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Quasiclassical Equations for ^{3}He–A*

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The quasiclassical equations for superfluid ^{3}He are studied in detail, with emphasis on the equations for the A phase. A careful discussion of the collision operators is given; the $\mathbf{l}\times\nabla T$ *term in the energy current is calculated; and the gradient expansion is discussed from the quasiclassical point of view.*

1. INTRODUCTION

Since the development of the quasiclassical approach by Eilenberger,[1] Larkin and Ovchinnikov,[2] and Eliashberg,[3] this technique has found many applications in the theory of nonequilibrium states in superconductors (see, e.g., Ref. 4). The reason behind this is that most superconductors are relatively dirty, i.e., the impurity mean free path is smaller than the coherence length, or, equivalently, the scattering rate is larger than the typical energy k_BT_c (T_c is the transition temperature). In such a situation, or whenever the variation in space and time happens on too small a scale, usual quasiparticle methods[5,6] are not adequate.

In contrast to the quasiparticle kinetic equations,[5,6] the quasiclassical technique only exploits the fact that the Fermi momentum p_{F} is large compared to the momenta of, say, external perturbations. In this case, the self-energies depend only weakly on the magnitude of the momentum, and the equations can be simplified considerably by introducing the quasiclassical or ξ_p integrated Green's function [$\xi_p = p^2/2m^* - \mu \simeq v_{\mathrm{F}}(|\mathbf{p}| - p_{\mathrm{F}})$; m^* is the effective mass, $\mu = p_{\mathrm{F}}^2/2m^*$, and $v_{\mathrm{F}} = p_{\mathrm{F}}/m^*$]. This leads to a description which is only limited by $\hbar\omega \ll \mu$ and $\hbar v_{\mathrm{F}}|\mathbf{q}| \ll \mu$, where ω and $\mathbf{q}$ are typical frequencies and wave vectors. However, cautionary remarks should be made. (i) The quasiclassical technique uses the fact that the normal density of states is independent of the energy, $N(\xi_p) \simeq N(0)$, and

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integrates the Green's function with respect to ξ_p *before* doing the appropriate frequency summation (to find the density, energy, etc.). This is the wrong order, and one has to ensure sufficient convergence by considering only the *difference* with respect to a suitable reference state* (most conveniently, the normal state in the same external field). (ii) In doing the ξ_p integration, approximations are made with regard to the momentum variable, and the resulting equations, in the original formulation, are not Galilei-invariant. These difficulties were overcome recently by defining a generalized quasiclassical approximation[7] (see below).

So far, quasiclassical methods have been used for superfluid ^{3}He only by a few authors (e.g., Refs. 7–13). One must emphasize that, although there seems to be an initial barrier against becoming acquainted with this technique, the equations are sufficiently simple to allow, e.g., the determination of strong coupling corrections,[8] or the behavior of the superfluid near walls.[13]

In this paper we demonstrate more details of the quasiclassical kinetic equations for the A phase of ^{3}He and for spin-independent situations only. Also, we work in the weak coupling approximation (however, collision processes are included!). Section 2 contains a brief summary of the generalized quasiclassical equations. In Section 3 we give a careful discussion of the collision operators in the kinetic equations for the symmetric and antisymmetric distribution functions, and in Section 4 we consider in detail the new terms due to the generalization of the technique. In particular, we discuss temperature gradients. The Appendix demonstrates the gradient expansion for the regular Green's functions.

2. THEORETICAL BACKGROUND†

The starting point in the derivation of the quasiclassical equations are Gorkov's equations as generalized according to the theory of Keldysh.[14] It is convenient to use a matrix Green's function $\bar{\hat{G}}$, defined by [15]

$$\bar{\hat{G}} = \begin{Bmatrix} \hat{G}^{\mathrm{R}} & \hat{G}^{\mathrm{K}} \\ 0 & \hat{G}^{\mathrm{A}} \end{Bmatrix} \tag{1}$$

where $\hat{G}^{\mathrm{R,A}}$ are the retarded and advanced functions, and $\hat{G}^{\mathrm{K}}$ is the expectation value of the anticommutator of the fermion field operators. The caret and overbar indicate matrices in particle–hole (p–h) and

*In this form, the argument is appropriate for a weakly interacting Fermi gas. For a strongly interacting system like ^{3}He, one argues that only the *quasiparticle part* of the Green's function can be described by the quasiclassical function, and one has to correct for the background contribution (from energies far from the Fermi surface).

†More details can be found in Refs. 7 and 12.

"Keldysh" space, respectively. We choose a mixed representation in which the Green's functions depend on momentum, energy, space, and time: $\mathbf{p}$, E, $\mathbf{r}$, t; here $\mathbf{p}$, E are the Fourier conjugate variables to the relative coordinates, and $\mathbf{r}$, t the center-of-mass coordinates. A functional "star" product between two functions A and B is defined by the expansion

$$A * B = AB + \frac{i\hbar}{2}\left(\frac{\partial A}{\partial E}\frac{\partial B}{\partial t} - \frac{\partial A}{\partial t}\frac{\partial B}{\partial E}\right) - \frac{i\hbar}{2}\left(\frac{\partial A}{\partial \mathbf{p}}\frac{\partial B}{\partial \mathbf{r}} - \frac{\partial A}{\partial \mathbf{r}}\frac{\partial B}{\partial \mathbf{p}}\right) + \cdots \tag{2}$$

With this definition, Gorkov's equations can be written in the form

$$[\check{Q} - \check{\Sigma}_c] * \check{G} = \check{1} \tag{3}$$

In this equation, we included in $\check{Q} = \hat{Q} \cdot \bar{1}$ the mean field parts of the self-energy and the external fields, and $\check{\Sigma}_c$ describes the effects of collisions. In particular,

$$\hat{Q} = E\hat{\tau}_3 - [\xi_p \hat{1} + \hat{U}(\mathbf{p}, \mathbf{r}, t)] + i\hat{\Delta}(\mathbf{p}, \mathbf{r}, t) \tag{4}$$

and $\hat{U}$ is the sum of external fields and Fermi liquid interactions; here

$$\hat{\Delta} = \begin{Bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{Bmatrix} \tag{5}$$

where Δ is the order parameter; and $\hat{\tau}_\alpha$ are the Pauli matrices. Finally, $\check{\Sigma}_c$ is given by the usual diagrammatic expansion,[16,8] and arranged as a matrix like

$$\check{\Sigma}_c = \begin{Bmatrix} \hat{\Sigma}_c^{\mathrm{R}} & \hat{\Sigma}_c^{\mathrm{K}} \\ 0 & \hat{\Sigma}_c^{\mathrm{A}} \end{Bmatrix} \tag{6}$$

We consider now the equation of motion (3) together with its adjoint equation. Then it is clear that $\check{G}$ is also a solution of the commutator equation

$$[\check{Q} - \check{\Sigma}_c \overset{*}{,} \check{G}] = 0 \tag{7}$$

One realizes, however, that the strongly momentum-dependent term $\xi_p \check{G}$ has dropped out of the equation. We now define the generalized quasiclassical function $\check{g}$ by

$$[\check{Q} - \check{\Sigma}_c \overset{*}{,} \check{g}] = 0 \tag{8}$$

$$\check{g} * \check{g} = \check{1} \tag{9}$$

Equation (9) is called the "normalization condition," and it is evident that this Ansatz is a solution of (8). The quasiclassical function $\check{g}$ may be interpreted as the ξ_p integrated Green's function, with the generalization of Ref. 7.

