf{p} , energy E , space \mathbf{r} , and time t , where $\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2)/2$, $t = (t_1 + t_2)/2$, and \mathbf{p} and E are obtained by Fourier transformation with respect to the relative coordinates. Then we find

$$\hat{Q}_0(\mathbf{p}, E, \mathbf{r}, t) = E\hat{p}_3 - (\mathbf{p}^2/2m) + \mu - U_{\text{ext}}(\mathbf{r}, t), \quad (2.10)$$

and the “*” is in this representation given by

$$A * B = \{\exp((i/2)[\partial_E^A \cdot \partial_t^B - \partial_t^A \cdot \partial_E^B - \partial_p^A \cdot \partial_r^B + \partial_r^A \cdot \partial_p^B])\}(AB), \quad (2.11)$$

where A and B are functions of \mathbf{p} , E , \mathbf{r} , t , and ∂_E^A means differentiation with respect to E of A only, etc. Up to first order, one finds:

$$A * B = A \cdot B + \frac{i}{2} \left[\frac{\partial A}{\partial E} \frac{\partial B}{\partial t} - \frac{\partial A}{\partial t} \frac{\partial B}{\partial E} - (\nabla_p A)(\nabla_r B) + (\nabla_r A)(\nabla_p B) \right], \quad (2.12)$$

and it should be noted that this product is associative which follows also from Eq. (2.8):

$$A * (B * C) = (A * B) * C, \quad (2.13)$$

where A , B , C are three arbitrary functions.

Up to now, we only have rewritten the microscopic equations into a form which will be convenient for the following argument. Like the original Hamilton operator, the equation of motion, Eq. (2.9), possesses the full symmetry of the system under consideration, and one may easily deduce all conservation laws, provided one chooses a conserving approximation for the self energy. Particularly, the equations are translation and rotation invariant. As an example, we study now the latter case in more detail.

We consider time dependent rotations of the coordinate system characterized by a vector $\Theta(t)$, whose absolute value is the angle of rotations around an axis given by $\Theta / |\Theta|$; the corresponding rotation matrix is denoted by $\mathcal{R}(t)$, with the property

$$\frac{\partial}{\partial t} \mathcal{R} \mathbf{a} = \boldsymbol{\omega} \times (\mathcal{R} \mathbf{a}) \quad (2.14)$$

for any time independent vector \mathbf{a} , where $\boldsymbol{\omega} = \dot{\Theta}$. The unitary rotation operator is then given by

$$T = \exp[-i(\mathbf{r} \times \mathbf{p}) \cdot \Theta] \quad (2.15)$$

and, from the definition, Eq. (2.11), we immediately find:

$$\begin{aligned} \tilde{E} &= T^+ * E * T = E + \boldsymbol{\omega}(\mathbf{r} \times \mathbf{p}), \\ \tilde{\mathbf{p}} &= T^+ * \mathbf{p} * T = \mathcal{R} \mathbf{p}, \quad \tilde{\mathbf{r}} = \mathcal{R} \mathbf{r}, \end{aligned} \quad (2.16)$$

and we denoted the quantities in the rotating frame by \tilde{E} , etc. Furthermore, it is clear that the self energy, although a complicated functional of the Green function, has the property

$$\tilde{\Sigma}(\mathbf{p}, E, \mathbf{r}, t) = \tilde{\Sigma}[\tilde{\mathcal{G}}] = \tilde{\Sigma}(\tilde{\mathbf{p}}, \tilde{E}, \tilde{\mathbf{r}}, t),$$

which can be studied explicitly for the model to be discussed below, and we conclude that the Green function in the rotating system, $\tilde{\mathcal{G}}$, is connected with the function in the rest frame, \mathcal{G} , by

$$\tilde{\mathcal{G}}(\mathbf{p}, E, \mathbf{r}, t) = \mathcal{G}(\tilde{\mathbf{p}}, \tilde{E}, \tilde{\mathbf{r}}, t). \quad (2.17)$$

We want to stress at this point that we need the equation of motion for the full Green function for these considerations, with the full expansion in ∂_t and ∂_r ; similar considerations apply for Galilean transformations. We return to a discussion of this question after introducing the quasiclassical approximation.

(B) The Model

Although ^3He is a system of strongly interacting particles, most of its properties at low temperatures $T \ll \mu$ may be understood in the independent quasiparticle model derived by Landau [9]. The key to its success is the observation that, in the low temperature region, and provided the external perturbation varies slowly in space and time ($\omega \ll \mu$, $|\mathbf{q}| \ll p_F$), all excitations are located near the Fermi surface. All high energy contributions may be absorbed into a set of parameters to be taken from experiment, and into effective vertices.

Below the transition temperature, the situation is more complicated, and strong coupling corrections (deviations from the simple BCS-approximation) are known to be essential for the qualitative and quantitative understanding of static properties. It seems, however, that these corrections are less important for the description of dynamic phenomena, provided one performs the "trivial" correction, namely inserting the correct (experimental) temperature dependence of the absolute value of the order parameter. Consequently, to construct a dynamical response theory, we rely strongly on the quasiparticle concept of Landau, and assume that the Fermi liquid interactions and the scattering amplitude are not changed by the superfluid transition. Furthermore, an additional effective pairing interaction depending only on the momenta near the Fermi surface is introduced.

Consequently, the equation of motion for the (normalized) *quasiparticle part of the Green function* assumes the form:

$$[\hat{Q} - \hat{\Sigma}_c; \tilde{\mathcal{G}}] = 0, \quad (2.18)$$

where \hat{Q} is given by

$$\hat{Q} = E\hat{\rho}_3 - (\mathbf{p}^2/2m^*) - \hat{U}(\mathbf{p}, \mathbf{r}, t) + i\hat{\Delta}(\mathbf{p}, \mathbf{r}, t); \quad (2.18a)$$

m^* is the effective mass, and \hat{U} includes the external field as well as Fermi liquid interactions

$$\hat{U}(\mathbf{p}, \mathbf{r}, t) = U_{\text{ext}}(\mathbf{r}, t) + \hat{U}_{\text{F.L.}}(\mathbf{p}, \mathbf{r}, t) \quad (2.18b)$$

and the parameters F_0, F_1, \dots , are defined as usual:

$$\hat{U}_{\text{F.L.}}(\mathbf{p}, \mathbf{r}, t) = F_0 \left(\frac{\partial \mu}{\partial \rho_0} \right) \delta \rho(\mathbf{r}, t) + \frac{F_1}{3} \frac{m}{m^* \rho_0} \mathbf{p} \cdot \mathbf{j}(\mathbf{r}, t) \hat{\rho}_3 + \dots, \quad (2.18c)$$

where $(\partial\mu/\partial\rho_0)^{-1} = 2N_0$, N_0 is the density of states at the Fermi surface for one spin, $N_0 = m^*p_F/2\pi^2$, and ρ_0 is the equilibrium number density given by $\rho_0 = p_F^3/3\pi^2$. Furthermore, $\delta\rho = \rho - \rho_0$, where ρ is the density, and \mathbf{j} the current density. The order parameter $\hat{\Delta}(\mathbf{p}, \mathbf{r}, t)$ is by definition off-diagonal in particle-hole space, and it has to be determined self-consistently (see below). Finally $\check{\Sigma}_c$ describes collision processes between quasiparticles, and the scattering probability is connected with the Landau parameters in the usual way. We will not give a detailed discussion of the collision operator and refer the reader to Ref. [11] (for some details of the derivation in the Keldysh technique, see also [23]). We remark only that the structure of the collision operator is very much the same as in the normal state, the most important quantitative corrections arising from the singular character of the density of states below T_c . Approximations to the collision operator will be guided by the conservation laws, and in all situations we consider in the following, we may replace the $*$ -product by the usual product:

$$[\check{\Sigma}_c * \check{\mathcal{G}}] \rightarrow [\check{\Sigma}_c, \check{\mathcal{G}}] \quad (2.19)$$

in the collision part of the equation of motion.

(C) The Quasiclassical Approach

Crucial for the quasiclassical approach is the observation that the quasiparticle self energy, more precisely: $\check{U}_{\text{F.L.}} + \check{\Sigma}_c$, is approximately independent of the absolute value of the momentum (the scale for variations is set by p_F). We realize, furthermore, that the direct kinetic energy term has dropped out of the equation of motion; namely,

$$\left[\frac{\mathbf{p}^2}{2m^*} * \check{\mathcal{G}} \right] = -i \frac{\mathbf{p}}{m^*} \cdot \nabla_r \check{\mathcal{G}}, \quad (2.20)$$

and that we may neglect the space derivatives in working out

$$[\check{U} - i\hat{\Delta} * \check{\mathcal{G}}],$$

provided $|\mathbf{q}| \ll p_F$ (see below). In the resulting equation, we may then replace \mathbf{p} by \mathbf{p}_F (a vector in direction of \mathbf{p} with length p_F): this is what we call the Eilenberger approximation [10]. Obviously, we then have obtained the equation of motion for the quasiclassical or ξ_p -integrated Green function, where its precise definition is²

$$\check{G}_p(E, \mathbf{r}, t) = (i/\pi) \int_{-\epsilon_c}^{+\epsilon_c} d\xi_p \check{\mathcal{G}}(\mathbf{p}, E, \mathbf{r}, t), \quad (2.21)$$

where ϵ_c is an intermediate cut-off between T_c and the Fermi energy. Defining also a \circ -product which is the expansion Eq. (2.11), neglecting the \mathbf{r} -derivatives,

$$A \circ B = \left\{ \exp \frac{i}{2} [(\partial_E A \partial_t B - \partial_t A \partial_E B)] \right\} (AB), \quad (2.22)$$

² The index p indicates that \check{G}_p depends only on the direction of the vector \mathbf{p} .

the equation for the quasiclassical function has the form

$$[\hat{Q}(\mathbf{p}_F, E, \mathbf{r}, t) - \hat{\Sigma}_c(\mathbf{p}_F, E, \mathbf{r}, t); \check{G}_v] + i \frac{\mathbf{p}_F}{m^*} \cdot \nabla_r \check{G}_v = 0. \quad (2.23)$$

To study in more detail the range of validity of the Eilenberger approximation, we discuss, as an example, the external potential. In Eq. (2.6), this contributes a typical term $U_{\text{ext}}(\mathbf{r}_1) \check{\mathcal{G}}(\mathbf{r}_1, \mathbf{r}_2)$, concentrating for the moment on the space coordinates. Analyzing this expression with respect to the momenta: $\mathbf{r}_1 \rightarrow \mathbf{p} + \mathbf{q}/2$, $\mathbf{r}_2 \rightarrow \mathbf{p} - \mathbf{q}/2$, we arrive at

$$\int \frac{d^3k}{(2\pi)^3} U_{\text{ext}}(\mathbf{k}) \check{\mathcal{G}}\left(\mathbf{p} - \frac{\mathbf{k}}{2}, \mathbf{q} - \mathbf{k}\right),$$

which, integrated with respect to ξ_p , results in

$$\int \frac{d^3k}{(2\pi)^3} U_{\text{ext}}(\mathbf{k}) \check{G}_{v'}(\mathbf{q} - \mathbf{k}), \quad \mathbf{p}' = \mathbf{p} - \mathbf{k}/2. \quad (2.24)$$

In the quasiclassical limit, $|\mathbf{k}| \ll p_F$, the vector \mathbf{p}' almost points in the same direction as \mathbf{p} , and it may be replaced by the latter. Actually, we have neglected the change in the direction of the trajectory of the particle: For example, the Lorentz force in a system of charged particles is not included in this approximation. We recognize then that (2.24) is a convolution integral, and we find in coordinate representation:

$$U_{\text{ext}}(\mathbf{r}) \check{G}_v(\mathbf{r}),$$

in the quasiclassical regime; the same approximations are used to simplify the expressions for the self energy, with the result indicated above.

