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The Decay of a Metastable State in a Multidimensional Configuration Space

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1. Introduction

Metastable states with large, though finite, lifetimes occur rather frequently in nature. As a model of a metastable state, one takes a particle in a potential well which is separated from states of lower energy by a potential barrier. Such a metastable state may decay by thermal activation and in many systems as they occur in condensed matter physics, this is presumably the most frequent decay mode. However, as one succeeds in reaching lower and lower temperatures, there appears also the possibility that quantum tunneling may play a role.

It was Leggett (1978, 1980) who drew attention to the quantum decay mode in large systems. In particular, he had argued that it may serve as a demonstration of peculiarities connected with macroscopic quantum phenomena. Consider, e.g., a system which features a metastable state in connection with a collective coordinate of macroscopic dimensions. In general, this collective coordinate will be coupled more or less strongly to the remaining degrees of freedom of the system; thus, a situation emerges which may be best described by a macroscopic object coupled to a dissipative environment.

Subsequently, Caldeira and Leggett (1981, 1983, 1984) have modelled such a system as follows: The macroscopic object is represented by a particle in a potential with a metastable minimum as shown in fig. 1, and the environment is taken to be a set of harmonic oscillators (see section 3.6) which are coupled to the object.*

Let us consider first the case of zero temperature, $T = 0$. In this case, and for their model, Caldeira and Leggett have calculated the quantum decay rate Γ using a field-theoretic technique (instanton technique), which was invented by Langer (1967) and independently somewhat later by Lifshitz and Kagan (1972). The instanton technique was subsequently discussed by Coleman (1979, 1985) in his most illuminating and elegant paper. Note that this technique is of quasi-classical accuracy, in the sense that Planck's constant $\hbar$ is considered there to be a small quantity.

The instanton technique has sometimes met with skepticism. The seemingly heuristic treatment of those quantum fluctuation modes that produce zero, or even negative contributions to the action, has been questioned. More

* Frequently, we will call this ensemble a dissipative object.

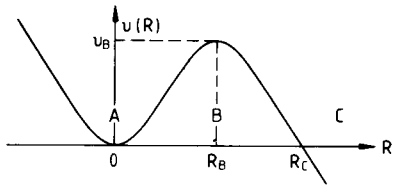

Fig. 1. Potential energy of an object featuring a metastable minimum. Regions A, B, and C represent, respectively, the metastable well, the barrier, and the outside region, where the object is found after the decay.

specifically, it has been questioned whether the instantons of the Caldeira–Leggett model, which feature a long-range interaction, allow the dilute-gas approximation.*

Such questions motivated an investigation (Schmid 1986) by one of the authors. As an alternative to the instanton technique, it has been proposed there to calculate the wave function of the decaying mode in quasiclassical approximation, which is an approximation where the phase of the wave function is expanded in powers of $\hbar$. The quasiclassical approximation has been used frequently in one-dimensional tunneling problems, but difficulties arise in systems of several degrees of freedom. In the last case, it seems that Kapur and Peierls (1937) have been the first ones to investigate the properties of quantum decay in systems with several degrees of freedom. Later, specific calculations of wave functions in quasiclassical accuracy have been carried out by Banks et al. (1973), Banks and Bender (1973) and de Vega et al. (1978).

However, the investigations of Banks et al. (1973) and Banks and Bender (1973) on quantum decay were concerned mostly with systems of a rather specific symmetry, whereas de Vega et al. (1978) were interested predominantly in tunneling between degenerate minima of the potential energy. Since the wave function of a decaying state exhibits some unique features, it seemed to be worthwhile to carry out a quasiclassical calculation (Schmid 1986) for a general model in a multidimensional configuration space.

From a technical point of view, differences in method seem to separate the decay at zero temperature and that at finite temperatures. At $T = 0$, e.g., Coleman (1979, 1985) has been able to demonstrate explicitly that the probability of the object to remain in the metastable well decreases exponentially with time, i.e., is $\propto \exp(-\Gamma t)$ (dilute gas of instantons). For finite temperatures $T > 0$, on the other hand, the calculation of the decay rate Γ proceeds by a more or less direct identification of Γ with an imaginary part of the free energy, as was first suggested by Langer (1967) and later also by Lifshitz and Kagan (1974).

* We cannot disperse completely this suspicion. See the last paragraph of section 3.6.

This identification has also been emphasized by Affleck in his lucid and transparent article (Affleck 1981).

A generalization of Affleck's work to a dissipative object was formulated by Waxman and Leggett (1985), but their work has been left uncompleted. Later, this program was taken up by Ludviksson (1989) in his thesis, where it has been carried out successfully.

This chapter consists mostly of a review of the papers by Schmid (1986) and Ludviksson (1989), with an emphasis on a comprehensive and unified presentation. The formal procedure is based on the quasiclassical approximation either of wave functions (Landau and Lifshitz 1975) or of Feynman's path integrals (Feynman and Hibbs 1965, Schulman 1981).

At this point, we wish to acknowledge the work of Miller (1975) on semiclassical methods in chemical physics. We should also not fail to recall the pioneering work of Kramers (1940) on the classical decay problem 50 years ago, which started and motivated many other investigations on the decay problem. In this context, we mention recent review articles on the decay problem by Hänggi et al. (1990) and Mel'nikov (1991).

The chapter is organized as follows. Section 2 is meant to introduce the fundamental concepts, which are illustrated explicitly for a one-dimensional system. The multidimensional quantum decay for a wave function, i.e., at zero temperature, will be discussed in section 3. Finally, section 4 is devoted to the discussion of the statistical matrix and to the multidimensional decay at finite temperatures, and a conclusion is given in section 5.

A detailed discussion of a one-dimensional system is given in sections 2.1–2.6. In section 2.1, the concept of a quasistationary state is introduced. This concept is the basis of all the following calculations; in fact, it seems difficult to work out a general theory without this assumption. The matching of the quasiclassical wave function is discussed in section 2.2; and in section 2.3, it is shown how this matching procedure can be reduced to comparatively simple operations. The decay at finite temperatures as put forward by Affleck (1981) is reviewed in section 2.4. The statistical matrix appropriate to a decaying state is introduced in section 2.5 and it is also shown how it can be obtained in quasiclassical approximation from a Feynman path integral. This approximation is based on complex extremal paths and section 2.6 provides a demonstration that complex orbits of arbitrary multiplicity are required for a unified theory. It is also shown that the generalization to multiple orbits leads to quantitative, if not qualitative, changes as compared to the Affleck ansatz.

Sections 3.1–3.6 discuss the multidimensional quasiclassical wave function. The direct approach of section 2.3 for calculating wave functions decaying from a metastable well is applied in section 3.1 to quantum decay for a N-dimensional system. The quasiclassical approximation there leads to nonlinear partial differential equations of first order which can be solved by the method of characteristics. This method, however, requires the evaluation of a classical equation of

motion in N dimensions; it is shown in section 3.2 how this rather complicated equation can be solved in the small-fluctuation approximation provided that the escape path is known. In section 3.3, the analytical properties of the principal wave are investigated near the caustic (which is the generalization of a turning point to N dimensions) and it is shown how the transmitted wave can be obtained by analytical continuation. There, an unpublished work of Landauer (1950) has been most helpful to the authors in clarifying important concepts. The decay rate is then calculated in section 3.4; and complete agreement is found with the result of the instanton technique. Section 3.5 discusses the construction of the reflected wave and the validity of the applied procedure in N dimensions. For illustration, the Caldeira–Leggett model is presented in section 3.6 as an example of quantum decay in a multidimensional system.

The theory of the multidimensional statistical matrix is explained through sections 4.1–4.3. As pointed out in section 4.1, the quasiclassical approximation is built on a set of extremal and complex paths which incorporates periodic orbits which can be seen as a generalization of the escape path referred to above. In section 4.2, we include the Gaussian fluctuations about these paths and calculate their contribution to the decay current in the outside region; thus, we obtain a generalization to N dimensions of Afflecks result for the decay rate. However, the summation over multiple orbits is a nontrivial problem. This will be demonstrated in section 4.3 and we will find qualitative changes as compared to the standard theory. For the Caldeira–Leggett model a comparison with the results of the standard theory is carried out at the end.

A discussion in section 4.4 summarizes the main results.

There are various appendices which are meant to remove detailed calculations from the main body. In Appendix A we calculate the quantum-mechanical transmission of a smooth barrier in the quasiclassical approximation; the result confirms the idea of summing up the contribution of multiple orbits. Appendix B contains some results for the one-dimensional decay. In Appendix C, the nontrivial problem of Gaussian fluctuations about complex paths is investigated. In Appendix D we extend the work of Larkin and Ovchinnikov (1983 a, b, 1984) on the pre-factor for a heavily damped object to multiple orbits. Finally, conceptual and computational details on the decay rate of a heavily damped object are contained in Appendix E.

2. *Decay in one dimension*

2.1. Quasistationary state

The concept of a quasistationary state is essential to all considerations which will follow in this chapter. It is only for the sake of an easy demonstration that we discuss this concept here for the one-dimensional decay. Let us recall the

standard model of a metastable state, which consists of a particle ("object") in a potential well (region A in fig. 1), separated from the decayed state (region C) by a potential barrier (region B). Characteristic parameters of this model are the mass m of the object; the height v_B; and width R_C of the barrier. Of interest is also the oscillation frequency ω_A at A and ω_B at B for the inverted potential, respectively. Later (section 3.6), we will also specify the interaction of the object with its environment but for the moment we will do without it.

In extreme cases, the decay may occur entirely either by quantum tunneling (quantum decay; QD) or by thermal activation (classical decay; CD). If the condition for a quasistationary state applies, the theory leads to an exponential decay law

$$P(t) = \exp(-\Gamma t), \tag{2.1}$$

where $P(t)$ is the probability for the object to remain in the well. Furthermore, it is commonly found that the decay rate Γ is of the form

$$\Gamma = \mathscr{A} \exp(-\mathscr{B}). \tag{2.2}$$

For orientation, we give some simple relations

$$\mathscr{B} \propto v_B/\hbar\omega_B \quad \text{(QD)},$$

$$\mathscr{B} = v_B/kT \quad \text{(CD)},$$

$$\mathscr{A} \propto \omega_A. \tag{2.3}$$

Within the limits of this simple picture, the condition for a quasistationary decaying state* is given by

$$\Gamma \ll \omega_A. \tag{2.4}$$

Physically, this condition means that the metastable state may be considered to be internally in equilibrium. Using eqs. (2.2) and (2.3), we may rewrite this condition as

$$\mathscr{B} \gg 1. \tag{2.5}$$

2.2. *Matching of the quasiclassical wave function*

At zero temperature, the concept of a quasistationary decaying state allows us to look for a wave function $\psi(R)$ which is a solution of the time-independent Schrödinger** equation

$$\mathscr{H}\psi = E\psi, \tag{2.6}$$

*See, e.g., Landau and Lifschitz (1975) for a discussion of quasistationary states and the Breit–Wigner formula.