The equations of motion are closed by expressing in turn the self-energies in terms of the quasiclassical function. This is simple for the mean field parts, which are parametrized by the Landau parameters and a pairing interaction; the Landau parameters and the strength of the pairing interaction λ (or T_c) are an input into the theory. In particular,

$$\hat{\Delta}(\mathbf{p}, \mathbf{r}, t) = -\frac{i\lambda}{4}\left\langle\frac{3\mathbf{p}\cdot\mathbf{p}'}{pp'}\int dE\,[\hat{g}^{\mathrm{K}}(\mathbf{p}', E, \mathbf{r}, t)]_{\mathrm{o.d.}}\right\rangle_{\mathbf{p}'} \tag{10}$$

where o.d. refers to the off-diagonal part in p–h space, and $\langle\cdots\rangle_{\mathbf{p}'}$ is the angular average with respect to $\mathbf{p}'$. Furthermore, the particle number density and current are given by*

$$\rho = \rho_0 - \frac{N(0)}{4}\operatorname{Tr}\left\langle\int dE\,\hat{g}^{\mathrm{K}}\right\rangle \tag{11}$$

$$\mathbf{j} = \mathbf{j}_0 - \frac{N(0)}{4}\operatorname{Tr}\hat{\tau}_3\left\langle\frac{\mathbf{p}}{m}\int dE\,\hat{g}^{\mathrm{K}}\right\rangle \tag{12}$$

and ρ_0 and $\mathbf{j}_0$ appear because of the subtraction procedure discussed in Section 1. In particular,

$$\rho_0 = p_{\mathrm{F}}^3/3\pi^2\hbar^3 - 2N(0)U^{\mathrm{s}}; \qquad \mathbf{j}_0 = -\partial U^{\mathrm{a}}/\partial\mathbf{p}\cdot m^*\rho_0/m$$

and we have introduced $\hat{U} = U^{\mathrm{s}}\hat{1} + U^{\mathrm{a}}\hat{\tau}_3$. In addition, the energy and the energy current are found to be†

$$e = -\frac{N(0)}{4}\operatorname{Tr}\hat{\tau}_3\left\langle\int dE\,\hat{g}^{\mathrm{K}}E\right\rangle + N(0)\frac{\langle|\Delta|^2\rangle}{\lambda} - U^{\mathrm{s}}\rho - N(0)[U^{\mathrm{s}}]^2 \tag{13}$$

$$\mathbf{j}_e = -\frac{N(0)}{4}\operatorname{Tr}\left\langle\left[\frac{\mathbf{p}}{m^*} - i\frac{\partial\hat{\Delta}}{\partial\mathbf{p}}\right]\int dE\,\hat{g}^{\mathrm{K}}E\right\rangle - U^{\mathrm{s}}\mathbf{j} \tag{14}$$

The energy conservation law then has the form

$$\dot{e} + \nabla\cdot\mathbf{j}_e = -\mathbf{j}\cdot\nabla U^{\mathrm{s}} \tag{15}$$

Note that (15) can be brought easily into the form where only the external field appears in the dissipated energy. Finally, the collision self-energy will be discussed in the following section.

*Here and in the following, Tr denotes the p–h trace, since we consider only spin-independent problems.

†We neglect terms connected with U^{a}. Those can usually be determined by simple reasoning, e.g., by considering Galilei transformations.

In the next step, one connects $\hat{g}^{\mathrm{K}}$ with a distribution function. Writing out explicitly the normalization condition (9), one finds

$$\hat{g}^{\mathrm{R}} * \hat{g}^{\mathrm{R}} = \hat{g}^{\mathrm{A}} * \hat{g}^{\mathrm{A}} = \hat{1}; \qquad \hat{g}^{\mathrm{R}} * \hat{g}^{\mathrm{K}} + \hat{g}^{\mathrm{K}} * \hat{g}^{\mathrm{A}} = 0 \tag{16}$$

The second equation is solved by

$$\hat{g}^{\mathrm{K}} = \hat{g}^{\mathrm{R}} * \hat{h} - \hat{h} * \hat{g}^{\mathrm{A}} \tag{17}$$

and a certain freedom* in the choice of $\hat{h}$ allows us to choose

$$\hat{h} = h^{\mathrm{a}}\hat{1} + h^{\mathrm{s}}\hat{\tau}_3 \tag{18}$$

We also define†

$$h^{\mathrm{a}} = \mathrm{th}(E/2k_{\mathrm{B}}T) - 2\delta f^{\mathrm{a}}; \qquad h^{\mathrm{s}} = -2\delta f^{\mathrm{s}} \tag{19}$$

where δf^{a} and δf^{s} are antisymmetric and symmetric with respect to E, $\mathbf{p} \to -E, -\mathbf{p}$, respectively; it is now straightforward to derive a transport equation for these quantities by taking $i/8\,\mathrm{Tr}\,(\cdots)$ and $i/8\,\mathrm{Tr}\,(\hat{\tau}_3 \cdots)$ of the equation of motion for $\hat{g}^{\mathrm{K}}$. For example, the "time derivative term" follows from

$$[E\hat{\tau}_3 \overset{*}{,} \hat{g}^{\mathrm{K}}] = E(\hat{\tau}_3\hat{g}^{\mathrm{K}} - \hat{g}^{\mathrm{K}}\hat{\tau}_3) + \frac{i\hbar}{2}\frac{\partial}{\partial t}(\hat{\tau}_3\hat{g}^{\mathrm{K}} + \hat{g}^{\mathrm{K}}\hat{\tau}_3) \tag{20}$$

by inserting (17), and (for simplicity) using the equilibrium result for $\hat{g}^{\mathrm{R,A}}$, namely

$$[\hat{g}^{\mathrm{R}} - \hat{g}^{\mathrm{A}}]_{\mathrm{diag}} = 2\mathcal{N}_1\hat{\tau}_3 \tag{21}$$

where $\mathcal{N}_1$ is the (normalized) density of states for a given energy and direction of the momentum; then

$$[E\hat{\tau}_3 \overset{*}{,} \hat{g}^{\mathrm{K}}]_{\mathrm{diag}} = -4i\mathcal{N}_1\hbar\frac{\partial}{\partial t}(\delta f^{\mathrm{a}}\hat{1} + \delta f^{\mathrm{s}}\hat{\tau}_3) \tag{22}$$

This also demonstrates that, in general, the equation for the regular functions have to be solved in addition to the transport equation.

The most complicated part of the equations is, of course, the collision contribution to the self-energy. It appears in the equation for $\hat{g}^{\mathrm{K}}$ in the form [see (8); also, we may often replace the $*$ by the ordinary product]

$$[\check{\Sigma}_c, \check{g}]^{\mathrm{K}} = \hat{\Sigma}_c^{\mathrm{R}}\hat{g}^{\mathrm{K}} - \hat{g}^{\mathrm{K}}\hat{\Sigma}_c^{\mathrm{A}} - \hat{g}^{\mathrm{R}}\hat{\Sigma}_c^{\mathrm{K}} + \hat{\Sigma}_c^{\mathrm{K}}\hat{g}^{\mathrm{A}} \tag{23}$$

*The replacement $\hat{h} \to \hat{h} + \hat{g}^{\mathrm{R}} * \hat{b} + \hat{b} * \hat{g}^{\mathrm{A}}$ leaves $\hat{g}^{\mathrm{K}}$ unchanged, for *any* $\hat{b}$; see Ref. 4.

†The antisymmetric and symmetric parts (a, s) are sometimes[4,7] called longitudinal (L) and transverse(T), respectively.

The quantities K^a and K^s, defined by

$$\frac{i}{4\hbar}\{[\check{\Sigma}_c, \check{g}]^K\}_{\text{diag}} = K^a\hat{1} + K^s\hat{\tau}_3 \tag{24}$$

then play the role of collision operators in the kinetic equations. Note that the weak coupling approximation replaces $\hat{\Sigma}_c^R$ and $-\hat{\Sigma}_c^A$ by $(\hat{\Sigma}_c^R - \hat{\Sigma}_c^A)/2$.