For consistency, we must show that we may in turn express $\hat{U}_{F.L.}$, $\hat{\Delta}$, and $\hat{\Sigma}_c$ as functionals of the quasiclassical Green function. First we present the expressions for the density and the current which are connected with the Keldysh function:

$$\begin{aligned} \delta\rho &= -N_0 \text{Tr} \left\langle \int \frac{dE}{4} \hat{G}_v^K + \hat{U} \right\rangle, \\ \mathbf{j} &= -N_0 \text{Tr} \hat{\rho}_3 \left\langle \frac{\mathbf{p}_F}{m} \left[\int \frac{dE}{4} \hat{G}_v^K + \hat{U} \right] \right\rangle, \end{aligned} \quad (2.25)$$

where $\langle \rangle$ denotes the angular average. Note the appearance of \hat{U} in these expressions, which results from contributions far from the Fermi surface not correctly described by the ξ_p -integrated functions [11]. In other words, before performing the ξ_p -integration (thus assuming a constant normal density of states, N_0), one has to ensure sufficient convergence of the expressions by certain subtractions. We may rewrite Eq. (2.25), using Eq. (2.18c), and obtain

$$\begin{aligned}
\delta\rho &= -\frac{N_0}{1+F_0} \text{Tr} \left\langle \int \frac{dE}{4} \hat{G}_p^K + U_{\text{ext}} \right\rangle \\
\mathbf{j} &= -\frac{N_0}{1+F_1/3} \text{Tr} \hat{\rho}_3 \left\langle \frac{\mathbf{p}_F}{m} \int \frac{dE}{4} \hat{G}_p^K \right\rangle \\
&= -N_0 \text{Tr} \hat{\rho}_3 \left\langle \frac{p_F}{m^*} \int \frac{dE}{4} \hat{G}_p^K \right\rangle;
\end{aligned} \tag{2.26}$$

Eqs. (2.26) are valid to a linear approximation in the external potential. For the considerations concerning the collision self energy, compare [11]. Finally, it is straightforward (in the considered model) to express the gap equation in the quasiclassical function, with the result:

$$\hat{\Delta}_p(\mathbf{r}, t) \equiv \hat{\Delta}(\mathbf{p}_F, \mathbf{r}, t) = -\frac{i}{4} \lambda \int \frac{d\Omega_{p'}}{4\pi} \frac{3\mathbf{p} \cdot \mathbf{p}'}{|\mathbf{p}| |\mathbf{p}'|} \int dE [\hat{G}_p^K(E, \mathbf{r}, t)]_{\text{o.d.}}; \tag{2.27}$$

$\lambda(>0)$ denotes the dimensionless coupling constant, and “o.d.” indicates the off-diagonal part in particle-hole space.

An important consequence of the commutator-like equation of motion, Eq. (2.23), is that the quasiclassical Green function is normalized in the following sense: We directly conclude that $\check{G}_p \circ \check{G}_p$ is also a solution of (2.23) provided \check{G}_p is a solution. A special solution is then $\check{G}_p \circ \check{G}_p = \text{const}$; furthermore, since the equation is a first order differential equation in the space coordinate, its solution is uniquely determined by the boundaries which we may choose in thermal equilibrium. An explicit calculation for this case shows that

$$\check{G}_p \circ \check{G}_p = \check{1}, \tag{2.28a}$$

and we conclude that this condition holds everywhere. In components:

$$\begin{aligned}
\hat{G}_p^R \circ \hat{G}_p^R &= \hat{G}_p^A \circ \hat{G}_p^A = \hat{1}, \\
\hat{G}_p^R \circ \hat{G}_p^K + \hat{G}_p^K \circ \hat{G}_p^A &= 0.
\end{aligned} \tag{2.28b}$$

Obviously, the normalization condition allows to define a quasiparticle distribution function $\hat{f}(\mathbf{p}_F, E, \mathbf{r}, t)$:

$$\begin{aligned}
\hat{G}_p^K &= \hat{G}_p^R \circ \hat{h} - \hat{h} \circ \hat{G}_p^A, \\
\hat{h} &= 1 - 2\hat{f}.
\end{aligned} \tag{2.29}$$

Such a Keldysh function \check{G}_p^K fulfills the normalization condition for arbitrary \hat{f} . Furthermore, \hat{f} agrees with the usual convention since in equilibrium, we find $\hat{h} = \tanh E/2T \hat{1}$. In general, \hat{h} will be of the form

$$\hat{h} = h^a \hat{1} + h^s \hat{\rho}_3, \tag{2.30}$$

where h^a and h^s correspond to the antisymmetric and symmetric part of the distribution function upon interchanging $E, \mathbf{p}_F \rightarrow -E, -\mathbf{p}_F$, respectively.

In summary, the equation of motion Eq. (2.23), together with the self-consistency relations for the molecular fields $\hat{U}_{F.L.}$ and the order parameter $\hat{\Delta}$, Eqs. (2.18c, 2.26, 2.27), and the expression for the collision self energy form the basis for a description of superfluid ^3He in the quasiclassical regime, i.e., in lowest order in T_c/μ , ω/μ , $|\mathbf{q}|/q_F$. An equation of motion for the distribution function, defined by Eq. (2.29), is found by studying the equation for \hat{G}_p^K .

As is clear from the above derivation, an essential feature of the resulting kinetic equation is that the momenta are fixed at the Fermi surface, and that time and space derivatives are not treated consistently. Consequently, this equation is not Galilean- or rotation-invariant; equivalently, the equation for the quasiclassical Green function does not imply momentum or angular momentum conservation (although number conservation is a direct consequence of Eqs. (2.23, 2.26)). This is clear since, as we discussed above, the conservation laws in general are found by a consistent treatment of space and time, i.e., the equation for the full Green function. To resolve this difficulty, we suggest the following (more intuitive than rigorous) procedure: Study the system of equations defined by

$$[\hat{Q}(\mathbf{p}, E, \mathbf{r}, t) - \hat{\Sigma}_c(\mathbf{p}, E, \mathbf{r}, t) * \check{G}_p] = 0, \quad \check{G}_p * \check{G}_p = \check{1}, \quad (2.31)$$

which is the equation of motion for the full Green function supplemented by a normalization condition. This system has obviously the same symmetry properties as the microscopic Hamiltonian, the solutions, however, are expected to be "close" to the solutions found in the Eilenberger approximation except that they are depending weakly on the absolute value of \mathbf{p} . Thus we obtain kinetic equations which have all the invariance properties and, treating the $|\mathbf{p}|$ -dependence somewhat intuitively (and putting $|\mathbf{p}| \simeq p_F$ at the end), this system implies the conservation laws; an illustration will be given below (see also [24]).

(D) The Kinetic Equations Near T_c , and Application to the Orbital Dynamics

In this section, we study in more detail the structure of the kinetic equations in the Eilenberger approximation, restricting ourselves from the beginning to low frequencies ($\omega \ll \Delta, T$) and wave vectors ($|\mathbf{q}| \ll \xi_0^{-1} \sim T_c/v_F$). The following considerations apply to the A- and the B-phase, the A-phase having a momentum dependent absolute value and phase of the order parameter. Although the calculational details are more complicated for the B-phase, we remark that it may be treated formally similar to a s-wave superconductor for the cases considered here.

In the low frequency regime, we may expand with respect to the external time variation in definition (2.22), collecting terms up to the order under consideration. A formal prescription, however, cannot be given easily since the solution of the kinetic equation may also be proportional to a derivative. For consistency, one must also treat the time derivative of the overall phase as a zero order quantity since it is connec-

ted with the chemical potential of the pairs. We introduce the following notation:

$$\hat{\Delta}_p = \begin{Bmatrix} 0 & \Delta_p \\ \Delta_p^* & 0 \end{Bmatrix}, \quad (2.32a)$$

and $\Delta_p = |\Delta_p| \exp\{-i\Theta_p\}$ is the order parameter. Furthermore, the absolute value $|\Delta_p|$ has nodes in the A-phase for \mathbf{p} pointing in certain directions on the Fermi surface characterized by a unit vector \mathbf{l} :

$$|\Delta_p| = \Delta_0 [1 - (\mathbf{n} \cdot \mathbf{l})^2]^{1/2}, \quad (2.32b)$$

where $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$, and $\Delta_0 = \Delta_0(T)$ is the maximum gap over the Fermi surface, connected with the angular averaged gap by

$$\langle |\Delta_p|^2 \rangle = \Delta^2 = \frac{2}{3} \Delta_0^2. \quad (2.32c)$$

On the other hand, $|\Delta_p|$ is independent of the momentum in the B-phase. Finally, we denote the overall phase by φ ; the pair chemical potential μ_s is thus given by $\mu_s = \dot{\varphi}/2$.

Consider first the equation of motion for the regular (retarded and advanced) Green functions. Working out the “ \circ ” in Eq. (2.23) up to first order in the time derivative, we find

$$\begin{aligned} (E + i\Gamma)[\hat{\rho}_3, \hat{G}^R] - [\hat{U}, \hat{G}^R] + i[\hat{\Delta}, \hat{G}^R] \\ + \frac{i}{2} \left[\hat{\rho}_3, \frac{\partial \hat{G}^R}{\partial t} \right]_+ + \frac{i}{2} \left[\frac{\partial \hat{U}}{\partial t}, \frac{\partial \hat{G}^R}{\partial E} \right]_+ + \frac{1}{2} \left[\frac{\partial \hat{\Delta}}{\partial t}, \frac{\partial \hat{G}^R}{\partial E} \right]_+ \\ + \frac{i}{2} \mathbf{v}_F \cdot \nabla_r \hat{G}^R + O(\partial_t^2) = 0, \end{aligned} \quad (2.33)$$

where $[\ , \]$; $[\ , \]_+$ denote the commutator and the anticommutator, respectively, and $\mathbf{v}_F = \mathbf{p}_F/m^*$. Furthermore, the equation for G^A is given by reversing the sign of the pair breaking parameter Γ . In the simple situation under consideration, Γ is connected with the self energy and the inelastic scattering rate τ_E^{-1} by

$$\Gamma = i\hat{\Sigma}_c^R \hat{\rho}_3 = -i\hat{\Sigma}_c^A \hat{\rho}_3 = 1/2\tau_E.$$

Treating $\hat{U} = U^s + U^a \hat{\rho}_3$, where U^s and U^a denote the symmetric and antisymmetric parts of \hat{U} , respectively, as first order, we find in zero order the well known result:

$$\hat{G}^R = \alpha^R \hat{\rho}_3 + \beta^R (\hat{\Delta}_p / |\Delta_p|), \quad (2.34)$$

where we introduced

$$\begin{aligned} \alpha^R &= \frac{-iE + \Gamma}{[(-iE + \Gamma)^2 + |\Delta_p|^2]^{1/2}} = \mathcal{N}_1 + i\mathcal{R}_1 = -(\alpha^A)^*, \\ \beta^R &= \frac{|\Delta_p|}{[(-iE + \Gamma)^2 + |\Delta_p|^2]^{1/2}} = \mathcal{N}_2 + i\mathcal{R}_2 = (\beta^A)^*, \end{aligned} \quad (2.35)$$

the square root defined such that $\text{Re } \sqrt{} > 0$. \mathcal{N}_1 , for example, is the reduced density of states, including pair breaking. In first order, the corrections due to \hat{U} are taken into account by replacing $E \rightarrow E - U^a$ in α and β (note that $U^s \cdot \hat{1}$ drops out of the equation). Furthermore, one finds that in the diagonal part of $\hat{G}^{R,A}$, E has to be replaced by $E - \hat{\Theta}_p \hat{\rho}_3/2$; this could be expected for general reasons, since it is always possible to choose a gauge in which the overall phase is zero. Equivalently,³ one can measure the energy relative to the actual pair potential which results into $E \rightarrow E + (\mu_s - \hat{\Theta}_p/2) \hat{\rho}_3$, canceling just $\hat{\phi}/2$ as expected. In this gauge, \hat{G}^R has the simplest form (note, however, the additional contribution if \mathbf{l} is time dependent: $\hat{\phi} - \hat{\Theta}_p \sim \mathbf{l}$); finally, there are no corrections to $\hat{G}^{R,A}$ proportional to $|\Delta_p|$. The expansion in the space derivative is straightforward: for example, if we assume for simplicity that $\varphi = \varphi(\mathbf{r})$ only, we find in first order $E \rightarrow E - \mathbf{p}_F \cdot \mathbf{v}_s m/m^*$ in α and β , where the superfluid velocity \mathbf{v}_s is given by $\mathbf{v}_s = -\nabla\varphi/2m$ (note that in such a case U^a also has to be calculated consistently).

