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Quasiclassical Green's Function in the BCS Pairing Theory

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We study a generalization of the quasiclassical Green's function which allows us to include the transfer of momentum to the particles. In this approach, we may handle Galilei transformations, rotations, and gauge transformations in a systematic way. As an example, we calculate the quasiparticle flow pattern which arises during the motion of the orbital vector in the ABM phase, and discuss the meaning of the intrinsic angular momentum of the Cooper pairs. Finally, we consider charged particles in a magnetic field, and derive a Boltzmann equation for a superconductor which applies to the Hall effect in the case of moving vortices.

1. INTRODUCTION

The pairing theory of Bardeen, Cooper, and Schrieffer has been found extremely successful for superconductors as well as for the low-temperature phases of ^{3}He. In general terms, this theory owes its success in quantitative predictions to the fact that nature has made the pairing energy so much smaller than the Fermi energy. Another reason for its success results from the ease (comparatively) by which calculations can be performed. Indeed, most calculations can be simplified by means of approximations which exploit the fact that the Fermi momentum is extremely large in comparison with the momenta of, say, external perturbations; in other words, it is possible to simplify calculations since the Fermi wavelength is negligibly small compared with any other lengths.

A theory which incorporates systematically such approximations from the beginning has been developed independently by Eilenberger[1] and

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Larkin and Ovchinnikov.[2] Let us call this approach the theory of the quasiclassical Green's function; and, as a short introduction, let us remind the reader that, as Gor'kov[3] has shown, the BCS pairing theory can be formulated most conveniently in terms of Green's functions, say, of $G(\mathbf{p}, E)$. The fact that the Fermi momentum is extremely large can be understood in the sense that the length of the momentum p in the Green's function is of no importance. Consequently, one may argue that it is possible to integrate $G(\mathbf{p}, E)$ with respect to the magnitude of $\mathbf{p}$ with no loss of information. Thus, we obtain the quasiclassical* Green's function $g(\mathbf{p}, E)$, which depends only on the direction of $\mathbf{p}$. This theory has been found extremely useful in nonequilibrium problems of superconductivity† and of superfluid ^{3}He. A survey of this quasiclassical theory is presented in Section 2.

The quasiclassical theory in this form, however, seems to neglect a transfer of momentum to the particles entirely.‡ Such neglect has most severe consequences in cases where the transfer of momentum occurs in a systematic form. An example for a systematic transfer is the motion of a charged particle in a magnetic field which, no matter how small, leads to closed orbits. Another example is provided by the transition to a moving frame of reference where each particle acquires an additional momentum which leads to essentially different trajectories. Thus, there are cases where the standard quasiclassical Green's function fails to give an adequate description. Therefore, we have developed a generalized quasiclassical theory, which is presented in Section 3. There, the Fermi energy, that is, the chemical potential, is considered as an irrelevant variable and consequently the generalized quasiclassical Green's function $g(\mathbf{p}, E)$ is obtained by integrating $G(\mathbf{p}, E)$ with respect to the chemical potential. It is shown at the end of that section that the classical Boltzmann equation can be derived easily with this new technique.

The generalized quasiclassical theory possesses the same symmetries as the original problem. This is demonstrated in Section 4 explicitly for Galilei transformations, rotations, and gauge transformations. In addition, energy and momentum conservation are investigated in that section.

As a specific example, we study in Section 5 the quasiparticle flow pattern during the motion of the orbital vector in an anisotropic pairing state, which is commonly known as the ABM state.[6] We also learn in what sense one may assign an intrinsic angular momentum to the Cooper pairs.

We show in Section 6 that we are now able to recover the Lorentz force on a charged particle in a magnetic field. Eventually, we derive a

*This term is meant to indicate that the de Broglie wavelength of the particles is unimportant.
†For a review see Ref. 4.
‡Except perhaps, for a sudden transition during a collision.

Boltzmann equation for a superconductor which is expected to provide a proper basis for the Hall effect in the case of moving vortices.

The Appendix gives results on the quasiclassical Green's function in some simple cases.

2. QUASICLASSICAL GREEN'S FUNCTION

Our starting point* is the BCS pairing theory in the form of Gor'kov's equation, which is an equation of motion for a system of Green's functions including anomalous expectation values. In its conventional form, however, this equation of motion can only be integrated for thermal equilibrium. Since our main concern is with nonthermal states, we have to generalize Gor'kov's equation according to the theory of Keldysh such that it contains an enlarged set of Green's functions. Thus we are led to define a matrix Green's function

$$\check{\hat{G}} = \begin{pmatrix} \hat{G}^{\mathrm{R}} & \hat{G}^{\mathrm{K}} \\ 0 & \hat{G}^{\mathrm{A}} \end{pmatrix} \tag{1}$$

where $\hat{G}^{\mathrm{R(A)}}$ is the retarded (advanced) Green's function and where we may call $\hat{G}^{\mathrm{K}}$ Keldysh's function. These quantities are defined by

$$\begin{aligned} \hat{G}^{\mathrm{R}}(x_1, x_2) &= [\hat{G}^{>}(x_1, x_2) - \hat{G}^{<}(x_1, x_2)]\theta(t_1 - t_2) \\ \hat{G}^{\mathrm{A}}(x_1, x_2) &= -[\hat{G}^{>}(x_1, x_2) - \hat{G}^{<}(x_1, x_2)]\theta(t_2 - t_1) \\ \hat{G}^{\mathrm{K}}(x_1, x_2) &= [\hat{G}^{>}(x_1, x_2) + \hat{G}^{<}(x_1, x_2)] \end{aligned} \tag{2a}$$

In the case of s-wave pairing (which we consider here for simplicity), $\hat{G}^{>}$ and $\hat{G}^{<}$ are matrices in particle hole space as follows:

$$\begin{aligned} \hat{G}^{>}(x_1, x_2) &= -i\left\langle \begin{pmatrix} \psi_\uparrow(x_1)\psi_\uparrow^+(x_2) & \psi_\uparrow(x_1)\psi_\downarrow(x_2) \\ -\psi_\downarrow^+(x_1)\psi_\uparrow^+(x_2) & -\psi_\downarrow^+(x_1)\psi_\downarrow(x_2) \end{pmatrix} \right\rangle \\ \hat{G}^{<}(x_1, x_2) &= i\left\langle \begin{pmatrix} \psi_\uparrow^+(x_2)\psi_\uparrow(x_1) & \psi_\downarrow(x_2)\psi_\uparrow(x_1) \\ -\psi_\uparrow^+(x_2)\psi_\downarrow^+(x_1) & -\psi_\downarrow(x_2)\psi_\downarrow^+(x_1) \end{pmatrix} \right\rangle \end{aligned} \tag{2b}$$

where $\psi_s(x) \equiv \psi_s(\mathbf{r}, t)$ are the fermion destruction operators.

Using these definitions, we may write the generalized form of Gor'kov's equation as follows:

$$\left\{ \check{\tau}_3 i\frac{\partial}{\partial t_1} - \check{\hat{1}}[H_0(x_1) - \mu] + i\check{\hat{\Delta}}(x_1) \right\} \check{\hat{G}}(x_1, x_2) = \check{\hat{1}}\,\delta(x_1 - x_2) \tag{3}$$

In this relation, $H_0(x_1) = -(1/2m)\,\partial^2/\partial \mathbf{r}_1^2 + U(x_1)$ is the single-particle

*For a broader presentation of the contents of this section, see Ch. II of Ref. 4.

Hamiltonian and $\check{\tau}_z = \bar{1} \otimes \hat{\tau}_z$, and $\hat{\tau}_\alpha$ denotes the Pauli matrix. Furthermore,

$$\check{\Delta} = \bar{1} \otimes \hat{\Delta}; \qquad \hat{\Delta} = \begin{pmatrix} 0 & \Delta \\ \Delta^* & 0 \end{pmatrix} \tag{4a}$$

is the matrix system of the complex order parameter Δ, which has to be determined self-consistently according to

$$\hat{\Delta}(x) = \tfrac{1}{2} v \{\hat{G}^K(x, x)\}_{\text{o.d.}} \tag{4b}$$

Above (and in the following), the subscript o.d. means the off-diagonal part and v denotes the strength of the (attractive) interaction potential.

The quasiclassical approximation is based on the fact that in a degenerate Fermi system, the range of spatial variation in the physical quantities is much larger than the Fermi wavelength. This means that, as a function of the space coordinates, $\check{G}(\mathbf{r}_1, \mathbf{r}_2)$ is equal to $\exp[i\mathbf{p}(\mathbf{r}_1 - \mathbf{r}_2)]$, where $|\mathbf{p}| = p_F$ is the Fermi momentum, times a slowly varying amplitude. Consequently, in the equation of motion (3) we may approximate

$$H_0(x_1) \simeq \frac{p^2}{2m} - i\frac{\mathbf{p}}{2m}\frac{\partial}{\partial \mathbf{r}} + U(\mathbf{r}, t_1); \qquad \hat{\Delta}(x_1) \simeq \hat{\Delta}(\mathbf{r}, t_1) \tag{5}$$

where $\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2)2$ is the center-of-mass coordinate. In other words, the quasiclassical approximation (5) can be justified by the fact that $\check{G}$ is a rather compact object as a function of its relative space coordinate $\mathbf{r}' = \mathbf{r}_1 - \mathbf{r}_2$.

For a further development of the theory, it is advantageous to introduce a mixed representation which displays the center-of-mass coordinates $\mathbf{r}$ and $t = (t_1 + t_2)/2$, together with momentum $\mathbf{p}$ and energy E, which are the Fourier conjugate variables to the relative coordinates $\mathbf{r}'$ and $t' = t_1 - t_2$. Therefore,

$$\check{G}(\mathbf{p}, E; \mathbf{r}, t) = \int d^3r'\, dt' \,[\exp(-i\mathbf{p}\mathbf{r}' + iEt') \times \check{G}(\mathbf{r}_1, t_1 : \mathbf{r}_2, t_2) \tag{6}$$

where $\mathbf{r}_{1(2)} = \mathbf{r} + (-)\mathbf{r}'/2$: $\qquad t_{1(2)} = t + (-)\, t'/2$.

In contrast to the approximation in the spatial coordinates, no simple procedure is possible with regard to the time coordinates. From a general point of view, the equation of motion for the Green's function connects two quantities in the form of a matrix multiplication. Hence, this connection is of the type

$$\int dt_3\, A(t_1, t_3) B(t_3, t_2) \Rightarrow A(E; t) \circ B(E; t) \tag{7a}$$

where the transition is from the coordinate representation on the left side to the mixed representation on the right side. The "dot product" which

we have introduced above in the mixed representation is associative and it can be expressed as an infinite series of differentiations,

$$A \circ B = \left(\exp\left\{\frac{i}{2}\left[\partial_E^A \partial_t^B - \partial_t^A \partial_E^B\right]\right\}\right) AB \tag{7b}$$

$$= A \cdot B + \frac{i}{2}\left[\frac{\partial A}{\partial E}\frac{\partial B}{\partial t} - \frac{\partial A}{\partial t}\frac{\partial B}{\partial E}\right] + \cdots$$

where ∂_E^A means a differentiation with respect to E which acts only on A, etc.