3. LINEWIDTH(S) AND COLLISION OPERATOR(S)

3.1. Pair-breaking Effects

We study first the effect of collisions on the regular functions $\hat{g}^{R,A}$ and consider a spatially homogeneous equilibrium situation. (Slow spatial variations are discussed in the Appendix.) In this case, we have to solve the equation

$$[E\hat{\tau}_3 + i\hat{\Delta}(\mp)\tfrac{1}{2}(\hat{\Sigma}_c^R - \hat{\Sigma}_c^A), \hat{g}^{R(A)}] = 0 \tag{25}$$

together with the normalization condition (9). Keeping in mind that Δ depends on the direction of the momentum, we introduce the magnitude and the phase via $\Delta = |\Delta| \exp(-i\Theta)$, and

$$\begin{gathered} \Delta = \Delta^{(1)}\hat{\tau}_1 + \Delta^{(2)}\hat{\tau}_2 \\ \Delta^{(1)} = |\Delta| \cos\Theta; \qquad \Delta^{(2)} = |\Delta| \sin\Theta \end{gathered} \tag{26}$$

note that $\{\hat{\tau}_\Theta = \cos\Theta\hat{\tau}_1 + \sin\Theta\hat{\tau}_2, \hat{\tau}_{\Theta+\pi/2}, \hat{\tau}_3\}$ form the usual algebra. Defining also

$$\tfrac{1}{2}(\hat{\Sigma}_c^R - \hat{\Sigma}_c^A) = -i\Gamma_3\hat{\tau}_3 + \Gamma_1\hat{\tau}_1 + \Gamma_2\hat{\tau}_2 \tag{27}$$

it is clear that solving (25) with collisions is no more complicated than without. The result is of the form*

$$\begin{aligned} \hat{g}^R &= \alpha^R\hat{\tau}_3 + \beta_1^R\hat{\tau}_1 + \beta_2^R\hat{\tau}_2 \\ &= \frac{(-iE + \Gamma_3)\hat{\tau}_3 + (\Delta^{(1)} + i\Gamma_1)\hat{\tau}_1 + (\Delta^{(2)} + i\Gamma_2)\hat{\tau}_2}{[(-iE + \Gamma_3)^2 + (\Delta^{(1)} + i\Gamma_1)^2 + (\Delta^{(2)} + i\Gamma_2)^2]^{1/2}} \end{aligned} \tag{28}$$

the square root being defined as $\mathrm{Re}\sqrt{\;\;} > 0$. Furthermore, we introduce spectral functions according to

$$\alpha^R = \mathcal{N}_1 + i\mathcal{R}_1; \qquad \beta_{1,2}^R = \mathcal{N}_2^{(1,2)} + i\mathcal{R}_2^{(1,2)} \tag{29}$$

The BCS limits of these quantities (Γ_i small) are easily found: in the physical region, i.e., for $E^2 \geqslant |\Delta|^2$, one obtains

$$\mathcal{N}_1 \simeq |E|/\xi, \qquad \mathcal{R}_2^{(1,2)}/\mathcal{N}_1 \simeq \Delta^{(1,2)}/E \tag{30}$$

*The advanced function follows from changing $\Gamma_i \to -\Gamma_i$.

with $\xi = [E^2 - |\Delta|^2]^{1/2}$. The other functions are zero for zero pair-breaking in this region, and one has to be more careful. First, in an expansion with respect to the Γ_i, one may define an effective linewidth Γ by rewriting the denominator of (28)

$$[\cdots]^{1/2} \simeq [(-iE + \Gamma)^2 + |\Delta|^2]^{1/2} \tag{31}$$

where

$$\mathcal{N}_1\Gamma = \mathcal{N}_1\Gamma_3 - \mathcal{R}_2^{(1)}\Gamma_1 - \mathcal{R}_2^{(2)}\Gamma_2 \tag{32}$$

$\mathcal{N}_1$ and $\mathcal{R}_2^{(1,2)}$ are just the density of states and the function familiar from the BCS gap equation, respectively. (Note that $\mathcal{R}_2 = \mathcal{R}_2^{(1)} \cos\Theta + \mathcal{R}_2^{(2)} \sin\Theta \simeq |\Delta|\mathcal{N}_1/E$.) The more unusual $\mathcal{N}_2^{(1,2)}$, or rather the combination

$$\mathcal{N}_2 = \mathcal{N}_2^{(1)} \cos\Theta + \mathcal{N}_2^{(2)} \sin\Theta \tag{33}$$

plays an important role in the kinetic equation for the symmetric distribution function (the "conversion term" introduced in Ref. 17; see Section 4). One finds

$$|\Delta|\mathcal{N}_2 \simeq \frac{|\Delta|^2}{\xi^2}\,\mathcal{N}_1\Gamma - \mathcal{R}_2^{(1)}\Gamma_1 - \mathcal{R}_2^{(2)}\Gamma_2 \tag{34}$$

for $E^2 \gtrsim |\Delta|^2$; note the sharp increase for $\xi \to 0$.

3.2. Linewidth and Scattering Rate

It is now straightforward to connect the linewidth with the scattering rate of the kinetic equation. Considering the linearized version of the collision operator, Eq. (24), and inserting the definition of the quasiparticle distribution function (17), we may identify the scattering out contribution of the collision operators, and define two scattering times τ_E^{a} and τ_E^{s}. They are given by

$$\begin{aligned}
\frac{\hbar}{\tau_E^{\mathrm{a}}}\mathcal{N}_1 &= \frac{i}{8}\operatorname{Tr}\{(\hat{\Sigma}_c^{\mathrm{R}} - \hat{\Sigma}_c^{\mathrm{A}}), (\hat{g}^{\mathrm{R}} - \hat{g}^{\mathrm{A}})\}_+ \\
\frac{\hbar}{\tau_E^{\mathrm{s}}}\mathcal{N}_1 &= \frac{i}{8}\operatorname{Tr}\hat{\tau}_3\{(\hat{\Sigma}_c^{\mathrm{R}} - \hat{\Sigma}_c^{\mathrm{A}}), (\hat{g}^{\mathrm{R}}\hat{\tau}_3 - \hat{\tau}_3\hat{g}^{\mathrm{A}})\}_+
\end{aligned} \tag{35}$$

where $\{\,,\}_+$ is the anticommutator. We consider again the BCS (weak scattering) limit: then, in the definition of $1/\tau_E^{\mathrm{s}}$, we need only take into account the diagonal part of $(\hat{g}^{\mathrm{R}}\hat{\tau}_3 - \hat{\tau}_3\hat{g}^{\mathrm{A}})$—the off-diagonal contributions will be $\sim\mathcal{N}_2$ and can be neglected. The result is

$$\hbar/2\tau_E^{\mathrm{a}} = \Gamma; \qquad \hbar/2\tau_E^{\mathrm{s}} = \Gamma_3 \tag{36}$$

The considerations in this section so far have been quite general, and apply in similar form to the B phase as well as to electron–phonon scattering in superconductors.* Now (34) takes the form†

$$\frac{2|\Delta|\mathcal{N}_2}{\hbar} \simeq \left[\frac{E^2}{\xi^2}\frac{1}{\tau_E^{\mathrm{a}}} - \frac{1}{\tau_E^{\mathrm{s}}}\right]\mathcal{N}_1 \tag{37}$$

3.3. The Collision Operator

As a final task, we now want to derive the expressions for the scattering rates and the collision operators. As usual, we have to study the standard collision diagram, generalized to include the particle–hole index.[8,16] Although the perturbation theory can be formulated with Keldysh matrices, we avoid this and start with the imaginary time version, namely

$$\begin{aligned}\hat{\Sigma}_c(\mathbf{p}, t, t') = \frac{1}{3!}\int \frac{d^3p_2}{(2\pi\hbar)^3}\int \frac{d^3p_3}{(2\pi\hbar)^3}[\tilde{T}]^2 \\ \times \hat{G}(\mathbf{p}_2, t', t)\hat{G}(\mathbf{p}_3, t, t')\hat{G}(\mathbf{p}_4, t, t')\end{aligned} \tag{38}$$

We use an obvious mixed representation, suppressing the spatial variable; $\tilde{T}$ denotes the vertex function, which, since $\hat{\Sigma}_c$ is of third order in the T_c/μ classification,[8] can be evaluated in the normal state, for zero temperature and frequencies, and with all momenta fixed at the Fermi surface. All quantities $\hat{\Sigma}_c$, $\tilde{T}$, and $\hat{G}$ also depend on spin and p–h indices (see below). Momentum conservation takes the form $\mathbf{p}+\mathbf{p}_2=\mathbf{p}_3+\mathbf{p}_4$, and the factor $1/3!$ takes care of the additional p–h index.