Using now the expansion of the Keldysh function, Eq. (2.29), with respect to the time variation, we may easily derive from (2.23) the equations of motion for h^a and h^s , the antisymmetric and symmetric parts of the distribution function (recall that $\hat{h} = 1 - 2\hat{f}$). Inserting the regular Green functions in zero order, we derive the Boltzmann equation by taking the trace of (2.23) and taking the trace after multiplying by $\hat{\rho}_3$; it is convenient to use the equation of motion for $\hat{G}^{R,A}$ in terms connected with h^a . Then we find (see also [13]):

$$\mathcal{N}_1 \frac{\partial h^a}{\partial t} - K^a + \mathbf{v}_F \cdot \nabla \mathcal{N}_1 h^s = -\mathcal{R}_2 |\Delta_p| \frac{\partial h^a}{\partial E} - \mathcal{N}_1 \dot{U}^a \frac{\partial h^a}{\partial E}, \quad (2.36a)$$

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{N}_1 h^s - K^s + \mathcal{N}_1 \mathbf{v}_F \cdot \nabla h^a + 2 |\Delta_p| \left(\mathcal{N}_2 - \frac{1}{2} \frac{\partial \mathcal{R}_2}{\partial E} \frac{\partial}{\partial t} \right) h^s \\ = - |\Delta_p| \frac{\partial h^a}{\partial E} \left(\mathcal{N}_2 \hat{\Theta}_p - \frac{1}{2} \frac{\partial \mathcal{R}_2}{\partial E} \hat{\Theta}_p \right) - \mathcal{N}_1 \dot{U}^s \frac{\partial h^a}{\partial E}, \end{aligned} \quad (2.36b)$$

and we neglected certain coupling terms in these equations since h^s is typically small. K^s and K^a are the symmetric and the antisymmetric part of the particle-particle collision operator (for some details of the corresponding K^s and K^a for electron-phonon collisions, see [25]). Also remarkable is the appearance of a “conversion term” which is in lowest order given by $2 |\Delta_p| \mathcal{N}_2 h^s$, and the corresponding expression on the r.h.s., namely, $- |\Delta_p| (\partial h^a / \partial E) \mathcal{N}_2 \hat{\Theta}_p$. We realize that for

$$h^s = - \frac{\hat{\Theta}_p}{2} \frac{\partial h^a}{\partial E}, \quad (2.37)$$

both terms cancel each other, and for $h^a = th(E/2T)$ and $\hat{\Theta}_p$ independent of the momentum, such a h^s describes a shifted Fermi function and is an eigenfunction of the

³ In this form, the argument is valid for spatially homogeneous situations only. In general, we are faced again with the problem of a consistent expansion in ∂_t and ∇_r .

linearized collision operator, reflecting again the gauge symmetry connected with the overall phase (of course, the argument is not restricted to the linearized theory).

The two Boltzmann-like equations (2.36) are supplemented by the self-consistency equation (2.27) which we consider for the moment only near T_c and in the case where the deviations from thermal equilibrium are small. If we define $\delta f^s = -h^s/2$, $\delta f^a = -(h^a - th(E/2T))/2$, we are able to rewrite (2.27) into the form of a time dependent Ginzburg–Landau equation:

$$\frac{\pi}{8T_c} \dot{\Delta}_p + \left\langle \int dE [\mathcal{R}_2 \delta f^a + i\mathcal{N}_2 \delta f^s] e^{-i\Theta_p} \right\rangle' = -\frac{1}{N_0} \left\langle \frac{\delta F_{GL}}{\delta \Delta_p^*} \right\rangle', \quad (2.38)$$

where F_{GL} is the usual Ginzburg–Landau free energy, and $\langle \cdot \rangle' = \langle 3\mathbf{p} \cdot \mathbf{p}' / |\mathbf{p}| |\mathbf{p}'| \cdot \rangle_{p'}$ implies that the quantities in the brackets depend on \mathbf{p}' . As is well known, the additional time derivative is of special importance for fluctuation effects above T_c . Other orientation energies, like the dipolar interaction, have to be included on the r.h.s. of this equation. Of course, the TDGL equation together with the Boltzmann equation directly implies particle conservation: Separating from (2.38) the equation for the overall phase, we find that the r.h.s. will be proportional $\sim \text{div } \mathbf{j}_s$, while the gradient term in the Boltzmann equation will contribute the divergence of the normal current (upon integration).

As an example, we discuss now the viscous motion of the \mathbf{l} -vector near T_c ($|\Delta_p| \ll T_c$, $\omega\tau_E \ll 1$, $\mathbf{q} = 0$). In such a situation, the deviations from equilibrium are small, and we may solve the Boltzmann equation in a relaxation approximation [12, 26], since the inhomogenities are rapidly decreasing in E over a scale given by $|\Delta_p|$

$$-K^s \simeq \frac{1}{\tau_E} \mathcal{N}_1 \delta f^s, \quad -K^a \simeq \frac{1}{\tau_E} \mathcal{N}_1 \delta f^a,$$

where τ_E is the inelastic scattering time (taken for $T = T_c$ and $E = 0$). Then we can solve (2.36) directly and find (Fermi liquid effects are not important!):

$$\begin{aligned} \delta f^a &= \tau_E \frac{\mathcal{R}_2}{\mathcal{N}_1} |\dot{\Delta}_p| \frac{1}{4Tch^2E/2T}, \\ \delta f^s &= \frac{|\Delta_p| \tau_E \mathcal{N}_2}{\mathcal{N}_1 + 2|\Delta_p| \tau_E \mathcal{N}_2} \dot{\Theta}_p \frac{1}{4Tch^2E/2T}. \end{aligned} \quad (2.39a)$$

As long as $\Delta_0 \tau_E \gg 1$, one may insert the BCS-limits of the special quantities, namely,

$$\begin{aligned} \mathcal{N}_1 &= E/\xi, \quad \mathcal{R}_2 = (|\Delta_p|/E) \mathcal{N}_1, \\ |\Delta_p| \mathcal{N}_2 &= (|\Delta_p|^2/\xi^2) \Gamma \mathcal{N}_1; \end{aligned}$$

more precisely, these forms are valid for $\xi^2 = E^2 - |\Delta_p|^2 \gg |\Delta_p| \Gamma$. Then we find:

$$\begin{aligned} \delta f^a &\simeq \tau_E \frac{|\Delta_p| |\dot{\Delta}_p|}{E} \frac{1}{4Tch^2(E/2T)}; \\ \delta f^s &\simeq \frac{|\Delta_p|^2 \dot{\Theta}_p}{2E^2} \frac{1}{4Tch^2(E/2T)}. \end{aligned} \quad (2.39b)$$

Obviously, δf^a (parity $(- +)$ for interchanging E , $\mathbf{p}_F \rightarrow -E$, $-\mathbf{p}_F$) describes the Cross-Anderson mechanism [27, 26] for the damping of the \mathbf{l} -motion. This is well known and does not need to be discussed further. On the other hand, δf^s (parity $(+ +)$) results partly from the type of disequilibrium connected with the “pair angular momentum” as considered in detail by Leggett and Takagi [28]. In other words, δf^s describes a local (on the Fermi surface) difference between Cooper-pair and quasi-particle chemical potential. (This will become clearer in the case of the spin dynamics where the interpretation is more straightforward.)

Inserting these results into the time dependent Ginzburg-Landau equation (2.38), we derive the expression for the orbital viscosity. Some integrals are useful:

$$\begin{aligned} \int dE \mathcal{N}_2 &= \pi |\Delta_p|; \\ \int dE \mathcal{N}_2^2 / \mathcal{N}_1 &= \pi [|\Delta_p|^2 + \Gamma^2]^{1/2} - \pi \Gamma; \\ \int dE \frac{\mathcal{N}_1 \mathcal{N}_2}{\mathcal{N}_1 + 2 |\Delta_p| \tau_E \mathcal{N}_2} &= \frac{\pi |\Delta_p| (\Gamma/2\tau_E)^{1/2}}{[|\Delta_p|^2 + \Gamma/2\tau_E]^{1/2}}; \end{aligned} \quad (2.40)$$

note that $\Gamma = 1/2\tau_E$ in the situation under consideration. Inserting these results, we then find the equation of motion in the usual form:

$$\mu \mathbf{l} \times \dot{\mathbf{l}} = \mathbf{l} \times \frac{\partial F}{\partial \mathbf{l}}, \quad (2.41)$$

where F includes all orienting energies. There are three contributions to the orbital viscosity μ : from the antisymmetric and the symmetric part of the distribution function, and from the additional time dependence in Eq. (2.38), which represents some sort of quantum-mechanical coupling between pairs and quasiparticles. It is, however, not convenient to discuss these contributions separately. Indeed, we consider first the limit $\Delta_0 \tau_E \gg 1$, and we find, upon expansion in the inverse of this quantity, the two first terms of the form:

$$\mu = \mu_{\text{C.A.}} + \frac{\chi_{\text{orb}}}{\tau_E}, \quad \Delta_0 \tau_E \gtrsim 1, \quad (2.42)$$

where $\mu_{\text{C.A.}}$ is the Cross-Anderson viscosity [27]

$$\mu_{\text{C.A.}} = N_0 (\pi^2 \Delta_0^3 / 64 T_c) \tau_E \quad (2.42a)$$

and the “orbital susceptibility” [28] χ_{orb} is given by:

$$\chi_{\text{orb}} = N_0 \frac{3\pi^2}{128} \frac{\Delta_0}{T_c}, \quad (2.42b)$$

in agreement with Nagai [29].⁴ On the other hand, the time derivative in the TDGL dominates in the gapless region,

$$\mu = N_0(\pi\Delta_0^2/12T_c), \quad \Delta_0\tau_E \lesssim 1, \quad (2.42c)$$

and the viscosity is independent of the inelastic scattering rate in this limit. In the intermediate range, both expressions join smoothly.

It should be noted that the corrections to (2.41) in second order in the time derivative are easily found in the linearized regime (and near T_c ; $\Delta_0\tau_E \gg 1$) by taking the time derivative of the distribution function in (2.36) into account. This leads to the replacement $\tau_E^{-1} \rightarrow -i\omega + \tau_E^{-1}$ in (2.42), in agreement with the phenomenological treatment [28].

Finally, we consider briefly rotations of the system, and we use the generalization of the Eilenberger approximation as outlined in part (C) of this chapter, i.e., a consistent treatment of time and space derivatives. In this scheme, the distribution function is defined by (note the $*$ - instead of the \circ -product):

$$\begin{aligned} \hat{G}^K &= \hat{G}^R * \hat{h} - \hat{h} * \hat{G}^A, \\ \hat{h} &= \hat{1} - 2\hat{f}. \end{aligned} \quad (2.43)$$

As an example, we consider again the orbital dynamics, and the antisymmetric part of \hat{h} only (Eq. (2.36a)). The r.h.s. of this equation has to be modified into

$$-\mathcal{R}_2 \left[\frac{\partial |\Delta_p|}{\partial t} \frac{\partial h^a}{\partial E} + \frac{\partial |\Delta_p|}{\partial \mathbf{p}} \cdot \frac{\partial h^a}{\partial \mathbf{r}} - \frac{\partial |\Delta_p|}{\partial \mathbf{r}} \cdot \frac{\partial h^a}{\partial \mathbf{p}} \right]. \quad (2.44)$$

From expressions of this type, one may draw several conclusions: If the \mathbf{l} -vector is constant in time ($= \mathbf{l}_0$), and h^a has its equilibrium form, $h^a = h_0 = th(E/2T)$, expression (2.44) is obviously zero. Looking then onto this system from a rotating reference frame (compare Eqs. (2.14)–(2.17)), we find that

$$|\tilde{\Delta}(\mathbf{p})| = |\Delta(\tilde{\mathbf{p}}, \mathbf{l}_0)| = |\Delta(\mathbf{p}, \mathbf{l}(t))|,$$

where $\tilde{\mathbf{p}} = \mathcal{R}\mathbf{p}$, $\mathbf{l}(t) = \mathcal{R}^{-1}\mathbf{l}_0$, and \mathcal{R} is the rotation matrix as discussed (part (A) of this chapter; a “ \sim ” denotes the quantities in the rotating frame).

Furthermore,

$$\tilde{h}_0(E) = h_0(\tilde{E}) = h_0(E + \boldsymbol{\omega}(\mathbf{r} \times \mathbf{p}));$$

note that $\tilde{h}_0(E)$ as well as $h_0(E)$ is an eigenfunction of the collision operator, due

⁴ Apparently, Kopnin [30] has calculated the contribution to χ_{orb} resulting from the higher order terms neglected in (2.38). This contribution is a factor Δ_0/T_c smaller than (2.42b).

to energy and momentum conservation. Then we obtain a cancelation in (2.44) as expected, since

$$(\boldsymbol{\omega} \times \mathbf{p}) \cdot \frac{\partial |\Delta_p|}{\partial \mathbf{p}} = -(\boldsymbol{\omega} \times \mathbf{l}) \cdot \frac{\partial |\Delta_p|}{\partial \mathbf{l}}.$$

Consider now a situation, in which \mathbf{l} is moving in the *rest* frame. If we look onto the system from the rotating frame with the same (possibly time dependent) angular velocity, we find obviously that $|\tilde{\Delta}_p|$ is time independent, and the second term in (2.44) is the drive term in this situation:

$$\frac{\partial |\tilde{\Delta}_p|}{\partial \mathbf{p}} \cdot \frac{\partial \tilde{h}_0}{\partial \mathbf{r}} = -(\boldsymbol{\omega} \times \mathbf{p}) \cdot \frac{\partial |\tilde{\Delta}_p|}{\partial \mathbf{p}} \frac{\partial \tilde{h}_0}{\partial E}$$

(the third term is important if \mathbf{l} varies also in space); of course, the physical content is the same.