It is a consequence of the approximation made above that the momentum appears only as a parameter. This allows us to introduce a quasiclassical Green's function $\check{g}$ as follows:

$$\check{g}(\mathbf{p}, E; \mathbf{r}, t) = \frac{i}{\pi} \fint d\xi_p \, \check{G}(\mathbf{p}, E; \mathbf{r}, t) \tag{8}$$

where $\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ is the center-of-mass coordinate. In other words, the taken. The momentum variable on the left side has to be interpreted as a dependence on the direction of the momentum only; the magnitude of $\mathbf{p}$ is understood to be fixed at $|\mathbf{p}| = p_F$. One expects no loss of information; for instance, it is known that the transitions in a Fermi system always occur close to the Fermi surface such that it is only necessary to specify the direction of the momentum.

The equation of motion for the quasiclassical Green's function can be obtained as follows. We form the difference between the equation of motion (3) and the adjoint equation,

$$\left[\check{\tau}_3 i\frac{\partial}{\partial t_1} \cdots + i\check{\Delta}(x_1)\right]\check{G}(x_1, x_2)$$

$$-\check{G}(x_1, x_2)\left[\check{\tau}_3 i\frac{\partial}{\partial t_2} + \cdots + i\check{\Delta}(x_2)\right] = 0 \tag{9}$$

where $\check{G}(\partial/\partial t_2)$ means $-(\partial/\partial t_2)\check{G}$. Then, we realize that the strongly momentum-dependent terms $(p^2/2m)\check{G}$ cancel. Furthermore, $\mathbf{p}(\partial\check{G}/\partial\mathbf{r})$ depends only weakly on the magnitude of the momentum, and we are allowed to put $|\mathbf{p}| = p_F$. Thus, an integration of Eq. (9) with respect to ξ_p leads to the equation for the quasiclassical Green's function, namely

$$[\check{Q}_0(\mathbf{p}, E; \mathbf{r}, t) \circ \check{g}(\mathbf{p}, E; \mathbf{r}, t)]_- = 0 \tag{10}$$

where we have introduced the notation

$$\check{Q}_0 = \check{\tau}_3 E + \check{1}\left(i\frac{\mathbf{p}}{m}\frac{\partial}{\partial \mathbf{r}} - U(\mathbf{r}, t)\right) + i\check{\Delta}(\mathbf{r}, t) = \bar{1} \otimes \hat{Q}_0 \tag{11}$$

Note that the commutator in Eq. (10) is based on the dot product; this means, for instance,

$$[\check{\tau}_3 E \circ \check{g}] = E[\check{\tau}_3, \check{g}]_- + \tfrac{1}{2} i[\check{\tau}_3, \check{g}]_-$$

Furthermore, the space differentiation acts as an operator of the usual type, that is, $[\partial/\partial \mathbf{r} \circ \check{g}] = 2(\partial \check{g}/\partial \mathbf{r})$.

The following remark is most important. Since Eq. (10) is homogeneous in $\check{g}$, it cannot determine the quasiclassical Green's function completely. The necessary information is supplied by the normalization condition

$$\check{g} \circ \check{g} = \check{1} \tag{12}$$

One may convince oneself that this condition is valid by the following arguments. In thermal equilibrium, and in a spatially homogeneous state, one finds by explicit calculation that Eq. (12) is true. In the general case, we consider the equation $[\check{Q}_0, \circ\, \check{g} \circ \check{g}]_- = 0$, which can be derived from Eq. (10) by an obvious algebra. Evidently, ansatz (12) solves this equation; and since this solution joins up smoothly to the equilibrium solution, we conclude that this is the only possible solution.

The formal aspects of the normalization can be illuminated by the following considerations. In the quasiclassical approximation, the solution of Eq. (3) can be written as

$$\check{G} = [\check{Q}_0 - \check{1}\xi_p]^{-1} \tag{13}$$

Clearly, $\check{G}$ can be considered as the resolvent of the operator $\check{Q}_0$. In the definition (8) of $\check{g}$, the principal value integral along the real ξ_p axis is equal to the sum of two integrals along contours which consist of the real axis (from $-\infty$ to $+\infty$) closed by semicircles in the upper and lower half-planes,

$$\int = \frac{1}{2}\oint_{+} + \frac{1}{2}\oint_{-} \tag{14a}$$

This means that

$$\check{g} = \check{P}_+ - \check{P}_- \tag{14b}$$

where $\check{P}_{+(-)}$ are projection operators such that

$$\check{P}_{+(-)} \circ \check{P}_{+(-)} = \check{P}_{+(-)}, \qquad \check{P}_+ \circ \check{P}_- = \check{0}, \qquad \check{P}_+ + \check{P}_- = \check{1} \tag{14c}$$

provided that the spectrum of $\check{Q}_0$ is separated from the real axis and from the infinite points.[7] In general, quasiparticle collisions (which will be included later) and the restriction to slow changes in space and time will guarantee that $\check{Q}_0$ satisfies this condition. Clearly, it follows from Eq. (14) that the normalization (12) is valid.

The quasiclassical approximation is self-contained in the sense that all physical quantities can be calculated once the quasiclassical Green's function $\check{g}$ is known. There appears, however, one peculiarity which can be understood best if one recalls how one calculates these quantities (which are as a rule equal-time expectation values) from the Green's function $\check{G}$. Due to their slow decrease for large values of E and ξ, it is necessary to observe the correct sequence of integration, which is E first and then ξ. Obviously, we have reversed this sequence when introducing quasiclassical Green's functions. We can repair the error which arises from this procedure if we improve the convergence of the integral by calculating difference quantities between the superfluid state of interest and say, the normal state in equilibrium.

Therefore, the following formula for the mass density ρ and the mass current (momentum) density $\mathbf{j}$ are meant to be valid for the calculation of just this difference.*† With this interpretation

$$\begin{aligned} \rho(\mathbf{r}, t) &= -2mN(0)\left\{\tfrac{1}{8}\int dE\ \mathrm{Sp}\langle\hat{g}^{\mathrm{K}}(\mathbf{p}, E; \mathbf{r}, t)\rangle + U(\mathbf{r}, t)\right\} \\ \mathbf{j}(\mathbf{r}, t) &= -\tfrac{1}{4}N(0)\int dE\ \mathrm{Sp}\ \hat{\tau}_3\langle\mathbf{p}\hat{g}^{\mathrm{K}}(\mathbf{p}, E; \mathbf{r}, t)\rangle \end{aligned} \tag{15}$$

where $\langle\cdots\rangle$ denotes the average with respect to the directions of $\mathbf{p}$, and where $N(0)$ is the density of states at the Fermi level. Note the potential energy U, which anticipates results of the subtraction procedure discussed above.

For the sake of completeness, we also rewrite the order parameter self-consistency condition (4b) valid for s-wave pairing,

$$\hat{\Delta}(\mathbf{r}, t) = \frac{\nu N(0)}{4i}\int dE\ \{\langle\hat{g}^{\mathrm{K}}(\mathbf{p}, E; \mathbf{r}, t)\rangle\}_{\mathrm{o.d.}} \tag{16}$$

The quasiclassical Green's function allows us to define the particle distribution function as follows. Consider first the normalization (12) in the detailed form $\hat{g}^{\mathrm{R}} \circ \hat{g}^{\mathrm{R}} = \hat{g}^{\mathrm{A}} \circ \hat{g}^{\mathrm{A}} = 1$ and $\hat{g}^{\mathrm{R}} \circ \hat{g}^{\mathrm{K}} + \hat{g}^{\mathrm{K}} \circ \hat{g}^{\mathrm{A}} = 0$. One recognizes that the last relation can be solved by the ansatz

$$\hat{g}^{\mathrm{K}} = \hat{g}^{\mathrm{R}} \circ \hat{h} - \hat{h} \circ \hat{g}^{\mathrm{A}} \tag{17a}$$

*The calculation of difference quantities is an important concept of the present theory, and it serves to justify one or the other procedure later. As a rule, the values of the quantities in the normal state can be found by direct reasoning. For instance, there are cases where obviously the mass density is uniform and where the currents are zero.

†Landau's Fermi liquid theory shows some resemblance to the features discussed above. There, one may distinguish between a direct contribution of the quasiparticles to the physical quantities on the one hand and an indirect contribution which may be attributed to the molecular fields arising from the quasiparticle interaction. Generally, the latter does not depend on minute details of the quasiparticle distribution and hence this contribution is the same in the normal and superfluid states. Thus, superfluid and normal state differ by the direct quasiparticle contribution.[6]

Furthermore, it is possible to require that $\hat{h}$ be a diagonal matrix in particle–hole space. Therefore, we put*

$$\hat{h} = \hat{1}h^{(L)} + \hat{\tau}_3 h^{(T)} \tag{17b}$$

and simultaneously we introduce $\hat{f} = (\hat{1} - \hat{h})/2$, where

$$f = f^{(L)} + f^{(T)}; \qquad f^{(L)} = \tfrac{1}{2}(1 - h^{(L)}); \quad f^{(T)} = -\tfrac{1}{2}h^{(T)} \tag{18a}$$

has the meaning of a distribution function. By explicit calculation, one can show that in thermal equilibrium, $f = f_{th}$, where

$$f_{th} = \frac{1}{e^{E/T} + 1}; \qquad \left(h_{th} = \text{th}\,\frac{E}{2T}\right) \tag{18b}$$

and where $f^{(T)} = 0$. We interpret f such that $f = f(\mathbf{p}, E; \mathbf{r}, t)$ is the number of particles in a state† labeled by the momentum direction $\mathbf{p}$ and energy E.

The Boltzmann equation for f can be established as follows. In the equation of motion (10), we insert the ansatz (17) for $\hat{g}^K$. Using the relation

$$[A \circ B \circ C]_- = B \circ [A \circ C]_- + [A \circ B]_- \circ C$$

as well as $[\hat{Q}_0 \circ \hat{g}^R]_- = [\hat{Q}_0 \circ \hat{g}^A]_- = 0$, we obtain

$$[\hat{\tilde{Q}}_0 \circ \hat{g}^K]_- = \hat{g}^R \circ [\hat{\tilde{Q}}_0 \circ \hat{h}]_- - [\hat{\tilde{Q}}_0 \circ \hat{h}]_- \circ \hat{g}^A = 0 \tag{19}$$

Of this equation, only the diagonal part is needed. Its detailed structure depends very much on the form of $\hat{g}^R$ and $\hat{g}^A$ and therefore we will only write it down schematically. At the same time, we will also add a term which takes care of the inelastic collisions, on which we will comment in a moment. Thus,

$$\frac{1}{2}\left\{\hat{g}^R \circ \left(\hat{\tau}_3 \dot{\hat{f}} + \frac{\mathbf{p}}{m}\frac{\partial \hat{f}}{\partial \mathbf{r}} + [\hat{1}iU + \hat{\Delta} \circ \hat{f}]_-\right) - (\hat{\tau}_3 \dot{\hat{f}} + \cdots) \circ g^A\right\}_{d.} - \hat{K} = 0 \tag{20}$$

where $\hat{K} = \hat{1}K^{(L)} + \hat{\tau}_3 K^{(T)}$ is called the collision integral.