Following Eliashberg,[3] we introduce in the first step the quasiclassical Green's functions, and obtain

$$\hat{\Sigma}_c(\mathbf{p}, t, t') = \mathcal{L}_0\{\hat{g}(\mathbf{p}_2, t', t)\hat{g}(\mathbf{p}_3, t, t')\hat{g}(\mathbf{p}_4, t, t')\} \tag{39}$$

with

$$\mathcal{L}_0\{\cdots\} = \frac{1}{3!}\frac{[N(0)]^2}{v_{\mathrm{F}}p_{\mathrm{F}}}\left(\frac{\pi}{i}\right)^3\int \frac{d\Omega_2\, d\Omega_3}{(4\pi)^2}[\tilde{T}]^2\delta\left(\frac{|\mathbf{p}_4|}{p_{\mathrm{F}}}-1\right)\{\cdots\} \tag{40}$$

In the next step, this expression is connected with the real-time $\gtrless$ functions, using

$$(\hat{g}\hat{g}\hat{g})^{\gtrless} = \hat{g}^{\lessgtr}\hat{g}^{\gtrless}\hat{g}^{\gtrless}$$

*Compare with Ref. 17, or the recent discussion by Beyer–Nielson *et al.*[18]

†In Ref. 12, $\Gamma_{1,2}$ were neglected in order to simplify the discussion. Careful consideration shows that those terms *are needed* in order to make the connection with the Boltzmann equation for Bogoliubov quasiparticles.

and we find the R, A, K, functions from

$$\hat{\Sigma}_c^> \mp \hat{\Sigma}_c^< = \begin{cases} \hat{\Sigma}_c^{\mathrm{R}} - \hat{\Sigma}_c^{\mathrm{A}} \\ \hat{\Sigma}_c^{\mathrm{K}} \end{cases} \tag{41}$$

Similar relations hold for $\hat{g}^{\mathrm{R,A,K}}$; for example,

$$\hat{g}^{\gtrless} = \tfrac{1}{2}[\hat{g}^{\mathrm{K}} \pm (\hat{g}^{\mathrm{R}} - \hat{g}^{\mathrm{A}})] \tag{42}$$

Finally, we Fourier-transform with respect to the time difference and introduce the distribution functions according to (17)–(19). Although quite cumbersome, it is then straightforward to derive the collision operators from (24).

Consider first the collision operator of the antisymmetric equation, K^{a}, defined by

$$K^{\mathrm{a}} = \frac{i}{8\hbar} \operatorname{Tr}[(\hat{\Sigma}_c^{\mathrm{R}} - \hat{\Sigma}_c^{\mathrm{A}})\hat{g}^{\mathrm{K}} - \hat{\Sigma}_c^{\mathrm{K}}(\hat{g}^{\mathrm{R}} - \hat{g}^{\mathrm{A}})] \tag{43}$$

It is convenient to define

$$f = \frac{1}{2}\left[1 - \operatorname{th}\left(\frac{E}{2k_{\mathrm{B}}T}\right)\right]; \qquad f^{\mathrm{a}} = f + \delta f^{\mathrm{a}} \tag{44}$$

and also use the abbreviations

$$f^{\mathrm{a}} = f^{\mathrm{a}}(E, \mathbf{p}); \qquad f_i^{\mathrm{a}} = f^{\mathrm{a}}(E_i, \mathbf{p}_i) \tag{45}$$

with $i = 2, 3, 4$ $(E + E_2 = E_3 + E_4)$. Finally, introduce also the dimensionless scattering amplitude T by $T = 2N(0)\tilde{T}$. The general structure of K^{a} is then easily derived, with the result

$$K^{\mathrm{a}} = \mathscr{L}\{M^{\mathrm{a}}[(1-f^{\mathrm{a}})(1-f_2^{\mathrm{a}})f_3^{\mathrm{a}}f_4^{\mathrm{a}} - f^{\mathrm{a}}f_2^{\mathrm{a}}(1-f_3^{\mathrm{a}})(1-f_4^{\mathrm{a}})]\} \tag{46}$$

where now $(1 \equiv (E, \mathbf{p})$, etc.)

$$\mathscr{L}\{\cdots\} = \frac{\pi\mathcal{N}_1(1)}{2\hbar v_{\mathrm{F}} p_{\mathrm{F}}} \int \frac{dE_2\, dE_3\, d\Omega_2\, d\Omega_3}{(4\pi)^2} \times \mathcal{N}_1(2)\mathcal{N}_1(3)\mathcal{N}_1(4)\delta\left(\frac{|\mathbf{p}_4|}{p_{\mathrm{F}}} - 1\right)\{\cdots\} \tag{47}$$

and E_4, $\mathbf{p}_4$ have to be inserted according to the conservation laws. K^{a} has evidently the same structure as in the normal state. The quantity M^{a} is given by

$$\mathcal{N}_1(2)\mathcal{N}_1(3)\mathcal{N}_1(4)M^{\mathrm{a}}=\frac{1}{3!}\left(\frac{1}{2}\right)^5 \sum_{\substack{\rho_1\rho_2\rho_3\rho_4\\ \rho_1'\rho_2'\rho_3'\rho_4'}} \sum_{\sigma_2\sigma_3\sigma_4}$$
$$\times T_{\substack{\uparrow\sigma_2\sigma_3\sigma_4\\ \rho_1\rho_2\rho_3\rho_4}}(\mathbf{p},\mathbf{p}_2;\mathbf{p}_3,\mathbf{p}_4)T_{\substack{\sigma_4\sigma_3\sigma_2\uparrow\\ \rho_4\rho_3\rho_2\rho_1}}(\mathbf{p}_4,\mathbf{p}_3;\mathbf{p}_2,\mathbf{p})$$
$$\times[\hat{g}^{\mathrm{R}}(1)-\hat{g}^{\mathrm{A}}(1)]_{\rho_1'\rho_1}[\hat{g}^{\mathrm{R}}(2)-\hat{g}^{\mathrm{A}}(2)]_{\rho_2'\rho_2}$$
$$\times[\hat{g}^{\mathrm{R}}(3)-\hat{g}^{\mathrm{A}}(3)]_{\rho_3\rho_3'}[\hat{g}^{\mathrm{R}}(4)-\hat{g}^{\mathrm{A}}(4)]_{\rho_4\rho_4'} \tag{48}$$

We used the fact that the Green's functions are diagonal in spin space (let σ_1 be $\uparrow$, e.g.), and also introduced the p–h components of the scattering amplitude.[16] The symmetry properties of T follow directly from the time-ordered two-particle Green's function. Also, if we denote the p–h index by $(+,-)$, we have to remember that $c_{\mathbf{p}+}=c_{\mathbf{p}}$, $c_{\mathbf{p}-}=c^{+}_{-\mathbf{p}}$, etc. Then it is clear that the $(++++)$ element of T is the usual amplitude of normal Fermi liquid theory, namely

$$T_{\substack{\sigma_1\sigma_2\sigma_3\sigma_4\\ ++++}}(\mathbf{p},\mathbf{p}_2;\mathbf{p}_3,\mathbf{p}_4)=T^{\mathrm{s}}\delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4}+T^{\mathrm{a}}(\sigma_\mu)_{\sigma_1\sigma_3}(\sigma_\mu)_{\sigma_2\sigma_4} \tag{49}$$

while the others are closely related:

$$T_{\substack{\sigma_1\sigma_2\sigma_3\sigma_4\\ +-+-}}(\mathbf{p},\mathbf{p}_2;\mathbf{p}_3,\mathbf{p}_4)=T_{\substack{\sigma_1\sigma_2\sigma_3\sigma_4\\ ++++}}(\mathbf{p},-\mathbf{p}_4;\mathbf{p}_3,-\mathbf{p}_2)$$
$$T_{\substack{\sigma_1\sigma_2\sigma_3\sigma_4\\ +--+}}(\mathbf{p},\mathbf{p}_2;\mathbf{p}_3,\mathbf{p}_4)=T_{\substack{\sigma_1\sigma_2\sigma_3\sigma_4\\ ++++}}(\mathbf{p},-\mathbf{p}_3;-\mathbf{p}_2,\mathbf{p}_4) \tag{50}$$
$$T_{+-+-}=T_{-+-+};\qquad T_{+--+}=T_{-++-};\qquad T_{++++}=T_{----}$$