Another interesting question is the conservation of angular momentum, and, for simplicity, we consider a situation where the system is not influenced by the walls of the container. Then, if \mathbf{l} is time dependent, there is an angular momentum input given by

$$\frac{\partial \mathcal{L}}{\partial t} = \mu \mathbf{l} \times \dot{\mathbf{l}} \cdot \text{Vol.},$$

which leads, if \mathbf{l} is rotated at constant angular velocity (by an external force), after long times (of order $\sim \rho R^2 / \mu \sim$ hours, where R is a typical dimension of the container) to a solid body rotation of the normal component, and, in thermal equilibrium, a vortex lattice will be found in the superfluid component. As in the case of momentum conservation, the conservation law for the total angular momentum is only found in a consistent treatment of space and time derivatives, i.e., by going back to the equation for the full Green function or by generalizing the quasiclassical approach as discussed above.

3. SPIN DYNAMICS OF THE A- AND B-PHASE

(A) Generalization of the Kinetic Equations to the Spin Dynamics

In this chapter, we want to study in more detail the spin modes of the A- and B-phase in the low frequency ($\omega \ll \Delta, T$) and homogeneous limit. As is well known, the order parameter is a 2×2 -matrix in spin space, and it is parameterized conveniently by a complex momentum dependent vector $\mathbf{d}(\mathbf{n})$ (\mathbf{n} a unit vector along \mathbf{p}):

$$\Delta_p = \Delta(T) i \hat{\sigma}_\eta \hat{\sigma}_2 d_\eta, \quad (3.1)$$

sum over indices implied, where \mathbf{d} has the property $\langle |\mathbf{d}|^2 \rangle = 1$, and $\mathbf{d} \times \mathbf{d}^* = 0$ for unitary states (like the ABM- and BW-state). Consequently,

$$\langle \Delta_p \Delta_p^+ \rangle = \Delta(T)^2.$$

In the A- and B-phase (ABM- and BW-state), the vector \mathbf{d} is given by

$$\begin{aligned} A: \quad \mathbf{d} &= \mathbf{d}_0[(\boldsymbol{\alpha}_1 + i\boldsymbol{\alpha}_2) \cdot \mathbf{n}], \\ B: \quad \mathbf{d} &= (\mathcal{R}\mathbf{n}) e^{-i\varphi}, \end{aligned} \quad (3.2)$$

where \mathbf{d}_0 , $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$ are real unit vectors, and \mathcal{R} is a rotation matrix, characterized by an axis $\boldsymbol{\omega}$ and an angle Θ . Furthermore, $\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2 = 0$, and the \mathbf{l} -vector is defined by $\mathbf{l} = \boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2$.

As above, a distribution function may be defined through the normalization condition (2.28) by:

$$\hat{G}_p^{\mathbf{K}} = \hat{G}_p^{\mathbf{R}} \circ \hat{h} - \hat{h} \circ \hat{G}_p^{\mathbf{A}}, \quad (3.3)$$

where summation over spin indices is now understood. Thus, \hat{h} will be a matrix in spin- and Nambu-space, and we define the 4×4 -spin matrices α_μ [31, 32]:

$$\alpha_1 = \hat{\rho}_3 \hat{\sigma}_1, \quad \alpha_2 = \hat{1} \hat{\sigma}_2, \quad \alpha_3 = \hat{\rho}_3 \hat{\sigma}_3. \quad (3.4)$$

In the following, we consider only the symmetric part of the spin distribution function which we denote by $\delta\mathbf{f}$. It is defined by (compare Eq. (2.30)):

$$\hat{h} = -2 \delta f_\mu \alpha_\mu. \quad (3.5)$$

With these generalization, the kinetic equation can be derived following the same lines as in Chapter 2; the result is, however, still a matrix equation in spin space, and we transform (similar to Eq. (3.1)) to a vector notation. Then we obtain the following Boltzmann equation, in the homogeneous case, and neglecting second order time derivatives which can be justified near T_c :

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{N}_1 \delta\mathbf{f} - K(\delta\mathbf{f}) + 2 |\Delta_p| \mathcal{N}_2 \delta\mathbf{f}_\perp + \gamma \mathcal{N}_1 \mathbf{H}_{\text{eff}} \times \delta\mathbf{f} \\ = \frac{1}{4Tch^2(E/2T)} \left[\mathcal{N}_2 |\Delta_p| \text{Re} \frac{\mathbf{d}^* \times \mathbf{d}}{|\mathbf{d}|^2} + \mathcal{N}_1 \dot{\mathbf{U}} \right]. \end{aligned} \quad (3.6)$$

As above, \mathcal{N}_1 , \mathcal{N}_2 denote the spectral quantities which are connected with the (time independent) zero order regular Green functions (2.34), and $K(\delta\mathbf{f})$ denotes the (linearized) spin conserving collision operator. An external magnetic field \mathbf{H} is taken into account in (3.6) by

$$\mathbf{H}_{\text{eff}} = \mathbf{H}/(1 + Z_0/4), \quad (3.7)$$

where Z_0 ($\simeq -3$) is the usual Landau parameter, and

$$\begin{aligned} \mathbf{U} &= \mathbf{U}_{\text{ext}} + \mathbf{U}_{\text{F.L.}}, \\ \mathbf{U}_{\text{ext}} &= -\frac{1}{2}\gamma\mathbf{H}, \quad \mathbf{U}_{\text{F.L.}} = (1/4N_0) Z_0\mathbf{S}; \end{aligned} \quad (3.8)$$

\mathbf{S} is the total spin, and γ the gyromagnetic ratio. Near T_c , the spin is connected with the distribution function by:

$$\begin{aligned}\mathbf{S} &= N_0 \left[\left\langle \int dE \mathcal{N}_1 \delta \mathbf{f} \right\rangle - \mathbf{U} \right] \\ &= \frac{N_0}{1 + Z_0/4} \left[\left\langle \int dE \mathcal{N}_1 \delta \mathbf{f} \right\rangle - \mathbf{U}_{\text{ext}} \right].\end{aligned}\quad (3.9)$$

Finally, it is remarkable that only the part of the distribution function “perpendicular to \mathbf{d} ,” $\delta \mathbf{f}_\perp$, appears in the conversion term in (3.6):

$$\delta \mathbf{f} = \delta \mathbf{f}_\parallel + \delta \mathbf{f}_\perp; \quad \delta \mathbf{f}_\perp = \mathbf{d}^* \times (\delta \mathbf{f} \times \mathbf{d}) / |\mathbf{d}|^2. \quad (3.10)$$

This is clear, however, from an intuitive point of view: At any given point of the Fermi surface, the pairs are condensed into a spin state which is an eigenstate of the spin projection along some axis with eigenvalue zero, and \mathbf{d} represents this axis. Consequently, we can convert quasiparticle spin “perpendicular to \mathbf{d} ” only.

The Boltzmann equation is supplemented by a time dependent Ginzburg–Landau equation, which has in the present situation the form (near T_c):

$$\frac{\pi}{8T_c} \dot{\mathbf{d}} - \left\langle \int dE \mathcal{N}_2 (\delta \mathbf{f} \times \mathbf{d}) / |\Delta_p| \right\rangle' = - \frac{1}{N_0 \Delta^2} \left\langle \frac{\delta F_D}{\delta \mathbf{d}^*} \right\rangle', \quad (3.11)$$

where we included the dipole energy on the r.h.s. (F_{GL} is minimized since $\Delta = \Delta(T)$ assumes its equilibrium value). A direct consequence of this coupled system is the spin conservation law: Integrating the Boltzmann equation and noting that

$$\left\langle \int dE K(\delta \mathbf{f}) \right\rangle = 0, \quad (3.12)$$

we find immediately

$$\dot{\mathbf{S}} + \gamma \mathbf{H} \times \mathbf{S} = N_0 \left\langle (\pi \Delta^2 / 4 T_c) \text{Re}(\mathbf{d}^* \times \dot{\mathbf{d}}) - 2 |\Delta_p| \int dE \mathcal{N}_2 \delta \mathbf{f}_\perp \right\rangle;$$

on the other hand, from (3.11), we obtain

$$\frac{\pi \Delta^2}{8 T_c} \mathbf{d}^* \times \dot{\mathbf{d}} - \left\langle |\Delta_p| \int dE \mathcal{N}_2 \delta \mathbf{f}_\perp \right\rangle = - \frac{1}{N_0} \left\langle \mathbf{d}^* \times \frac{\delta F_D}{\delta \mathbf{d}^*} \right\rangle, \quad (3.13)$$

which gives the final result:

$$\dot{\mathbf{S}} + \gamma \mathbf{H} \times \mathbf{S} = - \left\{ \left\langle \mathbf{d}^* \times \frac{\delta F_D}{\delta \mathbf{d}^*} \right\rangle + \text{c.c.} \right\}, \quad (3.14)$$

namely, the first of Leggett’s famous equations [3, Eq. (11.35)]. For completeness,

we add the expression for the dipole energy. Up to an additive constant, it is given by

$$F_D = g_D \langle 3 | \mathbf{n} \cdot \mathbf{d}(\mathbf{n})|^2 \rangle, \quad (3.15)$$

where g_D is temperature dependent ($\sim \Delta^2(T)$). In the A-phase, the dipole energy depends on the orientation of \mathbf{d}_0 relative to \mathbf{l} , namely,

$$F_D^A = -\frac{3}{5} g_D (\mathbf{d}_0 \cdot \mathbf{l})^2, \quad (3.16a)$$

while in the B-phase, it depends only on the angle of rotation Θ (see (3.2)):

$$F_D^B = \frac{4}{5} g_D (\cos \Theta + 2 \cos^2 \Theta), \quad (3.16b)$$

which is minimal at $\Theta_0 \simeq 104^\circ$. Of course, g_D differs in the two phases due to the different temperature dependence of $\Delta(T)$. In the following sections, we study in detail the solution of the kinetic equations.

(B) Longitudinal A-Phase near T_c : Resonance in the Hydrodynamic Limit

In a longitudinal resonance experiment, one applies a time dependent magnetic field parallel to the static field \mathbf{H}_0 , which defines the \mathbf{z} -direction in the following, and measures the absorbed energy for a given external frequency ω . For a description of the observed collective mode, we may assume as usual that the static external field has oriented the spin quantization axis \mathbf{d}_0 , which is for the A-phase independent of the direction on the Fermi surface, such that it lies in the \mathbf{x} - \mathbf{y} -plane. Furthermore, \mathbf{l} is assumed to be fixed (for example, in the \mathbf{y} direction), and we consider the hydrodynamic limit $\omega \tau_E \ll 1$, where τ_E is the temperature dependent inelastic scattering rate. For simplicity, we neglect the time dependent field and restrict ourselves first to temperatures near T_c .

We parametrize the spin quantization axis by an angle Θ :

$$\mathbf{d}_0 = (\sin \Theta, \cos \Theta, 0), \quad (3.17)$$

and since $\delta \mathbf{f}$ has in this case only a component in the \mathbf{z} direction, namely,

$$\delta \mathbf{f} = (0, 0, \delta f^z), \quad (3.18)$$

the dynamics is described by equations similar to the phase modes of superconductors. From Eq. (3.6), we obtain

$$\mathcal{N}_1 \delta f^z - K(\delta f^z) + 2 |\Delta_p| \mathcal{N}_2 \delta f^z = \frac{1}{4Tch^2E/2T} \{ \mathcal{N}_2 |\Delta_p| \dot{\Theta} + \mathcal{N}_1 \dot{U}^z \}; \quad (3.19)$$

note that the static field does not affect the equation for this component. The Ginzburg–

Landau equation has also a simple form, since we may get rid of the momentum dependence by averaging (3.13) over angles. Then we obtain:

$$\frac{\pi}{8T_c} \Theta - \left\langle \frac{|\Delta_p|}{\Delta^2} \int dE \mathcal{N}_2 \delta f^z \right\rangle = - \frac{1}{2N_0 \Delta^2} \frac{\partial F_D}{\partial \Theta}. \quad (3.20)$$

The solution of the Boltzmann equation represents, as usual, a rather complicated problem since it involves an unhandy collision operator. In fact, exact solutions can only be found in the normal state [33]. On the other hand, there is a well defined approximation which was already used by Abrikosov and Khalatnikov [34] and which we elsewhere called the “reduced operator” [25]. Wölfle used this approximation extensively [35], and the agreement with exact results is excellent.

