Quantum Dynamics of Tunneling between Superconductors: Microscopic Theory Rederived via an Oscillator Model

V. Ambegaokar* and U. Eckern
Institut für Theorie der Kondensierten Materie, Universität Karlsruhe, Federal Republic of Germany

Results of the microscopic theory of the dissipative quantum mechanics of a tunnel junction are obtained from a model in which non-linear functions of the phase variable are coupled to two independent sets of oscillators describing the environment. Both the statistical mechanics ("imaginary time effective action") and real time correlations are treated. The spectral densities of the oscillators are discussed in detail.

1. Introduction

The study of the dissipative quantum mechanics of the phase variable associated with weakly coupled superconductors raises technically interesting, and perhaps deep, theoretical questions, requires non-trivial experiments, and is enjoying a well deserved vogue. The relevant dynamical equations have been derived in two ways. One method, due to Caldeira and Leggett [1–3], very cleverly starts from a "classical" equation for the motion of the phase and adds, on well motivated physical grounds, an environment of oscillators which when coupled to the phase provides the (not necessarily weak) damping mechanism of the latter variable.

In the other method, pioneered by Schön and the present authors [4–7], the junction and the superconductors on either side are modelled microscopically, order parameters are introduced via a functional representation, and the dynamics of the phase variable is obtained by explicitly tracing out the electronic degrees of freedom. This method is not completely free of technical worries – in particular, the operator properties and the boundary conditions of the phase have to be put in by hand – but it has the merit of generating the effective potential for the phase and the steady and fluctuating currents across the junction at one and the same time. This second method yields results that are both similar and significantly different from those obtained from the first. Notably, it contains as a special case – in the limit of small quantum fluctuations – the full Werthamer calculation of junction dynamics [8], including the infamous "$\cos\phi$" dissipative current (see [9] for a review).

Although it is true that in many present experiments circuitry external to the junction is the most important dissipative mechanism, and that this is best modelled by the phenomenological method, it would seem useful to keep in mind for delicate tests of the theory that the "washboard" potential, $U(\phi) = -(\hbar/2e)\,[I_J \cos\phi + I_{\mathrm{ext}} \cdot \phi]$, is not the only way in which pairing correlations enter the problem. [In the formula, ϕ is the phase, and I_J and I_{ext} are the Josephson and external currents, respectively.] Unfortunately, the microscopic derivation from first principles seeems to require mathematical language so specialized as to discourage casual reading. We have therefore thought it useful in this paper to expand on the observation ([7], Sect. 4) that the results of that method can, after the fact, be obtained from a model in which the phase variable is coupled in a special way to two independent sets of oscillators. This rather cursory hint seems not to have been widely appreciated, and its generality has been questioned (see [10], Sect. 2).

In outline, the program of this article is as follows. In the next section, we summarize the Caldeira-Leggett method in a manner sufficiently general to permit the easy working out of our oscillator model. In Sect. 3, we use this model to obtain the statistical operator. In Sect. 4, we discuss the equivalent of the Feynman-Vernon theory [11, 12], i.e. the real time

* Permanent address: Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, NY 14853, USA

dynamics. The final section contains remarks about the physical consequences of the results.

2. Preliminaries

Caldeira and Leggett have carefully described their work in a long paper [3]. In brief, their method proceeds by imagining a quantum object ("particle") moving in a given potential, designing an environment that provides the required dissipation in the classical limit without changing the potential, and working out the quantum mechanics of the object plus environment under the assumption that the latter is characterized by a temperature but is otherwise unobserved. A key idea is that the environment is very large, so that any one of its many degrees of freedom is only weakly perturbed. Without loss of generality, the environment can then be thought of as a infinite set of oscillators. Thus, Caldeira and Leggett consider a Hamiltonian of the form

$$H=H_0+H_{\text{int}}+H_{\text{osc}} \tag{1}$$

where H_0 depends only on the (momentum and position coordinates of the) object, H_{osc} describes the environment, and

$$H_{\text{int}}=\sum_\alpha F_\alpha(q)\,x_\alpha+\sum_\alpha \frac{F_\alpha^2(q)}{2m_\alpha\,\omega_\alpha^2} \tag{2}$$

is the coupling between object and environment, taken linear in $\{x_\alpha\}$ and as a function of the "position" coordinate of the object, q, only. The second term is the counter term which cancels a renormalization of the potential in H_0 (due to the first term in (2)). For this model, let us consider the path integral expression for the partition function, Z:

$$Z=\int \mathscr{D}q\int\mathscr{D}\{x_\alpha\}\exp(-S/\hbar) \tag{3}$$

where $S=S_0+S_{\text{int}}+S_{\text{osc}}$ is the imaginary time action of the model (1), and path integration with respect to $\{x_\alpha\}$ is under the restriction* $x_\alpha(\tau=0)=x_\alpha(\tau=\hbar\beta)$. As usual, $\beta=(kT)^{-1}$, where T is the temperature. For the present model, it is only a little exercise in Gaussian integrals to show that Z can be expressed in terms of an effective action involving only the coordinate of the object, namely

$$Z=\int\mathscr{D}q\exp(-S_{\text{eff}}/\hbar) \tag{4}$$

with $S_{\text{eff}}[q]=S_0[q]+S_1[q]$, where S_1 is a quadratic form in $F_\alpha(q(\tau))$. It is sufficient to specialize to the case of a "separable" interaction, namely

$$F_\alpha(q)=C_\alpha f(q) \tag{5}$$

such that $f(q)$ is independent of the environment. Then we obtain

$$S_1[q]=\tfrac{1}{2}\int_0^{\hbar\beta}\mathrm{d}\tau\int_0^{\hbar\beta}\mathrm{d}\tau'\,\alpha(\tau-\tau')[f(q(\tau))-f(q(\tau'))]^2 \tag{6}$$

where the kernel $\alpha(\tau-\tau')$ can be expressed by the spectral density of the environment, $J(\omega)$, defined by:

$$J(\omega)=\frac{\pi}{2}\sum_\alpha\frac{C_\alpha^2}{m_\alpha\,\omega_\alpha}[\delta(\omega-\omega_\alpha)-\delta(\omega+\omega_\alpha)] \tag{7}$$

through the relation

$$\alpha(\tau-\tau')=\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}J(\omega)[-b(-\omega)]\,\mathrm{e}^{-\omega|\tau-\tau'|}. \tag{8}$$