In the normal state, M^{a} reduces to the well-known result

$$M^{\mathrm{a}}=[T^{\mathrm{s}}]^2+3[T^{\mathrm{a}}]^2,\qquad T>T_c \tag{51}$$

In the superfluid state, the expression for M^{a} is rather complicated. Working out the p–h sum, we find

$$M^{\mathrm{a}}=\frac{1}{2}\sum_{\sigma_2\sigma_3\sigma_4}\left\{\frac{1}{2}T^2_{++++}\left[1+\frac{\Delta\Delta_2\Delta_3^*\Delta_4^*}{EE_2E_3E_4}\right]\right.$$
$$\left.-T_{++++}T_{+-+-}\left[\frac{\Delta_2\Delta_4^*}{E_2E_4}+\frac{\Delta\Delta_3^*}{EE_3}\right]+\text{c.c.}\right\} \tag{52}$$

where we have inserted the BCS limit for simplicity; the arguments of the T's are $\uparrow\sigma_2\sigma_3\sigma_4/\mathbf{p}, \mathbf{p}_2; \mathbf{p}_3, \mathbf{p}_4$. In the limit where the scattering amplitude can be parametrized by a point interaction of strength v, i.e.,

$$T_{++++} = v(\delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4} - \delta_{\sigma_1\sigma_4}\delta_{\sigma_2\sigma_3}) \tag{53}$$

we obtain a result similar to Kopnin,[10] namely*

$$M^{\mathrm{a}} = \frac{v^2}{2}\left(1 + \frac{\Delta\Delta_2\Delta_3^*\Delta_4^*}{EE_2E_3E_4} - \frac{\Delta_2\Delta_4^*}{E_2E_4} - \frac{\Delta\Delta_3^*}{EE_3} + \mathrm{c.c.}\right) \tag{54}$$

On the other hand, the expression for the symmetric operator K^{s} cannot be brought into a simple form [like (46)]. However, if we work in the BCS limit and also linearize with respect to δf^{s}, we find

$$K^{\mathrm{s}} = -\mathscr{L}\left\{\frac{1}{2}F\sum_{\sigma_2\sigma_3\sigma_4}\left[T^2_{++++}(\phi^{\mathrm{s}} + \phi_2^{\mathrm{s}} - \phi_3^{\mathrm{s}} - \phi_4^{\mathrm{s}}) - T_{++++}T_{+-+-}\frac{\Delta_2\Delta_4^* + \mathrm{c.c.}}{E_2E_4}(\phi^{\mathrm{s}} - \phi_3^{\mathrm{s}})\right]\right\} \tag{55}$$

with $F = ff_2(1-f_3)(1-f_4)$, and $\delta f^{\mathrm{s}} = f(1-f)\phi^{\mathrm{s}}$, etc. In the potential scattering limit, this simplifies to[10],†

$$K^{\mathrm{s}} = -\mathscr{L}\left\{Fv^2\left[1 - \frac{\Delta_2\Delta_4^* + \mathrm{c.c.}}{2E_2E_4}\right](\phi - \phi_3^{\mathrm{s}})\right\} \tag{56}$$

Keeping in mind the results of the previous sections, especially Eqs. (32) and (36), we can easily check the results for the scattering rates with Ref. 16.

A brief inspection of the above expressions shows that a precise calculation of the scattering rates, for example, is hardly possible. A simple argument shows the order of magnitude: at low temperatures the rates will be proportional to the number of excitations, namely $\sim(T/T_c)^2$. and we expect

$$\tau_E^{\mathrm{a}} \sim \tau_E^{\mathrm{s}} \sim (T_c/T)^2\tau_N(T) \tag{57}$$

which might serve as an interpolating formula for all T [$\tau_N(T)$ is the normal state scattering time on the Fermi surface]. On the other hand, this argument also makes clear that the momentum dependence of τ_E^{a}, τ_E^{s} will reflect in detail the momentum dependence of the scattering amplitudes—at low T, we have $\mathbf{p}_2 \| \pm \mathbf{l}$ to a good approximation.

As an illustration, we mention electron–phonon processes in superconductors.[17] In that case, the expressions for $K^{\mathrm{a,s}}$ are relatively simple, and

*We find the terms quadratic in Δ larger by a factor two than in Ref. 10.
†See preceding footnote.

the low-temperature limits of τ_E^{a}, τ_E^{s} were calculated.[19] Note that, due to a coherence factor of the form

$$\left[1-\frac{|\Delta|^2}{EE_2}\right] \sim \frac{k_B T}{|\Delta|}, \qquad T \ll T_c$$

in the expression for $1/\tau_E^{\mathrm{a}}$, one finds the relation $\tau_E^{\mathrm{s}} \sim T\tau_E^{\mathrm{a}}/T_c$.

4. THE KINETIC EQUATIONS

4.1. The Structure of the Equations

Following the prescriptions given at the end of Section 2, one derives the kinetic equations from the equation of motion for the Keldysh function $\hat{g}^{K}$ into which Eq. (17) is inserted. For slow spatial and temporal variations, we need only low-order terms in the expansion of the $*$ product, and obtain

$$\begin{aligned}\mathcal{N}_1\, \partial_t h^{\mathrm{a}} + 2K^{\mathrm{a}} + \mathbf{v}\cdot\nabla\mathcal{N}_1 h^{\mathrm{s}} \\ = -\mathcal{R}_2\{\{|\Delta|, h^{\mathrm{a}}\}\} - \mathcal{N}_1\{\{U^{\mathrm{a}}, h^{\mathrm{a}}\}\} + \cdots\end{aligned} \tag{58}$$

and

$$\begin{aligned}\partial_t\mathcal{N}_1 h^{\mathrm{s}} + 2K^{\mathrm{s}} + \mathcal{N}_1\mathbf{v}\cdot\nabla h^{\mathrm{a}} + 2|\Delta|\mathcal{N}_2 h^{\mathrm{s}} \\ = -|\Delta|\mathcal{N}_2\{\{\Theta, h^{\mathrm{a}}\}\} - \mathcal{N}_1\{\{U^{\mathrm{s}}, h^{\mathrm{a}}\}\} + \cdots\end{aligned} \tag{59}$$

We have defined $\mathbf{v} = \mathbf{p}/m^*$, and

$$\{\{A, B\}\} = \frac{\partial A}{\partial t}\frac{\partial B}{\partial E} + \frac{\partial A}{\partial \mathbf{p}}\frac{\partial B}{\partial \mathbf{r}} - \frac{\partial A}{\partial \mathbf{r}}\frac{\partial B}{\partial \mathbf{p}} \tag{60}$$

and one should emphasize the Poisson bracket term in (60) which is due to the generalization of the quasiclassical technique, as discussed above. The dots in (58) and (59) indicate that we are not trying to be systematic in the expansion; for example, the time derivative of the overall phase should be considered as being "zero order." As has been pointed out by many authors, a systematic expansion is easiest in a gauge in which the overall phase is time and space independent. Also, certain terms—like $\{\{\Theta, \mathcal{N}_2 h^{\mathrm{s}}\}\}$—were neglected in (58) since h^{s} is usually small; and the equation of motion for $\hat{g}^{R,A}$ was used to simplify terms connected with h^{a} (see Ref. 4).

Consider first, as an illustration, the terms characteristic for the symmetric equation, namely the terms $\sim\mathcal{N}_2$. Also, we insert for h^{a} the equilibrium solution with a finite normal fluid velocity*

$$h^{\mathrm{a}} = \mathrm{th}\left(\frac{E - \mathbf{p}\cdot\mathbf{v}_n}{2T}\right) \tag{61}$$

*$\hbar = k_B = 1$ in the following.

and we obtain

$$\{\{\Theta, h^{a}\}\} \simeq \frac{1}{4T\,\mathrm{ch}^2(E/2T)}\left[\frac{\partial\Theta}{\partial t} - \frac{\partial\Theta}{\partial \mathbf{p}}\cdot\frac{\partial(\mathbf{p}\cdot\mathbf{v}_n)}{\partial \mathbf{r}} + \frac{\partial\Theta}{\partial \mathbf{r}}\cdot\mathbf{v}_n - \frac{\partial\Theta}{\partial \mathbf{p}}\cdot\frac{\partial T}{\partial \mathbf{r}}\frac{E}{T}\right] \tag{62}$$

We realize that the first three terms have symmetry $(++)$, and the last one has $(--)$, with respect to $E, \mathbf{p} \to -E, -\mathbf{p}$. Thus, while the first three lead to a change in the density—and have to be treated carefully, since they are proportional to the eigenfunction with eigenvalue zero of K^{s}—the last one leads to a contribution to the energy current (see below).