**The time-dependent Schrödinger equation has been solved repeatedly in the past for simple models of decay. See Ludviksson (1987) for a recent publication.

where the Hamiltonian

$$\mathscr{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial R^2} + v(R) \tag{2.7}$$

includes a potential energy $v(R)$ of a type shown in fig. 1. The ansatz above also requires that the wave function has to satisfy the following conditions:

(i) Far to the right, ψ must carry an outgoing probability current.
(ii) ψ must vanish far to the left.
(iii) Near the metastable minimum, ψ must resemble the ground state wave function of a harmonic oscillator.

For such a wave function, the Schrödinger equation (2.6) can be solved only for a complex energy $E = \frac{1}{2}\hbar(\omega_A - i\Gamma)$; it follows that in a time representation, the probability decays exponentially in time at a rate Γ, namely $|\psi(t)|^2 \propto \exp(-\Gamma t)$. Obviously,

$$\Gamma = -2\,\mathrm{Im}\,E/\hbar \tag{2.8}$$

has to be identified with the decay rate.

In a region near the origin, $R = 0$, as well as in a region near to the classical turning point $R = R_C$, the general solution of the differential equation (2.6) may be expressed by known functions. This follows from the fact that there $v(R)$ can be approximated by a harmonic and a linear potential, respectively. Accordingly,

$$v(R) = \begin{cases} \frac{1}{2}m\omega_A^2 R^2, & |R| \ll R_B, \\ -F_C(R - R_C), & |R - R_C| \ll R_C, \end{cases} \tag{2.9}$$

where we have also indicated the regions of validity of the approximation for a smooth potential. The simplest of such a smooth potential is

$$v(R) = \tfrac{1}{2}m\omega_0^2 R^2(1 - R/R_C), \tag{2.10a}$$

where

$$\omega_A = \omega_B = \omega_0, \qquad R_B = \tfrac{2}{3}R_C,$$
$$v_B = \tfrac{1}{6}m\omega_0^2 R_B^2, \qquad F_C = \tfrac{1}{2}m\omega_0 R_C. \tag{2.10b}$$

Concerning the region near the classical turning point R_C, we recognize that a finite energy in eq. (2.6) means only a redefinition of R_C. Therefore, we may put $E = 0$ (i.e., $\mathrm{Re}\,E = \mathrm{Im}\,E = 0$). Then the solution which satisfies boundary condition (i) is of the form

$$\psi(R) = \mathscr{N}\left[\mathrm{Bi}\left(\frac{R_C - R}{a_C}\right) + i\mathrm{Ai}\left(\frac{R_C - R}{a_C}\right)\right], \quad a_C = (\hbar^2/2mF_C)^{1/3}, \tag{2.11a}$$

where Ai and Bi are Airy functions (Abramowitz and Stegun 1968). To the right of R_C and for $a_C \ll |R - R_C|$, the asymptotic form of $\psi(R)$ is

$$\tilde{\psi}(R) = \mathcal{N}\pi^{-1/2}\left(\frac{R - R_C}{a_C}\right)^{-1/4} \exp\left\{i\left[\frac{2}{3}\left(\frac{R - R_C}{a_C}\right)^{3/2} + \frac{\pi}{4}\right]\right\}, \quad (2.11b)$$

which represents clearly an outgoing wave. The asymptotic form of $\psi(R)$ to the left of R_C may be written as $\psi_0(R) + \psi_1(R)$, which is a sum of what we wish to call a principal and a reflected wave. Asymptotically, their form is*

$$\psi_0(R) = \mathcal{N}\pi^{-1/2}\left(\frac{R_C - R}{a_C}\right)^{-1/4} \exp\left[\frac{2}{3}\left(\frac{R_C - R}{a_C}\right)^{3/2}\right],$$

$$\psi_1(R) = \frac{i}{2}\mathcal{N}\pi^{-1/2}\left(\frac{R_C - R}{a_C}\right)^{-1/4} \exp\left[\frac{-2}{3}\left(\frac{R_C - R}{a_C}\right)^{3/2}\right]. \quad (2.11c)$$

Considering the differential equation (2.6) in the region near the origin, we find that a solution satisfying the boundary conditions (ii) and (iii) is

$$\psi(R) = D_\nu(-R/a_0), \quad a_0 = (\hbar/2m\omega_0)^{1/2}, \quad \nu = -i\Gamma/2\omega_0, \quad (2.12a)$$

where D_ν is a parabolic cylinder function (Abramowitz and Stegun (1981)). The asymptotic form of eq. (2.12a) to the right of the origin and for $a_0 \ll R$ is again of the form $\psi_0(R) + \psi_1(R)$, where

$$\psi_0(R) = |R/a_0|^\nu \exp(-R^2/4a_0^2),$$

$$\psi_1(R) = -\nu(2\pi)^{1/2}(a_0/R)^{1+\nu} \exp(R^2/4a_0^2). \quad (2.12b)$$

The asymptotic form of $\psi(R)$ to the left of the origin is identical to $\psi_0(R)$ in eq. (2.12b). [Note that $a_C/R_C \equiv (2^{1/2}a_0/R_C)^{4/3}$.]

Outside the regions $R \sim 0$ and $R \sim R_C$ considered above, the wave function can be calculated in the quasiclassical approximation.** In fact, we will find that the condition for quasistationarity $\mathcal{B} \gg 1$ of eq. (2.5) guarantees the validity of this approximation.

In this approximation, the asymptotic waves of eq. (2.11) and of eq. (2.12) can be connected as follows. We consider first the principal wave $\psi_0(R)$. In the classically inaccessible region we expect it to be a real quantity; therefore, we put

$$\psi_0(R) = \exp\left\{-\frac{1}{\hbar}[\mathcal{W}(R) + \hbar\mathcal{W}^{(1)}(R) + O(\hbar^2)]\right\}, \quad (2.13)$$

* Since ψ_1 is purely imaginary, it is legitimate to retain this subdominant contribution next to the real dominant part ψ_0. See also the interesting discussion in Coleman (1979, 1985).

** An excellent discussion of the quasiclassical approximation for one-dimensional systems can be found in Landau and Lifschitz (1975). Frequently, this approximation is called the WKB or WKBJ approximation.

where the expression $\mathscr{W}(R)$ – the first term in the square bracket – may be called the (abbreviated) Euclidean action. We insert the ansatz (2.13) into the Schrödinger equation (2.6) and then equate separately the terms of equal power in $\hbar$, observing that $E = \hbar\omega_0(\frac{1}{2} + v)$ is of order $\hbar$. Thus, we obtain
*in leading order the eikonal equation**

$$\frac{1}{2m}\left(\frac{\partial \mathscr{W}}{\partial R}\right)^2 - v(R) = 0, \tag{2.14}$$

in next order the transport equation

$$\frac{1}{m}\frac{\partial \mathscr{W}^{(1)}}{\partial R}\frac{\partial \mathscr{W}}{\partial R} - \frac{1}{2m}\frac{\partial^2 \mathscr{W}}{\partial R^2} + \omega_0(\tfrac{1}{2} + v) = 0. \tag{2.15}$$

The solution of the eikonal equation (2.14) is

$$\mathscr{W}(R) = \int_0^R \mathrm{d}R'[2mv(R')]^{1/2}, \tag{2.16a}$$

where the sign of the square root has been chosen such that ψ_0 decreases exponentially with increasing R. For convenience, an arbitrary integration constant has been fixed by the condition $\mathscr{W}(0) = 0$. In the case where $v(R)$ is given by eq. (2.10), we obtain

$$\frac{1}{\hbar}\mathscr{W}(R) = \frac{1}{2}\mathscr{B}\left[1 - \left(1 + \frac{3R}{2R_C}\right)\left(1 - \frac{R}{R_C}\right)^{3/2}\right] \tag{2.16b}$$

where

$$\mathscr{B} = \frac{8}{15}\frac{m\omega_0^2 R_C^2}{\hbar\omega_0} = \frac{36}{5}\frac{v_B}{\hbar\omega_0}. \tag{2.17}$$

Limiting expressions for $\mathscr{W}(R)$ are as follows:

$$\frac{1}{\hbar}\mathscr{W}(R) = \begin{cases} \dfrac{R^2}{4a_0^2}, & |R| \ll R_C, \\ \dfrac{1}{2}\mathscr{B} - \dfrac{2}{3}\left(\dfrac{R_C - R}{a_C}\right)^{3/2}, & |R_C - R| \ll R_C, \end{cases} \tag{2.16c}$$

where a_C and a_0 are given by eqs. (2.11) and (2.12a).

The transport equation (2.15) can be solved by a straightforward integration; choosing a suitable integration constant, one obtains

$$\mathscr{W}^{(1)}(R) = \frac{1}{2}\ln\left[\frac{R}{4R_C}\left(1 - \frac{R}{R_C}\right)^{1/2}\right] - \left(\frac{1}{2} + v\right)\ln\frac{1 - \sqrt{1 - R/R_C}}{1 + \sqrt{1 - R/R_C}} - v\ln\frac{4R_C}{a_0}. \tag{2.18a}$$

* This terminology has been used, e.g., in Banks and Bender (1973) and Banks et al. (1973).