At this point, we should recall that Eq. (3) is only a special case of Dyson's equation

$$[\check{G}_0^{-1} - \check{\Sigma}]G = \check{1} \tag{21}$$

*(L) and (T) are abbreviations for longitudinal and transverse, respectively. This terminology stems from the interpretation of the complex order parameter Δ as a vector in the complex plane. The changes $\delta\Delta$ are then parallel and perpendicular to Δ in the (L) and (T) modes, respectively.

†This state is not a quasiparticle state of the Bogoliubov type, but rather a weighted sum of such states.

where $\hat{\tilde{\Sigma}}$ is a matrix of the same structure as $\hat{\tilde{G}}$ defined by Eq. (1). In general, the diagonal part of $(\hat{\Sigma}^{R}+\hat{\Sigma}^{A})/2$ can be absorbed by defining suitably renormalized quantities, whereas the off-diagonal part contains the pair correlations and equals $-i\hat{\Delta}$ in the weak coupling limit.

Inelastic collisions are contained in $\hat{\Sigma}^{R}-\hat{\Sigma}^{A}$ (scattering out) and in $\hat{\Sigma}^{K}$ (scattering in). In the spirit of Boltzmann's "stosszahlansatz" we may calculate $\hat{\mathbf{K}}$ by replacing the dot product by simple multiplication. Thus, we obtain

$$\hat{K}=\tfrac{1}{4}i\{\tfrac{1}{2}[\hat{\Sigma}^{R}-\hat{\Sigma}^{A},\hat{g}^{K}]_{+}+\hat{\Sigma}^{K}\hat{g}^{A}-\hat{g}^{R}\hat{\Sigma}^{K}\}_{d.} \tag{22}$$

In general, the collision integral is such that it conserves mass, energy, and momentum.

3. GENERALIZED QUASICLASSICAL APPROXIMATION

We will now remove some of the restrictions which the standard quasiclassical approximation imposes on the momentum variable. Let us first rewrite Gor'kov's equation (3) in the mixed representation defined by Eq. (6) Then it will be necessary to generalize the dot product of Eq. (7) such that it also includes the momentum and space variables. It will be called the star product and it is defined by

$$\begin{aligned} A*B&=\left(\exp\left\{\frac{i}{2}[\partial_E^A\partial_t^B-\partial_{\mathbf{p}}^A\partial_{\mathbf{r}}^B-\partial_t^A\partial_E^B+\partial_{\mathbf{r}}^A\partial_{\mathbf{p}}^B]\right\}\right)_{A\cdot B}\\ &=A\cdot B+\frac{i}{2}\left[\frac{\partial A}{\partial E}\frac{\partial B}{\partial t}-\frac{\partial A}{\partial \mathbf{p}}\frac{\partial B}{\partial \mathbf{r}}-\frac{\partial A}{\partial t}\frac{\partial B}{\partial E}+\frac{\partial A}{\partial \mathbf{r}}\frac{\partial B}{\partial \mathbf{p}}\right]\\ &\quad+\cdots \end{aligned} \tag{23}$$

Note that the above expansion in powers of derivatives means in conventional units an expansion in powers of $\hbar$; for convenience, we will henceforth refer to it in this sense. For instance, the terms that are explicitly written down in Eq. (23) mean an expansion of the star product through first order in $\hbar$.

The notation of the star product allows us to write Gor'kov's equation in the form

$$[\hat{\tilde{Q}}(\mathbf{p},E;\mathbf{r},t)+\hat{\tilde{1}}\mu]*G(\mathbf{p},E;\mathbf{r},t)=\hat{\tilde{1}} \tag{24}$$

where

$$\hat{\tilde{Q}}=\tau_3E-\hat{\tilde{1}}[p^2/2m+U(\mathbf{r},t)]+i\hat{\tilde{\Delta}}(\mathbf{r},t)=\tilde{1}\otimes\hat{Q} \tag{25}$$

replaces $\check{Q}_0$ of Eq. (11). Formally, the solution of Eq. (24) is given by

$$\check{G} = [\check{Q} + \check{1}\mu]^{-1} \tag{26}$$

For our purposes we find it is necessary to define the quasiclassical Green's function, different from Eq. (8), as follows:

$$\check{g}(\mathbf{p}, E; \mathbf{r}, t) = \frac{i}{\pi} \oint d\mu \, \check{G}(\mathbf{p}, E; \mathbf{r}, t) \tag{27}$$

Roughly, the μ integral collects most of its contribution at $\mu \sim p^2/2m$ where $\check{G}$ has a pronounced maximum. This will lead to a result similar to the previous one if $|\mathbf{p}|$ is about the Fermi momentum. In particular, for spatially homogeneous states, both definitions agree. Further illustrations are given in the Appendix. In general, however, $\check{g}$ will now depend weakly on the magnitude of the momentum, though we expect this dependence to be weak. Of course, there will be some (subtle) effects which rely on this weak momentum dependence and which are of concern in the present paper. Nevertheless, we will continue to use the symbol $\check{g}$, since no confusion may arise.

The difference between Eq. (24) and its adjoint equation leads, upon μ integration, without further approximation to the equation of motion

$$[\check{Q}(\mathbf{p}, E; \mathbf{r}, t) \overset{*}{,} \check{g}(\mathbf{p}, E; \mathbf{r}, t)]_- = 0 \tag{28}$$

which has to be supplemented by the normalization condition

$$\check{g} * \check{g} = \check{1} \tag{29}$$

The arguments which show that this condition is valid are similar to the ones discussed in connection with Eqs. (12) and (14), and only a few obvious replacements have to be carried through. On the other hand, it might be possible that the increased complexity of $\check{Q}$ may require more restrictions on the states for which Eq. (29) is valid.

As far as the expressions for the densities of mass and current are concerned, we could almost adopt the formulation of Eq. (15), where averages with respect to the directions of $\mathbf{p}$ occur at fixed length $|\mathbf{p}| \simeq p_F$. However, we need more flexibility with respect to small changes in the momentum of the particles, particularly with respect to the shift $\mathbf{p} \to \mathbf{p} + m\mathbf{u}$. Therefore, we define

$$\begin{aligned} \rho(\mathbf{r}, t) &= -\tfrac{1}{4} m \int d\sigma_p \int dE \, \mathrm{Sp} \, \hat{g}^K(\mathbf{p}, E; \mathbf{r}, t) \\ \mathbf{j}(\mathbf{r}, t) &= -\tfrac{1}{4} \int d\sigma_p \int dE \, \mathrm{Sp} \, \hat{\tau}_3 \mathbf{p} \hat{g}(\mathbf{p}, E; \mathbf{r}, t) \end{aligned} \tag{30}$$

where the σ_p integration acts on the momentum according to

$$\int d\sigma_p\, a(\mathbf{p}) = N(0)\langle a(\mathbf{p})\rangle; \qquad \int d\sigma_p \frac{\partial}{\partial \mathbf{p}} a(\mathbf{p}) = 0 \tag{31}$$

In particular, the last relation means $\int d\sigma_p\, a(\mathbf{p}+m\mathbf{u}) = \int d\sigma_p\, a(\mathbf{p})$. It can be justified if one goes back to the original definition of the densities in terms of $\hat{G}^{\mathrm{K}}$. There, the three-dimensional $\mathbf{p}$ integration extends over the whole momentum space and a shift in the integration variable is possible if the integral is sufficiently convergent. This condition requires the subtraction of, let us say, a background contribution which is insensitive to the details of the state and which we take as the normal state contribution as discussed in the previous section. Consequently, Eq. (30) is valid for the evaluation of difference quantities.* This rule has to be observed in a somewhat stricter sense than before, and therefore we have not included the potential energy correction to the mass density ρ. With the present notation, the self-consistency condition (16) assumes the form

$$\hat{\Delta}(r, t) = \frac{v}{4i} \int d\sigma_p \int dE\, \{\hat{g}^{\mathrm{K}}(\mathbf{p}, E; \mathbf{r}, t)\}_{\mathrm{o.d.}} \tag{32}$$

We define a distribution function similar to relations (17) and (18), with the difference that the star product now replaces the dot product. For instance, we have

$$\hat{g}^{\mathrm{K}} = \hat{g}^{\mathrm{R}} * \hat{h} - \hat{h} * \hat{g}^{\mathrm{A}} \tag{33}$$

and $\hat{f} = (\hat{1} - \hat{h})/2$. The distribution function, specifically $f = \{\hat{f}\}_{11} = f(\mathbf{p}, E; \mathbf{r}, t)$, is expected to depend weakly on the magnitude of the momentum. However, we interpret f as before, namely as the number of particles in a state labeled by the momentum direction $\mathbf{p}$ and the energy E.

From the definition (2), one may deduce symmetry properties of the Green's function with respect to the interchange of the arguments. For instance,

$$\{\hat{g}^{\mathrm{K}}(-\mathbf{p}, -E)\}_{22} = \{\hat{g}^{\mathrm{K}}(\mathbf{p}, E)\}_{11} \tag{34a}$$

From this and related relations, it follows that

$$h^{(\mathrm{L})}(-\mathbf{p}, -E) = -h^{(\mathrm{L})}(\mathbf{p}, E), \qquad h^{(\mathrm{T})}(-\mathbf{p}, -E) = h^{(\mathrm{T})}(\mathbf{p}, E) \tag{34b}$$

The symmetry properties of $f^{(\mathrm{L})}$ and $f^{(\mathrm{T})}$ differ only by minor changes.

*See footnotes cited just prior to Eqs. (15).

For sake of completeness let us note that the Boltzmann equation (20) appears now in the generalized form

$$\frac{1}{2}\left\{\hat{g}^{\mathrm{R}} * \left(\hat{\tau}_3 \dot{\hat{f}} + \frac{\mathbf{p}}{m}\frac{\partial \hat{f}}{\partial \mathbf{r}} + [\hat{1} i\dot{U} + \hat{\Delta} \overset{*}{,} \hat{f}]_-\right) - (\hat{\tau}_3 \dot{\hat{f}} + \cdots) * \hat{g}^{\mathrm{A}}\right\}_{\mathrm{d.}} - \hat{K} = 0 \tag{35}$$

Again, the contribution $\hat{K}$ of inelastic collisions may be evaluated in the approximation of simple multiplication, and hence its expression agrees with Eq. (22).

As an example, we consider a system in the normal state where $\Delta = 0$. It follows from Eqs. (25)–(27) that

$$\hat{g}^{\mathrm{R}}(\mathbf{p}, E; \mathbf{r}, t) = \hat{\tau}_3 \tag{36a}$$

and that $\hat{g}^{\mathrm{A}} = -\hat{g}^{\mathrm{R}}$ (see also Appendix A). It follows from this relation and from Eq. (33) that

$$\hat{g}^{\mathrm{K}}(\mathbf{p}, E; \mathbf{r}, t) = 2\hat{\tau}_3 \hat{h}(\mathbf{p}, E; \mathbf{r}, t) \tag{36b}$$

We note in passing that this is an example of how condition (34) can be satisfied.