Considering the $(++)$ contribution first, we note that

$$\begin{aligned} \partial\Theta/\partial t &= \partial\varphi/\partial t + O(\partial_t \mathbf{l}) \\ |\Delta|^2\,\partial\Theta/\partial\mathbf{p} &= \Delta_0^2(\hat{\mathbf{p}}\times\mathbf{l})/p_{\mathrm{F}} \\ \partial\Theta/\partial\mathbf{r} &= -2m\mathbf{v}_s + O(\partial_{\mathbf{r}}\mathbf{l}) \end{aligned} \tag{63}$$

where φ denotes the overall phase, $|\Delta|^2 = \Delta_0^2[1-(\hat{\mathbf{p}}\cdot\mathbf{l})^2]$, $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$, and $\mathbf{v}_s = -\nabla\varphi/2m$ is the superfluid velocity. Inserting these expressions into (62) and taking the angular average,* we obtain

$$\langle\{\{\Theta, h^{a}\}\}\rangle = \frac{1}{4T\,\mathrm{ch}^2(E/2T)}\left[\frac{\partial\varphi}{\partial t} - 2m\mathbf{v}_s\cdot\mathbf{v}_n - \frac{1}{2}\mathbf{l}\cdot\nabla\times\mathbf{v}_n\right] \tag{64}$$

Comparing now the $\mathcal{N}_2$ terms, we are led to define a local equilibrium piece of the distribution function:

$$\delta f^{s} = \delta f^{s}_{\mathrm{l.e.}} + \delta\tilde{f}^{s} \tag{65}$$

and

$$\delta f^{s}_{\mathrm{l.e.}} = \frac{1}{4T\,\mathrm{ch}^2(E/2T)}\left[\frac{1}{2}\frac{\partial\varphi}{\partial t} - m\mathbf{v}_s\cdot\mathbf{v}_n - \frac{1}{4}\mathbf{l}\cdot\nabla\times\mathbf{v}_n\right] \tag{66}$$

The last term in this expression, $\sim\mathbf{l}\cdot\nabla\times\mathbf{v}_n$, has attracted much attention in connection with rotating equilibrium solutions (see, e.g., Ref. 20). A careful discussion of these questions was given recently by Nagai.[21]

4.2. The Heat Current

In the next step, we consider in more detail the $(--)$ mode, which is evidently involved whenever a temperature gradient is imposed on the

*Strictly speaking, we have to study this term as it appears in the transport equation, i.e., multiplied by $|\Delta|\mathcal{N}_2$. However, integrating with respect to E and then taking the angular average leads to the same result.

system. In the case where $U^s = 0$, and for time-independent situations, Eq. (59) leads to

$$-K^s + 2|\Delta|\mathcal{N}_2\,\delta f^s = -\frac{\left\{\mathcal{N}_1(\mathbf{v}\cdot\nabla T) + |\Delta|\mathcal{N}_2\left[\left(\frac{\partial\Theta}{\partial\mathbf{p}}\right)\cdot\nabla T\right]\right\}E}{4T^2\,\mathrm{ch}^2(E/2T)} \tag{67}$$

Having found δf^s, we can calculate the energy current from [see (14); the term $\sim\mathrm{Tr}\,\hat{g}^K\,\partial\hat{\Delta}/\partial\mathbf{p}$ is small, $O(\Delta_0/\mu)$, and can be neglected]

$$\mathbf{j}_e = 2N(0)\left\langle\frac{\mathbf{p}}{m^*}\int dE\,E\mathcal{N}_1\,\delta f^s\right\rangle \tag{68}$$

However, finding an approximate solution of (67) does not pose a particular problem; we may use a simple relaxation time approximation for K^s, namely

$$K^s \to -(1/\tau_E)\mathcal{N}_1\,\delta f^s$$

(In view of the discussion in Section 3, we ignore in the following the difference between τ_E^a and τ_E^s.) Then we find

$$\begin{aligned}\delta f^s &= \delta f^s_{(1)} + \delta f^s_{(2)}\\ &= -\frac{\mathcal{N}_1(\mathbf{v}\cdot\nabla T) + |\Delta|\mathcal{N}_2[(\partial\Theta/\partial\mathbf{p}\cdot\nabla T]}{(1/\tau_E)\mathcal{N}_1 + 2|\Delta|\mathcal{N}_2}\,\frac{E}{4T^2\,\mathrm{ch}^2\,(E/2T)}\end{aligned} \tag{69}$$

In the first term, which gives just the ordinary thermal conductivity contribution, we may insert the BCS limit of $\mathcal{N}_2$, namely $2|\Delta|\mathcal{N}_2 \simeq |\Delta|^2\mathcal{N}_1/\xi^2\tau_E$, and obtain the simple result

$$\delta f^s_{(1)} = -\tau_E(\mathbf{v}\cdot\nabla T)\frac{\xi^2}{E^2}\frac{E}{4T^2\,\mathrm{ch}^2\,(E/2T)} \tag{70}$$

On the other hand, $\delta f^s_{(2)}$ has a finite contribution also for $E^2 \leqslant |\Delta|^2$; the result is

$$\delta f^s_{(2)} = -\frac{1}{2}\left(\frac{\partial\Theta}{\partial\mathbf{p}}\cdot\nabla T\right)\frac{E}{4T^2\,\mathrm{ch}^2\,(E/2T)}W \tag{71}$$

with $W \simeq 1$ for $E^2 \leqslant |\Delta|^2$ and $W \simeq |\Delta|^2/E^2$ for $E^2 \geqslant |\Delta|^2$. It should be emphasized that $\delta f^s_{(2)}$ is independent of the scattering rate (although we expect that a more accurate solution of the Boltzmann equation might slightly change the energy dependence of $\delta f^s_{(2)}$).

The heat current calculated from (70) and (71) is then found to be of the form

$$\mathbf{j}_e = -\boldsymbol{\kappa}\cdot\nabla T - C\mathbf{l}\times\nabla T \tag{72}$$

with ($\mathcal{N}_1\, dE \equiv d\xi$)

$$\kappa_{ij} = \frac{2N(0)v_F^2\tau_E}{T}\left\langle \hat{p}_i\hat{p}_j \int d\xi \frac{\xi^2}{4T\,\mathrm{ch}^2\,(E/2T)}\right\rangle \tag{73}$$

which is just the well-known result.[22] For low temperatures, one estimates $\kappa_\|(T) \sim \kappa(T_c)T_c/T$ and $\kappa_\perp(T) \sim \kappa(T_c)T/T_c$, where $\|$ and $\perp$ refer to the direction of $\mathbf{l}$. For the second contribution, we obtain [C in (72) is the $\perp$ component of C_{ij}]

$$C_{ij} = \frac{N(0)v_F\Delta_0^2}{Tp_F}\left\langle \hat{p}_i\hat{p}_j \int d\xi \frac{E^2 \cdot W/|\Delta|^2}{4T\,\mathrm{ch}^2\,(E/2T)}\right\rangle \tag{74}$$

Near T_c, C is easily evaluated, and we obtain

$$C = N(0)v_F\Delta_0^2/3T_cp_F, \qquad T \to T_c \tag{75}$$

while we estimate the low-temperature behavior as

$$C \sim \frac{N(0)v_F}{p_F}\frac{T^3}{T_c^2}, \qquad T \ll T_c \tag{76}$$

As far as order of magnitude is concerned (say, for $T \sim 0.9T_c$), one finds

$$C/\kappa \sim (\tau_E \cdot \mu)^{-1} \sim 10^{-4}$$

Near T_c, the coefficient C was also recently calculated by Nagai,[23] using the method of the matrix kinetic equation (or *energy* integrated Green's function). In that approach, the drive terms leading to the analog of $\delta f^s_{(2)}$ are off-diagonal in Nambu space, and special care seems necessary to solve the kinetic equation.*

Finally, we mention that the "drive term" in Eq. (58) leads to an additional contribution in the momentum stress tensor. A simple calculation, making use of the fact that $\mathcal{R}_2$ depends only through $|\Delta|$ on $\mathbf{p}$ and $\mathbf{r}$, shows that

$$\mathcal{R}_2\left[\frac{\partial|\Delta|}{\partial \mathbf{p}}\frac{\partial h^a}{\partial \mathbf{r}} - \frac{\partial|\Delta|}{\partial \mathbf{r}}\frac{\partial h^a}{\partial \mathbf{p}}\right] = \frac{\partial}{\partial \mathbf{r}}\left[\mathcal{R}_2\frac{\partial|\Delta|}{\partial \mathbf{p}}h^a\right] - \frac{\partial}{\partial \mathbf{p}}\left[\mathcal{R}_2\frac{\partial|\Delta|}{\partial \mathbf{r}}h^a\right] \tag{77}$$

Operating then with $-N(0)\langle \mathbf{p}/m \int dE \cdots\rangle$ on Eq. (58), and making use of the self-consistency equation for the order parameter, one obtains the contribution to Π which was called[7] $\Pi^{(2)}$, namely

$$\Pi_{ij}^{(2)} = -\frac{2}{5}\frac{N(0)\Delta_0^2}{m\lambda}(2\delta_{ij} - l_il_j) \tag{78}$$

*Note that, in our technique, we may use a relaxation time approximation without difficulties for the determination of C.