Limiting expressions for $\mathscr{W}^{(1)}(R)$ are as follows:

$$\exp[-\mathscr{W}^{(1)}(R)] = \begin{cases} \left(\dfrac{R}{a_0}\right)^{\nu}, & |R| \ll R_C, \\ \left(\dfrac{4R_C}{a_0}\right)^{\nu}\left(\dfrac{R_C - R}{16R_C}\right)^{1/4}, & |R_C - R| \ll R_C. \end{cases} \tag{2.18b}$$

Comparing the limiting expressions, we recognize that $\psi_0 = \exp[-(\mathscr{W} + \hbar\mathscr{W}^{(1)})/\hbar]$ agrees with ψ_0 of eq. (2.12) and also that it agrees with ψ_0 of eq. (2.11c) if we choose

$$\mathscr{N} = 2\pi^{1/2}\left(\frac{R_C}{a_C}\right)^{1/4}\left(\frac{4R_C}{a_0}\right)^{\nu}\exp(\tfrac{1}{2}\mathscr{B}). \tag{2.19}$$

The quasiclassical ansatz for the reflected wave $\psi_1(R)$ is chosen similar to eq. (2.13) with an Euclidean action $\mathscr{W}_1(R) + \hbar\mathscr{W}_1^{(1)}(R)$. Clearly, the structure of the eikonal equation and of the transport equation remains unchanged. The different boundary condition, however, requires the opposite sign of the square root; hence,

$$\mathscr{W}_1(R) = -\mathscr{W}(R). \tag{2.20}$$

As a consequence, $\mathscr{W}_1^{(1)}(R)$ is similar to eq. (2.18), with, however, a change in the sign of $(\frac{1}{2} + \nu)$; therefore,

$$\mathscr{W}_1^{(1)}(R) = \frac{1}{2}\ln\frac{R}{R_C}\left(1 - \frac{R}{R_C}\right)^{1/2} + \left(\frac{1}{2} + \nu\right)\ln\frac{1 - \sqrt{1 - R/R_C}}{1 + \sqrt{1 + R/R_C}}$$

$$+ (1 + \nu)\ln\frac{4R_C}{a_0} - \ln(2\pi)^{1/2}(-\nu), \tag{2.21}$$

where a suitable change in the integration constant has been included. Thus, the limiting expressions are

$$\exp[-\mathscr{W}_1^{(1)}(R)]$$

$$= \begin{cases} -(2\pi)^{1/2}\nu\left(\dfrac{a_0}{R}\right)^{1+\nu}, & |R| \ll R_C, \\ -(2\pi)^{1/2}\nu\left(\dfrac{a_0}{4R_C}\right)^{1+\nu}\left(\dfrac{R_C - R}{16R_C}\right)^{-1/4}, & |R_C - R| \ll R_C, \end{cases} \tag{2.22}$$

Comparing again the limiting expressions, we recognize that $\psi_1 = \exp[-(\mathscr{W}_1 + \hbar\mathscr{W}_1^{(1)})]/\hbar$ agrees with ψ_1 of eq. (2.12b). We also recognize that it agrees with ψ_1 of eq. (2.11c) if we choose

$$\nu = -\mathrm{i}\frac{\Gamma}{2\omega_0} = -\mathrm{i}\frac{1}{2(2\pi)^{1/2}}\left(\frac{4R_C}{a_0}\right)^{1+2\nu}\exp(-\mathscr{B}), \tag{2.23}$$

and $\mathscr{N}$ according to eq. (2.19).

If $|v| \ll 1$, the v-dependence on the right-hand side can be neglected. Then, eq. (2.23) can be rewritten in the standard form [eq. (2.2)], $\Gamma = \mathscr{A} \exp(-\mathscr{B})$, where $\mathscr{B}$ is given by eq. (2.17) and where

$$\mathscr{A} = 60^{1/2} \left(\frac{\mathscr{B}}{2\pi} \right)^{1/2} \omega_0. \tag{2.24}$$

This expression for the decay rate is similar to eq. (2.3) except for numerical constants and a factor $\mathscr{B}^{1/2}$ contributing to $\mathscr{A}$.

It is known (Landau and Lifschitz 1975) that the quasiclassical approximation relies on the inequality

$$\hbar \left| \frac{\partial^2 \mathscr{W}}{\partial R^2} \right| \ll \left| \frac{\partial \mathscr{W}}{\partial R} \right|^2. \tag{2.25}$$

In the present case, where $\partial \mathscr{W}/\partial R = \pm m\omega_0 R(1 - R/R_C)^{1/2}$, this inequality reduces either to $a_0 \ll R_C$ or to $a_C \ll |R - R_C|$. Therefore, we conclude that the present method of solving the Schrödinger equation by matching exact solutions for an approximate potential with approximate solutions for the exact potential requires the inequalities

$$a_0 \ll R_C, \qquad a_C \ll R_C. \tag{2.26}$$

Inserting the appropriate definitions, we find that inequalities (2.26) are satisfied provided that eq. (2.5), i.e., $\mathscr{B} \gg 1$, is satisfied. Therefore, the concept of a quasi-stationary state and the quasiclassical approximation are valid in the quasiclassical limit.

2.3. A simple approach to quantum decay

We have put the discussion of the one-dimensional quantum decay on a broad basis in order to demonstrate how so many intricate details conspire to bring forth the simplicity of the final result [eqs. (2.17) and (2.24)]. One might wonder whether there is a much simpler scheme leading to the same end. Indeed, there is an alternative method which ultimately relies on the fact that the principal wave ψ_0 can be found, to a sufficient degree of accuracy, without prior knowledge of Γ.

Considering the Schrödinger equation (2.6), we conclude that $\operatorname{Im} \psi^* [\mathscr{H} - E] \psi = 0$. Integrating this expression from $-\infty$ to R, and recalling eq. (2.8), we obtain

$$\Gamma = \left[\int_{-\infty}^{R} \mathrm{d}R' |\psi(R')|^2 \right]^{-1} J(R), \tag{2.27}$$

where the probability current

$$J(R) = (\hbar/m) \operatorname{Im} \psi^* \partial \psi / \partial R. \tag{2.28}$$

Later we will take $R \sim R_C$ for the current $J(R)$. On the other hand, the choice $R' \ll R_B$ in the denominator of eq. (2.27) is sufficient since it includes most of the normalization of a decaying state.

One might quite easily argue that the ratio (2.27) of probability current divided by the probability of the object to be found in the region of the well may a priori be considered as the definition of the decay rate and that no reference to eq. (2.8) is required. We will make use of this idea later.

Consider the limit $\Gamma = \nu = 0$, where eq. (2.12b) assumes the simple form $\psi_0(R) = \exp(-R^2/4a_0^2)$. Note also that in the same limit the quasiclassical approximation to the wave function as given implicitly by eqs. (2.16) and (2.18) agrees with $\psi(R)$ in the region $|R| \ll R_C$. Therefore, we conclude that the following statement is true:

The quasiclassical solution can be found directly from the eikonal equation (2.14) *and the transport equation* (2.15) *if we put* $\Gamma = \nu = 0$ *and if we extend these equations to the origin* $R = 0$, *supplementing them by the boundary condition*

$$\mathscr{W}(0) = \mathscr{W}^{(1)}(0) = 0, \qquad \mathscr{W}(R) \geqslant 0. \tag{2.29}$$

Thus, the quasiclassical method leads us directly from the harmonic oscillator wave function near the origin to the asymptotic form of the principle wave $\psi_0(R)$ near the classical turning point R_C. According to eqs. (2.16) and (2.18), we obtain for $R \approx R_C$,

$$\psi_0(R) = 2\left(\frac{R_C - R}{R_C}\right)^{-1/4} \exp\left[\frac{2}{3}\left(\frac{R_C - R}{a_C}\right)^{3/2} - \frac{\mathscr{B}}{2}\right]. \tag{2.30}$$

In principle, we could use the connection formula contained in eq. (2.11) in order to construct the outgoing wave $\tilde{\psi}(R)$. However, it has been found* that the breakdown of the quasiclassical approximation near the turning point R_C and the matching of the wave function can be avoided if one generalizes the eikonal equation and the transport equation to complex R. Then, it becomes possible to pass from one side of the turning point to the other side on a semicircle in the complex plane which remains at a sufficiently large distance from R_C.

In order to obtain a wave with outgoing probability current, we have to perform an analytical continuation via the upper R-half plane. Thus, we are led to substitute

$$(R_C - R) \to e^{-i\pi}(R - R_C) \tag{2.31}$$

* In a recent edition of Landau and Lifschitz (1975), the method of analytical continuation is attributed to Zwaan (1929). There are doubts whether it is valid in general. Presently, one might say that it depends on some analytical properties of the Airy functions. See also the discussion in section 3.8 of Bender and Orszag (1978).

in the principal wave of eq. (2.30), whence we obtain the outgoing, i.e., the transmitted wave

$$\tilde{\psi}(R) = 2\left(\frac{R - R_C}{R_C}\right)^{-1/4} \exp\left[\frac{2i}{3}\left(\frac{R - R_C}{a_C}\right)^{3/2} - \frac{\mathscr{B}}{2} + \frac{i\pi}{4}\right] \tag{2.32}$$

which agrees with eq. (2.11b) together with eq. (2.19).

Calculating the probability current (2.28) from $\tilde{\psi}(R)$, we need to differentiate only with respect to the argument of the exponential. Thus, we obtain $J(R) = 4\omega_0 R_C \exp(-\mathscr{B})$ which is independent of R. Concerning the normalization, we note that it is given by $\int dR|\psi_0|^2 = (2\pi)^{1/2} a_0$ by a high degree of accuracy. Inserting these expressions in eq. (2.27), we recover the same result for Γ as given in eqs. (2.2), (2.17), and (2.24).

As a final comment on the possibilities and ambiguities enclosed in the concept of analytical continuation, we remark that a substitution

$$(R_C - R) \to e^{-2i\pi}(R_C - R) \tag{2.33}$$

in the principal wave of eq. (2.30) does lead to the reflected wave $\psi_1(R)$ as given by eq. (2.11c) except for a factor of $\frac{1}{2}$. It is possible to implement this factor by the additional requirement of current conservation.

2.4. *Decay at finite temperatures*

In this section, we review the theory which has long roots but which Affleck (1981) has presented lucidly in his paper. Consider again the metastable potential as shown in the pictogram of fig. 2. In quasiclassical approximation, the energies E_n of the levels in the well are implicitly given by (Bohr–Sommerfeld rule)

$$W(E_n) = 2\pi\hbar(n + \tfrac{1}{2}), \tag{2.34}$$

where the abbreviated action W for a closed orbit in the well is calculated according to

$$W(E) = \oint P\, dR,$$

$$P(R) = \{2m[E - v(R)]\}^{1/2}. \tag{2.35}$$

In the simplest quasiclassical approximation, the transmission probability $D(E)$ through the barrier is given by

$$D(E) = \exp\left[-\frac{1}{\hbar}\mathscr{W}(E)\right], \tag{2.36}$$

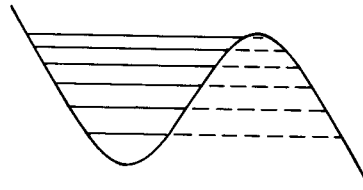

Fig. 2. Quasistationary levels in the well (full lines) and the tunneling distances (broken lines).

where $\mathscr{W}(E)$ is the abbreviated action for a closed orbit in the inverted potential $-v(R)$ of the barrier. Hence,

$$\mathscr{W}(E) = \oint \mathscr{P}\,\mathrm{d}R,$$

$$\mathscr{P}(R) = \{2m[v(R) - E]\}^{1/2}. \tag{2.37}$$

Recall that, classically, $\partial W/\partial E$ is the time the object needs to complete the closed orbit in the metastable well. Therefore,

$$\Gamma_n = \left(\frac{\partial W}{\partial E}\right)^{-1} D(E)\bigg|_{E=E_n} \tag{2.38}$$

is the rate by which the object leaves the nth level of the well.