Here, $b(\omega)=[\exp(\hbar\beta\omega)-1]^{-1}$ denotes the Bose-Einstein distribution function. Equation (8) holds in the range $0\leqq|\tau-\tau'|\leqq\hbar\beta$; outside this interval, $\alpha(\tau)$ can be periodically continued. This concludes our brief summary of the Caldeira-Leggett method. The special case $f(q)=q$, for ohmic dissipation, i.e. $J(\omega)=\eta\,\omega$, has been discussed in great detail [1–3] (see also [10], and references therein).

3. Tunneling between Superconductors – Effective Action

In the oscillator model for the tunneling between superconductors, we start like Caldeira and Leggett from a Hamiltonian of the form given by (1), however, modified in two ways. First, we take H_0 to contain, besides the kinetic energy of the object, only the harmonic part of the potential, $U_h(q)$; in the case of a current biased tunnel junction $U_h(q)=-F\cdot q$, where $F=\hbar I_{\text{ext}}/2e$, and q is identified with the phase difference across the junction. On the other hand, for a superconducting ring interrupted by an oxide barrier or a weak link, we have to consider the quadratic potential due to the inductance of the ring, L, instead, i.e. $U_h(q)=a\cdot(q-q_{\text{ext}})^2$, where $a=\Phi_0^2/8L\pi^2$. Here, Φ_0 is the flux quantum, $q=2\pi\Phi/\Phi_0$, and $q_{\text{ext}}=2\pi\Phi_{\text{ext}}/\Phi_0$, where Φ and Φ_{ext} are the magnetic flux in the ring, and the external magnetic flux, respectively.

Secondly, the interaction of the object with the environment is chosen in the form

$$H_{\text{int}}=\sum_{m=1}^{2}f_m(q)\sum_\alpha C_\alpha^{(m)}\,x_\alpha^{(m)} \tag{9}$$

* The boundary conditions for $q(\tau)$ need not be specified for the moment, and will be discussed below

i.e. the object is coupled in two different ways, characterized by $f_1(q)$ and $f_2(q)$, to two independent sets of oscillators, $\{x_\alpha^{(1)}\}$ and $\{x_\alpha^{(2)}\}$. Since in contrast to (2) there is no counterterm in (9), the periodic potential as well as the dissipative contributions to the junction dynamics, are treated on an equal footing. In particular, we choose

$$f_1(q)=\sin(q/2); \quad f_2(q)=\cos(q/2). \tag{10}$$

Why should we use (10) to describe the tunneling between superconductors? The final criterion is the agreement with the results of the microscopic theory. A hint towards such a choice can also be found in discussions of the classical equation of motion [13]. Since q is a phase variable, we expect trigonometric functions to be adequate. In addition, since the superconductors are described by complex order parameters, and the order parameter and its complex conjugate are independent variables, a coupling of two functions to two independent sets of oscillators is indicated. Finally, the fact that electrons tunnel singly leads to (10) as the consistent choice. [Higher order terms, like $\sin q$, $\sin 2q$, etc., are unimportant under most circumstances.]

Obviously, the trace over the environment is similar to the one of Sect. 2, and the partition function is written in the same form as in (4), with $S_{\text{eff}}=S_0+S_1$. As already mentioned, S_0 contains the harmonic contribution to the action only, and

$$S_1[q]=\sum_{m=1}^{2}\int_0^{\hbar\beta}\mathrm{d}\tau\int_0^{\hbar\beta}\mathrm{d}\tau'\,\alpha_m(\tau-\tau')f_m(q(\tau))f_m(q(\tau')). \tag{11}$$

Here $\alpha_m(\tau-\tau')$ is given by (8) with $J(\omega)$ replaced by $J_m(\omega)$, and $J_1(\omega)$ and $J_2(\omega)$ are the spectral densities of the two sets of oscillators. Of couse, (11) follows from (9) for any choice of $f_m(q)$.

Inserting (10), and also defining

$$\begin{pmatrix}\alpha(\tau-\tau')\\ \beta(\tau-\tau')\end{pmatrix}=\frac{1}{2}[\alpha_1(\tau-\tau')\pm\alpha_2(\tau-\tau')] \tag{12}$$

we arrive at the following result:

$$S_1[q]=\int_0^{\hbar\beta}\mathrm{d}\tau\int_0^{\hbar\beta}\mathrm{d}\tau'\left[\beta(\tau-\tau')\cos\frac{q(\tau)+q(\tau')}{2}-\alpha(\tau-\tau')\cos\frac{q(\tau)-q(\tau')}{2}\right]. \tag{13}$$

This agrees with the result of the microscopic theory ([7], (25)), provided $\alpha(\tau)$ and $\beta(\tau)$, or equivalently $J_1(\omega)$ and $J_2(\omega)$, are chosen as follows:

$$\begin{pmatrix}J_1(\omega)\\ J_2(\omega)\end{pmatrix}=\frac{\hbar}{e}[I_n(\omega)\pm I_c(\omega)]. \tag{14}$$

Here $I_n(\omega)$ and $I_c(\omega)$ are given, for example, in [7], (41). From previous definitions, it follows that*

$$\begin{pmatrix}\alpha(\tau)\\ \beta(\tau)\end{pmatrix}=\frac{\hbar}{e}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\begin{pmatrix}I_n(\omega)\\ I_c(\omega)\end{pmatrix}[-b(-\omega)]\,\mathrm{e}^{-\omega|\tau|}. \tag{15}$$

Note that $\alpha(\tau)$, $\beta(\tau)$ here are $\hbar$ times the corresponding quantities in [7]. $I_{n(c)}(\omega)$ is related to the normal (super) current across the junction. More precisely, if the junction is driven at a constant voltage, V, the normal current and the quasiparticle-pair amplitude (the prefactor of the "$\cos\phi$" in the current) are given by $I_n(eV/\hbar)$ and $I_c(eV/\hbar)$, respectively.