In fact, these considerations as well as the study of the conservation laws are easier if the equations for $\hat{g}^{\mathrm{K}}$ are used *before* introducing the distribution functions.

APPENDIX

In this appendix, we review the gradient expansion for the regular Green's function. The equation of motion, duscussed in Section 2, is given by

$$[\hat{Q}^{\mathrm{R,A}} \overset{*}{,} \hat{g}^{\mathrm{R,A}}] = 0 \tag{A1}$$

where, in the simple approximation for the linewidth (see Section 3.1)

$$\hat{Q}^{\mathrm{R,A}} = E^{\mathrm{R,A}}\hat{\tau}_3 - [\xi_p + \hat{U}] + i\hat{\Delta}; \qquad E^{\mathrm{R(A)}} = E(\pm)i\Gamma \tag{A2}$$

In addition, we have the normalization condition

$$\hat{g}^{\mathrm{R}} * \hat{g}^{\mathrm{R}} = \hat{g}^{\mathrm{A}} * \hat{g}^{\mathrm{A}} = \hat{1} \tag{A3}$$

In the following, we drop the superscripts R, A when no confusion can arise.

Working out (A1) for stationary situations, we find

$$\begin{aligned}(E - U^{\mathrm{a}})[\hat{\tau}_3, \hat{g}] + i[\hat{\Delta}, \hat{g}] + i\mathbf{v}\cdot\nabla\hat{g} \\ + \frac{i}{2}\left\{\frac{\partial(\hat{U} - i\hat{\Delta})}{\partial\mathbf{p}}, \frac{\partial\hat{g}}{\partial\mathbf{r}}\right\}_+ - \frac{i}{2}\left\{\frac{\partial(\hat{U} - i\hat{\Delta})}{\partial\mathbf{r}}, \frac{\partial\hat{g}}{\partial\mathbf{p}}\right\}_+ + \cdots\end{aligned} \tag{A4}$$

Then it is clear that the terms in the second line are $O(\nabla/p_{\mathrm{F}})$ smaller than those in the first line, except

$$-\frac{i}{2}\left\{\frac{\partial U^{\mathrm{s}}}{\partial\mathbf{r}}, \frac{\partial\hat{g}}{\partial\mathbf{p}}\right\}_+ \tag{A5}$$

which is the only one of its kind. Neglecting the small terms and similar ones in the normalization condition, we then have to consider

$$\begin{gathered}(E - U^{\mathrm{a}})[\hat{\tau}_3, \hat{g}] + i[\hat{\Delta}, \hat{g}] + i\mathscr{D}\hat{g} = 0 \\ \hat{g}\hat{g} = \hat{1}\end{gathered} \tag{A6}$$

with

$$\mathscr{D} = \mathbf{v}\cdot\nabla - (\nabla U^{\mathrm{s}})\cdot\frac{\partial}{\partial\mathbf{p}} \tag{A7}$$

The significance of the ∇U^s term was first pointed out by Mermin and Muzikar,[24] who used an external potential to describe states with a spatially varying density.

Starting from the zeroth-order solution

$$\hat{g}^{(0)} = \alpha\hat{\tau}_3 + \beta\hat{\tau}_\Theta$$
$$\alpha = \frac{-iE}{[(-iE)^2 + |\Delta|^2]^{1/2}}, \qquad \beta = \frac{|\Delta|}{-iE}\alpha \tag{A8}$$

it is now straightforward to iterate (A6) and obtain the Green's function in desired order. Also, we may then find the current $\mathbf{j}$ from

$$\begin{aligned}\mathbf{j} &= -\frac{N(0)}{4} \operatorname{Tr} \hat{\tau}_3 \left\langle \frac{\mathbf{p}}{m^*} \int dE\, (\hat{g}^{\mathrm{R}} - \hat{g}^{\mathrm{A}}) \operatorname{th} \frac{E}{2T} \right\rangle \\ &= -i\pi N(0) \operatorname{Tr} \hat{\tau}_3 \left\langle \frac{\mathbf{p}}{m^*} T \sum_n \hat{g}(\omega_n) \right\rangle\end{aligned} \tag{A9}$$

where $\hat{g}(\omega_n)$ is the Green's function defined for Matsubara frequencies $\omega_n[\hat{g}(\omega_n \to -iE(\pm)\Gamma) = \hat{g}^{\mathrm{R(A)}}(E)]$. Furthermore, the gradient free energy f_g can be calculated using[25]

$$f_g = \frac{N(0)}{2i} \left\langle \int dE\, (z^{\mathrm{R}} - z^{\mathrm{A}}) \operatorname{th} \frac{E}{2T} \right\rangle = 2\pi N(0) \left\langle T \sum_n z(\omega_n) \right\rangle \tag{A10}$$

where the function z is introduced via

$$\frac{1}{2} \operatorname{Tr} \hat{\tau}_3 \hat{g} = -i \frac{\partial z}{\partial E} = -\frac{\partial z(\omega_n)}{\partial \omega_n} \tag{A11}$$

In proceeding with the gradient expansion, we realize that it is convenient to expand the Green's function with respect to the matrices $\hat{\tau}_\Theta, \hat{\tau}_{\Theta+\pi/2}, \hat{\tau}_3$ [note that Θ is momentum dependent; see (26)]. We also find, from the normalization condition, that $\operatorname{Tr} \hat{g} = 0$; thus we introduce

$$\hat{g} = a\hat{\tau}_3 + b\hat{\tau}_\Theta + c\hat{\tau}_{\Theta+\pi/2} \tag{A12}$$

Keeping in mind that $\mathscr{D}\hat{\tau}_\Theta = (\mathscr{D}\Theta)\hat{\tau}_{\Theta+\pi/2}$ and $\mathscr{D}\hat{\tau}_{\Theta+\pi/2} = -(\mathscr{D}\Theta)\hat{\tau}_\Theta$, we obtain the following equations:

$$\begin{gathered} 2\tilde{E}b - 2i|\Delta|a + \mathscr{D}c = 0 \\ 2\tilde{E}c = \mathscr{D}b; \qquad 2i|\Delta|c = -\mathscr{D}a \end{gathered} \tag{A13}$$

with $\tilde{E}=E-U^{a}+\mathscr{D}\Theta/2$. It is clear that in an expansion with respect to the $\mathscr{D}$ *explicit* in (A13), a and b contain only even, and c contains only odd, powers in $\mathscr{D}$. This fact simplifies the algebra considerably.

Writing then

$$a=a^{(0)}+a^{(1)}+a^{(2)}+\cdots \tag{A14}$$

we find the following results:

$$a^{(1)}=-\frac{\partial\alpha}{\partial E}(U^{a}-\mathscr{D}\Theta/2) \tag{A15}$$

$$\begin{aligned} a^{(2)}=-i\frac{\partial}{\partial E}\bigg\{\frac{1}{8}\bigg[i\frac{\partial\alpha}{\partial E}(\mathscr{D}\Theta)^2 \\ +(-iE)^2[(-iE)^2+|\Delta|^2]^{-5/2}(\mathscr{D}|\Delta|)^2\bigg] \\ +\frac{i}{2}\frac{\partial\alpha}{\partial E}[(U^{a})^2-U^{a}\mathscr{D}\Theta]\bigg\} \end{aligned} \tag{A16}$$

with $i\partial\alpha/\partial E=|\Delta|^2[(-iE)^2+|\Delta|^2]^{-3/2}$; from $a^{(2)}$, we directly read off $z^{(2)}$.