To explain the reduced operator, we discuss some properties of the collision operator. As can be easily seen, $K(\delta f^z)$ is a negative semi-definite integral operator, i.e., defining the usual scalar product and denoting δf^z by $|F\rangle$,

$$\langle F | K | F \rangle \leq 0 \quad (3.21)$$

for any function $|F\rangle$. Furthermore, K is Hermitian; i.e., for any two functions $|F_1\rangle$, $|F_2\rangle$,

$$\langle F_1 | K | F_2 \rangle = \langle F_2 | K | F_1 \rangle, \quad (3.22)$$

and it has one solution with eigenvalue zero which corresponds to a shift of the chemical potential of the spin-up-particles relative to the spin-down-particles, reflecting the conservation of spin in collision processes.

Quite generally, the collision operator consists of two parts which we call “scattering-in” and “scattering-out” contributions. The scattering-out term is given by

$$K_{\text{out}}(\delta f^z) = -(1/\tau_E) \mathcal{N}_1 \delta f^z, \quad (3.23)$$

while the scattering-in is an integral operator.

The reduced operator is then defined as the integral operator, which agrees with the exact operator in the scattering-out contribution, and which has the same eigensolution with eigenvalue zero. A possible choice is:

$$K_{\text{red}}(\delta f^z) = -\frac{1}{\tau_E} \mathcal{N}_1 \left[\delta f^z - \frac{1}{4Tch^2(E/2T)} \frac{\langle \int dE \mathcal{N}_1 \delta f^z / \tau_E \rangle}{\langle \int dE (\mathcal{N}_1 / \tau_E) / 4Tch^2(E/2T) \rangle} \right]. \quad (3.24)$$

From the above discussion, it is clear that the reduced operator will treat the lowest eigenfunction correctly, while “higher” eigenfunctions are treated in a relaxation time approximation.

In contrast to the normal state, however, the energy and temperature dependence of the inelastic scattering rate cannot be calculated analytically. Furthermore, a numerical calculation in the A-phase will hardly be possible, and we can give only an

estimate for the temperature dependence of $1/\tau_E$ on the Fermi surface. At low temperatures, the quasiparticles are located in small regions on the Fermi surface along $\pm \mathbf{l}$, since in these directions the gap is zero. Furthermore, the inelastic scattering rate will be roughly given by the number of quasiparticles available for scattering processes, and this number is proportional to the occupied area on the Fermi surface relative to the total surface, namely $(T/\Delta_0)^2$. Thus, we obtain

$$\frac{1}{\tau_{E \approx 0}} \simeq \frac{1}{\tau_N(T)} \left(\frac{T}{\Delta_0} \right)^2, \quad T \ll \Delta_0,$$

where $\tau_N(T)$ is the normal state scattering time at the Fermi surface. Near T_c , on the other hand, we find from a comparison with experimental results (see below) that the inelastic scattering rate is to a good approximation equal to its normal state value

$$\frac{1}{\tau_{E \approx 0}} \simeq \frac{1}{\tau_N(T)} = \frac{1}{\tau_N(T_c)} (T/T_c)^2 \quad (3.25)$$

for temperatures $0.8T_c \lesssim T \lesssim T_c$.

In view of the uncertainties in calculating τ_E , we will use the reduced operator with energy independent scattering rate. We remark that the general trend of this approximation—neglecting the energy dependence—is to underestimate slightly the collision processes.

For the solution of the kinetic equation, Eq. (3.19), we have to remember that $\bar{\theta}/2$ is the chemical potential of the spin-up pairs, and $-\bar{\theta}/2$ the chemical potential of the spin-down pairs. Consequently, we will separate in δf^z the part corresponding to a gauge transformation, and we redefine the distribution function:

$$\delta f^z \rightarrow \frac{\bar{\theta}/2}{4Tch^2(E/2T)} + \delta f^z; \quad (3.26)$$

δf^z now describes the non-equilibrium between pairs and quasiparticles. By inspection of the kinetic equation, where K is replaced by the reduced operator, we find immediately ($\omega\tau_E \ll 1$):

$$\delta f^z = \frac{\mathcal{N}_1}{\mathcal{N}_1 + 2|\Delta_\nu| \tau_E \mathcal{N}_2} \frac{\eta}{4Tch^2(E/2T)}, \quad (3.27)$$

and the constant η can be determined by integration. Defining further:

$$\begin{aligned} J_1 &= 1/\eta \left\langle \int dE \mathcal{N}_1 \delta f^z \right\rangle; \\ 2\Delta J_2 &= 1/\eta \left\langle 2|\Delta_\nu| \int dE \mathcal{N}_2 \delta f^z \right\rangle; \\ \tau_R &= J_1/2\Delta J_2, \end{aligned} \quad (3.28)$$

we obtain from (3.19), after Fourier transforming with respect to the time:

$$\left[\frac{-i\omega\tau_R}{1 + Z_0/4} + 1 \right] 2 \Delta J_2 \eta = \frac{\omega^2 \Theta / 2}{1 + Z_0/4}. \quad (3.29)$$

Note that all quantities are defined analogous to the case of a superconductor; near T_c ,

$$J_1 \simeq 1 \quad (3.30)$$

$$2 \Delta J_2 \simeq \left\langle \frac{\pi |\Delta_p|}{4T\tau_E} \left[1 + \frac{1}{(2 |\Delta_p| \tau_E)^2} \right]^{-1/2} \right\rangle.$$

Explicitly, we find for the relaxation time

$$\tau_R = \frac{3T_c}{\pi \Delta_0^2}, \quad \Delta_0 \tau_E \ll 1, \quad (3.31a)$$

$$= \frac{16T_c}{\pi^2 \Delta_0} \tau_E = \frac{4T_c}{\pi \langle |\Delta_p| \rangle} \tau_E, \quad \Delta_0 \tau_E \gg 1. \quad (3.31b)$$

Equation (3.31.b) differs slightly from the result in superconductors, since $|\Delta_p|$ depends on the direction; note that $\tau_E \simeq \tau_N(T_c)$. In the limit of a well defined gap, the distribution function is of the form

$$\delta f^z = \frac{\xi^2}{E^2} \frac{\eta}{4Tch^2(E/2T)}, \quad E^2 \gtrsim |\Delta_p|^2, \quad (3.32)$$

which describes indeed a chemical potential difference between pairs and quasi-particles (see Chapter 4). The result (3.31b) is found with equal success using this expanded form.

Inserting the above results into the time dependent Ginzburg–Landau equation, we realize first that the gauge transformation cancels the first order time derivative term (which is, in fact, true for all temperatures), and the TDGL equation assumes the form

$$2 \Delta J_2 \eta = N_0^{-1} \left(\frac{\partial F_D}{\partial \Theta} \right)_\omega, \quad (3.33)$$

from which we obtain the following equation:

$$\omega^2 \Theta - \left(1 + \frac{-i\omega\tau_R}{1 + Z_0/4} \right) \frac{2(1 + Z_0/4)}{N_0} \left(\frac{\partial F_D}{\partial \Theta} \right) = 0. \quad (3.34)$$

Linearized around the equilibrium position, $\Theta = 0$, we find the well known collective mode at a frequency ω_A , where

$$\omega_A^2 = (1 + Z_0/4) \frac{2}{N_0} \frac{\partial^2 F_D}{\partial \Theta^2} \Big|_{\Theta=0} = (1 + Z_0/4) \frac{12g_D}{5N_0}, \quad (3.35)$$

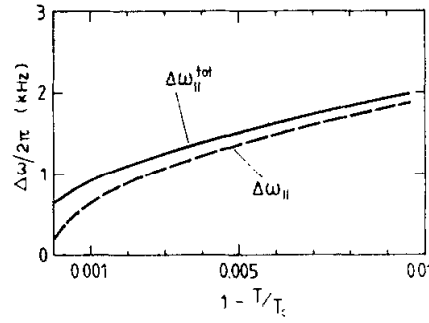


FIG. 1. Total line width of the longitudinal A-phase resonance (solid line), and that under the assumption of a fixed l -vector (dashed line), at the melting pressure.

and a line width given by

$$\Delta\omega_{\parallel} = \frac{\omega_A^2 \tau_R}{1 + Z_0/4} \quad (\Delta_0 \ll T). \quad (3.36)$$

In the case of a well defined gap, this result is well known [36, 37]; we emphasize, however, that we obtained (3.34) directly from microscopic theory, with the correct identification of the relaxation time.

Finally, we study the influence of the orbital viscosity on the line width of the longitudinal resonance by relaxing the assumption that l is fixed in a NMR experiment. We write $\Theta = \psi - \chi$, where ψ and χ are angles characterizing \mathbf{d}_0 and l , respectively, and consider the equations linearized around the equilibrium position $\psi = \chi$, and in the temperature region near T_c where this effect may be of importance. Instead of (3.34), we have to consider then a coupled system of equations, found from (2.41) and a generalization of (3.34) (compare [28]):

$$i\omega(1 + Z_0/4) 2\mu N_0^{-1} \chi = \omega_A^2 (\chi - \psi), \quad (3.37)$$

$$\left(\omega^2 + \frac{i\omega \cdot \Delta\omega_{\parallel}}{1 + Z_0/4} \right) \psi = \omega_A^2 (\psi - \chi).$$

Then we find the total line width to be given by

$$\Delta\omega_{\parallel}^{\text{tot}} = \frac{\omega_A^2}{1 + Z_0/4} \left(\tau_R + \frac{N_0}{2\mu} \right), \quad (3.38)$$

and, as expected, $\tau_R \gg N_0/2\mu$ for the case of a well defined gap ($\Delta_0 \tau_E \gg 1$). Very near T_c in the gapless regime, however, the orbital viscosity contributes twice as much as the intrinsic mechanism to the damping, as can be seen from Fig. 1, where we plot $\Delta\omega_{\parallel}^{\text{tot}}$, taking the data for the melting pressure. In the limit $T \rightarrow T_c$, we obtain from Eqs. (2.42c, 3.31a):

$$\lim_{T \rightarrow T_c} \Delta\omega_{\parallel}^{\text{tot}} = \frac{\omega_A^2}{1 + Z_0/4} \frac{9T_c}{\pi \Delta_0^2}. \quad (3.39)$$

which is $\simeq 2\pi \times 0.6$ kHz. However, we do not find an increase and a final decrease

in this regime ($T_c - T \simeq 10^{-3}T_c$) as discussed by Leggett and Takagi [28]; instead, the theoretical temperature dependence is rather undramatic and cannot explain the experimentally observed increase [38].

(C) *A-Phase Resonance at Lower Temperatures and for Arbitrary $\omega\tau_E$*

To extend the results to low temperatures and the collisionless regime, $\omega\tau_E \gg 1$, we have to include higher order terms in the Boltzmann equation (3.6), and we have to use the regular Green functions expanded up to second order in ω in the Ginzburg-Landau equation. The most important correction to (3.6) is, however, of a simple type, provided we have performed the transformation (3.26). As long as $\omega \ll \Delta$, it consists of including in \mathcal{N}_2 the external frequency [12] by replacing $1/\tau_E$ by $-i\omega + 1/\tau_E$. This can be seen as follows. From (2.36), we realize that the conversion term, for example, is modified into

$$2 |\Delta_p| \left[\mathcal{N}_2 - \frac{1}{2} \frac{\partial \mathcal{R}_2}{\partial E} \frac{\partial}{\partial t} \right] \delta f^z, \quad (3.40a)$$

which is written (after Fourier transformation, $\partial/\partial t \rightarrow -i\omega$) as

$$2 |\Delta_p| \mathcal{N}_2(\omega) \delta f^z, \quad (3.40b)$$

where $\mathcal{N}_2(\omega)$ is given by

$$\mathcal{N}_2(\omega) = \frac{1}{2} [\beta^R(E + \omega/2) + \beta^A(E - \omega/2)].$$

For a well-defined gap, we may again use the expanded form of \mathcal{N}_2 , namely,

$$2 |\Delta_p| \mathcal{N}_2(\omega) \simeq \frac{|\Delta_p|^2}{\xi^2} \mathcal{N}_1 \left(-i\omega + \frac{1}{\tau_E} \right), \quad (3.40c)$$

and the distribution function will be of the same form as above, (3.32), for arbitrary $\omega\tau_E$, provided $\omega, \tau_E^{-1} \ll \Delta, T$. Consequently, the non-linear longitudinal resonance equation near T_c is directly given by replacing τ_R by $\tau_R(\omega)$ in (3.34), where

$$\tau_R(\omega) = \tau_R / (1 - i\omega\tau_E), \quad (3.41)$$

and the resulting equation is easily transformed back to the time variable after multiplying by $1 - i\omega\tau_E$.