Finally, to arrive at the most familiar form of the washboard potential, we introduce the quantity

$$\gamma(\tau-\tau')=\beta(\tau-\tau')+g\cdot\delta(\tau-\tau') \tag{16}$$

where g is chosen such that the time average of $\gamma(\tau)$ vanishes. This leads to the identification $g=\hbar I_J/2e$, where I_J, the Josephson (critical) current, is given by

$$I_J=-\frac{2e}{\hbar}\int_0^{\hbar\beta}\mathrm{d}\tau\,\beta(\tau)=-\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{\pi}\frac{I_c(\omega)}{\omega}. \tag{17}$$

Then we obtain $S_1=S_1^{(1)}+S_1^{(2)}+S_1^{(3)}$, with

$$S_1^{(1)}=-\frac{\hbar I_J}{2e}\int_0^{\hbar\beta}\mathrm{d}\tau\cos q(\tau), \tag{18}$$

$$S_1^{(2)}=\int_0^{\hbar\beta}\mathrm{d}\tau\int_0^{\hbar\beta}\mathrm{d}\tau'\,\gamma(\tau-\tau')\cos\frac{q(\tau)+q(\tau')}{2}, \tag{19}$$

$$S_1^{(3)}=\int_0^{\hbar\beta}\mathrm{d}\tau\int_0^{\hbar\beta}\mathrm{d}\tau'\,\alpha(\tau-\tau')\left[1-\cos\frac{q(\tau)-q(\tau')}{2}\right]. \tag{20}$$

An unimportant constant has been added to $S_1^{(3)}$. The interpretation of the three terms is the following: $S_1^{(1)}$ describes the supercurrent across the junction; $S_1^{(2)}$ and $S_1^{(3)}$ are related to the quasiparticle-pair interference term, and the normal current, respectively. The quasiparticle-pair interference term seems to be of minor importance in many cases, and is omitted in most considerations.

We have thus demonstrated that the tunneling between superconductors, to the extent that it is given microscopically by the standard tunnel Hamiltonian, can be described by coupling $\sin(q/2)$ and $\cos(q/2)$

* Some details on the functions $\alpha(\tau)$ and $\beta(\tau)$ are summarized in the appendix

to two independent sets of oscillators. For completeness, we add some results of the microscopic theory. The spectral densities of the environment are found to be given by [7]

$$J_{1(2)} = \frac{\hbar}{e^2 R_N} \int_{-\infty}^{\infty} \mathrm{d}E\, \mathscr{M}_{1(2)}(E_+, E_-)[f_0(E_-) - f_0(E_+)] \tag{21}$$

where R_N is the normal state resistance of the junction, $f_0(E)$ the Fermi function, and $E_\pm = E \pm \hbar\omega/2$. In addition,

$$\mathscr{M}_{1(2)}(E_+, E_-) = \mathscr{N}_1(E_+)\mathscr{N}_1(E_-) \underset{(-)}{+} \mathscr{R}_2(E_+)\mathscr{R}_2(E_-) \tag{22}$$

where $\mathscr{N}_1$ and $\mathscr{R}_2$ are related to the diagonal and off-diagonal components of the (momentum integrated) Green's functions of the two superconductors [14]. Specializing to the tunneling between two identical superconductors, and assuming a finite linewidth, Γ, one obtains

$$\mathscr{N}_1(E) = \mathrm{Re}\,(E + i\Gamma)/[(E + i\Gamma)^2 - \Delta)^2]^{1/2}, \tag{23}$$

$$\mathscr{R}_2(E) = \mathrm{Re}\,\Delta/[(E + i\Gamma)^2 - \Delta^2]^{1/2} \tag{24}$$

where Δ is the magnitude of the order parameter; the square root has to be taken such that $\mathrm{Im}(.)^{1/2} > 0$. Thus $\mathscr{N}_1(E)$ and $\mathscr{R}_2(E)$ are even and odd functions of E, respectively. In the limit $\Gamma \to 0$, $\mathscr{N}_1$ reduces to the BCS density of states, while $\mathscr{R}_2 \simeq \Delta\,\mathscr{N}_1/E$.

In analogy to the ohmic dissipation case, we write

$$J_{1,2}(\omega, T) = \eta_{1,2}(\omega, T) \cdot \omega \tag{25}$$

and note the following results:

(i) $T \to T_c$ or $\hbar\omega \gg kT_c$

$$\eta_1 = \eta_2 = \frac{\hbar^2}{e^2 R_N} \equiv \eta_N \tag{26}$$

(ii) $\omega \to 0$, $T \to 0$

$$\eta_1 = \eta_2 = \frac{\Gamma^2}{\Gamma^2 + \Delta^2} \cdot \eta_N \tag{27}$$

(iii) $\omega \to 0$, $\Gamma \to 0$

$$\eta_2/\eta_N = 2 f_0(\Delta). \tag{28}$$

Note that η_1 has a logarithmic divergence for ω, $\Gamma \to 0$. To complete the picture, we show the temperature and frequency dependence of η_1 and η_2, as well as of $\eta_\pm = (\eta_1 \pm \eta_2)/2$, in Figs. 1–4. The linewidth has been taken to be a constant, namely $\Gamma = 0.3\ kT_c$.