The normal state Boltzmann equation can be derived from Eq. (35) as follows. First note that the equations for $f^{(\mathrm{L})}$ and $f^{(\mathrm{T})}$ are formally equivalent, since no off-diagonal quantities appear. This fact allows us to write down one equation for the total distribution $f = f^{(\mathrm{L})} + f^{(\mathrm{T})}$ at once. If we expand the star product $[U \overset{*}{,} f]_-$ through first order in $\hbar$, we obtain

$$\dot{f} + \frac{\mathbf{p}}{m}\frac{\partial f}{\partial \mathbf{r}} + \dot{U}\frac{\partial f}{\partial E} - \frac{\partial U}{\partial \mathbf{r}}\frac{\partial f}{\partial \mathbf{p}} - K[f] = 0 \tag{37}$$

In this Boltzmann equation, each term is already proportional to $\hbar$ and therefore it is exact in the classical limit $\hbar \to 0$.

A peculiar feature in the Boltzmann equations that can be derived from quasiclassical Green's functions is that the energy variable E always measures the local energy. In the present case, this means that

$$E = p^2/2m + U - \mu \tag{38a}$$

and one obtains the conventional type of Boltzmann equation if one introduces the distribution function

$$\tilde{f}(\mathbf{p}; \mathbf{r}, t) = f(\mathbf{p}, E = p^2/2m + U - \mu; \mathbf{r}, t) \tag{38b}$$

It follows from this relation that $\dot{f} + \dot{U}\,\partial f/\partial E = \dot{\tilde{f}}$ (and further relations).

Thus, we obtain the conventional result

$$\dot{\tilde{f}}+\frac{\mathbf{p}}{m}\frac{\partial \tilde{f}}{\partial \mathbf{r}}-\frac{\partial U}{\partial \mathbf{r}}\frac{\partial \tilde{f}}{\partial \mathbf{p}}-K[\tilde{f}]=0 \tag{38c}$$

Clearly, $\int d\sigma_p \int dE f$ is equivalent to $\int (2\pi)^{-3} d^3p \tilde{f}$ and therefore, in terms of $\tilde{f}$, the densities of mass and current assume their conventional form.

In the superfluid state substitutions of the type (38) are of little value since in general they lead to more complicated expressions.

4. SYMMETRIES AND CONSERVATION LAWS

The theory of the generalized quasiclassical Green's function possesses the same symmetries as the original problem from which it is derived. We will demonstrate this explicitly for Galilei transformations, rotations, and gauge transformations. Quite generally, conservation laws follow from the symmetries with respect to these transformations, but we will study specifically energy and momentum conservation in order to obtain explicit expressions for the energy and momentum currents.

As a rule, the collision integral is of a structure which allows a direct interpretation in physical terms and which displays clearly the symmetries and the conservation laws under consideration. Therefore, and for the sake of convenience, we will neglect it in this section. This means also that we do not need to make use of the representation (33) of $\hat{g}^{\mathrm{K}}$ in spectral quantities $\hat{g}^{\mathrm{R}}$ and $\hat{g}^{\mathrm{A}}$ and in the distribution function $\hat{f}$. In addition, the normalization to $\check{g}$ will not be used here. As a consequence, most relations obtained in this section are valid if $\check{g}$ is replaced by $\check{G}$.

4.1. Galilei Transformation

Consider a new reference frame which moves with velocity $\mathbf{u}$ with respect to the rest frame, and mark the physical quantities in the new reference frame by a tilde. Then,

$$\begin{aligned} &\tilde{\mathbf{r}}=\mathbf{r}-\mathbf{u}t; \qquad \tilde{\mathbf{p}}=\mathbf{p}\mp m\mathbf{u} \\ &\tilde{t}=t; \qquad \tilde{E}=E-\mathbf{pu}\mp\tfrac{1}{2}m\mathbf{u}^2 \end{aligned} \tag{39}$$

where the upper and lower signs distinguish between particles and holes. Note that $\pm E-\mathbf{p}^2/2m$ is an invariant. Furthermore, this transformation is unitary; it can be written as*

$$\check{\tilde{b}}=\check{S}^{+} * \check{b} * S; \qquad \check{S}=\exp(i\check{A}); \qquad \check{A}=\mathbf{pu}t-\check{\tau}_3 m\mathbf{ru} \tag{40}$$

*A different but equivalent expression, which is easily obtained by applying the Baker–Haussford formula, is $\check{S}=\{\exp[i\mathbf{pu}t+i\check{\tau}_3 m\mathbf{u}^2 t/2]\} * \{\exp[-i\check{\tau}_3 m\mathbf{ru}]\}$.

where we have employed a notation which allows us to operate directly on matrices in particle–hole and in Keldysh space. We recall that a unitary transformation of this type can be expressed as a series of commutators,

$$\check{\tilde{b}} = \check{b} + \frac{-i}{1!}[\check{A} \overset{*}{,} \check{b}]_- + \frac{(-i)^2}{2!}[\check{A} \overset{*}{,} [\check{A} \overset{*}{,} \check{b}]_-]_- + \cdots \tag{41}$$

Later, it will be necessary to use the fact that this transformation commutes with $\mathbf{p}$ and E integration. This statement can be demonstrated as follows:

$$\begin{aligned}
&\int d\sigma_p \int dE \{\check{\tilde{S}}^+ * \check{b}(\mathbf{p}, E; \mathbf{r}, t) * \check{\tilde{S}}\} \\
&\quad = \int d\sigma_p \int dE\, \check{b}(\mathbf{p} - \check{\tau}_3 m\mathbf{u}, E - \mathbf{pu} + \tfrac{1}{2}\check{\tau}_3 m\mathbf{u}^2; \mathbf{r} - \mathbf{u}t, t) \\
&\quad = \int d\sigma_p \int dE\, \check{b}(\mathbf{p}, E; \mathbf{r} - \mathbf{u}t, t) \\
&\quad = \check{\tilde{S}}^+ * \left\{ \int d\sigma_p \int dE\, \check{b}(\mathbf{p}, E; \mathbf{r}, t) \right\} * \check{\tilde{S}}
\end{aligned} \tag{42}$$

We remark that, strictly, the transition to the third line is possible only if the integral is convergent. This condition may necessitate the subtraction of a background contribution* as explained in the discussion following Eq. (14). In this sense Eq. (42) is consistent with the previous interpretation.

Suppose that in the rest frame the motion of the system is represented by the Green's function $\check{\tilde{g}}(\mathbf{p}, E; \mathbf{r}, t)$. Then Galilei's relativity principle is equivalent to the statement that

$$\check{\tilde{\tilde{g}}} = \check{\tilde{S}}^+ * \check{\tilde{g}} * \check{\tilde{S}} \tag{43}$$

also represents a possible state of motion of the system. There are two ways to interpret this relation. Namely, this state differs from the original state merely by the superposition of the constant velocity $\mathbf{u}$; that in a reference frame moving with velocity $[-\mathbf{u}]$, the original state of the system is described by $\check{\tilde{\tilde{g}}}$. Thus, we assert that†

$$[\check{\tau}_3 E - \mathbf{p}^2/2m + i\check{\tilde{\Delta}}' \overset{*}{,} \check{\tilde{g}}]_- = 0 \tag{44}$$

provided that the order parameter $\check{\tilde{\Delta}}'$ is determined self-consistently by Eq. (32) with $\check{\tilde{g}}^K$ replacing $\hat{g}^K$. Now, it follows from Eq. (42) and from the self-consistency condition for $\check{\tilde{\Delta}}$ in the original problem that

$$\check{\tilde{S}} \Delta * \check{\tilde{\Delta}}' * \check{\tilde{S}}^+ = \check{\tilde{\Delta}}(\mathbf{r}) \tag{45}$$

which concludes the proof. We emphasize that the simple quasiclassical

*See footnotes cited just prior to Eqs. (15).

†Clearly, the external potential has to be put equal to zero.

Green's function of Section 2 violates Galilei's relativity principle since there the momentum appears only as a parameter.

Let us now calculate the current density $\tilde{\mathbf{j}}$ for the new state, which is obtained from Eq. (30) by substituting $\tilde{\hat{g}}^{\mathrm{K}}$ for $\hat{g}^{\mathrm{K}}$. We encounter an integral of a type similar to Eq. (42) where the variables $\tilde{\mathbf{p}}$, $\tilde{E}$, etc. appear exclusively except for a prefactor $\mathbf{p}=\tilde{\mathbf{p}}+\tilde{\tau}_3 m\mathbf{u}$. Thus, we obtain

$$\tilde{\mathbf{j}}(\mathbf{r}, t)=\mathbf{j}(\mathbf{r}-\mathbf{u}t, t)+\mathbf{u}\rho(\mathbf{r}-\mathbf{u}t, t) \tag{46}$$

where $\mathbf{j}$ and ρ are the densities in the original state. It has been emphasized repeatedly that the densities when calculated according to Eq. (30) may necessitate corrections for background contributions. In the present case, it is evident that these contributions must cancel.

A comment is in order on the self-consistency condition in the case of a pairing different from the s-wave type. Then, we have to generalize the self-consistency relation (32) by the Galilei-invariant form

$$\hat{\Delta}(\mathbf{p}; \mathbf{r}, t)=\frac{-1}{4i}\int d\sigma_{p'}\int dE\upsilon\,(\mathbf{p}-\mathbf{p}')\{\hat{g}^{\mathrm{K}}(\mathbf{p}', E; \mathbf{r}, t)\}_{\mathrm{o.d.}} \tag{47}$$

where the matrices may now include spin indices. In a most common approximation, one replaces the interaction potential $(\mathbf{p}-\mathbf{p}')$ by an expression which factors in $\mathbf{p}$ and $\mathbf{p}'$. Clearly, such an approximation is not Galilei invariant and one must carefully examine the problem of whether it allows such a simplification.

4.2. Rotation

Consider a new reference frame which is obtained from the rest frame by a rotation $\boldsymbol{\theta}$ with axis parallel to $\boldsymbol{\theta}$ and of angle θ. The angle of rotation may depend on time, but for the sake of simplicity, we assume a constant axis of rotation, hence $\boldsymbol{\omega}=d\boldsymbol{\theta}/dt$ is parallel to $\boldsymbol{\theta}$. We mark the physical quantities in the new frame by a tilde; then

$$\begin{aligned} &\tilde{\mathbf{r}}=\mathscr{R}\mathbf{r}; \qquad \tilde{\mathbf{p}}_k=\mathscr{R}[\mathbf{p}-\tilde{\tau}_3 m(\boldsymbol{\omega}\times\mathbf{r})] \\ &\tilde{t}=t; \qquad \tilde{E}=E-(\boldsymbol{\omega}\times\mathbf{r})\mathbf{p} \end{aligned} \tag{48}$$

where $\mathscr{R}$ denotes the matrix of the rotation $[-\boldsymbol{\theta}]$. This transformation is not unitary. Here $\tilde{\mathbf{p}}_k$ is the kinematic momentum, which is connected with the canonical momentum $\tilde{\mathbf{p}}$ by

$$\tilde{\mathbf{p}}_k=\tilde{\mathbf{p}}-\tilde{\tau}_3 m(\boldsymbol{\omega}\times\tilde{\mathbf{r}}) \tag{49}$$

With the appropriate substitution in Eq. (48) the transformation is unitary

and of the type (40), where

$$\check{S} = \exp\left[i(\boldsymbol{\theta}\times\mathbf{r})\mathbf{p}\right] \tag{50}$$

It is evident that these are relations similar to Eqs. (41) and (42).