Let us now assume that the levels are populated as being in thermal equilibrium, i.e., with probability $\propto \exp(-E_n/kT)$. Then, the average decay rate is given by

$$\bar{\Gamma} = Z_0^{-1} \sum_n \Gamma_n \exp(-\tau_1 E_n/\hbar), \tag{2.39}$$

where Z_0 is the normalization and where, with respect to further developments, we have introduced the Euclidean time

$$\tau_1 = \hbar/kT \tag{2.40}$$

as a measure of the inverse temperature.

We emphasize that the ansatz (2.39) may have only a restricted validity since it ignores changes in the population of the levels which result from a level-dependent decay rate. On the other hand, we wish to emphasize that such an ansatz is completely in accordance with the concept of a quasistationary state introduced in section 2.

Concerning the normalization

$$Z_0 = \sum_n \exp(-\tau_1 E_n/\hbar), \tag{2.41}$$

one should observe that, effectively, only the lowest levels contribute. There, the well may be approximated by a harmonic potential; hence ($\omega_A = \omega_B = \omega_0$),

$$Z_0 = (2\sinh \omega_0 \tau_1 /2)^{-1}. \tag{2.42}$$

We consider now the case where the level spacing is so small that we may replace the summation in eq. (2.39) by an energy integration. Since the level density is $(\partial W/\partial E)/2\pi\hbar$, we recognize a cancellation of a corresponding term in the expression (2.38) for Γ_n. Thus, we arrive at

$$\bar{\Gamma} = Z_0^{-1} \int \frac{dE}{2\pi\hbar} \exp\left(-\frac{1}{\hbar}[\mathcal{W}(E) + \tau_1 E] \right). \tag{2.43}$$

In the quasiclassical limit, it is possible to evaluate the integral (2.43) by steepest descent. Thus, we obtain for the average decay rate a relation of the form [eq. (2.2)], which is $\bar{\Gamma} = \mathcal{A} \exp(-\mathcal{B})$, where the prefactor is given by

$$\mathcal{A} = Z_0^{-1}[2\pi\hbar \partial^2 \mathcal{W}(E)/\partial E^2]^{-1/2} \tag{2.44}$$

and where the argument of the exponential (except for a factor $\hbar$),

$$\begin{aligned} \hbar\mathcal{B} &= \mathcal{S}(\tau_1) = \mathcal{W}(E) + \tau_1 E, \\ \tau_1 &= -\partial \mathcal{W}(E)/\partial E, \end{aligned} \tag{2.45}$$

is the Legendre transform of $\mathcal{W}$. Alternatively, we may write the prefactor as follows:

$$\mathcal{A} = Z_0^{-1/2}[-2\pi\hbar \partial \tau_1(E)/\partial E]^{-1/2} = Z_0^{-1}(2\pi\hbar)^{-1/2}[\partial^2 \mathcal{S}(\tau_1)/\partial^2 \tau_1]^{1/2}. \tag{2.46}$$

An important point in Affleck's paper has been his demonstration that $\bar{\Gamma}$ is related with the imaginary part of the free energy [compare eq. (2.39)]

$$\bar{\Gamma} = -\frac{2}{\hbar} \operatorname{Im} \mathcal{F}, \tag{2.47}$$

which means a generalization of eq. (2.8) to finite temperatures. In the first step, Affleck proposes to calculate the full partition function Z by a path integral*

$$Z = \int d(R_\tau) \exp\left(-\frac{1}{\hbar} \mathcal{S}([R_\tau]; \tau_1) \right), \tag{2.48}$$

* For a general introduction to path integrals, see Feynman and Hibbs (1965) and Schulman (1981).

where the Euclidean action is given by*

$$\mathscr{S}([R_\tau];\tau_1)=\int_0^{\tau_1} d\tau\left[\frac{m}{2}(\partial_\tau R_\tau)^2+v(R_\tau)\right], \tag{2.49}$$

and where the integration includes all closed paths $R_{\tau_1}=R_{\tau=0}$.

In quasiclassical approximation (see the discussion by Dashen et al. (1974)), the predominant contribution to the path integral is from extremal paths r_τ that obey the equation

$$m\ddot{r}_\tau-\frac{\partial v(r_\tau)}{\partial r_\tau}=0. \tag{2.50}$$

Specifically, the relevant extremal paths that contribute to Z are closed orbits that are completed in time τ_1. There is the trivial orbit $r_\tau=0$ (metastable minimum) which provides, together with its Gaussian fluctuations (periodic in time τ_1), the normalization Z_0 of eq. (2.42). The nontrivial orbit $r_\tau\neq 0$ produces, first of all, the exponential

$$\hbar\mathscr{B}=\mathscr{S}([r_\tau];\tau_1), \tag{2.51}$$

which can easily be shown to agree in its value with eq. (2.45).

Concerning the Gaussian fluctuations about r_τ, one should observe that they are periodic in time with period τ_1. Furthermore, one should note that there is a mode $\propto \dot{r}_\tau$ with eigenvalue $\lambda_1=0$ and also one mode with a negative eigenvalue λ_0. In contrast to the Minkowski case, the negative eigenvalue poses a delicate problem in the Euclidean case. However, one can argue (Caldeira and Leggett 1981, 1983, 1984) that for the present case, it contributes with a factor $\frac{1}{2}(\lambda_0)^{-1/2}=-\frac{1}{2}(\mathrm{i})|\lambda_0|^{-1/2}$. For the remaining factors we refer to Dashen et al. (1974); accordingly, the overall contribution of the nontrivial orbit is

$$Z_1=Z_0\frac{\mathrm{i}\tau_1}{2}\mathscr{A}\exp(-\mathscr{B}), \tag{2.52}$$

where for the present case

$$\mathscr{A}=\left(\frac{\mathscr{S}_0([r_\tau];\tau_1)}{2\pi\hbar}\right)^{1/2}\left|\frac{\det[-\partial_\tau^2+\omega_0^2]}{\det'[-\partial_\tau^2+v''(r_\tau)/m]}\right|_{\mathrm{PB}}^{1/2}. \tag{2.53}$$

In the above relation

$$\mathscr{S}_0([r_\tau];\tau_1)=m\int_0^{\tau_1} d\tau\,(\partial_\tau r_\tau)^2 \tag{2.54}$$

and det means a determinant in τ-space (det': zero eigenvalue omitted). The subscript PB means that the eigenvalues are calculated for periodic boundary

* For convenience, we mark at appropriate places the time dependence by a subscript, e.g., $R_\tau=R(\tau)$.

conditions. If one makes use of relations given in Dashen et al. (1974), one finds that the prefactor of eq. (2.53) agrees exactly with the one given by eq. (2.44). Since the exponent $\mathscr{B}$ in eq. (2.52) is clearly identical to that given by (2.45), this completes the justification of eq. (2.47).

2.5. *Statistical matrix in quasiclassical approximation*

Again, we assume that the system is in a quasistationary state corresponding to thermal equilibrium. This means that we should look for a statistical matrix $\rho(R_1, R; \tau_1)$ which has to satisfy the following conditions:

(i) Far to the right, ρ must supply an outgoing probability current.
(ii) ρ must vanish far to the left.
(iii) ρ should contain a large part which represents the object localized in the metastable well.

Let us represent the statistical matrix by the path integral (Feynman and Hibbs 1965):

$$\rho(R_1, R; \tau_1) = \int \mathrm{d}[R_\tau] \exp\left(-\frac{1}{\hbar}\mathscr{S}([R_\tau]; \tau_1)\right), \tag{2.55}$$

where $\mathscr{S}$ is given by eq. (2.49) and where the integration above includes all paths R_τ which start from R at $\tau = 0$ and lead to R_1 at $\tau = \tau_1$. In the quasiclassical approximation, the path integral is dominated by extremal paths r_τ which obey eq. (2.50). There are different classes of paths (see fig. 3) but the final choice has to be in accordance with conditions (i)–(iii). In the discussion that follows, we will realize that in fig. 3, the class (a) provides the large part which is important for the normalization* whereas class (b) contributes to the outgoing probability current.** At first sight, there seem to be two different paths of class (b) which differ for $R_1 = R$ in the sense of circulation. However, the requirement of an outgoing current eliminates one of the possibilities.

The outgoing current is calculated from the statistical matrix according to the prescription

$$J(R) = \frac{\hbar}{2mi}\left(\frac{\partial}{\partial R_1} - \frac{\partial}{\partial R}\right)\rho(R_1, R; \tau_1)\Big|_{R_1 = R}. \tag{2.56}$$

The average decay rate then is given by

$$\bar{\Gamma} = Z_0^{-1} J(R), \tag{2.57}$$

*Note that in the present case the Gaussian fluctuations about extremal paths have to be calculated for zero boundary conditions. Alternatively, class (a) is equivalent to the trivial orbit including Gaussian fluctuations for periodic boundary conditions.

**It is this selection of paths by which we disagree with the work of Waxman and Leggett (1985).

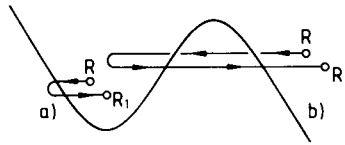

Fig. 3. Extremal path contributing to the statistical matrix $\rho(R_1, R; \tau_1)$ in quasiclassical approximation: (a) Contribution to the normalization. (b) Contribution to the outgoing probability current. At first sight, there seem to be two different paths (for $R_1 = R$, they differ in the sense of circulation). However, the requirement of an outgoing current eliminates one of the two possibilities.

where

$$Z_0 = \int_{-\infty}^{R} \mathrm{d}R' \rho(R', R'; \tau_1). \tag{2.58}$$

As already discussed following eq. (2.28), we take $R \gtrsim R_C$ in the expression for the current $J(R)$, and $R' \lesssim R_B$ for the normalization (see fig. 3).