It may be important to note that $J_1(\omega)$ and $J_2(\omega)$ are proportional to the frequency dependent attentuation of so-called case II and case I external perturbations, for example, electromagnetic radiation and ultrasound, respectively.* This guarantees, from very general considerations, that $J_{1,2}(\omega)$ is odd in ω, and positive for $\omega > 0$.

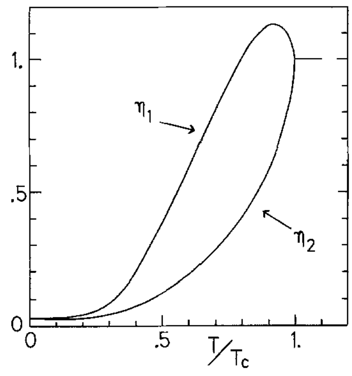

Fig. 1. η_1/η_N and η_2/η_N versus reduced temperature; $\omega = 0$, $\Gamma = 0.3\ kT_c$

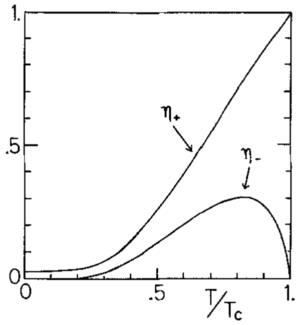

Fig. 2. $\eta_\pm/\eta_N = (\eta_1 \pm \eta_2)/2\eta_N$ versus reduced temperature; $\omega = 0$, $\Gamma = 0.3\ kT_c$

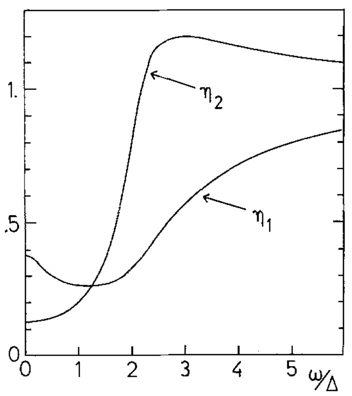

Fig. 3. η_1/η_N and η_2/η_N versus frequency; $T = 0.5\ T_c$, $\Gamma = 0.3\ kT_c$

* These quantities have been discussed in detail in the literature. See, for example, Tinkham's book [15], where further references can be found

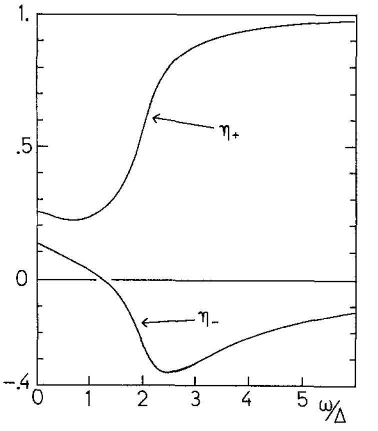

Fig. 4. η_+/η_N and η_-/η_N versus frequency; $T=0.5\ T_c$, $\Gamma=0.3\ kT_c$

4. Real Time Dynamics – Feynman-Vernon Theory

In the preceding section, we have demonstrated the complete equivalence of the results obtained from the oscillator model with the results from the microscopic theory for the statistical operator. Therefore, it is clear that the real time dynamics can be derived by well-known analytic continuation procedures. Nevertheless, we believe it to be instructive also to consider directly the Feynman-Vernon theory for the model we started with in Sect. 3. The quantity under consideration is the density matrix at time $t>0$, traced over the environment, $\rho(q_1, q_2, t)$, under the assumption that the density matrix of the total system at, say, time $t=0$ factorizes into the density matrix of the object, $\rho_0(q_1, q_2)$, times the equilibrium density matrix of the environment. Then one has the relation [11, 12, 2]

$$\rho(q_1, q_2, t)=\int d\bar{q}_1\, d\bar{q}_2\, J(q_1, q_2, t; \bar{q}_1, \bar{q}_2)\, \rho_0(\bar{q}_1, \bar{q}_2) \tag{29}$$

where

$$J=\int \mathscr{D}q_1\, \mathscr{D}q_2\, e^{i\mathscr{A}_0[q_1, q_2]/\hbar}\cdot \mathscr{F}[q_1, q_2]. \tag{30}$$

Here, $\mathscr{A}_0[q_1, q_2]=S_0[q_1]-S_0[q_2]$, S_0 is the usual (real time) action corresponding to H_0, and $\mathscr{F}[q_1, q_2]$ is called the influence functional. As before, since the interaction between object and environment is linear in the environment coordinate, and the environment is modelled hy harmonic oscillators, the influence functional can be computed without difficulty [11, 12]. The result is written as follows:

$$\mathscr{F}[q_1, q_2]=\exp \frac{i}{\hbar} \mathscr{A}_1[q_1, q_2] \tag{31}$$

where

$$\begin{pmatrix}\mathrm{Re}\,\mathscr{A}_1\\ \mathrm{Im}\,\mathscr{A}_1\end{pmatrix}=2\sum_{m=1}^{2}\int_0^t dt_1 \int_0^{t_1} dt_2 [f_m(q_1(t_1))-f_m(q_2(t_1))] \cdot\begin{pmatrix}-\alpha_m^I(t_1-t_2)\\ +\alpha_m^R(t_1-t_2)\end{pmatrix}[f_m(q_1(t_2))\pm f_m(q_2(t_2))]. \tag{32}$$