As previously, let $\check{g}(\mathbf{p}, E; \mathbf{r}, t)$ represent the state of motion in the rest frame. Next, we construct $\check{\tilde{g}}$ as shown in Eq. (43). There is, however, in the present case only one physical interpretation of this state, namely, that in a reference frame rotating with angular velocity $[-\boldsymbol{\omega}]$, the original state of the system is described by $\check{\tilde{g}}$. The invariance of the forces under a rotation and the transition to a noninertial frame now require that $[\check{Q}' * \check{\tilde{g}}]_- = 0$, where

$$\check{Q}' = \tau_3[E - (\boldsymbol{\omega}\times\mathbf{r})\mathbf{p}] - \mathbf{p}^2/2\mathrm{m} + i\Delta' \tag{51}$$

Clearly, the order parameter $\tilde{\Delta}'$ has to be determined self-consistently from $\check{\tilde{g}}^K$, and we find by the same reasoning as before that an equivalent of Eq. (45) holds, and that $\check{S} * \check{Q}' * \check{S}^+ = \check{Q}$. This concludes the proof.

In the case of a general type of pairing force, say $v = v(\mathbf{p}, \mathbf{p})$, rotational invariance requires

$$v(\mathcal{R}\mathbf{p}, \mathcal{R}\mathbf{p}') = v(\mathbf{p}, \mathbf{p}') \tag{52}$$

Thus, in the case of rotations with p-wave paired states, we may use the facterized form $v \sim (\mathbf{p}\cdot\mathbf{p}')/(|\mathbf{p}|\ |\mathbf{p}'|)$ without difficulties.

The current density $\tilde{\mathbf{j}}$ that results from $\check{\tilde{g}}$ is identical to the current density in the state $\check{g}$ when it is observed from a frame rotating with $[-\boldsymbol{\omega}]$. Note that the factor $\mathbf{p}$ in Eq. (30) has to be replaced by the kinematic momentum $\mathbf{p} + \hat{\tau}_3 m(\boldsymbol{\omega}\times\mathbf{r})$ in a rotating frame. Thus, we obtain

$$\tilde{\mathbf{j}}(\mathbf{r}, t) = \mathcal{R}^{-1}\mathbf{j}(\mathcal{R}\mathbf{r}, t) + (\boldsymbol{\omega}\times\mathbf{r})\rho(\mathcal{R}\mathbf{r}, t) \tag{53}$$

as should be the case.

4.3. Gauge Transformation

We consider a system characterized by

$$\check{\tilde{Q}} = \check{\tau}_3 E - e\varphi - \frac{1}{2m}(\mathbf{p} - \check{\tau}_3 e\mathbf{A})^2 + i\check{\tilde{\Delta}} \tag{54}$$

where $(\varphi, \mathbf{A})$ are the electromagnetic potentials. A gauge transformation $\varphi \to \varphi' = \varphi + \dot{\chi}/e$, $\mathbf{A} \to \mathbf{A}' = \mathbf{A} - \nabla\chi/e$, where $\chi = \chi(r, t)$, can be understood as a transformation of the form

$$\tilde{\mathbf{r}} = \mathbf{r}; \qquad \tilde{\mathbf{p}} = \mathbf{p} + \check{\tau}_3 \nabla\chi; \qquad \tilde{t} = t; \qquad \tilde{E} = E - \check{\tau}_3\dot{\chi} \tag{55}$$

This is a unitary transformation of the type (40), where

$$\check{S} = \exp\left[i\check{\tau}_3\chi(\mathbf{r}, t)\right] \tag{56}$$

Clearly, the two equations

$$[\check{Q}, \check{g}]_- = 0; \qquad [\check{\tilde{Q}}, \check{\tilde{g}}]_- = 0 \tag{57}$$

imply each other; thus the equation of motion is gauge covariant. The same is true for the self-consistency condition in the form (47). It is not difficult to show that the densities ρ and $\mathbf{j}$ are gauge invariant if we replace in the integrand of the expression for $\mathbf{j}$ the factor $\mathbf{p}$ by $\mathbf{p} - \check{\tau}_3 e\mathbf{A}$, that is, if we replace the canonical momentum by its kinematic counterpart.

For the sake of completeness, we remark that the theory of the simple quasiclassical Green's function is, in a proper formulation, also gauge covariant.

4.4. Continuity Equation for the Energy

We consider the diagonal part of the equation of motion for $\hat{g}^{\mathrm{K}}$,

$$\begin{aligned}
&\{-i[\hat{Q} \overset{*}{,} \hat{g}^{\mathrm{K}}]_-\}_{\mathrm{d.}} \\
&\quad = \{\hat{\tau}_3 \dot{\hat{g}}^{\mathrm{K}} + \frac{\mathbf{p}}{m} \frac{\partial}{\partial \mathbf{r}} \hat{g}^{\mathrm{K}} \\
&\qquad + \frac{i}{2}\left[\frac{\partial \hat{\Delta}}{\partial \mathbf{r}}, \frac{\partial \hat{g}^{\mathrm{K}}}{\partial \mathbf{p}}\right]_+ - \frac{i}{2}\left[\dot{\hat{\Delta}}, \frac{\partial \hat{g}^{\mathrm{K}}}{\partial E}\right]_+ \\
&\qquad - \frac{1}{8}\left[\frac{\partial^2 \hat{\Delta}}{\partial \mathbf{r}^2}, \frac{\partial^2 \hat{g}^{\mathrm{K}}}{\partial \mathbf{p}^2}\right]_- - \frac{1}{8}\left[\ddot{\hat{\Delta}}, \frac{\partial^2 \hat{g}^{\mathrm{K}}}{\partial E^2}\right]_- + \cdots\}_{\mathrm{d.}} = 0
\end{aligned} \tag{58}$$

and we operate on it by $-1/4 \int d\sigma_p \int dE \,\mathrm{Sp}\, E \cdots$. We recognize that the integration renders to zero all terms except a few. For instance, a nonvanishing contribution results from the last term in the third line of Eq. (58); integrating by parts with respect to E and using the self-consistency equation (32), we obtain a term $\propto \mathrm{Sp}\, \dot{\hat{\Delta}}\hat{\Delta}$. Eventually, we cast the result in the form

$$\dot{e} + \mathrm{div}\, \mathbf{j}_e = 0 \tag{59}$$

where

$$e = -\frac{1}{4} \int d\sigma_p \int dE \,\mathrm{Sp}\, \hat{\tau}_3 E \hat{q}^{\mathrm{K}}(\mathbf{p}, E; \mathbf{r}, t) + \frac{|\Delta|^2}{v} \tag{60}$$

obviously has the meaning of an energy density and where the energy current density is given by

$$\mathbf{j}_e = -\frac{1}{4m} \int d\sigma_p \int dE \,\mathrm{Sp}\, \mathbf{p} E \hat{g}^{\mathrm{K}}(\mathbf{p}, E; \mathbf{r}, t) \tag{61}$$

The above results have been derived for the case of an s-wave superfluid; however, a similar result is obtained in the more general case.

4.5. Continuity Equation for the Momentum

Considering the expression (30) for the momentum density $\mathbf{j}$, we operate by $-1/4\int d\sigma_p \int dE\ \mathrm{Sp}\,\mathbf{p}\cdots$ on the equation of motion (58). Again, most terms are of the form of a total derivative and vanish upon integration. There is an interesting contribution from the first term in the third line of Eq. (59). Integrating by parts with respect to $\mathbf{p}$ and using the self-consistency equation, we arrive at the continuity equation

$$\frac{\partial}{\partial t} j_\alpha + \frac{\partial}{\partial x_\beta} \Pi_{\alpha\beta} = 0 \tag{62}$$

where the momentum current density (stress tensor) $\Pi_{\alpha\beta} = \Pi^{(1)}_{\alpha\beta} + \Pi^{(2)}_{\alpha\beta}$ is given by

$$\begin{aligned} \Pi^{(1)}_{\alpha\beta} &= -\frac{1}{4m} \int d\sigma_p \int dE\ \mathrm{Sp}\, p_\alpha p_\beta \hat{g}^{\mathrm{K}}(\mathbf{p}, E; \mathbf{r}, t) \\ \Pi^{(2)}_{\alpha\beta} &= -\delta_{\alpha\beta} \frac{|\Delta|^2}{\nu} \end{aligned} \tag{63}$$

The case of an anisotropic superfluid requires a little bit more care. The $\mathbf{p}$ dependence of Δ leads to terms containing $\partial\Delta/\partial\mathbf{p}$, etc. and also keeps the expressions from being of the form of a total derivative. Nevertheless, one can show that only the terms that are first order in $\partial/\partial\mathbf{p}$ yield important contributions. Thus, we find

$$\Pi^{(2)}_{\alpha\beta} = \frac{i}{8} \int d\sigma_p \int dE\ \mathrm{Sp}\left[p_\alpha \frac{\partial \hat{\Delta}}{\delta_{p_\beta}} + p_\beta \frac{\partial \hat{\Delta}}{\delta_{p_\alpha}} + \delta_{\alpha\beta} \hat{\Delta} \right] \hat{g}^{\mathrm{K}} \tag{64}$$

where, if necessary, spin averages have to be included.

Consider now in particular the anisotropic ABM (Anderson–Brinkman–Morel) pairing state (p-wave pairing), where the orbital vector $\mathbf{l}$ points along the axis of anisotropy.[6,*] Furthermore, let us assume that the interaction potential in the self-consistency relation (47) can be replaced, with sufficient accuracy, by the factorized form $-(\nu_0/3)\mathbf{p}\cdot\mathbf{p}'/|\mathbf{p}|\,|\mathbf{p}'|$). Then we obtain for a thermal distribution

$$\Pi^{(2)}_{\alpha\beta} = -(2\Delta_0^2/5\nu_0)(2\delta_{\alpha\beta} - l_\alpha l_\beta) \tag{65}$$

where Δ_0 is the value of the gap perpendicular to $\mathbf{l}$. It appears that other

*See also Eq. (69).

plausible approximations to the interaction potential lead to similar results with only a change in the numerical coefficient.

5. ORBITAL MOTION AND QUASIPARTICLE DRAG

In this section, we will study the quasiparticle flow pattern generated by the motion of the orbital vector $\mathbf{l}$ in the ABM superfluid state (realized in the ^{3}He-A phase).