Furthermore, including the corrections of the regular functions in the Ginzburg-Landau equation, we find that the non-linear longitudinal resonance for all temperatures is described by

$$\omega^2 \Theta - i\omega^3 \tau_R(\omega)(1 - Y) \Theta - \left\{ 1 - i\omega\tau_R(\omega) \left[1 - \frac{YZ_0/4}{1 + Z_0/4} \right] \right\} \frac{\gamma^2}{\chi} \left(\frac{\partial F_D}{\partial \Theta} \right)_\omega = 0,$$

which is transformed into a third order differential equation of the form:

$$\ddot{\Theta} + \frac{1}{\tau_{\text{LT}}} \dot{\Theta} + \frac{\gamma^2}{\chi_p} \Theta \frac{\partial^2 F_D}{\partial \Theta^2} + \frac{1}{\tau_{\text{LT}}} \frac{\gamma^2}{\chi} \frac{\partial F_D}{\partial \Theta} = 0. \quad (3.42)$$

In this equation, χ is the magnetic susceptibility of the normal state, given by $\chi_0(1 + Z_0/4)^{-1}$, $\chi_0 = N_0\gamma^2/2$, and χ_p is the ‘‘Cooper-pair susceptibility’’ as introduced by Leggett and Takagi [17]. Our result is in complete agreement with the more or less phenomenological treatment of the non-linear spin dynamics with dissipation [17, 18, 39] since we find $\chi_p^{-1} = \chi^{-1} - \chi_0^{-1} + ((1 - Z)\chi_0)^{-1}$, and the connection with the phenomenological time τ_{LT} and the inelastic scattering time τ_E is given by

$$\tau_{\text{LT}} = \frac{(1 - Z) Y}{Y - Z} \tau_E; \quad (3.43)$$

Y and Z are defined below (see also Fig. 2). We emphasize strongly that we obtained this result directly from microscopic theory, solving explicitly the Boltzmann equation in a reasonable approximation, and the equation is applicable for all temperatures and arbitrary $\omega\tau_E$, provided $\omega, 1/\tau_E \ll \Delta, T$. On the other hand, τ_{LT} may be regarded as the typical time in which a non-equilibrium distribution decays in a closed system, and we will discuss this further in Chapter 4.

Finally, by linearizing (3.42), we obtain as usual the collective mode. The resonance frequency, (3.35), is not modified, and the line width is given by

$$\Delta\omega_{\parallel} = \frac{\omega_A^2 Y}{1 + Z_0/4} \tau_R, \quad (3.44)$$

where $Y = Y(\Delta/T)$ is the Yoshida function defined by

$$Y = \left\langle \int dE \frac{\mathcal{N}_1}{4Tch^2(E/2T)} \right\rangle. \quad (3.45)$$

For a well defined gap, we calculate then τ_R , using the distribution function Eq. (3.32), and obtain

$$\tau_R = (Z/(Y - Z)) \tau_E. \quad (3.46)$$

Furthermore, $Z = Z(\Delta/T)$ is defined by

$$Z = \left\langle \int dE \frac{\mathcal{N}_1}{4Tch^2(E/2T)} \frac{\xi^2}{E^2} \right\rangle \quad (3.47)$$

(which was denoted $f(T)$ in [36]). A plot of Y and Z is given in Fig. 2, and one easily finds the high and low temperature limits:

$$\begin{aligned} Y &\simeq 1 - \frac{7\zeta(3)}{4\pi^2} \left(\frac{\Delta}{T}\right)^2, \\ Z &\simeq 1 - \left(\frac{3}{2}\right)^{1/2} \frac{\pi^2}{16} \frac{\Delta}{T} + \frac{7\zeta(3)}{4\pi^2} \left(\frac{\Delta}{T}\right)^2, \quad \Delta \ll T; \end{aligned} \quad (3.48)$$

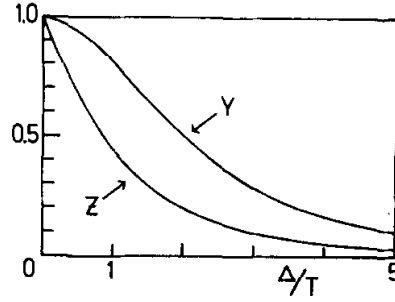


FIG. 2. The characteristic functions Y and Z for the A-phase vs Δ/T .

and

$$Y \simeq \frac{2\pi^2}{9} \left(\frac{T}{\Delta} \right)^2, \quad (3.49)$$

$$Z \simeq \frac{1}{3}Y, \quad T \ll \Delta.$$

The unusual temperature dependence for $T \rightarrow 0$ is due to the non-singular behavior of the angular averaged density of states and its inverse, which appear in the definition of Y and Z . The result for the line width is, of course, in complete agreement with the result of Leggett and Takagi who find

$$\Delta\omega_{\parallel} = \frac{\omega_A^2}{1 + Z_0/4} \frac{Z}{1 - Z} \tau_{LT}, \quad (3.50)$$

provided the above identification. In the low temperature limit, τ_{LT} is equal to $3\tau_E/2$.

An experimental determination of the longitudinal line width has been performed by Gully *et al.* [38] at the melting pressure ($T_c \simeq 2.6$ mK, $\Delta C/C_N \simeq 2.0$, $Z_0 \simeq -3$). For a comparison of the results (see Fig. 3), we have to calculate the temperature dependence of $\Delta(T)$, taking approximately the strong coupling corrections into account through the relation

$$\Delta(T) = 1.76T_c \left\{ \frac{\pi}{1.76} \left[\frac{2\Delta C}{3C_N} \left(\frac{T_c}{T} - 1 \right) \right]^{1/2} \right\},$$

which gives the correct specific heat jump near T_c and tends to the weak coupling limit for $T \rightarrow 0$. We then find that the temperature dependence of the line width is in good agreement with the experiment if we choose $\tau_E = \tau_N(T)$, and $\tau_N(T_c) \simeq 3.8 \times 10^{-8}$ sec; this is in good agreement with the result of Gully *et al.* who extracted a value of 4.8×10^{-8} sec using the phenomenological theory. Very near T_c , however, the agreement with experiment is not satisfactory—the limiting value for the line width is given by $\Delta\omega_{\parallel}/2\pi \simeq 0.6$ kHz. Of course, the finite limit for $T = T_c$ is in better qualitative agreement than previous theories.

With the described methods, we are able to handle in the same way the transverse resonance. As long as $\omega_L \tau_E \ll 1$, where $\omega_L = \gamma H_0$ is the Larmor frequency, the distribution function is not modified, and we find, besides the usual shift of the resonance

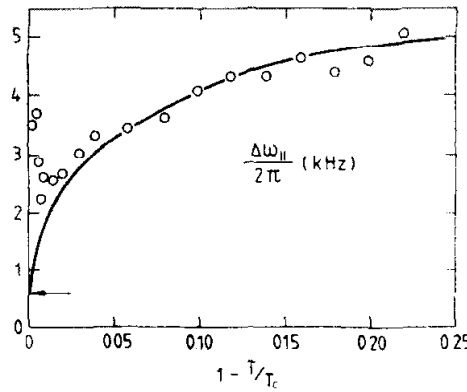


FIG. 3. Line width of the longitudinal A-phase resonance at the melting pressure (solid line), compared with experimental results (dots) from [38]. The arrow indicates the limit for $T = T_c$.

frequency, that the line width of the transverse mode, for all temperatures, is given by

$$\Delta\omega_{\perp} = ((\omega^2 - \omega_L^2)/\omega^2) \Delta\omega_{\parallel}, \quad (3.51)$$

in agreement with other calculations.

(D) Longitudinal Resonance in the B-Phase

Before applying the framework of the kinetic equations to the B-phase resonance, we want to discuss briefly the concept of splitting the distribution function $\delta\mathbf{f}$ into “parallel and perpendicular to $\mathbf{d}(\mathbf{n})$.” In the A-phase resonance, the spin-quantization axis \mathbf{d}_0 is independent of the direction of the momentum, and the splitting of $\delta\mathbf{f}$ and the kinetic equations is, since independent of \mathbf{n} , straightforward (compare (3.6, 3.11)), and note that for the A-phase $\delta\mathbf{f}_{\perp} = \mathbf{d}_0 \times (\delta\mathbf{f} \times \mathbf{d}_0)$, $\delta\mathbf{f}_{\parallel} = \mathbf{d}_0(\mathbf{d}_0 \cdot \delta\mathbf{f})$. This concept is very convenient in analyzing the structure of the solutions, since the “conversion term,” $2\frac{1}{2} \Delta_{\perp} \mathcal{V}_2 \delta\mathbf{f}_{\perp}$, appears only in the equation for the perpendicular component: quasiparticle spin can only be converted perpendicular to \mathbf{d}_0 . Consequently, the distribution function has the typical energy dependence given by (after the gauge transformation):

$$\begin{aligned} \delta\mathbf{f}_{\perp} &= \frac{\xi^2}{E^2} \frac{\boldsymbol{\eta}_{\perp}}{4Tch^2(E/2T)}, \quad \Delta_0\tau_E \gg 1, \\ \delta\mathbf{f}_{\parallel} &= \frac{\boldsymbol{\eta}_{\parallel}}{4Tch^2(E/2T)}, \end{aligned} \quad (3.52)$$

where $\boldsymbol{\eta}_{\perp}$, $\boldsymbol{\eta}_{\parallel}$ are constant vectors perpendicular and parallel to \mathbf{d}_0 ; note the “hole” in $\delta\mathbf{f}_{\perp}$ compared to $\delta\mathbf{f}_{\parallel}$.

In the B-phase, however, the spin-quantization axis is different for each \mathbf{n} , and we must perform a local splitting of $\delta\mathbf{f}$. Within the approximation of the reduced operator, this can be performed easily, and we consider then the angular averaged distribution function whose energy dependence will be similar to (3.52).

We consider now in more detail the longitudinal resonance, in which case we may neglect the external field from the beginning (it only defines the \mathbf{z} -direction in the following). As mentioned above, the spin-quantization axis $\mathbf{d}(\mathbf{n})$ is given by ($\omega = \mathbf{z}$ can be assumed, and $\varphi = 0$):

$$\mathbf{d}(\mathbf{n}) = \mathcal{R}(\omega, \Theta)\mathbf{n}$$

and we only have to consider oscillations of Θ (around its equilibrium value, $\Theta_0 \simeq 104^\circ$).