In this expression, α_m^R and α_m^I are the real and imaginary part, respectively, of

$$\alpha_m^{>}(t)=\alpha_m^R(t)+i\alpha_m^I(t)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi} J_m(\omega)[-b(-\omega)]\, e^{-i\omega t} \tag{33}$$

where $J_1(\omega)$ and $J_2(\omega)$ have been introduced above. We define, analogously to (12), the quantities $\alpha_I=[\alpha_1^I+\alpha_2^I]/2$, and similarly α_R, and β_I and β_R. It is also convenient to introduce center of mass and relative variables, according to

$$x=\tfrac{1}{2}(q_1+q_2), \qquad y=q_1-q_2. \tag{34}$$

Inserting these definitions into (32), it follows that:

$$\begin{aligned}\mathrm{Re}\,\mathscr{A}_1[x, y] &=8\int_0^t dt_1 \int_0^t dt_2\, \Theta(t_1-t_2) \sin\frac{y(t_1)}{4} \cos\frac{y(t_2)}{4}\\ &\cdot\Big[\alpha_I(t_1-t_2)\sin\frac{x(t_1)-x(t_2)}{2}\\ &-\beta_I(t_1-t_2)\sin\frac{x(t_1)+x(t_2)}{2}\Big]\end{aligned} \tag{35}$$

where $\Theta(t_1-t_2)$ is the step function, and

$$\begin{aligned}\mathrm{Im}\,\mathscr{A}_1[x, y]&=4\int_0^t dt_1 \int_0^t dt_2 \sin\frac{y(t_1)}{4}\sin\frac{y(t_2)}{4}\\ &\cdot\Big[\alpha_R(t_1-t_2)\cos\frac{x(t_1)-x(t_2)}{2}\\ &+\beta_R(t_1-t_2)\cos\frac{x(t_1)+x(t_2)}{2}\Big].\end{aligned} \tag{36}$$

These expressions are in agreement with the microscopic theory ([7], (45)). Note also that

$$\mathscr{A}_0[x, y]=\int_0^t dt[-m\ddot{x}+F]\cdot y \tag{37}$$

where the "mass" is given by $m=\hbar^2 C/4e^2$, with C the capacitance of the junction, and F the external "force" $\hbar I_{\mathrm{ext}}/2e$ (see above). As we have previously remarked [4, 5, 7], (35) and (36) agree with the results of [2] if we formally replace the trigonometric function by their small argument expansion, and take the limit $|\Delta| \to 0$ in the spectral densities.

In order to develop some physical intuition for the above the results, we proceed to discuss the quasiclassical limit (see [16]), or,more precisely, the quasiclassical equation of motion. As is well known, it is not possible to apply a least action principle directly to $\mathscr{A} = \mathscr{A}_0 + \mathscr{A}_1$, since this quantity has an imaginary part, and one proceeds by introducing *two* Gaussian fields, $\xi_1(t)$ and $\xi_2(t)$, such that

$$\exp(-\operatorname{Im} \mathscr{A}_1/\hbar) = \langle \exp \mathrm{i} \mathscr{A}_\xi/\hbar \rangle_{\xi_1, \xi_2} \tag{38}$$

where $\langle\rangle_{\xi_1,\xi_2}$ denotes the average with respect to ξ_1, ξ_2. Then the quasiclassical equation of motion is obtained from the condition

$$\left. \frac{\delta \tilde{\mathscr{A}}[x, y, \xi_1, \xi_2]}{\delta y} \right|_{y=0} = 0 \tag{39}$$

where $\tilde{\mathscr{A}} = \mathscr{A}_0 + \operatorname{Re} \mathscr{A}_1 + \mathscr{A}_\xi$. The result is a stochastic equation of motion, which is called the quantum Langevin equation, since the noise includes quantum fluctuations as well. From the condition (38), one is lead to the following choice:

$$\mathscr{A}_\xi = 4 \int_0^t \mathrm{d}t' \left[\xi_1(t') \cos \frac{x(t')}{2} + \xi_2(t') \sin \frac{x(t')}{2} \right] \sin \frac{y(t')}{4} \tag{40}$$

which involves the coupling of $\sin(x/2)$ and $\cos(x/2)$ to two independent random fields, analogous to the interaction between object and environment as discussed above. To satisfy (38), ξ_1 and ξ_2 have to be chosen independent of each other, with average equal to zero, and correlations given by

$$\langle \xi_{1(2)}(t)\, \xi_{1(2)}(t') \rangle = \hbar [\alpha_R(t-t') \underset{(-)}{+} \beta_R(t-t')]/2. \tag{41}$$

Applying (39), we obtain the following quasiclassical equation of motion:

$$m\ddot{x} - 2 \int_{-\infty}^{t} \mathrm{d}t' \left[\alpha_I(t-t') \sin \frac{x(t)-x(t')}{2} - \beta_I(t-t') \sin \frac{x(t)+x(t')}{2} \right] = F + F_\xi \tag{42}$$

where the stochastic force, F_ξ, is a nonlinear function of the coordinate, and given by

$$F_\xi = \xi_1(t) \cos \frac{x}{2} + \xi_2(t) \sin \frac{x}{2}. \tag{43}$$

Note that the initial time has been shifted to $(-\infty)$ in (42); on the left of this equation, the quasiparticle and the supercurrent in their general non-local form are apparent. An interesting aspect is the form in which the noise enters. As it turns out, the non-linear dependence on x is the characteristic feature of shot noise, i.e. related to the fact that charge is transported is units of the elementary charge across a junction, and not continuously as in the presence of a shunt resistor. More detailed investigations have been done for the tunneling between normal metals [17, 18]; however, the difference between shot noise and (ordinary) Johnson-Nyquist noise seems to be small in many cases*.