It is known (and it will also be confirmed by the following considerations) that the motion of $\mathbf{l}$ is hindered by viscous forces such that

$$\mu \mathbf{l} \times \dot{\mathbf{l}} = -\mathbf{l} \times \partial E / \partial \mathbf{l} \tag{66}$$

where $\partial E/\partial \mathbf{l}$ represents the torque which drives the orbital vector $\mathbf{l}$. Close to T_c, the coefficient of viscosity is given by[5,8]

$$\mu = \begin{cases} \mu_c = \pi^2 N(0) \Delta_0^3 \tau / 64 T_c; & \Delta_0 \tau \gg 1 \\ \mu_q = \pi N(0) \Delta_0^2 / 12 T_c; & \Delta_0 \tau \ll 1 \end{cases} \tag{67}$$

where τ is the inelastic quasiparticle scattering time. This result can be obtained from a time-dependent Ginzburg–Landau equation, valid for changes in the modulus of the order parameter if one incorporates quasiparticle relaxation by replacing the "simple" time derivative according to the rule*

$$\frac{\pi}{8T} \dot{\Delta} \rightarrow \frac{\pi}{8T} (1 + 2\tau \Delta b) \dot{\Delta} \tag{68}$$

where the two terms on the right-hand side correspond to the two limiting expressions for μ.

What concern us in the following are the details in the quasiparticle dynamics that do not appear explicitly when only forces on the order parameter are considered. Indeed, the result for μ has been obtained within the simple quasiclassical approximation, which, as we will see, fails to give the right answers to the quasiparticle problem.

We briefly review the basic properties of the ABM state in thermal equilibrium.[5,6] In this state, only fermions with parallel spin are paired; this property allows us to eliminate the spin indices and we are left only with the particle–hole index similar to an s-wave superfluid. Thus, $\hat{\Delta}$ is of the form (4a), where $\Delta = \Delta(\mathbf{p}) = |\Delta(\mathbf{p})| \exp[-i\theta(\mathbf{p})]$ now denotes the momentum-dependent order parameter. In the ABM state, we have

$$\Delta(\mathbf{p}) = \Delta_0 [\boldsymbol{\alpha}_1 + i\boldsymbol{\alpha}_2] \mathbf{p} / |\mathbf{p}|, \qquad |\Delta(\mathbf{p})|^2 = \Delta_0^2 [1 - (\mathbf{p}\mathbf{l})^2 / \mathbf{p}^2] \tag{69}$$

where $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, and $\mathbf{l} = \boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2$ are a triad of orthonormal vectors. The

*See, for instance, Section III.I of Ref. 4. The numerical coefficient b depends on some angular averages.

equilibrium form of the Green's function is given by*

$$\hat{g}_{\mathrm{eq}}^{\mathrm{R(A)}}(\mathbf{p}, E)=\frac{(E_{(}^{\pm})i/2\tau)\hat{\tau}_3+i\hat{\Delta}(\mathbf{p})}{[(E_{(}^{\pm})i/2\tau)^2-|\Delta(\mathbf{p})|^2]^{1/2}} \tag{70}$$

where the square root is defined by $\mathrm{Im}[\cdots]^{1/2}>0$. Various expressions can be derived from $\hat{g}^{\mathrm{R(A)}}$; the most important are

$$\begin{aligned} N_1(\mathbf{p}, E)&=\mathrm{Re}\frac{E+i/2\tau}{[(E+i/2\tau)^2-|\Delta(\mathbf{p})|^2]^{1/2}} \\ N_2(\mathbf{p}, E)&=\mathrm{Re}\frac{i|\Delta(\mathbf{p})|}{[(E+i/2\tau)^2-|\Delta(\mathbf{p})|^2]^{1/2}} \\ R_2(\mathbf{p}, E)&=\mathrm{Im}\frac{i|\Delta(\mathbf{p})|}{[(E+i/2\tau)^2-|\Delta(\mathbf{p})|^2]^{1/2}} \end{aligned} \tag{71}$$

All these quantities are even functions in $\mathbf{p}$; in their dependence on E, however, R_2 is an odd function and N_1 and N_2 are even functions. In the limit $1/\tau\to 0$, the quantities N_1 and R_2 vanish in the gap region $|E|<|\Delta(\mathbf{p})|$, whereas $N_2=0$ in the accessible range $|E|>|\Delta(\mathbf{p})|$.

In the case of slow orbital motion, the structure (69) of the order parameter remains unaffected; only the triad of vectors become space–time dependent. If $\boldsymbol{\omega}=\boldsymbol{\omega}(\mathbf{r}, t)$ is the angular velocity of the local rotation, then

$$\dot{\mathbf{l}}=\boldsymbol{\omega}\times\mathbf{l} \tag{72}$$

etc., and we may calculate $\dot{\Delta}(\mathbf{p})$ accordingly. We will assume that $\boldsymbol{\omega}$ is given; then the time-dependent order parameter appears as a drive in the Boltzmann equation (35).

This equation is considerably simplified if we restrict ourselves to the linear regime, that is, to the response linear in $\boldsymbol{\omega}$. It is easy to see that in this case, only the equilibrium form (70) for $\hat{g}^{\mathrm{R(A)}}$ appears in the kinetic equation. In this approximation, the collision integrals assume the form of linear operators which operate on the deviations $\delta f^{(\mathrm{L})}=f^{(\mathrm{L})}-f_{\mathrm{th}}$ and $f^{(\mathrm{T})}$ of the distribution functions from equilibrium. It is important to note that conservation of energy, momentum, and particle number implies that there are eigenfunctions of zero eigenvalue as follows:

$$\begin{aligned} &\text{(i)} && \delta f^{(\mathrm{L})}=E\cdot\partial f_{\mathrm{th}}/\partial E \\ &\text{(ii)} && \delta f^{(\mathrm{L})}=\mathbf{p}\cdot\partial f_{\mathrm{th}}/\partial E \\ &\text{(iii)} && f^{(\mathrm{T})}=\partial f_{\mathrm{th}}/\partial E \end{aligned} \tag{73}$$

*Inelastic collisions are responsible for the damping terms $(\pm)i/2\tau$. In these expressions it has been taken into account that close to T_c the collisions affect mostly the diagonal parts of the self-energy, whereas their off-diagonal contributions are negligible.

It is possible to approximate the collision operators K by a reduced form K_r which incorporates the conservation laws on one side but which reduces to the relaxation approximation otherwise.[9] This reduced form is of sufficient accuracy for the present problem. Specifically, only momentum conservation plays a central role, and therefore we may put*

$$K_{rm}^{(\mathrm{L})} = \frac{N_1}{\tau}\left[\delta f^{(\mathrm{L})} - (T^{-1})_{\alpha\beta}\frac{p_\alpha (j_{\mathrm{qp}})_\beta}{4T\,\mathrm{ch}^2(E/2T)}\right]$$
$$K_{rm}^{(\mathrm{T})} = \frac{N_1}{\tau} f^{(\mathrm{T})} \tag{74a}$$

where

$$T_{\alpha\beta} = 2N(0)\int dE\left\langle N_1 \frac{p_\alpha p_\beta}{4T\,\mathrm{ch}^2(E/2T)}\right\rangle$$
$$\mathbf{j}_{\mathrm{qp}} = 2N(0)\int dE\,\langle N_1 \mathbf{p}\delta f^{(\mathrm{L})}\rangle \tag{74b}$$

Clearly, the form (ii) of Eq. (73) is an eigenfunction of $K_{rm}^{(\mathrm{L})}$ with eigenvalue zero. We will call j_{qp} the quasiparticle contribution to the current density.

The separation of the physical quantities into a pair and a quasiparticle contribution is formally equivalent to an appropriate decomposition of $\hat{g}^{\mathrm{K}} = \hat{g}^{\mathrm{K}} * \hat{h} - \hat{h} * \hat{g}^{\mathrm{A}}$. In general, there may be no meaningful answer to this problem; in a linear approximation, the definition

$$(\delta\hat{g}^{\mathrm{K}})_p = \delta\hat{g}^{\mathrm{R}} * \hat{h}_{\mathrm{th}} - \hat{h}_{\mathrm{th}} * \delta\hat{g}^{\mathrm{A}}$$
$$(\delta\hat{g}^{\mathrm{K}})_{\mathrm{qp}} = \hat{g}_{\mathrm{eq}}^{\mathrm{R}} * \delta\hat{h} - \delta\hat{h} * \hat{g}_{\mathrm{eq}}^{\mathrm{A}} \tag{75}$$

seems to meet the intuitive expectations. For instance, the quasiparticle contribution depends in an essential way on the non-equilibrium part of the distribution function.

Concerning the star product, we should note that in a linearized form, there is only one factor in such a product which depends on space and time. Therefore, it is possible to express the star product in a closed form if one chooses a Fourier representation.† However, it is more instructive to proceed on the basis of expansion (23) in powers of $\hbar$.

In terms of this expansion, the leading contribution to the quasiparticle current is the expression $\mathbf{j}_{\mathrm{qp}}$ of Eq. (74b). There is a similar expression for the quasiparticle mass density ρ_{qp}. These approximations are sufficient, since we will calculate the distribution function only to leading order.

*The additional subscript m means that this reduced form includes only momentum conservation.

†For instance, $\hat{g}_{\mathrm{eq}}^{\mathrm{R}} * \delta\hat{h} \to \hat{g}_{\mathrm{eq}}^{\mathrm{R}}(\mathbf{p}+\mathbf{k}/2;\ E+\Omega/2)\ \delta\hat{h}(\mathbf{p}, E)$, if $\delta\hat{h}$ oscillates in space-time $\propto\exp(i\mathbf{kr} - i\Omega t)$].

5.1. Classical Quasiparticle Drag

In the limit $\Delta_0\tau \gg 1$, it is only necessary to expand the star products in the Boltzmann equation through first order in $\hbar$. Thus, we obtain*

$$\begin{aligned} &N_1\delta\dot{f}^{(\mathrm{L})} + N_1\frac{\mathbf{p}}{m}\frac{\partial f^{(\mathrm{T})}}{\partial \mathbf{r}} \\ &\quad + R_2\left[\frac{\partial|\Delta|}{\partial t}\frac{\partial f_{\mathrm{th}}}{\partial E} + \frac{\partial|\Delta|}{\partial \mathbf{p}}\frac{\partial \delta f^{(\mathrm{L})}}{\partial \mathbf{r}}\right] - K^{(\mathrm{L})} = 0 \\ &N_1\dot{f}^{(\mathrm{T})} + N_1\frac{\mathbf{p}}{m}\frac{\partial \delta f^{(\mathrm{L})}}{\partial \mathbf{r}} \\ &\quad + N_2|\Delta|\left[\frac{\partial\Theta}{\partial t}\frac{\partial f_{\mathrm{th}}}{\partial E} + 2f^{(\mathrm{T})}\right] - K^{(\mathrm{T})} = 0 \end{aligned} \tag{76}$$

We wish to emphasize that the second term in the square brackets of the (L)-mode equation is decisive in what follows; it owes its existence to a generalization of the quasiclassical theory. In the (T)-mode equation, some terms of $O(\hbar^1)$ have been omitted, as they are unimportant.

In this (and in the next) subsection we consider the case where $\boldsymbol{\omega}$ is perpendicular to $\mathbf{l}$. Then the time derivative of the phase may be neglected. Furthemore, in the accessible range we approximate N_2 by $N_1^3(|\Delta|/2\tau E^2)$; replacing $K^{(\mathrm{T})}$ by $K_{rm}^{(\mathrm{T})}$, we solve the (T)-mode Boltzmann equation with the result

$$f^{(\mathrm{T})} = -\tau N_1^{-2}\frac{\mathbf{p}}{m}\frac{\partial \delta f^{(\mathrm{L})}}{\partial \mathbf{r}} \tag{77a}$$

which holds if the time variation is sufficiently slow.