In calculating the current from (A15) we use that $U^{a}=F_1 m\mathbf{p}\cdot\mathbf{j}/(3m^*\rho_0)$, $\rho_0=p_F^3/3\pi^2$, and

$$\begin{aligned} \frac{\partial\Theta}{\partial\mathbf{p}}&=\frac{\Delta_0^2(\hat{\mathbf{p}}\times\mathbf{l})}{|\Delta|^2 p_F} \\ \nabla\Theta&=\nabla\varphi-\frac{(\hat{\mathbf{p}}\cdot\mathbf{l})(\hat{\mathbf{p}}\times\mathbf{l})_m\nabla l_m}{(\hat{\mathbf{p}}\times\mathbf{l})^2} \end{aligned} \tag{A17}$$

Our result is identical to the one obtained in Ref. 24, or its generalization to finite temperatures[23,26]—this is not surprising, however, since all calculations are based on Gorkov's equations (we only have chosen to do the momentum integral at the beginning).

As was recently pointed out by Volovik and Mineev,[27] an expansion with respect to $\mathbf{v}\cdot\nabla\Theta$ might be problematic at very low temperatures. Consider again the expression for the current ($U^{s}=U^{a}=0$ for simplicity), but in the presence of a finite $\mathbf{v}_n$:

$$\mathbf{j}=-N(0)\left\langle\frac{\mathbf{p}}{m}\int dE\,\mathcal{N}_1(E+\mathbf{v}\cdot\nabla\Theta/2)\,\mathrm{th}\frac{E-\mathbf{p}\cdot\mathbf{v}_n}{2T}\right\rangle \tag{A18}$$

$\alpha(\tilde{E})$ was used to obtain this equation. Consequently, we find the normal density to be given by

$$(\rho_n)_{ij} = 3\rho_0 \left\langle \hat{p}_i \hat{p}_j \int dE \frac{\mathcal{N}_1(E + \mathbf{v} \cdot \nabla\Theta/2)}{4T \operatorname{ch}^2(E/2T)} \right\rangle \tag{A19}$$

This can be evaluated easily at zero temperature: $[4T \operatorname{ch}^2(E/2T)]^{-1} \to \delta(E)$, and we obtain

$$(\rho_n)_{ij} = 3\rho_0 \left\langle \hat{p}_i \hat{p}_j \frac{|\frac{1}{2}\mathbf{v} \cdot \nabla\Theta|}{[(\frac{1}{2}\mathbf{v} \cdot \nabla\Theta)^2 - |\Delta|^2]^{1/2}} \right\rangle \tag{A20}$$

this expression has to be integrated over angles such that the argument of the square root is positive (i.e., over momenta with direction almost parallel to $\mathbf{l}$; if $\hat{\mathbf{p}} \cdot \mathbf{l} = \cos\gamma$, then $\gamma^2, (\gamma - \pi)^2 \leq v_F|(\mathbf{l} \cdot \nabla)\mathbf{l}|/\Delta_0$). The result is[27,28]

$$(\rho_n)_{ij} = \rho_0 \frac{3v_F}{4\Delta_0} l_i l_j |(\mathbf{l} \cdot \nabla)\mathbf{l}| \tag{A21}$$

On the other hand, the order of magnitude is not unexpected. The same calculation, using instead a finite pair breaking Γ, gives

$$\frac{(\rho_n)_{ij}}{\rho_0} = \left\langle 3\hat{p}_i \hat{p}_j \frac{\Gamma}{|\Delta|} \right\rangle = \frac{3\pi\Gamma}{8\Delta_0} (\delta_{ij} + l_i l_j) \tag{A22}$$

provided $\Gamma \ll \Delta_0$. For scattering at the surface, we might insert $\Gamma \sim v_F/L$, where L is the size of the system: then (A21) and (A22) are of the same order.

The question of the nonanalytic contributions to the current was further investigated by Muzikar and Rainer.[28] They find that in general a careful solution of the quasiclassical equations near the singular points becomes necessary. As a result, the term given in Eq. (A21) receives a different numerical coefficient.

The calculation of the gradient free energy is straightforward, given $a^{(2)}$ in Eq. (A16). In particular, it is possible to eliminate the gradient of U^s with respect to the gradient of the density, and find the corresponding contributions to f_g (see Ref. 21). However, some uncertainty remains concerning the numerical coefficients of some of the terms.*

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*Garg[29] finds coefficients B_1 and B_2 with different numerical factors compared to Nagai.[21]

REFERENCES

1. G. Eilenberger, *Z. Phys.* **214**, 195 (1968).
2. A. I. Larkin and Yu. N. Ovchinnikov, *Zh. Eksp. Teor. Fiz.* **55**, 2262 (1968) [*Sov. Phys. JETP* **28**, 1200 (1969)].
3. G. M. Eliashberg, *Zh. Eksp. Teor. Fiz.* **61**, 1254 (1971) [*Sov. Phys. JETP* **34**, 668 (1972)].
4. A. Schmid, in *Nonequilibrium Superconductivity, Phonons, and Kapitza Boundaries* (Proceedings of the NATO Advanced Study Institute), K. E. Gray, ed. (Plenum, New York 1981), and references therein.
5. J. Bardeen, G. Rickayzen, and L. Tewordt, *Phys. Rev.* **113**, 982 (1959).
6. O. Betbeder-Matibet and P. Nozières, *Ann. Phys.* (*N.Y.*) **51**, 392 (1969).
7. U. Eckern and A. Schmid, *J. Low Temp. Phys.* **45**, 137 (1981).
8. D. Rainer and J. W. Serene, *Phys. Rev. B* **13**, 4745 (1976); J. W. Serene and D. Rainer, *Phys. Rev. B* **17**, 2901 (1978).
9. L. J. Buchholtz and D. Rainer, *Z. Phys. B* **35**, 151 (1979).
10. N. V. Kopnin, *Zh. Eksp. Teor. Fiz.* **74**, 1538 (1978) [*Sov. Phys. JETP* **47**, 804 (1978)].
11. N. Schopohl and L. Tewordt, *J. Low Temp. Phys.* **41**, 305 (1980); N. Schopohl, *J. Low Temp. Phys.* **41**, 409 (1980).
12. U. Eckern, *Ann. Phys.* (*N.Y.*) **133**, 390 (1981).
13. L. J. Buchholtz and G. Zwicknagl, *Phys. Rev. B* **23**, 5788 (1981).
14. L. V. Keldysh, *Zh. Eksp. Teor. Fiz.* **47**, 1515 (1964) [*Sov. Phys. JETP* **20**, 1018 (1965)].
15. A. I. Larkin and Yu. N. Ovchinnikov, *Zh. Eksp. Teor. Fiz.* **73**, 299 (1977) [*Sov. Phys. JETP* **46**, 155 (1977)].
16. P. Wölfle, *J. Low Temp. Phys.* **22**, 157 (1976).
17. A. Schmid and G. Schön, *J. Low Temp. Phys.* **20**, 207 (1975).
18. J. Beyer–Nielson, C. J. Pethick, J. Rammer, and H. Smith, *J. Low Temp. Phys.* **46**, 565 (1982).
19. U. Eckern and G. Schön, *J. Low Temp. Phys.* **32**, 821 (1978).
20. T.-L. Ho and N. D. Mermin, *Phys. Rev. B* **21**, 5190 (1980).
21. K. Nagai, *Progr. Theor. Phys.* **65**, 793 (1981).
22. P. Wölfle, *Rep. Progr. Phys.* **42**, 269 (1979).
23. K. Nagai, *J. Low Temp. Phys.* **38**, 677 (1980).
24. N. D. Mermin and P. Muzikar, *Phys. Rev. B* **21**, 980 (1980).
25. G. Eilenberger, *Z. Phys.* **182**, 427 (1965).
26. P. Muzikar, Ph.D. thesis, Cornell University (1980).
27. G. E. Volovik and V. P. Mineev, *Zh. Eksp. Teor. Fiz.* **81**, 989 (1981) [*Sov. Phys. JETP* **54**, 524 (1981)].
28. P. Muzikar and D. Rainer, preprint (August 1982).
29. A. Garg, private communication.