5. Discussion and Conclusion

In this article, we have demonstrated in detail that the results of the microscopic theory of the tunneling between superconductors, can be re-derived from a model in which the phase variable is coupled in a special way, employing certain trigonometric functions of the phase, to two independent sets of oscillators. The spectral densities of these oscillators, which we denote by $J_1(\omega)$ and $J_2(\omega)$, have been characterized in detail for the tunneling between two superconductors, which are weakly coupled through a thin oxide barrier, i.e. a situation which can be modelled microscopically by the standard tunneling Hamiltonian. In general, J_1 and J_2 are strongly temperature and frequency dependent, because of the gap in the single particle excitation spectrum; they are also sensitive to whether a finite level broadening or other pairbreaking mechanisms, such as paramagnetic impurities, are present. Specializing to the case of a finite level broadening, we have given illustrations of the temperature and the frequency dependence in Figs. 1 and 3. Most notable is the fact that $J_1(\omega)$ and $J_2(\omega)$, for the tunneling between identical superconductors, are proportional to the attentuation of case II and case I external perturbations, i.e. perturbations which are odd and even under time reversal, respectively. [These are characterized by different coherence factors.] In addition, J_1 and J_2 are closely related to $I_n(\omega)$ and $I_c(\omega)$, where $I_n(\omega)$ is the frequency dependent normal current, and $I_c(\omega)$ is related to the quasiparticle-pair interference term, and the supercurrent across the junction.

The result for the (imaginary time) effective action is given in Eq. (13); in fact, we expect that such a form is also applicable for tunneling between superconductors which are weakly coupled, for example, through a weak link. For such a situation, the characteristic functions $\alpha(\tau)$ and $\beta(\tau)$ still have to be investi-

* A detailed discussion can be found in the review of Schön [19]

gated (see, however, [20]). If, however, quasiparticle non-equilibrium populations play an important role, as in the case of weak links at high temperatures [21], we believe it to be more appropriate to work directly with the real-time representation, e.g. the Keldysh technique [22, 23].

For the remainder of the discussion, let us summarize the result for the effective action, in the approximation which ignores the quasiparticle-pair interference term, i.e. with $\beta(\tau) \to -g \cdot \delta(\tau)$ in (13):

$$S = \int_0^{\hbar\beta} d\tau \left[\frac{m}{2}\dot{q}^2 + U(q)\right] + 2\int_0^{\hbar\beta} d\tau \int_0^{\hbar\beta} d\tau' \,\alpha(\tau-\tau') \sin^2 \frac{q(\tau)-q(\tau')}{4}. \tag{44}$$

For a current-biased junction, the potential is given by $U(q) = -g\cos q - F \cdot q$. Even at this level of approximation, our result significantly differs from the one obtained by Caldeira and Leggett [1–3]. In particular, the kernel $\alpha(\tau-\tau')$ is related to the normal current, which is strongly temperature and frequency dependent as discussed above. More generally, we think that Eq. (44) can be applied to describe the quantum dynamics of a *realistic* junction, provided we consider $I_n(\omega)$ to be an input to the theory, which has to be determined *experimentally*. Depending on the junction, it then may be sufficient to characterize the behavior of $I_n(\omega)$ by a (temperature independent) sub-gap conductance, i.e. $I_n(\omega) \simeq \hbar\omega/eR_{\mathrm{eff}}$, provided frequencies and voltages small compared to the gap play the only role.

The appearance of the trigonometric function in the dissipative part of the action is worth re-emphasizing. This is related to the fact that in our model, dissipation is due to *electron tunneling*, in contrast to what Caldeira and Leggett had in mind, namely dissipation due to an *external shunt resistor*. [In the latter case, one finds a result which follows from (44) by the replacement $\sin^2 x \to x^2$, $x \equiv (q(\tau)-q(\tau'))/4$.] In the classical limit, the two cases differ only in the way in which noise enters the equation of motion, i.e. the noise being shot-noise or Johnson-Nyquist noise, respectively. Thus, in principle, we may extend the argument of Caldeira and Leggett who also start from the classical equation of motion: Given a certain tunnel junction, and given the *experimental* observation that its properties can be described, in the classical limit, by the standard RSJ model, with the tilted washboard potential, and with ohmic dissipation. Then, in addition, one has to characterize the noise in order to decide which model should be applied in the quantum regime. We suspect this to be difficult in general except, of course, in cases in which a shunt resistor has been explicitly attached to the junction.

Fortunately, for a discussion of macroscopic quantum tunneling, the difference between electron tunneling and a shunt resistor is unimportant, because when the external current is close to the critical current, only small ($\ll 2\pi$) excursions of the phase need to be considered. As another interesting example, the difference between the two types of dissipation is also found to be unimportant for the problem of quantum coherence in the double-well potential [24], which may be realizable in a SQUID with an external bias of half a flux quantum.*

Physically, the difference between dissipation due to electron tunneling, and due to a shunt resistor, becomes most transparent if one works with the variable conjugate to the phase, namely the charge (difference) across the junction [19]. Clearly, in the case of electron tunneling, the charge changes in discrete units ($\pm e$ for single electron, and $\pm 2e$ for pair tunneling, respectively), while the charge transfer across a shunt resistor in continuous. In fact, the allowed charge states (discrete or continuous), are intimately related to which boundary conditions are chosen in evaluating the path integral with respect to the phase [19]: If the phase is considered an extended coordinate, which applies to a current biased junction, to a SQUID, and to situations in which a shunt resistor is present (see also [25]), then the charge is continuous. On the other hand, in an isolated junction in which dissipation is due to electron tunneling, one has to restrict the phase to the interval $0 \ldots 4\pi$, and the charge is an integer times the elementary charge.**

A more detailed discussion of the consequence of the discreteness of the charge states is beyond the scope of the present article (see Refs. 26 and 27, and also Ref. 19). In brief, for a junction coupled very weakly to the external circuitry (see also [28]), it seems possible to introduce the concept of a (continuous) external charge, which is the analog of a Bloch wave vector, and to calculate the corresponding energy bands. As an important result [27], the energy bands and thus the response of the system ("Bloch oscillations"), is strongly modified by the presence of single electron tunneling. Of course, these ideas have to be tested experimentally. Finally, we wish to mention that electron tunneling may be relevant [29, 30] for an understanding of recent experiments [31] on granular superconducting films.