We choose a coordinate system $\boldsymbol{\alpha}_1 = (1, 0, 0)$, $\boldsymbol{\alpha}_2 = (0, 1, 0)$, $\mathbf{l} = (0, 0, 1)$, where $T_{\alpha\beta} = T_\alpha\delta_{\alpha\beta}$; and in approximation (74) the (L)-mode equation assumes the form

$$\begin{aligned} &\left[\frac{\partial}{\partial t} - \tau\left(\frac{\mathbf{p}}{mN_1}\frac{\partial}{\partial \mathbf{r}}\right)^2 + \frac{1}{2E}\frac{\partial|\Delta|^2}{\partial \mathbf{p}}\frac{\partial}{\partial \mathbf{r}} + \frac{1}{\tau}\right]\delta f^{(\mathrm{L})} \\ &\quad = \left[\frac{1}{2E}\frac{\partial|\Delta|^2}{\partial t} + \frac{1}{\tau}T_\alpha^{-1}p_\alpha(j_{\mathrm{qp}})_\alpha\right]\frac{1}{4T\,\mathrm{ch}^2(E/2T)} \end{aligned} \tag{77b}$$

*So far, we have not considered Fermi liquid effects, which are important for ^{3}He. They can be taken into account systematically[5] and do not affect the general discussion, except for trivial modifications: for example, the rhs of (80b) has to be multiplied by m/m^*, where m^* is the effective mass; μ has to be calculated with the true density of states, and λ is the mean free path $\tau p/m^*$.

The following steps are straightforward. We assume a space–time dependence $\propto\exp(i\mathbf{kr}-i\Omega t)$, solve Eq. (77) for $\delta f^{(L)}$, and then establish an equation for $\mathbf{j}_{qp}$. For the sake of convenience, we restrict our attention to the hydrodynamic regime where $\tau\Omega \ll 1$ and $k\lambda \ll 1$, where $\lambda = \tau p/m$ is the mean free path. Furthermore, we have

$$\int \mathrm{dE}\frac{N_1}{\mathrm{ch}^2(E/2T)} \simeq \int dE\frac{N_1^{-1}}{\mathrm{ch}^2(E/2T)} \simeq 4T$$

$$\int dE\, N_1(E\,\mathrm{ch}\frac{E}{2T})^{-2} \simeq \frac{\pi}{|\Delta|}$$

provided that $|\Delta| \ll T$. For simplicity, we assume that $\boldsymbol{\omega} = (0, \omega, 0)$; and we obtain*

$$(j_{qp})_\alpha = \frac{1}{-i\tau\Omega + \frac{1}{5}\lambda^2(\mathbf{k}^2 + 2k_\alpha^2)} \frac{-i\tau^2\pi^2\Delta_0^3 N(0)}{128T}(k_z, 0, -k_x)_\alpha \omega \qquad (78)$$

Let us first examine the behavior of the total angular momentum of the quasiparticles,

$$\mathbf{M}_{qp} = \int d^3r\, \mathbf{r} \times \mathbf{j}_{qp} \qquad (79a)$$

which in terms of Fourier transforms is $\mathbf{M}_{qp} = i[(\partial/\partial\mathbf{k}) \times \mathbf{j}_{qp}]_{\mathbf{k}\to 0}$. Hence, it follows from Eq. (78) that

$$\frac{\partial}{\partial t}\mathbf{M}_{qp} = \mu_c \int d^3r\boldsymbol{\omega} \qquad (79b)$$

where μ_c is the classical viscosity coefficient of Eq. (67). Comparing Eq. (79) with Eq. (66), we conclude that the angular momentum initially imparted to the pairs will be transferred to the quasiparticle system.†

For illustration, let us consider an angular velocity with the following profile:

$$\omega(z) = \begin{cases} \omega_0 & \text{if} \quad |z| < L \\ 0 & \text{else} \end{cases} \qquad (80a)$$

In the quasistationary limit $|z| \ll p/m\Omega$, we obtain a quasiparticle flow parallel to the x axis of the following form:

$$j_{qp} = \frac{5\tau\mu_c}{2\lambda^2}\begin{cases} z & \text{if} \quad |z| < L \\ L \cdot \operatorname{sgn} z & \text{else.} \end{cases} \qquad (80b)$$

*Since $\operatorname{div} \mathbf{j}_{qp} \neq 0$, there are fluctuations in the phase of the order parameter such that particle number is conserved. These fluctuations contribute little to $\mathbf{j}_{qp}$ and thus we may neglect them.

†Obviously, Eq. (66) neglects the angular momentum of the pairs.

This result demonstrates that quasiparticle diffusion carries the transferred momentum quickly away to distant regions. Such an interpretation is also confirmed by the fact that the rotation of the quasiparticle flow velocity, namely $\mathbf{j}_{qp}/\rho$, is smaller by the extremely small factor Δ_0^3/TE_F^2 than the local angular velocity.

5.2. Quantum Drag

As a result of the orbital motion, the value of the gap in a given direction changes with time. This fact necessitates a redistribution in energy of the quasiparticles, which is ultimately responsible for the drag in the classical region. However, if collision broadening $\sim 1/\tau$ becomes larger than Δ_0, an energy gap ceases to exist, and the classical drag is zero* in the limit $\Delta_0 \tau \ll 1$.

On the other hand, since $\Delta_0 \neq 0$, there are still pair correlations which have to be adjusted if the wave function of the pairs changes in time. The quasiparticle drag due to this effect can be recovered from the Boltzmann equation if the star product is expanded through second order in $\hbar$. An inspection shows that most of the terms of this order lead only to a correction of the previous result; thus it is sufficient to replace the (L)-mode equation (76) by the following expression:

$$N_1 \delta \dot{f}^{(L)} + N_1 \frac{\mathbf{p}}{m} \frac{\partial \delta f^{(L)}}{\partial \mathbf{r}} - K^{(L)} - \frac{1}{2} \operatorname{Re} \frac{\partial}{\partial \mathbf{r}} \left(\frac{\partial}{\partial \mathbf{p}} N_2 e^{i\Theta} \right) \left[\dot{\Delta} \frac{\partial f_{th}}{\partial E} - 2i \Delta f^{(T)} \right] = 0 \tag{81}$$

We are still considering the situation where $\boldsymbol{\omega} = (0, \omega, 0)$ is perpendicular to $\mathbf{l}$. In such a case, the second term $f^{(T)}$ in the square brackets of Eq. (81) can be neglected and the solution (77a) for $f^{(T)}$ is still valid. Therefore, we may solve the Boltzmann equation in the same way as before with the result

$$(\mathbf{j}_{qp})_\alpha = \frac{1}{-i\tau\Omega + \frac{1}{5}\lambda^2(\mathbf{k}^2 + 2k_\alpha^2)} \frac{-i\tau\pi\Delta_0^2 N(0)}{60T} (k_z, 0, -4k_x)_\alpha \omega \tag{82}$$

The rate of change of the quasiparticle angular momentum assumes the same form (79b) as previously; however, the quantum viscosity coefficient μ_q now appears in this relation.

Comparing the flow pattern, we conclude that in the classical case, the directions $\boldsymbol{\alpha}_1$ and $\mathbf{l}$ perpendicular to $\boldsymbol{\omega}$ are essentially equivalent, whereas in the quantum case, the flow parallel to $\mathbf{l}$ is much stronger than

*Such a result would have been obtained in the previous calculation if spectral quantities (71) of finite τ had been used.

the flow parallel to $\boldsymbol{\alpha}_1$. This peculiarity can perhaps be explained by the fact that the pair correlations are largest in the direction perpendicular to $\mathbf{l}$, and that small oscillations in $\Delta(\mathbf{p})$ rock these correlations up and down, with a subsequent increase in momentum transfer in the $\mathbf{l}$ direction.

5.3. Gauge Wheel

We consider here the peculiar case where $\boldsymbol{\omega} = (0, 0, \omega)$; i.e., where the axis of rotation is parallel to the orbital vector. Clearly, $\mathbf{l} = 0$ and no orbital motion takes place; however, since there is an interesting transfer of angular momentum between pairs and quasiparticles, we will study the ensuing phenomena in this section.

With $\boldsymbol{\omega}$ parallel to $\mathbf{l}$, the two basis vectors $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ rotate in their common plane and the order parameter changes only by a phase factor according to $\dot{\Delta}(\mathbf{p}, t) = -i\dot{\Theta}\Delta(\mathbf{p}, t)$, where $\dot{\Theta} = \omega$. These features have led to the name[10] which heads this subsection.

We will not determine here the details of the quasiparticle flow, since this would require a more careful solution of the (T)-mode Boltzmann equation. It suffices to say that it is of the same form as in the classical case except that now the flow takes place in the x–y plane. Thus, we consider only the angular momentum balance. We operate on Eq. (81) by

$$2N(0)\int d^3r\,\mathbf{r}\int dE\,\langle \mathbf{p}\cdots\rangle$$

and obtain in a first step

$$\frac{\partial}{\partial t}\mathbf{M}_{\mathrm{qp}} = -\mathbf{l}N(0)\int d^3r\int dE\left\langle |\Delta|N_2\left[\dot{\Theta}\frac{\partial f_{\mathrm{th}}}{\partial E} + 2f^{(\mathrm{T})}\right]\right\rangle \tag{83}$$

where we have used momentum conservation of $K^{(\mathrm{L})}$ and a symmetry of $f^{(\mathrm{T})}$, which implies that

$$\int d^3r\int dE\,\langle (p_x p_z/p^2)|\Delta|N_2 f^{(\mathrm{T})}\rangle = 0$$

We consider now the (T)-mode equation of (76) and operate by $2N(0)\int d^3r\int dE\,\langle\cdots\rangle$. In the present case, it is absolutely necessary to observe particle conservation, which means that $\int dE\,\langle K^{(\mathrm{T})}\rangle = 0$. The result allows us to cast Eq. (83) in the form

$$\frac{\partial}{\partial t}\mathbf{M}_{\mathrm{qp}} = \frac{1}{2}\mathbf{l}\frac{\partial}{\partial t}N_{\mathrm{qp}} \tag{84a}$$

where $N_{\mathrm{qp}} = 2N(0)\int d^3r\int dE\,\langle N_1 f^{(\mathrm{T})}\rangle$ is the quasiparticle number. Since the total angular momentum and the total particle number are conserved, we may equally write

$$\frac{\partial}{\partial t}\mathbf{M}_p = \frac{1}{2}\mathbf{l}\frac{\partial}{\partial t}N_p \tag{84b}$$

where $\mathbf{M}_p$ and N_p are the angular momentum and number of particles bound, respectively, in the Cooper pairs. In other words, this result means that in the ABM state, the formation of a Cooper pair requires an angular momentum equal to $\hbar\mathbf{l}$ (in ordinary units) which is supplied by the quasiparticle system. Obviously, this statement has a local character and does not allow conclusions on the total angular momentum of the ABM state.