In quasiclassical accuracy, the decay current can be calculated by taking the derivatives only with respect to the argument of the exponential, i.e., with respect to the action. Considering this action as a function of the coordinates, $\mathscr{S}([r_\tau]; \tau_1) = \mathscr{S}(R_1, R; \tau_1)$, we find that

$$\begin{aligned} J(R) &= -\frac{1}{2mi}\left[\frac{\partial \mathscr{S}}{\partial R_1} - \frac{\partial \mathscr{S}}{\partial R}\right]_{R_1 = R} \rho(R, R; \tau_1) \\ &= -\frac{1}{2mi}[\mathscr{P}_1 - (-\mathscr{P})]_{R_1 = R} \rho(R, R; \tau_1), \end{aligned} \tag{2.59}$$

where $\mathscr{P}_i = \mathscr{P}(R_i)$ is given by eq. (2.37) with R_i in the outside region (C). Clearly, the requirement of an outgoing probability current is met if we choose the sign of the square root such that $\mathrm{Im}\, \mathscr{P}_i < 0$. Observing that $m\dot{r}_{\tau_1} = m\dot{r}_0 = \mathscr{P}_1 = \mathscr{P}$, we may write

$$J(R) = |\dot{r}_0| \rho(R, R; \tau_1). \tag{2.60}$$

In retrospect, we realize that we need to calculate only extremal paths r_τ that are closed. For the one-dimensional case we are presently considering, these paths are also periodic since $\dot{r}_{\tau_1} = \dot{r}_0$. It is worth noting that it will also be a periodic path which will play a prominent role in multidimensional systems.

A further property of a periodic path can easily be demonstrated in one dimension and for the cubic potential [eq. (2.10)]. There, the extremal paths r_τ are found to be doubly periodic functions in the complex τ-plane; more precisely, they are found to be Weierstrass functions (Abramowitz and Stegun 1968). Specifically, the (real) period τ_1 fixes the energy as well as the second period which is purely imaginary (and which is the period of the periodic motion

in the classically accessible region). Now, if one allows also complex extremal paths $r_\tau^* \neq r_\tau$, where τ is real and $0 \leqslant \tau \leqslant \tau_1$, then one can see that for any τ_1, there exists a complex path where $r_0 = r_{\tau_1} = R$ is real and larger than R_C. In the complex R-plane such a path is completely characterized by its topology with respect to the branch points (classical turning points; see fig. 4). Also, one recognizes at once that any deformation of the path in the complex plane leaves the action $S([r_\tau]; \tau_1)$ and the final momentum $\mathscr{P}_i = -\mathrm{i}(2m[E - v(R_i)])^{1/2}$ unchanged.

In order to elucidate a basic feature of the complex path, let us compare it with a real path which will, however, require a complex time. Considering fig. 5, we recognize that the real paths $R_0 \to R_2$ and $R_3 \to R_1$ in the classical accessible region of fig. 5a run along the imaginary τ-axis in fig. 5b. Of importance is that at the classical turning points R_2 and R_3, the velocities vanish; and it is this very fact which allows us to introduce corners in the contour of the complex τ-plane.

We anticipate the fact that in multidimensional problems the turning points are replaced by caustics where the velocity does not vanish; in this case, only the concept of complex paths carries through without modifications.

Of importance is also the fact that complex paths cannot be avoided in the case $E > v_B$ (high temperatures); see fig. 4c for illustration. Concerning corrections due to Gaussian fluctuations about the extremal paths, useful relations can

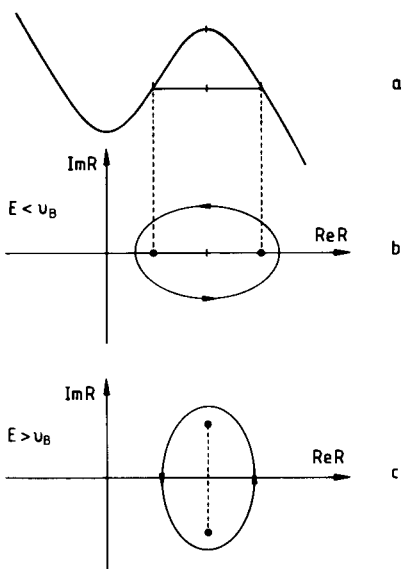

Fig. 4. Extremal orbits completed in time $\tau_1 = \hbar/kT$ under the barrier: (a) Potential energy and turning points; (b) complex extremal path which encloses the two branch points for $E < v_B$ ($T < T_B$); (c) the same for $E > v_B$ ($T > T_B$).

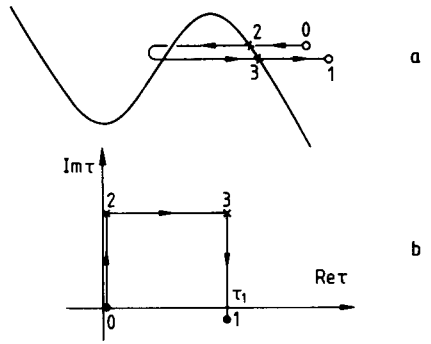

Fig. 5. (a) Real path under the barrier ($2 \to 3$) in time τ_1 and real paths in the outside region ($0 \to 2$, $3 \to 1$) for imaginary τ. (b) The corresponding contour in the complex τ-plane. (For $R_1 = R_0 \equiv R$, the endpoint of the contour is on the real axis.) Note the corners at the classical turning points.

be found in Dashen et al. (1974). There, they are derived for a real path but we will show in Appendix C that they are also valid for complex paths with appropriate interpretation. Accordingly,

$$\rho(R, R; \tau_1) = \sum_{\text{CP}} [2\pi\hbar\,|\dot{r}_0|^2(-\,\partial\tau_1/\partial E)]^{-1/2} \exp\left(-\frac{1}{\hbar}\mathscr{S}([r_\tau]; \tau_1)\right), \quad (2.61)$$

where the sum includes all closed paths. Eventually, we arrive at the relation

$$J(R) = \sum_{\text{PP}} [2\pi\hbar(-\,\partial\tau_1/\partial E)]^{-1/2} \exp\left(-\frac{1}{\hbar}\mathscr{S}([r_\tau]; \tau_1)\right), \quad (2.62)$$

where now only periodic paths have to be included.

We recognize that eqs. (2.62) and (2.57) are equivalent to eqs. (2.45) and (2.46), except for contributions of all periodic paths that satisfy the requirements.

2.6. Unified theory by multiple orbits

As pointed out by Affleck (1981), that part of the theory which we have presented in section 2.4 is not applicable to higher temperatures. In what follows, we present a unified theory which comprises the separate parts of Affleck's theory which are meant to cover different temperature ranges. In order to emphasize at least one of the differences between our theory and Affleck's one, we note that the trivial path $r_\tau = R_B$ cannot contribute to an outgoing probability current; hence, it does not play a role in the foundation of our theory.

As already mentioned, there are nontrivial complex orbits (Weierstrass functions) even for small values of τ_1, i.e., for large temperatures (see fig. 4c). Therefore, we may also look for periodic orbits that are completed in time

(primitive period)

$$\tau_p = \frac{1}{p}\tau_1, \quad p = 1, 2, \ldots, \tag{2.63}$$

and which are traversed p times. Recalling the summation of eq. (2.62), we write

$$\bar{\Gamma} = \sum_{p=1}^{\infty} \bar{\Gamma}_p, \tag{2.64}$$

where according to eqs. (2.57) and (2.62) we have

$$\bar{\Gamma}_p = Z_0^{-1}(-1)^{p+1}[2\pi\hbar p(-\partial\tau_p/\partial E)]^{-1/2}\exp\left(-\frac{p}{\hbar}\mathscr{S}([r_\tau];\tau_p)\right). \tag{2.65}$$

The factor $(-1)^{p+1}$ above deserves a comment. In a saddle point approximation, the overall sign depends on the direction in which the "mountain pass" is being traversed. Quite generally, this direction has to be found from a proper deformation of the original integration contour. For an integration in function space, however, it seems to be difficult to ascertain the details of the proper deformation. At least, this appears to be the case with Euclidean functional integrals and we have been unable to find general sign rules in the literature. On the other hand, the factor $(-1)^{p+1}$ above does lead to correct results in cases where we have been able to check it.*

Since $\mathscr{S}([r_\tau];\tau_p) \to -\infty$ for $\tau_p \to 0$, the sum given by eq. (2.64) is divergent.** However, it is possible to interpret this summation in a meaningful way as follows. In a first step, we calculate the Laplace transform in quasiclassical approximation

$$\begin{aligned}\bar{\Gamma}_p(E) &= \int_0^\infty d\tau_1 \bar{\Gamma}_p(\tau_1)\exp\left(-\frac{1}{\hbar}E\tau_1\right) \\ &= Z_0^{-1}(-1)^{p+1}\exp\left[-\frac{1}{\hbar}p\mathscr{W}(E)\right],\end{aligned} \tag{2.66}$$

where $\mathscr{W}(E)$ is defined by eq. (2.37) – or by the inverse of the Laplace transform (2.45). (Note that τ_p is the same function of E independent of p.) Summation of the series leads to

$$\bar{\Gamma}(E) = \sum_{p=1}^{\infty} \bar{\Gamma}_p(E) = Z_0^{-1} D(E), \tag{2.67}$$

* For an illustration see Appendix B.

** This is a general property of asymptotic expansions (see section 3.8 of Bender and Orszag 1978).

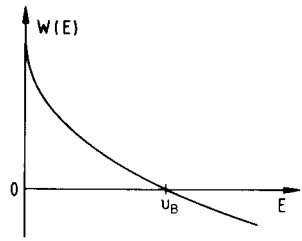

Fig. 6. Abbreviated Euclidean action $\mathscr{W}(E)$ for a closed orbit under the barrier. We have for $\tau = -\partial\mathscr{W}(E)/\partial E$ the values ∞ and $\tau_B = \hbar/kT_B$ at $E = 0$ and $E = v_B$, respectively.

where

$$D(E) = \left[1 + \exp\frac{1}{\hbar}\mathscr{W}(E)\right]^{-1} \tag{2.68}$$

means a generalization of the barrier transmission probability of eq. (2.36). Next, we invert the Laplace transformation and obtain ($E\tau_1/\hbar = E/kT$)

$$\bar{\Gamma} = (2\pi\hbar Z_0)^{-1}\int \mathrm{d}E\, D(E)\mathrm{e}^{-E/kT}. \tag{2.69}$$

In his paper, Affleck (1981) has discussed three different regions of temperature. They correspond, to some extent, to the different types of approximations* which are suggested by some properties of eqs. (2.68) and (2.69). For an orientation, consider fig. 6 for a typical energy dependence of $\mathscr{W}(E)$. Note that

$$\mathscr{W}(v_B) = 0,$$
$$-\left.\frac{\partial\mathscr{W}(E)}{\partial E}\right|_{E=v_B} = \tau_B = \frac{\hbar}{kT_B}, \tag{2.70}$$

where T_B is called the crossover temperature. (In the one-dimensional case we are considering here $kT_B = \hbar\omega_0/2\pi$.)