We start then from the Boltzmann equation (3.6) and redefine $\delta\mathbf{f}$ by

$$\delta\mathbf{f} \rightarrow \frac{\dot{\Theta}/2}{4Tch^2(E/2T)} \mathbf{z} + \delta\mathbf{f}, \quad (3.53)$$

$\mathbf{z} = (0, 0, 1)$, which leads, since

$$\dot{\Theta}\mathbf{d} \times (\mathbf{z} \times \mathbf{d}) = \mathbf{d} \times \dot{\mathbf{d}},$$

to the expected cancelation. In the resulting equation, we replace the collision operator by the reduced operator with energy independent scattering rate, split the equation in \parallel and \perp , and consider the angular average of the \mathbf{z} -component (which only is relevant). Note that for any momentum independent vector $\mathbf{a} = (0, 0, a)$,

$$\langle (\mathbf{a}_\parallel)^z \rangle = a/3, \quad \langle (\mathbf{a}_\perp)^z \rangle = 2a/3.$$

Defining further, for simplicity,

$$\begin{aligned} \langle \delta f^z \rangle &= \delta f_\parallel + \delta f_\perp, & \delta f_\parallel &= \langle (\delta \mathbf{f}_\parallel)^z \rangle, \\ & & \delta f_\perp &= \langle (\delta \mathbf{f}_\perp)^z \rangle, \end{aligned} \quad (3.54)$$

we obtain the Boltzmann equation in the form:

$$\begin{aligned} -i\omega \mathcal{N}_1 \delta f_\parallel + \frac{\mathcal{N}_1}{\tau_E} \left[\delta f_\parallel - \frac{1}{4Tch^2(E/2T)} \frac{\int dE \mathcal{N}_1 (\delta f_\parallel + \delta f_\perp)}{3Y} \right] \\ = \frac{1}{3} \frac{1}{4Tch^2(E/2T)} \left[\frac{\omega^2 \Theta}{2} - i\omega U^z \right] \\ -i\omega \mathcal{N}_1 \delta f_\perp + \frac{\mathcal{N}_1}{\tau_E} \left[\delta f_\perp - \frac{2}{4Tch^2(E/2T)} \frac{\int dE \mathcal{N}_1 (\delta f_\parallel + \delta f_\perp)}{3Y} \right] + 2 \Delta \mathcal{N}_2 \delta f_\perp \\ = \frac{2}{3} \frac{1}{4Tch^2(E/2T)} \left[\frac{\omega^2 \Theta}{2} - i\omega U^z \right], \end{aligned} \quad (3.55)$$

where the molecular field, U^z , is connected with the distribution function by

$$U^z = \frac{Z_0/4}{1 + \chi_0 Z_0/4} \left[\left\langle \int \mathcal{N}_1 \delta f^z \right\rangle + \frac{\Theta}{2} (1 - Y) \right], \quad (3.56)$$

and $\chi_0 = 2/3 + 1/3 Y$ is the normalized magnetic susceptibility of the B-phase, without Fermi-liquid corrections. Finally, the Ginzburg–Landau equation is of the form:

$$(-\omega^2\Theta + 2i\omega U^z) \frac{2(1-Y)}{3} + 4\Delta \int dE \mathcal{N}_2 \delta f_{\perp} = -\frac{2}{N_0} \left(\frac{\partial F_D^B}{\partial \Theta} \right)_{\omega} \quad (3.57)$$

According to the above discussion, we can immediately find the solution of the Boltzmann equation:

$$\begin{aligned} \delta f_{\parallel} &= \frac{\eta_{\parallel}}{4Tch^2(E/2T)}, \\ \delta f_{\perp} &= \frac{\mathcal{N}_1}{\mathcal{N}_1 + 2\Delta\tau_E(1-i\omega\tau_E)^{-1}\mathcal{N}_2} \frac{\eta_{\perp}}{4Tch^2(E/2T)}, \end{aligned} \quad (3.58)$$

and then it is straightforward to solve the system of equations. First of all, we find the usual result for the frequency of the collective mode which is given by

$$\omega_B^2 = \frac{\gamma^2}{\chi} \frac{\partial^2 F_D^B}{\partial \Theta^2} \Big|_{\Theta_0}, \quad (3.59)$$

where χ is the susceptibility of the B-phase

$$\chi = \frac{N_0\gamma^2}{2} \frac{\chi_0}{1 + \chi_0 Z_0/4}.$$

For the determination of the line width, we restrict ourselves first to the hydrodynamic regime $\omega\tau_E \ll 1$, and obtain

$$\Delta\omega_B = \frac{\omega_B^2}{1 + \chi_0 Z_0/4} \frac{Y}{2\chi_0} (3\tau_R + \tau_E), \quad (3.60)$$

where τ_R is defined as above (and is now identical to the τ_R discussed for superconductors). This is in exact agreement with the result of Leggett and Takagi [17] (for $\Delta\tau_E \gg 1$) provided we identify:

$$\tau_{LT} = \frac{Y}{\chi_0} \frac{1-Z}{Y-Z} \tau_E. \quad (3.61)$$

The temperature dependence of τ_E , taken for typical energies $E \simeq \Delta$, is quite different from the A-phase behavior since the probability of finding a partner for a collision process is decreasing rapidly at low temperatures. The result is [40]

$$\frac{1}{\tau_{E=\Delta}} \simeq \frac{1}{\tau_N(T)} \frac{3}{(2\pi)^{1/2}} \left(\frac{\Delta}{T} \right)^{3/2} e^{-\Delta/T}, \quad (3.62)$$

and a numerical calculation by Einzel and Wölfle [41] shows that the low temperature

limit is reached for $T \lesssim 0.5 T_c$. Since $Z \simeq (T/\Delta)Y$ for $T \ll \Delta$, we obtain in this limit

$$\tau_{LT} \simeq \frac{3}{2}\tau_E,$$

which was also recently reported by Einzel and Wölfle [42]. It is amusing to notice that the relation between the phenomenological relaxation time of Leggett and Takagi, and the inelastic scattering time appears to be the same in both phases for low temperatures.

It is, however, not realistic to neglect the external frequency in the expression for the line width at low temperatures, since $\omega \gtrsim \tau_E^{-1}$ for temperatures below $\lesssim 0.4T_c$. Thus, we have investigated also this case, and we find ($T \ll \Delta$)

$$\Delta\omega_B = \frac{\omega_B^2}{1 + \chi_0 Z_0/4} \frac{Y}{2} \frac{\tau_{LT}}{1 + (\omega\tau_{LT})^2} \quad (3.63)$$

with the above identification of τ_{LT} . This is also in agreement with the work of Combescot and Ebisawa [36] since the relaxation time introduced by them ad hoc in a kinetic equation is connected with τ_{LT} by

$$\tau_{CE} = \frac{2 + Y}{2(1 - Z)} \tau_{LT} \simeq \tau_{LT}, \quad T \ll \Delta.$$

(see, for example, the discussion in [17]). The temperature dependence of the line width at low temperatures has been measured recently, and it is in good agreement with the theoretical result [43], for intermediate temperatures. The reason for the temperature independent plateau for $T \rightarrow 0$, however, is still unclear.

In concluding this section, we want to remark that we have, also for the B-phase, given a complete justification of the phenomenological description of the damping of the longitudinal magnetic resonance, and of the proper identification of the phenomenological relaxation times. We must remark, however, that we found such a simple result only by the approximation of the reduced operator with *constant* inelastic scattering rate. We know, in fact, from work on similar problems in superconductors [25] that this introduces possibly an error of some 10 % in the final results (at lower temperatures).

4. CONNECTION WITH OTHER WORK ON NON-EQUILIBRIUM PHENOMENA

(A) *Spin Dynamics and the "Branch-Imbalance" in Superconductors*

Typical "branch-imbalance" situations have been considered several years ago, experimentally as well as theoretically [44, 45]. In these experiments, a tunneling current from a normal metal into a superconducting film generates a potential difference between pairs and quasiparticles in the superconductor, and this difference can be sensed by a second normal metal film, coupled via a tunnel barrier to the reverse

side of the superconductor. The theory was developed further by Schmid and Schön [12], near T_c , and extended to low temperatures by Schön and the author [25]. Reviews were given recently [19, 20, 46]. We want to clarify in this and the following section the connection between the two approaches, which may be called “particle representation” and “excitation representation,” since this is actually also the difference between our approach to superfluid ^3He , and other work using kinetic equations.

As was shown by Tinkham and Clarke [45], the voltage V_p in the detecting probe is given by

$$V_p = Q^*/2eN_0g_{\text{NS}}, \quad (4.1)$$

where $g_{\text{NS}}(T)$ is the measured normalized conductance of the probe junction in the low voltage limit, and the “quasiparticle charge” Q^* is given by

$$Q^* = 2N_0 \left\langle \int dE \mathcal{N}_1 \delta f^s \right\rangle \quad (4.2)$$

The quasiparticle distribution function has to be calculated from the stationary Boltzmann equation, which assumes for this situation the form (compare [12]):

$$-K(\delta f^s) + 2 \mid \Delta \mid \mathcal{N}_2 \delta f^s = P^{(T)}, \quad (4.3)$$

where $-K(\delta f^s)$ denotes the collision operator for electron–phonon collisions, and $P^{(T)}$ describes the symmetric part of the rate of change of δf^s due to the injection. Recall that δf^s is *even* with respect to the energy. The injection term is connected with the injected current I in the usual way,

$$I = 2N_0 \cdot \text{Vol} \cdot \int dE P^{(T)}; \quad (4.4)$$

thus we are led to *define* τ_R by:

$$\tau_R = 2N_0 \cdot \text{Vol} \cdot eV_p g_{\text{NS}}/I, \quad (4.5)$$

and it follows from the above definitions and the kinetic equation that

$$\tau_R = \left\langle \int dE \mathcal{N}_1 \delta f^s \right\rangle / \left\langle 2 \mid \Delta \mid \int dE \mathcal{N}_2 \delta f^s \right\rangle; \quad (4.6)$$

thus τ_R is directly related to the measured quantity. Comparing this result with Eq. (3.44), we realize that τ_R determines the damping of the nuclear magnetic resonance. In fact, the Boltzmann equations are identical for the tunnel injection in the low injection voltage limit, and for the determination of δf^z in the longitudinal A-phase resonance, and the result corresponds in both cases to a chemical potential difference between pairs and quasiparticles.

We remark that τ_R is, in a strict sense, not a relaxation time which can be measured directly in a real time-dependent relaxation process. To show this, we consider again the branch imbalance experiment and assume that the current is turned off suddenly, keeping, however, the chemical potential of the pairs ($= \Theta/2$) fixed—which corresponds thus to a situation where the superconductor is part of an open system. Recalling further that \mathcal{N}_2 is frequency dependent, and using the approximate result Eq. (3.40c) which is valid for $|\Delta| \tau_E \gtrsim 1$, namely

$$2|\Delta| \mathcal{N}_2 = \frac{|\Delta|^2}{\xi^2} \left(-i\omega + \frac{1}{\tau_E} \right) \mathcal{N}_1, \quad E^2 \gtrsim |\Delta|^2, \quad (4.7)$$

we may calculate the relaxation time by assuming that δf^s will not change during the relaxation process. Following Clarke *et al.* [46] in notation, we realize that the *relaxation time for fixed pair potential* is given by τ_{Q^*} :

$$\tau_{Q^*} = (Y/Z) \tau_R; \quad (4.8)$$

in fact, τ_{Q^*} was denoted τ_q by Pethick and Smith [20], and, from the above discussion, it has to be identified with the relaxation time introduced by Combescot and Ebisawa [36]:

$$\tau_{Q^*} = \tau_{CE}.$$

For low temperatures, τ_{Q^*} is equal to the inelastic scattering time for $E \simeq \Delta$. The above situation was also considered by Einzel and Wölfle [41], with the same result (remember that we neglected the energy dependence of τ_E throughout).

A rather different situation arises if we consider a closed superconductor such that the pair potential can rise during a relaxation process. Thus, the scattering processes only have to fill the “hole” of the distribution function [12]. The resulting time measures thus the *time of establishing local equilibrium between pairs and quasiparticles*, and has thus to be identified with the phenomenological time of Leggett and Takagi [17]. The result for all temperatures is [41]

$$\tau_{LT} = (1 - Z) \tau_{Q^*} = (1 - Z) Y / (Y - Z) \tau_E, \quad (4.9)$$

which we also found above in (3.43). The temperature dependence of this quantity is different, however, in a superconductor and in $^3\text{He-A}$. The above relations clarify further the interrelation between different relaxation times used in a variety of non-equilibrium phenomena in superfluid ^3He and in superconductors, and their relation to the inelastic scattering rate.

At this point, we would like to add a comment on the procedure we use to solve the kinetic equations, i.e., approximating the exact by the reduced collision operator with constant inelastic scattering rate. Although the eigenfunctions and eigenvalues of the exact operator are not known in the superconducting state, one may believe

that its spectrum has the same properties as in the normal state [33], namely the eigenvalue zero (connected with the conservation laws), and a continuous spectrum with eigenvalues $\leq -1/\tau_E$, with $E \simeq 0$ and $E \simeq \Delta$ for the A- and B-phase, respectively. Within the reduced operator, we treat the lowest eigenfunction exactly, and the higher eigenfunctions in a relaxation time approximation, with the lowest rate of the continuous spectrum. Thus, we believe to slightly underestimate $1/\tau_R$, in contrast to the variational principle which always gives an upper bound for $1/\tau_R$.

We remark finally that τ_R determines also the damping of fourth sound [47, 48] and is directly related to the second viscosity [48] which determines the dissipation in the acceleration equation of the superfluid velocity.

(B) Particle Representation–Excitation Representation

In this section, we briefly present the relation between usual kinetic equations for excitations (excitation representation), and the kinetic equations in the particle representation which we used throughout this work.