In concluding this section, we remark that most calculations on the transport properties of superfluid ^{3}He are based on kinetic equations which can be derived[11,12] if one chooses to integrate the Green's function $G(\mathbf{p}, E)$ with respect to the energy E. In this technique, it is not possible to deal adequately with the case of strong collision broadening. On the other hand, this theory requires no a priori approximations in the momentum variable. Therefore, if there have been questions on the correctness of some results, one has to put the blame on inadequate approximations in the course of the calculations. It seems to us that Nagai in his recent calculations[13] has overcome such difficulties. This remark applies in particular to his results concerning the gauge wheel effects.

6. BOLTZMANN EQUATION AND LORENTZ FORCE

We consider here systems where the external dynamics is governed by electromagnetic forces. Thus, there are electromagnetic potentials $(\varphi, \mathbf{A})$ which generate the field strength

$$\mathscr{E} = -\nabla\varphi - \dot{\mathbf{A}}; \qquad \mathscr{H} = \operatorname{curl} \mathbf{A} \tag{85}$$

We recall that gauge transformations on such a system have been studied in Section 4.

Let us first discuss the case of a normal metal, where $\Delta = 0$ and where $\hat{g}^{R(A)} = (\pm)\hat{\tau}_3$. This case is very similar to the one discussed at the end of Section 3 and we will now obtain a generalization of the classical Boltzmann equation (38). As before, it is possible to consider only one diagonal component of the particle–hole matrices; therefore, we may write

$$-i[E - e\varphi - (1/2m)(\mathbf{p} - e\mathbf{A})^2 \overset{*}{,} f]_- - K = 0 \tag{86}$$

In the classical limit, it is sufficient to expand the star product only through first order in $\hbar$. Furthermore, we introduce the kinetic momentum

$$\mathbf{p}_k = \mathbf{p} - e\mathbf{A} \tag{87a}$$

and the distribution function

$$\tilde{f}(\mathbf{p}_k; \mathbf{r}, t) = f(\mathbf{p} = \mathbf{p}_k - e\mathbf{A}, E = p_k^2/2m + e\varphi; \mathbf{r}, t) \tag{87b}$$

Then we obtain from (86)

$$\dot{\tilde{f}}+\frac{\mathbf{p}_k}{m}\frac{\partial\tilde{f}}{\partial\mathbf{r}}+e\left(\mathscr{E}+\frac{\mathbf{p}_k}{m}\times\mathscr{H}\right)\frac{\partial\tilde{f}}{\partial\mathbf{p}_k}-K=0 \tag{87c}$$

which is obviously of the standard form.

We turn our attention now to the case of a superconductor with s-wave pairing. Vortex motion in a type II superconductor is a prominent example of a situation where electromagnetic fields are present. In order to simplify matters, we will restrict our attention to the regime of linear response; this means in the example mentioned above that we wish to calculate the nonequilibrium quantities only in first order of the vortex velocity. Generally, the equilibrium state will be spatially inhomogeneous, and therefore we have to evaluate the star product by expansion. Such an expansion is better, the slower is the change in space–time of the quantities. Fortunately, we may choose a gauge such that the order Δ is of least variation, which is the case in a gauge where Δ is real ($\Theta=0$). In other words, we select new potentials such that

$$\begin{aligned} -e\varphi &\to -e\varphi'=\Phi=\tfrac{1}{2}\dot{\Theta}-e\varphi \\ -\frac{e}{m}\mathbf{A} &\to -\frac{e}{m}\mathbf{A}'=\mathbf{v}_s=-\frac{1}{2m}(\nabla\Theta+2e\mathbf{A}) \end{aligned} \tag{88}$$

We will call Φ the pair chemical potential and $\mathbf{v}_s$ the superfluid velocity.

Concerning the properties of the equilibrium quantities, it seems that an ansatz of the form

$$\hat{g}^{\mathrm{R}}=\hat{\tau}_3\alpha^{\mathrm{R}}+\hat{\tau}_1\beta^{\mathrm{R}}+\hat{\tau}_2\gamma^{\mathrm{R}} \tag{89a}$$

is sufficient and that the functions α, β, and γ can be determined from the equations

$$\begin{aligned} E(\beta\,(\pm)\,i\gamma)^{\mathrm{R}}-i\Delta\alpha^{\mathrm{R}}+\frac{\mathbf{p}}{2m}\left[(\pm)i\frac{\partial}{\partial\mathbf{r}}-2m\mathbf{v}_s\right](\beta\,(\pm)\,i\gamma)^{\mathrm{R}}=0 \\ (\alpha^{\mathrm{R}})^2+(\beta^{\mathrm{R}})^2+(\gamma^{\mathrm{R}})^2=1 \end{aligned} \tag{89b}$$

Note that these equations are exact in the ordinary quasiclassical theory. In the present case we should in addition require that* $\tilde{\Phi}=\Phi-m\mathbf{v}_s^2/2$ is constant in space. There are similar equations for the retarded quantities; and the generalized densities of state N_i and R_i are defined as follows

$$\alpha^{\mathrm{R(A)}}=(\pm)N_1+iR_1,\qquad \beta^{\mathrm{R(A)}}=N_2\,(\pm)\,iR_2,\qquad \gamma^{\mathrm{R(A)}}=N_3\,(\pm)\,iR_3 \tag{89c}$$

*In a more refined terminology, one should call Φ the mechanochemical potential, whereas $\tilde{\Phi}$ stands directly for the chemical potential.

When deriving the linearized Boltzmann equation, it is of some advantage to rewrite $\hat{Q}$ in the form

$$\hat{Q} = \hat{\tau}_3(E - \mathbf{p}\mathbf{v}_s) - \mathbf{p}^2/2m + \tilde{\Phi} + i\hat{\Delta} \tag{90}$$

and exploit the fact that $\tilde{\Phi}$ is constant in equilibrium. Then we obtain

$$\begin{aligned} N_1\Big\{ \delta\dot{f}^{(\mathrm{L})} + \mathbf{v}_s \frac{\partial \delta f^{(\mathrm{L})}}{\partial \mathbf{r}} - \frac{\partial \mathbf{p}\mathbf{v}_s}{\partial \mathbf{r}} \frac{\partial \delta f^{(\mathrm{L})}}{\partial \mathbf{p}} + \mathbf{p}\mathbf{v}_s \frac{\partial f_{\mathrm{th}}}{\partial E} \\ + \frac{\mathbf{p}}{m} \frac{\partial f^{(\mathrm{T})}}{\partial \mathbf{r}} \Big\} + R_2 \dot{\Delta} \frac{\partial f_{\mathrm{th}}}{\partial E} + 2\Delta R_3 f^{(\mathrm{T})} - K^{(\mathrm{L})} = 0 \\ N_1\Big\{ \dot{f}^{(\mathrm{T})} + \mathbf{v}_s \frac{\partial f^{(\mathrm{T})}}{\partial \mathbf{r}} - \frac{\partial \mathbf{p}\mathbf{v}_s}{\partial \mathbf{r}} \frac{\partial f^{(\mathrm{T})}}{\partial \mathbf{p}} + \frac{\mathbf{p}}{m} \frac{\partial \delta f^{(\mathrm{L})}}{\partial \mathbf{r}} \\ - \dot{\tilde{\Phi}} \frac{\partial f_{\mathrm{th}}}{\partial E} \Big\} + 2\Delta N_2 f^{(\mathrm{T})} - N_3 \dot{\Delta} \frac{\partial f_{\mathrm{th}}}{\partial E} - K^{(\mathrm{T})} = 0 \end{aligned} \tag{91}$$

These equations differ from the form which one obtains in the ordinary quasiclassical theory by the additional terms in the form of a classical Poisson bracket between $\mathbf{p}\mathbf{v}_s$ and $\delta f^{(\mathrm{L})}$, $f^{(\mathrm{T})}$ and by the appearance of a modified pair chemical potential $\tilde{\Phi}$.*

APPENDIX. QUASICLASSICAL GREEN'S FUNCTION IN SIMPLE CASES

Let us consider first the retarded Green's function for a noninteracting system. There it is necessary to study only the component

$$G^{\mathrm{R}}(x_1, x_2) = -i\langle [\psi_\uparrow(x_1), \psi_\uparrow^+(x_2)]_+ \rangle \theta(t_1 - t_2)$$

It is known that for free fields the expectation value of the anticommutator does not depend on the state of the system. This means that the chemical potential μ enters in G^{R} only through the time dependence of the field operators; for instance, we have $\psi(x_1) = e^{i\mu t_1}\psi(x_1; \mu = 0)$. Now,†

$$\int d\mu\, e^{i\mu t} = \pi[\delta(t+0) + \delta(t-0)] \tag{A1}$$

and therefore we obtain

$$g^{\mathrm{R}}(x_1, x_2) = \frac{i}{\pi} \int d\mu\, G^{\mathrm{R}}(x_1, x_2) = \delta(x_1 - x_2) \tag{A2}$$

A similar result can be derived for g^{A}, which is just $-g^{\mathrm{R}}$.

*See preceding footnote.

†In terms of Fourier transforms, Eq. (A1) is equivalent to the following statement: If $F(\omega) = \int dt\, e^{i\omega t} f(t)$, then $\int (d\omega/2\pi) F(\omega)\, e^{-i\omega t} = \frac{1}{2}[f(t+0) + f(t-0)]$.

In the case of an interacting system, one expects similar relations if the self-energy depends only weakly on the chemical potential.

Next, we study the properties of Keldysh's function for a noninteracting system. For the sake of simplicity, we assume that the simple particle Hamiltonian H_0 and the state of the system are independent of time. Accordingly, there is a complete set of eigenstates,

$$H_0|n\rangle = \varepsilon_n|n\rangle \tag{A3a}$$

and it will be convenient to introduce the field operators

$$\psi_n = \int d^3r' \langle n|\mathbf{r}'\rangle \psi(\mathbf{r}') \tag{A3b}$$

Using this notation, we obtain the following representation:

$$G^{\mathrm{K}}(x_1, x_2) = -i \sum_n \langle \mathbf{r}_1|n\rangle \times \{e^{-i(\varepsilon_n-\mu)(t_1-t_2)}\langle \psi_n\psi_n^+ - \psi_n^+\psi_n\rangle\}\langle n|\mathbf{r}_2\rangle$$

Clearly, $\langle \psi_n\psi_n^+ - \psi_n^+\psi_n\rangle = 1-2f_n$, where $f_n = \langle \psi_n^+\psi_n\rangle$ is the occupation number. In the case where $f_n = f(\varepsilon_n - \mu)$, we find the expected relation

$$G^{\mathrm{K}}(\mathbf{p}, E; \mathbf{r}, t) = 2[1-2f(E)] \tag{A5}$$

In thermal equilibrium, the dissipation-fluctuation theorem connects G^{K} and $G^{\mathrm{R}} - G^{\mathrm{A}}$ in a simple way for any interacting system. There, we have

$$G^{\mathrm{K}} = [G^{\mathrm{R}} - G^{\mathrm{A}}][1-2f_{\mathrm{th}}(E)]$$

where the argument of the Green's function is $(\mathbf{p}, E; \mathbf{r})$. Obviously, the same relation is valid for the quasiclassical Green's function.

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