(i) In the case of very low temperatures $T \ll T_B$, only small energies contribute in eq. (2.69) and we may approximate $D(E) \approx \exp[-(1/\hbar)\mathscr{W}(E)]$ as shown in eq. (2.36). Thus, we recover eqs. (2.44) and (2.45).

(ii) Next, we consider temperatures where T is distinctly larger than T_B. Then, we expect that energies $|E - v_B| \lesssim kT$ contribute mostly in eq. (2.69). This justifies the approximation $\mathscr{W} \approx \mathscr{W}_1$, where the first-order expression $\mathscr{W}_1$ is given [cf. eq. (2.70)] by

$$\frac{1}{\hbar}\mathscr{W}_1(E) = -(E - v_B)/kT_B. \tag{2.71}$$

*See also Hänggi and Hontscha (1988).

In this case, we obtain

$$\bar{\Gamma} = (2\hbar Z_0)^{-1} kT_B (\sin \pi T_B/T)^{-1} \exp(-v_B/kT). \tag{2.72}$$

For temperatures $kT \gg kT_B, \hbar\omega_0$, we may expand Z_0 of eq. (2.42) as well as the trigonometric function above. Thus, we obtain in leading order

$$\bar{\Gamma} = (\omega_0/2\pi) \exp(-v_B/kT), \tag{2.73}$$

which should be compared with eq. (2.3).

At this point we wish to draw attention to an alternative way to obtain eq. (2.72), which can already be found in Affleck (1981) and which has been worked out in some detail in Wolynes (1981) and Grabert and Weiss (1984). Starting from a modification of eq. (2.47), namely

$$\bar{\Gamma} = -\frac{2}{\hbar} \frac{T_B}{T} \operatorname{Im} \mathscr{F}, \tag{2.74}$$

one calculates the free energy $\mathscr{F} = -kT \ln Z$ from $Z = Z_0 + Z_1$, where Z_1 is the contribution of the trivial orbit $r_t = R_B$, including its Gaussian fluctuations. Thus,

$$Z_1 = \tfrac{1}{2} i [2 \sin(\hbar\omega_0/2kT)]^{-1} \exp(-v_B/kT), \tag{2.75}$$

where the factor i is a consequence of the unstable mode at R_B and where the factor $\frac{1}{2}$ needs a separate justification.

(iii) Close to the crossover temperature $T \approx T_B$, Affleck proposes the expression

$$\bar{\Gamma} = (2\pi\hbar Z_0)^{-1} \int_{-\infty}^{v_B} dE \exp\left\{ -\left[\frac{1}{\hbar} \mathscr{W}_2(E) + E/kT \right] \right\}, \tag{2.76}$$

where $\mathscr{W}_2(E)$ is the second-order expansion given by

$$\frac{1}{\hbar} \mathscr{W}_2(E) = \frac{1}{\hbar} \mathscr{W}_1(E) + \frac{1}{4} \left(\frac{E - v_B}{kT_2} \right)^2, \tag{2.77}$$

with

$$\left(\frac{1}{kT_2} \right)^2 = \frac{2}{\hbar} \frac{\partial^2 \mathscr{W}(E)}{\partial E^2} \bigg|_{E = v_B}. \tag{2.78}$$

[In the one-dimensional case, $(kT_2)^2 \sim kT_B v_B$.] This ansatz leads to*

$$\bar{\Gamma} = Z_0^{-1} \frac{kT_2}{2\pi\hbar} \pi^{1/2} \operatorname{erfc}\left(\frac{T_2}{T_B} - \frac{T_2}{T} \right) \exp\left[-\frac{v_B}{kT} + \frac{1}{2} \left(\frac{T_2}{T_B} - \frac{T_2}{T} \right)^2 \right], \tag{2.79}$$

* See also eq. (63) of Larkin and Ovchinnikov (1992).

where $\operatorname{erfc} z = 2\pi^{-1/2}\int_z^\infty \mathrm{d}t \exp(-t^2)$ is the (complementary) error function (Abramowitz and Stegun 1968).

For large values of $T_2/T_B - T_2/T$, we obtain from eq. (2.79)

$$\bar{\Gamma} = Z_0^{-1}\frac{1}{2\pi\hbar}\left[\frac{1}{kT_B} - \frac{1}{kT}\right]^{-1} \mathrm{e}^{-v_B/kT}, \tag{2.80}$$

which reduces to expression (2.72) for temperatures $T \gtrsim T_B$ above the crossover temperature.

In particular, we have for $T \to T_B$, where $\operatorname{erfc}(T_2/T_B - T_2/T) \to 1$,

$$\bar{\Gamma} = Z_0^{-1}\frac{kT_2}{2\pi^{1/2}\hbar} \mathrm{e}^{-v_B/T_B}. \tag{2.81}$$

A detailed discussion of the procedures above for a one-dimensional system is given in Appendix B.

3. *Quasiclassical wave function in multidimensional quantum decay*

3.1. *Quantum decay in N dimensions*

The following considerations concern a system with N degrees of freedom. Without loss of generality, one may assume all masses to be equal. Then the Hamiltonian may be written as

$$\mathscr{H} = -\frac{\hbar^2}{2m}\sum_{k=1}^{N}\frac{\partial^2}{\partial R_k^2} + V(\{R_k\}), \tag{3.1}$$

where the potential energy V is supposed to have one metastable minimum. We take this minimum to be at the origin such that

$$V = \tfrac{1}{2}m\, U^{(0)}_{kk'} R_k R_{k'} + 0(R^3), \qquad U^{(0)}_{kk'} = \frac{1}{m}\frac{\partial^2 V}{\partial R_k \partial R_{k'}}\bigg|_{R=0}. \tag{3.2}$$

Here and in what follows, summation over repeated indices is implied; we will also use vector notation $\boldsymbol{R} = \{R_k\} = (R_1, \ldots, R_N)$, if convenient.

Although the potential energy V is positive in a region which includes the origin, it should become negative in a range of directions and at larger distances. Consequently, there is a surface Σ_0, defined by $V(\boldsymbol{R}) = 0$, which separates the outside region C (classically accessible) from the well and barrier regions A + B (classically inaccessible), i.e., the regions where $V < 0$ and $V > 0$, respectively. See fig. 7 for illustration.

As pointed out in section 2.2, the wave function $\psi(\boldsymbol{R})$ of a quasistationary state has to be a solution of the Schrödinger equation [eq. (2.6)], where the Hamiltonian $\mathscr{H}$ now is given by eq. (3.1). However, conditions (i) and (ii) on the

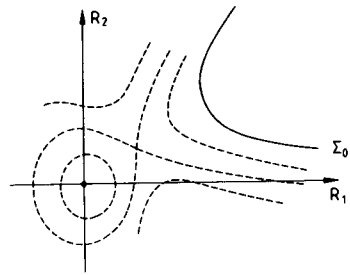

Fig. 7. Surfaces of constant potential energy $V(\boldsymbol{R})$ for a two-dimensional system with a metastable minimum (potential well A) at $\boldsymbol{R} = 0$. The surface Σ_0 defined by $V(\boldsymbol{R}) = V(0) = 0$ separates the well and barrier region (A + B) from the outside region (C).

behavior of the wave function at large distances are of little value on account of the missing information for $N - 1$ degrees of freedom. Therefore, we have to rely entirely on the simple approach of section 2.3.

We start from the quasiclassical ansatz (2.13) for the principal wave $\psi_0(\boldsymbol{R})$, where now the action is a function of $\boldsymbol{R}$. Inserting this ansatz into the Schrödinger equation, where $E = \frac{1}{2}\hbar\omega_T$ is real, we obtain
the eikonal equation

$$H = \frac{1}{2m} P_k^2 - V(\{R_k\}) = 0,$$

$$P_k = \frac{\partial \mathscr{W}(\{R_k\})}{\partial R_k}, \tag{3.3}$$

the transport equation

$$H^{(1)} = \frac{1}{m} P_k^{(1)} \frac{\partial \mathscr{W}}{\partial R_k} - \frac{1}{2m} \frac{\partial^2 \mathscr{W}}{\partial R_k^2} + \frac{1}{2} \omega_T = 0,$$

$$P_k^{(1)} = \frac{\partial \mathscr{W}^{(1)}(\{R_k\})}{\partial R_k}. \tag{3.4}$$

The equations above are nonlinear partial differential equations of first order which have to be solved for the boundary condition (2.29). Note also the special notation for the partial derivatives introduced above.

The solution to the eikonal equation (3.3) can be found by the method of characteristics.* The characteristics can be presented in parametric form as trajectories in the $2N$-dimensional $\{\boldsymbol{R}, \boldsymbol{P}\}$-space

$$R_k = R_k(\tau), \qquad P_k = P_k(\tau) \tag{3.5}$$

* A concise presentation of the method of characteristics can be found in section 2.13 of Whitham (1974). See also section II of Courant and Hilbert (1962) for a broad discussion.

such that they obey $H = 0$, together with the canonical equations of motion

$$\dot{R}_k = \frac{\partial H}{\partial P_k} = \frac{1}{m} P_k, \qquad \dot{P}_k = -\frac{\partial H}{\partial R_k} = \frac{\partial V}{\partial R_k}, \tag{3.6}$$

where the dot means differentiation with respect to τ.

In addition, we also introduce the concept of paths which are defined as projections of the trajectories on the N-dimensional $\{\boldsymbol{R}\}$-space. These paths obey the equation of motion

$$m\ddot{R}_k = \frac{\partial V}{\partial R_k}, \qquad \tfrac{1}{2} m\dot{R}_k^2 - V = 0, \tag{3.7}$$

where the condition $H = 0$ has been added for completeness. Clearly, eq. (3.7) corresponds to the classical equation of motion in the inverted potential for zero energy.

The change of $\mathscr{W}$ along a characteristic is given by

$$\dot{\mathscr{W}} = P_k \frac{\partial H}{\partial P_k} = m\dot{R}_k^2. \tag{3.8}$$

At this point, it is appropriate to recall that the relation $\partial \mathscr{W}/\partial \boldsymbol{R} = \boldsymbol{P} = m\dot{\boldsymbol{R}}$ means that the paths follow the lines of steepest descent of $\mathscr{W}(\boldsymbol{R})$.

Solving the transport equation (3.4) by the same method, one recognizes that the paths are the same as above. Furthermore,

$$\dot{\mathscr{W}}^{(1)} = P_k^{(1)} \frac{\partial H^{(1)}}{\partial P_k^{(1)}} = \frac{1}{2m} \frac{\partial^2 \mathscr{W}}{\partial R_k^2} - \frac{1}{2} \omega_T, \tag{3.9}$$

where the condition $H^{(1)} = 0$ has been used.