* Chakravarty [24], in fact, considered the action given in (44). Note, however, that the "critical" resistance is slightly different for the two cases

** Formally, in the path integral with respect to the phase, the restriction to a finite interval is taken into account by introducing the concept of a winding number

One of us (V.A.) is grateful to the Alexander von Humboldt Stiftung for supporting this collaboration through a Senior U.S. Scientist Award. He also thanks Professor Albert Schmid for the hospitality of his institute, and for many useful discussions. This work is also supported in part by the National Science Foundation under Grant DMR 84-17555.

Appendix

The functions $\alpha(\tau)$ and $\beta(\tau)$, defined above (Eq. (12)), which are characteristic for the tunneling between superconductors, can be expressed in terms of the momentum integrated Green's functions of the superconductors. For simplicity, we consider homogeneous equilibrium situations, and $\Gamma \to 0$, and define*

$$g_3(\tau) = T \sum_n e^{-i\omega_n \tau} \frac{\omega_n}{(\omega_n^2 + \Delta^2)^{1/2}}, \tag{A1}$$

$$g_1(\tau) = T \sum_n e^{-i\omega_n \tau} \frac{\Delta}{(\omega_n^2 + \Delta^2)^{1/2}} \tag{A2}$$

where $\omega_n = (2n+1)\pi T$ is the Matsubara (Fermi) frequency. [Note that the momentum integrated ("quasiclassical") matrix Green's function is given by $\hat{g} = g_3 \hat{\tau}_3 + g_1 \hat{\tau}_1$, where $\hat{\tau}_3$ and $\hat{\tau}_1$ are the Pauli matrices.] In general, in these equations, one has to replace Δ, the magnitude of the order parameter, by $\Delta_L (\Delta_R)$ for the "left" ("right") superconductor. Then the following result has been found [7]:

$$\alpha(\tau) = \frac{\pi}{2e^2 R_N} g_3(\tau) g_3(-\tau), \tag{A3}$$

$$\beta(\tau) = \frac{\pi}{2e^2 R_N} g_1(\tau) g_1(-\tau). \tag{A4}$$

In the following, we restrict ourselves to two identical superconductors. We introduce, for convenience, the functions $\bar{\alpha}(\tau)$ and $\bar{\beta}(\tau)$, whose Fourier transforms are dimensionless, by the relation

$$\alpha(\tau) = -\bar{\alpha}(\tau) \cdot \Delta / 2e^2 R_N, \tag{A5}$$

$$\beta(\tau) = -\bar{\beta}(\tau) \cdot \Delta / 2e^2 R_N. \tag{A6}$$

Since $\bar{\alpha}(\omega_m = 0)$ is irrelevant, it will be subtracted in the following. Some results are easily derived from the expressions given above:

(i) $\omega_m \gg \Delta$

$$\bar{\alpha}(\omega_m) \sim |\omega_m| / \Delta. \tag{A7}$$

Thus $\bar{\alpha}(\omega_m)$ approaches the familiar (linear) normal state result for high frequencies. Note that $\omega_m = 2m\pi T$ denotes a Bose frequency.

(ii) $\omega_m = 0$

$$\bar{\beta}(\omega_m = 0) = (\pi/2) \cdot \tanh\left(\frac{\Delta}{2T}\right). \tag{A8}$$

This quantity (see Eq. (17)) determines the temperature dependence of the critical current (see also [32]).

(iii) $\omega_m \ll \Delta$

$$\bar{\alpha}(\omega_m) \simeq \frac{3\pi}{32} \left(\frac{\omega_m}{\Delta}\right)^2, \tag{A9}$$

$$\bar{\beta}(\omega_m) - \bar{\beta}(\omega_m = 0) \simeq -\frac{\pi}{32} \left(\frac{\omega_m}{\Delta}\right)^2. \tag{A10}$$

In particular, for $T=0$, we conclude from these equations that the tunneling between ideal superconductors is dissipationless and, because of the quadratic dependence on frequency, the capacitance renormalization [7] can be recovered easily.

(iv) $T=0$

In this limit, it is straightforward to confirm that

$$\bar{\alpha}(\omega) = \frac{2\Delta}{\pi} \int_0^\infty d\tau [1 - \cos \omega\tau] \cdot [K_1(\Delta\tau)]^2 \tag{A11}$$

and

$$\bar{\beta}(\omega) = \frac{2\Delta}{\pi} \int_0^\infty d\tau \cdot \cos \omega\tau \cdot [K_0(\Delta\tau)]^2 \tag{A12}$$

where K_0 and K_1 are the modified Bessel functions. From these expressions, we obtain the following result:

$$\bar{\alpha}(\omega) = \frac{\omega(1+z)}{2\Delta z} \cdot E\left(\frac{2z^{1/2}}{1+z}\right) - \frac{\pi}{2} \tag{A13}$$

and

$$\bar{\beta}(\omega) = \frac{2\Delta z}{\omega} \cdot K(z) \tag{A14}$$

where $z^2 = \omega^2 / (\omega^2 + 4\Delta^2)$, and $K(\cdot)$ and $E(\cdot)$ denote the complete elliptic integral of the first and second kind, respectively.

Finally, the results of a numerical determination of $\bar{\alpha}(\omega_m)$ and $\bar{\beta}(\omega_m)$, for $\Delta / 2\pi T = 10$, are shown in Fig. 5, where these quantities are given for the first ~ 40 Matsubara frequencies. The curve labelled "*N*" is the normal state result. It is apparent that $\bar{\alpha}$ and $\bar{\beta}$ approach their asymptotic values only for very high frequencies. For completeness, $\bar{\alpha}$ and $\bar{\beta}$ are shown, in the zero temperature limit, in Fig. 6.