A very complete presentation of the kinetic equation in the excitation picture was given by Aronov and Gurevich [21], and we repeat some of their arguments here. The application of the excitation picture is restricted to situations for which the space and time variation of the various physical quantities is sufficiently slow such that

$$|\mathbf{q}| v_F \ll |\Delta| \quad \omega \ll |\Delta|, \quad (4.10)$$

where the first condition is necessary to ensure the existence of a local Bogoliubov transformation to almost stationary quasiparticles, while the second condition guarantees that the energies of the excitations are well defined. Actually, one has to require for the same reason

$$1/\tau_E \ll |\Delta|. \quad (4.11)$$

If these conditions are satisfied, the local spectrum is of the form (assume $\mathbf{v}_s = 0$):

$$\bar{E}_p = (\bar{\xi}_p^2 + |\Delta|^2)^{1/2}; \quad \bar{\xi}_p = \xi_p - \Theta/2; \quad \xi_p = \mathbf{p}^2/2m^* - \mu; \quad (4.12)$$

for simplicity, we consider only situations which can be characterized by one phase. Furthermore, the Bogoliubov amplitudes \bar{u}_p , \bar{v}_p assume the form

$$\begin{aligned} \bar{u}_p &= \frac{\bar{E}_p + \bar{\xi}_p}{[2\bar{E}_p(\bar{E}_p + \bar{\xi}_p)]^{1/2}}; \\ \bar{v}_p &= \frac{\Delta}{[2\bar{E}_p(\bar{E}_p + \bar{\xi}_p)]^{1/2}}; \end{aligned} \quad (4.13)$$

note that Δ can be a matrix in spin space (we avoid the index p here since the considerations apply also to superconductors).

In the following, we denote the distribution function of the excitations by $n_{\mathbf{p}}$, and we consider now, as a formal introduction, the correlation function

$$\langle a_{\mathbf{p}\alpha}^+(0) a_{\mathbf{p},\beta}(t) \rangle = e^{-i\bar{E}_{\mathbf{p}}t} (\bar{u}_{\mathbf{p}} n_{\mathbf{p}} \bar{u}_{\mathbf{p}})_{\alpha\beta} + e^{+i\bar{E}_{\mathbf{p}}t} (\bar{v}_{\mathbf{p}} [1 - n_{-\mathbf{p}}^T] \bar{v}_{\mathbf{p}}^+)_{\alpha\beta}, \quad (4.14)$$

where $a_{\mathbf{p}\alpha}^+$, $a_{\mathbf{p}\beta}$ are the usual Fermion creation and destruction operators for particles. Realize the different time dependences of the two contributions on the right-hand side of this equation; we may interpret this difference such that these two terms refer to particles in the upper and lower band, respectively, of a semiconductor model. Hence, we may introduce particle states which are labeled by the energy $E = +E_{\mathbf{p}}$ in the upper, and $E = -E_{\mathbf{p}}$ in the lower band as well as by the direction of the momentum, \mathbf{p}_F . Then it is consistent to define a distribution function $f(E, \mathbf{p}_F)$ of these particles by:

$$\mathcal{N}_1(E) f(E, \mathbf{p}_F) = \int d\xi_{\mathbf{p}} \{ \bar{u}_{\mathbf{p}} n_{\mathbf{p}} \bar{u}_{\mathbf{p}} \delta(\bar{E}_{\mathbf{p}} - E) + \bar{v}_{\mathbf{p}} [1 - n_{-\mathbf{p}}^T] \bar{v}_{\mathbf{p}}^+ \delta(\bar{E}_{\mathbf{p}} + E) \}, \quad (4.15)$$

where $T =$ transposed, and the $\xi_{\mathbf{p}}$ integration has to be performed for fixed \mathbf{p}_F . The Green function technique has led us to such a description in a very natural way. The description of a superconductor by $f(E, \mathbf{p}_F)$ and by $n_{\mathbf{p}} = n(\xi_{\mathbf{p}}, \mathbf{p}_F)$ are denoted particle representation and excitation representation, respectively. We will work this out more explicitly, and linearize (4.15) around the local equilibrium Fermi function. Writing

$$n_{\mathbf{p}} = n_0(\bar{E}) + \delta n_{\mathbf{p}}, \quad (4.16)$$

and omitting the bars for convenience, we find ($\xi \equiv \xi_{\mathbf{p}}$)

$$\mathcal{N}_1(E) \delta f(E, \mathbf{p}_F) = \int d\xi \{ u_{\mathbf{p}} \delta n_{\mathbf{p}} u_{\mathbf{p}} \delta(E_{\mathbf{p}} - E) - v_{\mathbf{p}} \delta n_{-\mathbf{p}}^T v_{\mathbf{p}}^+ \delta(E_{\mathbf{p}} + E) \}, \quad (4.17)$$

and we consider first the spinless case; i.e., $\delta n_{\mathbf{p}}$ is assumed to be proportional to the unit matrix in spin space. Then it is convenient to split the distribution functions according to their parity with respect to interchanging $E, \mathbf{p}_F \rightarrow -E, -\mathbf{p}_F$ in $f(E, \mathbf{p}_F)$ and with respect to $\xi, \mathbf{p}_F \rightarrow -\xi, -\mathbf{p}_F$ in $\delta n(\xi, \mathbf{p}_F)$. We introduce the notation $\delta f^{(\pm, \pm)}$ and $\delta n^{(\pm, \pm)}$. In a straightforward way, we obtain from Eq. (4.17) the following relations:

$$\begin{aligned} \delta f^{(++)} &= (\xi/|E|) \delta n^{(-+)}; & \delta f^{(+-)} &= \delta n^{(+-)}; \\ \delta f^{(-+)} &= \text{sgn } E \cdot \delta n^{(++)}; & \delta f^{(--)} &= (\xi/E) \delta n^{(--)}. \end{aligned} \quad (4.18)$$

Clearly, a branch imbalance situation is connected with $\delta n^{(-, +)}$, i.e., with the part of δn odd in ξ and even in \mathbf{p}_F [20]. Furthermore, in the low voltage limit, $\delta n^{(-, +)}$ is very well characterized by a chemical potential difference $\delta\mu$ between pairs and excitations. Consequently,

$$\delta n^{(-+)} = n_0(E(\mu + \delta\mu)) - n_0(E(\mu)),$$

from which we obtain

$$\delta n^{(-+)} = (-n'_0) \frac{\xi}{|E|} \delta \mu, \quad (4.19a)$$

and

$$\delta f^{(++)} = (-n'_0)(\xi^2/E^2) \delta \mu, \quad (4.19b)$$

in agreement with our result (3.32). Thus, with the above relations, the particle representation and the excitation representation can be brought into correspondence to each other. The conversion term in the particle representation is, in fact, connected with the peculiar form of the coherence factors in the excitation representation, which, for example, for electron-phonon collision processes in superconductors, are given by

$$[1 \pm (\xi\xi' - \Delta^2)/EE'].$$

Consequently, the scattering-out and scattering-in part of the excitation collision integral do not fit together, leading to conversion of excitations.

Finally, we discuss Eq. (4.17) for the spin case, and, restricting ourselves to distribution functions which are *even* with respect to \mathbf{p}_F , we find it again convenient to split them into parts parallel and perpendicular to $\mathbf{d}(\mathbf{n})$, namely

$$\begin{aligned} \delta \mathbf{n} &= \delta \mathbf{n}_{\parallel} + \delta \mathbf{n}_{\perp}; \\ \delta \mathbf{n}_{\parallel} &= \mathbf{d}(\mathbf{d}^* \cdot \delta \mathbf{n}) / |\mathbf{d}|^2, \end{aligned}$$

and in the same way for $\delta \mathbf{f}$. Then we find the following identification (positive parity with respect to $\mathbf{p}_F \rightarrow -\mathbf{p}_F$ understood):

$$\begin{aligned} \delta \mathbf{f}_{\parallel}^{(+)} &= (\xi/|E|) \delta \mathbf{n}_{\perp}^{(-)}; & \delta \mathbf{f}_{\parallel}^{(+)} &= \delta \mathbf{n}_{\parallel}^{(-)}; \\ \delta \mathbf{f}_{\perp}^{(-)} &= \text{sgn } E \cdot \delta \mathbf{n}_{\perp}^{(+)}; & \delta \mathbf{f}_{\parallel}^{(-)} &= (\xi/E) \delta \mathbf{n}_{\parallel}^{(-)}. \end{aligned} \quad (4.20)$$

This again reflects physically the fact that $\mathbf{d}(\mathbf{n})$ is the spin-quantization axis with eigenvalue zero; the consequences have been discussed in detail in the previous chapter.

5. SUMMARY AND CONCLUSION

In this paper, we described and applied the quasiclassical method to dynamical processes in superfluid ^3He . The basic quantity of this concept is the quasiclassical Green function, i.e., the Green function integrated with respect to $\xi_{\mathbf{p}}$. The enormous advantage is that a normalization condition allows the definition of a quasiparticle distribution function, labeled by the energy and the direction of the momentum (and by space and time). Consequently, one is able to treat non-equilibrium phenomena in the whole quasiclassical regime $\omega \ll \mu$, $v_F |\mathbf{q}| \ll \mu$, for example, one may study

strong pair breaking situations such as temperatures very near T_c or the influence of the walls of the container [49]. In the original Eilenberger approximation, however, the resulting equations are not Galilean and rotation invariant, since the derivation relies on a non-consistent treatment in the expansion with respect to the time and space variation (the latter being neglected in the self-energy part of the equation of motion; compare Chapter 2C). We propose to solve this problem by studying the equation of motion for the full Green function, supplemented by a normalization condition modified in the same way (see Eq. (2.31)). Obviously, this system has the required symmetry properties, and we expect that the solutions will depend only weakly on the absolute value of the momentum (compared to the strong ξ_p -dependence of the full Green function). Transformations are then easily performed by unitary operators like (2.15). In deriving the conservation laws from this equation of motion, we must, however, treat the $|\mathbf{p}|$ -dependence somewhat intuitively, allowing for partial integrations and setting $|\mathbf{p}| \simeq p_F$ at the end of a calculation. This procedure seems somehow artificial; we do not believe, however, that difficulties resulting from an inconsistent treatment of an expansion with respect to space and time can be overcome in the general case by choosing a certain reference frame [30, 50]—although it can be convenient in certain situations, of course.

In the preceding chapters, we applied the kinetic equations to the orbital and spin dynamics, in the homogeneous and low frequency ($\omega \ll \Delta$, T) limit in which most of the underlying physics is well understood: We have chosen these examples to illustrate the structure of the quasiclassical equations, and to show how they can be solved in a reasonable approximation for the collision operator. Thus we could give, directly from microscopic theory, a complete justification of the phenomenological theory, with the correct identification of the phenomenological relaxation time, for all temperatures including the gapless case very near T_c . Furthermore, with the generalization of the quasiclassical approach, we were able to study the properties of the equation under rotations (which have been of considerable interest recently [51, 52]).

In the last few years, a large amount of work has been done on kinetic equations for superfluid ^3He . In contrast to the quasiclassical approach, most of this work uses equations derived along the lines described by Betbeder–Matibet and Nozières [15]. In this technique, one also starts from the microscopic equation of motion, and then defines the distribution function by integration of the Green function with respect to the energy. In this case, one has to require that the inelastic scattering rate is small compared to the temperature (which is very well satisfied for superfluid ^3He), and that $\omega \ll T_c$, $v_F |\mathbf{q}| \ll T_c$ [53]. The resulting equation is a matrix equation in the additional particle–hole index characteristic for superconductivity. Furthermore, under the additional assumptions $\omega \ll \Delta$, $v_F |\mathbf{q}| \ll \Delta$, $\Delta \tau_E \gg 1$ which guarantee the existence of a local Bogoliubov transformation to almost stationary quasiparticles, a scalar Boltzmann equation for the excitations can be derived [21]. The system of equations is closed by the gap equation and the conservation laws. In this picture, the coherence factors in the collision operator play a crucial role in the relaxation of branch imbalance [20] and have to be handled very carefully. The connection of the excitation representation with the quasiclassical approach (which we also call particle

representation) has been discussed above, and we want to stress that in our approach, the crucial scattering processes appear in a natural way as a "conversion term" in the Boltzmann equation and are handled easily.

In conclusion, the quasiclassical approach to kinetic equations represent a very elegant and powerful method to study static as well as dynamic properties of superfluid ^3He . For example, since experiments in restricted geometries [54] become more and more important, we expect further application of this approach in the future.

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