Since $\mathscr{W} = \mathscr{W}^{(1)} = 0$ at the origin is given, we obtain the action $\mathscr{W}(\boldsymbol{R})$ and $\mathscr{W}^{(1)}(\boldsymbol{R})$ for other points in space by calculating the paths which connect the origin with these particular points. Being paths in the inverted potential, they will be confined to the classically inaccessible regions, A + B, acquiring large distances for sufficiently large τ. As an illustration, such paths are shown in fig. 8 for a two-dimensional system. The figure suggests that there is also a collection of paths which are reflected at some distance from Σ_0, having curvatures either to the left or to the right. Following this suggestion, we conclude that there exists, as a case on the border line, a single path which approaches Σ_0 perpendicularly and with zero velocity.

The particular role of such a singular path in quantum decay has been stressed by Banks et al. (1978). We follow their suggestion and assume that in the most general quantum decay problem, there always exists such a singular path – possibly a discrete number of such paths* – which connect the metastable

* One might suspect symmetries to be responsible for the appearance of several of such singular paths.

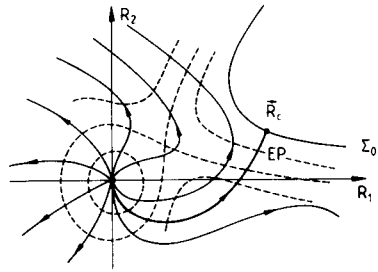

Fig. 8. The continuous lines (with arrow) represent paths that are solutions of the classical equation of motion in the *inverted* potential. These paths have left the unstable point at $\boldsymbol{R} = 0$ in the infinite past. There is a unique path, the escape path (EP), which connects $\boldsymbol{R} = 0$ with $\boldsymbol{R}_C$ on Σ_0.

minimum with the boundary between inaccessible and accessible regions. Investigations on specific models support this assumption. In fact, one has found one such path for the Caldeira–Leggett model (see section 3.6).

For simplicity, we consider the case where there is just one such path, say $\boldsymbol{R} = \boldsymbol{r}(\tau) = \boldsymbol{r}_\tau$. It obeys the equation of motion (3.7) together with the appropriate boundary conditions; explicitly,

$$m\ddot{\boldsymbol{r}}_\tau = \frac{\partial V}{\partial \boldsymbol{r}_\tau}, \qquad \frac{m}{2}\dot{\boldsymbol{r}}_\tau^2 - V = 0,$$

$$\boldsymbol{r}_{-\infty} = 0, \qquad \boldsymbol{r}_0 = \boldsymbol{R}_C, \tag{3.10}$$

where $\boldsymbol{R}_C$ is on Σ_0. We will call $\boldsymbol{r}_\tau$ the escape path* (EP) and $\boldsymbol{R}_C$ the escape point (see fig. 8).

Clearly, $\dot{\boldsymbol{r}}_\tau = 0$ for $\tau \to 0$; but there exists a unit vector

$$(\dot{\boldsymbol{r}}_\tau / |\dot{\boldsymbol{r}}_\tau|)_{\tau \to 0} = \boldsymbol{n}_C, \tag{3.11}$$

which is perpendicular to Σ_0 at $\boldsymbol{R}_C$.

If one follows the consequences of this assumption, a picture as shown in fig. 9 emerges. There is a dense bundle of paths which start (at $\tau = -\infty$) at the origin and which follow the escape path closely for a long distance. Eventually, they enter a region close to Σ_0 and then, it becomes evident that their velocity component parallel to Σ_0 remains finite preventing these paths from reaching Σ_0. Observe that in this region, the form of these paths is almost parabolic, and that there is an envelope Σ_C to these paths which may be called a caustic (Courant and Hilbert 1962). We conclude that, beyond Σ_C, there is no solution $\mathscr{W}(\boldsymbol{R})$ of the eikonal equation (3.3) which is real and which satisfies the boundary conditions at the origin.

* This is an abbreviation of the term "most probable escape path" found in Banks et al. (1978).

3.2. Small-fluctuation approximation

Let us assume that the wave function falls off quickly if one moves away perpendicular from the escape path.* In this case, we may restrict our attention to paths in the vicinity of the escape path. Thus, we are led to introduce the small-fluctuation approximation where $\boldsymbol{R}_\tau = \boldsymbol{r}_\tau + \boldsymbol{v}_\tau$, such that $\boldsymbol{v}_\tau$ is small. It follows from eq. (3.7) and from the initial condition discussed before, that, through first order in $\boldsymbol{v}$,

$$\ddot{\boldsymbol{v}}_\tau = \hat{U}_\tau \cdot \boldsymbol{v}_\tau, \qquad \boldsymbol{v}_{-\infty} = 0, \tag{3.12}$$

where the operator $\hat{U}_\tau$ is defined by**

$$U_{kk'}(\tau) = \frac{1}{m} \frac{\partial^2 V}{\partial R_k \partial R_{k'}}\bigg|_{\boldsymbol{R}=\boldsymbol{r}_\tau} \tag{3.13}$$

A quantity of most importance is the action along the escape path,

$$\mathscr{W}_{\mathrm{EP}}(\tau) = m \int_{-\infty}^{\tau} \mathrm{d}\tau' \, \dot{\boldsymbol{r}}_{\tau'}^2. \tag{3.14}$$

Concerning the action in the vicinity of the escape path, one may calculate it as follows, using eq. (3.8) and the condition $H = 0$:

$$\begin{aligned}
\mathscr{W}(\tau) &\equiv \int_{-\infty}^{\tau} \mathrm{d}\tau' \, m\dot{\boldsymbol{R}}_{\tau'}^2 \\
&= \int_{-\infty}^{\tau} \mathrm{d}\tau' \left\{ \tfrac{1}{2} m\dot{\boldsymbol{R}}_{\tau'}^2 - V(\boldsymbol{R}_{\tau'}) + \tfrac{1}{2} m\dot{\boldsymbol{R}}_{\tau'}^2 + V(\boldsymbol{R}_{\tau'}) \right\} \\
&= \int_{-\infty}^{\tau} \mathrm{d}\tau' \left\{ \tfrac{1}{2} m\dot{\boldsymbol{r}}_{\tau'}^2 + m\dot{\boldsymbol{r}}_{\tau'} \cdot \dot{\boldsymbol{v}}_{\tau'} + \tfrac{1}{2} m\dot{\boldsymbol{v}}_{\tau'}^2 + V(\boldsymbol{r}_{\tau'} + \boldsymbol{v}_{\tau'}) \right\}.
\end{aligned} \tag{3.15}$$

One expands $V(\boldsymbol{r} + \boldsymbol{v}) = V(\boldsymbol{r}) + (\boldsymbol{v} \cdot \partial/\partial \boldsymbol{r}) V + (1/2)(\boldsymbol{v} \cdot \partial/\partial \boldsymbol{r})^2 V$, then one performs partial integrations and observes that $\boldsymbol{v}_\tau \cdot m\ddot{\boldsymbol{r}}_\tau - (\boldsymbol{v}_\tau \cdot \partial/\partial \boldsymbol{r}_\tau) V = 0$ [cf. eqs. (3.12) and (3.13)] as well as $\boldsymbol{v}_\tau \cdot m\ddot{\boldsymbol{v}}_\tau - (\boldsymbol{v}_\tau \cdot \partial/\partial \boldsymbol{r}_\tau)^2 V = 0$. Thus, one obtains in quadratic accuracy

$$\mathscr{W}(\tau) = \mathscr{W}_{\mathrm{EP}}(\tau) + m\dot{\boldsymbol{r}}_\tau \cdot \boldsymbol{v}_\tau + \tfrac{1}{2} m(\dot{\boldsymbol{v}}_\tau \cdot \boldsymbol{v}_\tau). \tag{3.16}$$

We express $\mathscr{W}(\tau)$ in coordinate space as follows. By choosing τ, we select a point $\boldsymbol{r}_\tau$ on the escape path. A vicinity of this point will be represented by

$$\boldsymbol{R} = \boldsymbol{r}_\tau + \boldsymbol{\eta}, \tag{3.17}$$

* In section 3.3, we will show that this assumption is not really independent but fits in the framework of the quasiclassical approximation.

** In cases where the quantities carry Cartesian subscripts, we note the time dependence in the usual way, e.g., $(\hat{U}_\tau)_{kk'} = U_{kk'}(\tau)$.

where $|\boldsymbol{\eta}|$ is small. Of course, this representation is not unique. It could be made unique by requiring that $\boldsymbol{\eta}$ be locally perpendicular to the escape path (Banks and Bender 1973, Banks et al. 1973, Gervais and Sakita 1977) or by another convention, as will be shown in section 3.3. For the time being, however, it is not necessary to do so.

Next, we define a small-fluctuation operator $\hat{K}_\tau$ such that it obeys the equation of motion

$$[-\hat{1}\partial_\tau^2 + \hat{U}_\tau]\hat{K}_\tau = 0 \tag{3.18a}$$

and such that it vanishes for $\tau \to -\infty$. Note that in this limit $\hat{U}_\tau$ approaches the constant $\hat{U}^{(0)}$ introduced in eq. (3.2). Therefore, if

$$\hat{\omega}_0 = [\hat{U}^{(0)}]^{1/2} \tag{3.19}$$

is the positive square root, we have

$$\hat{K}_\tau \to \exp \hat{\omega}_0 \tau, \quad \tau \to -\infty. \tag{3.18b}$$

Since, $\hat{U}^{\mathrm{T}} = \hat{U}$, we may derive the relation

$$\dot{\hat{K}}_\tau^{\mathrm{T}} \hat{K} - \hat{K}_\tau^{\mathrm{T}} \dot{\hat{K}}_\tau = 0, \tag{3.20}$$

where the superscript T denotes the transpose of a matrix.