* In the appendix, we choose $\hbar = k = 1$

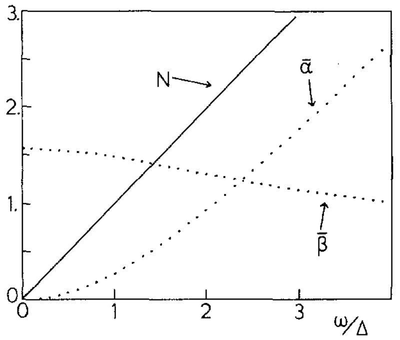

Fig. 5. $\bar{\alpha}(\omega_m)$ and $\bar{\beta}(\omega_m)$ versus ω_m/Δ for $\Delta/T=20\pi$. The line labelled "N" is the normal state limit of $\bar{\alpha}$, $\bar{\alpha}_N=|\omega_m|/\Delta$

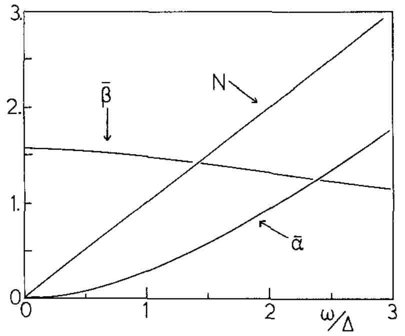

Fig. 6. Same as Fig. 5, in the limit $T\to 0$

References

1. Caldeira, A.O., Leggett, A.J.: Phys. Rev. Lett. **46**, 211 (1981)
2. Caldeira, A.O., Leggett, A.J.: Physica **121** A, 587 (1983)
3. Caldeira, A.O., Leggett, A.J.: Ann. Phys. (NY) **149**, 374 (1983); **153**, 445 (E) (1984)
4. Ambegaokar, V., Eckern, U., Schön, G.: Phys. Rev. Lett. **48**, 1745 (1982)
5. Ambegaokar, V.: Proceedings of the NATO Advanced Study Institute on Percolation, Localization, and Superconductivity, Les Arcs, 1983. Goldman, A.M., Wolf, S.A. (eds.), p. 43. New York, London: Plenum Press 1984
6. Larkin, A.I., Ovchinnikov, Yu.N.: Phys. Rev. B **28**, 6281 (1983)
7. Eckern, U., Schön, G., Ambegaokar, V.: Phys. Rev. B **30**, 6419 (1984)
8. Werthamer, N.R.: Phys. Rev. **147**, 255 (1966)
9. Langenberg, D.N.: Rev. Phys. Appl. **9**, 35 (1974)
10. Leggett, A.J., Chakravarty, S., Dorsey, A.T., Fisher, M.P.A., Garg, A., Zwerger, W.: Rev. Mod. Phys. **59**, 1 (1987)
11. Feynman, R.P., Vernon, F.L., Jr.: Ann. Phys. (NY) **24**, 118 (1963)
12. Feynman, R.P., Hibbs, A.R.: Quantum Mechanics and Path Integrals. Chap. 12. New York: McGraw-Hill 1965
13. Brunk, G.: J. Non-Equilib. Thermodyn. **5**, 339 (1980);
Brunk, G., Lübbig, H.: J. Low Temp. Phys. **42**, 179 (1981)
14. Schmid, A., Schön, G.: J. Low Temp. Phys. **20**, 207 (1975)
15. Tinkham, M.: Introduction to Superconductivity. New York: McGraw-Hill 1975
16. Schmid, A.: J. Low Temp. Phys. **49**, 609 (1982)
17. Ben-Jacob, E., Mottola, E., Schön, G.: Phys. Rev. Lett. **51**, 2064 (1983)
18. Schön, G.: Phys. Rev. B **32**, 4469 (1985)
19. Schön, G.: In: SQUID '85 – Superconducting Quantum Interference Devices and their Applications. Hahlbohm, H.D., Lübbig, H. (eds.), p. 251. Berlin, New York: de Gruyter 1985
20. Zaikin, A.D., Panyukov, S.V.: Zh. Eksp. Teor. Fiz. **89**, 242 (1985) [Sov. Phys. JETP **62**, 137 (1985)]
21. Schmid, A., Schön, G., Tinkham, M.: Phys. Rev. B **21**, 5076 (1980)
22. Keldysh, L.V.: Zh. Eksp. Teor. Fiz. **47**, 1515 (1964) [Sov. Phys. JETP **20**, 1018 (1965)]
23. Larkin, A.I., Ovchinnikov, Yu.N.: Zh. Eksp. Teor. Fiz. **87**, 1842 (1984) [Sov. Phys. JETP **60**, 1060 (1984)]
24. Chakravarty, S.: Phys. Rev. Lett. **49**, 681 (1982)
25. Zwerger, W., Dorsey, A.T., Fisher, M.P.A.: Phys. Rev. B **34**, 6518 (1986)
26. Likharev, K.K., Zorin, A.B.: J. Low Temp. Phys. **59**, 347 (1985)
27. Guinea, F., Schön, G.: Europhys. Lett. **1**, 585 (1986); Preprint
28. Büttiker, M.: Preprint
29. Chakravarty, S., Kivelson, S., Zimanyi, G.T., Halperin, B.I.: Phys. Rev. B**35**, 7256 (1987)
30. Kampf, A., Schön, G.: Preprint
31. Orr, B.G., Jaeger, H.M., Goldman, A.M.: Phys. Rev. B **32**, 7586 (RC) (1985);
Orr, B.G., Jaeger, H.M., Goldman, A.M., Kuper, C.G.: Phys. Rev. Lett. **56**, 378 (1986);
Jaeger, H.M., Haviland, D.B., Goldman, A.M., Orr, B.G.: Phys. Rev. B **34**, 4920 (RC) (1986)
32. Ambegaokar, V., Baratoff, A.: Phys. Rev. Lett. **10**, 486 (1963); **11**, 104 (E) (1963)

V. Ambegaokar
Laboratory of Atomic and
Solid State Physics
Cornell University
Ithaka, NY 14853
USA

U. Eckern
Institut für Theorie der
Kondensierten Materie
Universität Karlsruhe
Physikhochhaus
Postfach 6980
D-7500 Karlsruhe
Federal Republic of Germany