It is now easy to associate with each τ and $\boldsymbol{\eta}$ a path $\boldsymbol{v}_{\tau'}$ such that $\boldsymbol{v}_\tau = \boldsymbol{\eta}$ and

$$\boldsymbol{v}_{\tau'} = \hat{K}_{\tau'} \hat{K}_\tau^{-1} \cdot \boldsymbol{\eta}. \tag{3.21}$$

Therefore, in the small-fluctuation approximation, the action assumes the form

$$\mathscr{W}(\boldsymbol{\eta}, \tau) = \mathscr{W}_{\mathrm{EP}}(\tau) + m\dot{\boldsymbol{r}}_\tau \cdot \boldsymbol{\eta} + \tfrac{1}{2} m \boldsymbol{\eta} \cdot \hat{\Omega}_\tau \cdot \boldsymbol{\eta}, \tag{3.22}$$

where

$$\hat{\Omega}_\tau = \dot{\hat{K}}_\tau \hat{K}_\tau^{-1}. \tag{3.23}$$

We conclude from eq. (3.20) that $\hat{\Omega}_\tau^{\mathrm{T}} = \hat{\Omega}_\tau$. Furthermore, $\hat{\Omega}_{-\infty} = \hat{\omega}_0$. Considering eq. (3.18), one may derive for $\hat{\Omega}_\tau$ the following Riccati-type equation

$$\dot{\hat{\Omega}}_\tau = \hat{U}_\tau - \hat{\Omega}_\tau^2. \tag{3.24}$$

The redundancy of representation (3.17) can be used to calculate the derivative in the direction of the escape path in two ways. Accordingly, $\partial/\partial\tau = \dot{\boldsymbol{r}} \cdot \partial/\partial\boldsymbol{\eta}$, where the left-hand side is taken at $\boldsymbol{\eta} \equiv 0$ and the right-hand side requires the limit $\boldsymbol{\eta} \to 0$ to be taken only at the end. We apply this relation to $\partial\mathscr{W}/\partial\eta_k = \dot{r}_k + \Omega_{kk'}\eta_{k'}$, and obtain the useful relation

$$\ddot{\boldsymbol{r}}_\tau = \hat{\Omega}_\tau \cdot \dot{\boldsymbol{r}}_\tau \tag{3.25}$$

According to eq. (3.9), the calculation of the first-order correction $\mathscr{W}^{(1)}$ requires the evaluation of $\partial^2\mathscr{W}/\partial R_k^2$. In the small-fluctuation approximation,

this is equal to $\partial^2 \mathscr{W}/\partial \eta_k^2$; therefore, from eqs. (3.9) and (3.22),

$$\dot{\mathscr{W}}^{(1)}(\tau) = \tfrac{1}{2}\operatorname{Tr}\hat{\Omega}_\tau - \tfrac{1}{2}\omega_T \tag{3.26}$$

For $\mathscr{W}^{(1)}$ to be finite near the origin (where $\tau \to -\infty$), we require that

$$\omega_T = \operatorname{Tr}\hat{\Omega}_{-\infty} = \operatorname{Tr}\hat{\omega}_0. \tag{3.27}$$

Then it follows that

$$\mathscr{W}^{(1)}(\tau) = \tfrac{1}{2}\operatorname{Tr}\ln\{\hat{K}_\tau \exp(-\hat{\omega}_0\tau)\}. \tag{3.28}$$

Considering eqs. (2.13), (3.17), (3.22) and (3.28) and using the identity $\operatorname{Tr}\ln\hat{A} = \ln\det\hat{A}$, the principal wave function may be written as

$$\psi_0(\boldsymbol{\eta}, \tau) = \frac{\exp[-\{\mathscr{W}_{EP}(\tau) + m\dot{\boldsymbol{r}}_\tau\cdot\boldsymbol{\eta} + \tfrac{1}{2}m\boldsymbol{\eta}\cdot\hat{\Omega}_\tau\cdot\boldsymbol{\eta}\}/\hbar]}{[\det\{\hat{K}_\tau\exp(-\hat{\omega}_0\tau)\}]^{1/2}}. \tag{3.29}$$

Concerning the wave function near the origin, we assume that there is a sufficiently large environment where the potential is purely harmonic. This means that

$$\boldsymbol{r}_\tau = \hat{\omega}_0^{-1}\hat{K}_\tau\cdot\boldsymbol{\chi}_1^- = \hat{\omega}_0^{-1}\cdot\dot{\boldsymbol{r}}_\tau, \tag{3.30}$$

provided that $\boldsymbol{r}_\tau$ is sufficiently small ($\tau \to -\infty$). One may understand this relation by noting that $\boldsymbol{\chi}_1^- = \hat{\omega}_0^{-1}\hat{K}_\tau^{-1}\cdot\dot{\boldsymbol{r}}_\tau$ is a time-independent vector. It also follows that in the above limit, $\mathscr{W}_{EP}(\tau) = (m/2)\boldsymbol{r}_\tau\cdot\hat{\omega}_0\cdot\boldsymbol{r}_\tau$ and that

$$\mathscr{W}(\boldsymbol{\eta}, \tau) = \tfrac{1}{2}m[\boldsymbol{r}_\tau + \boldsymbol{\eta}]\cdot\hat{\omega}_0\cdot[\boldsymbol{r}_\tau + \boldsymbol{\eta}] = \tfrac{1}{2}m\boldsymbol{R}\cdot\hat{\omega}_0\cdot\boldsymbol{R}, \tag{3.31}$$

which means that eq. (3.29) reduces to the correct harmonic oscillator wave function $\psi_0 = \exp[-(m/2\hbar)\boldsymbol{R}\cdot\hat{\omega}_0\cdot\boldsymbol{R}]$. The normalization of the wave function is determined mostly by contributions from the environment near the origin; therefore,

$$\int \mathrm{d}^N R\,\psi_0^2 = \left[\frac{(\pi\hbar/m)^N}{\det\hat{\omega}_0}\right]^{1/2}. \tag{3.32}$$

3.3. Principal and transmitted wave near the caustic

For convenience, we introduce a coordinate system $\boldsymbol{\zeta} = \boldsymbol{R} - \boldsymbol{R}_C = (\zeta_1, \ldots, \zeta_N)$ such that the ζ_1-axis is parallel to $\boldsymbol{n}_C$, which is the tangent vector to the escape path at its end as given by eq. (3.11). Note that at the escape point, the force $\boldsymbol{F}_C = -(\partial V/\partial\boldsymbol{R})_{\boldsymbol{R}=\boldsymbol{R}_C} = (F_C, 0, \ldots, 0)$ points in the same direction.

Let us also shift the coordinates of the escape path according to $z_\tau = \boldsymbol{r}_\tau - \boldsymbol{R}_C$. Observe now that the solution to eq. (3.10) can be represented in the form of

a power series in τ^4:

$$z_k(\tau) = -\frac{1}{2m}\tau^2 F_{Ck} - \frac{1}{4!m}\tau^4 U^{C}_{kk'} F_{Ck'} + \cdots, \tag{3.33}$$

where* $\hat{U}^C = \hat{U}_{\tau=0}$. Note that in leading order, $z_1 = -\tau^2 F_C/2m$, and that $z_\perp \propto \tau^4 \propto z_1^2$, where $z_\perp$ is the component perpendicular to $\boldsymbol{n}_C$.

Consider now $\mathscr{W}(\boldsymbol{\eta}, \tau)$ as given by eq. (3.22), and observe that $\boldsymbol{\eta} = \boldsymbol{\zeta} - \boldsymbol{z}_\tau$. The redundancy in the representation (3.17) is used to choose $\tau = \tau_1$ such that

$$z_1(\tau_1) = \zeta_1, \quad \tau_1 = -(-2m\zeta_1/F_C)^{1/2} + \cdots, \tag{3.34}$$

where the second expression holds in leading order. Then,

$$\mathscr{W}(\zeta_1, \ldots, \zeta_N) = \mathscr{W}_{EP}(\tau_1) + m\dot{r}_k(\tau_1)[\zeta_k - z_k(\tau_1)] + \tfrac{1}{2} m\Omega_{kk'}(\tau_1)[\zeta_k - z_k(\tau_1)][\zeta_{k'} - z_{k'}(\tau_1)]. \tag{3.35}$$

Since $\zeta_1 - z_1(\tau_1) = 0$, the summation above includes only indices $k, k' > 1$. In particular, $\mathscr{W}$ depends on ζ_1 only through its dependence on τ_1.

One expects that $\mathscr{W}(\boldsymbol{\zeta})$ satisfies the eikonal equation (3.3). However, due to the small-fluctuation approximation, this will be true only through terms of order $(\boldsymbol{\zeta} - \boldsymbol{z})^2 \sim \zeta_\perp^2, \zeta_\perp \zeta_1^2, \zeta_1^4$. This can be checked by differentiating eq. (3.35) as follows. We introduce indices $j, j' \geqslant 2$, e.g., $\boldsymbol{\zeta} = (\zeta_1, \{\zeta_j\})$. Then we observe eqs. (3.14) and (3.25), whence we obtain

$$\frac{1}{m}\frac{\partial \mathscr{W}}{\partial \zeta_1} = \frac{d\tau_1}{d\zeta_1}\{\dot{r}_1^2 + \dot{r}_1 \Omega_{1j}[\zeta - z]_j + \tfrac{1}{2}\dot{\Omega}_{jj'}[\zeta - z]_j[\zeta - z]_{j'}\},$$

$$\frac{1}{m}\frac{\partial \mathscr{W}}{\partial \zeta_j} = \dot{r}_j + \Omega_{jj'}[\zeta - z]_{j'}. \tag{3.36}$$

As an abbreviation, we introduce again $\eta_j = \zeta_j - z_j(\tau_1)$. Then one obtains through $O(\eta_j^2)$

$$\frac{1}{m^2}\left(\frac{\partial \mathscr{W}}{\partial \zeta_k}\right)^2 = \dot{r}_k^2 + 2\dot{r}_k \Omega_{kj}\eta_j + \dot{\Omega}_{jj'}\eta_j\eta_{j'} + \Omega_{kj}\Omega_{kj'}\eta_j\eta_{j'}$$

$$= \dot{r}_k^2 + 2\ddot{r}_j\eta_j + U_{jj'}\eta_j\eta_{j'}. \tag{3.37}$$

Note that the last line follows from eqs. (3.24) and (3.25). Using eq. (3.10), we find through order η^2 that $(\partial \mathscr{W}/\partial \zeta_k)^2 - 2mV = 0$, as expected.

Of central importance is the fact that $\boldsymbol{r}_{\tau_1}$ and $\hat{\Omega}_{\tau_1}$ can be expanded in powers of τ_1. This follows from the equations of motion (3.10) and (3.24); eq. (3.33) may serve as an example. On the other hand, it is also possible to express τ_1 in a series of powers in $(-\zeta_1)^{1/2}$ the leading term of which is shown in eq. (3.34). Hence, it follows that all quantities can be represented by power series in $(-\zeta_1)^{1/2}$. Thus,

* Note that $\hat{U}^C$ has (at least) one negative eigenvalue.

we may write eq. (3.35) as follows:

$$\mathscr{W}(\zeta) = \mathscr{W}_{\mathrm{EP}}(0) - \tfrac{2}{3}(2mF_{\mathrm{C}})^{1/2}(-\zeta_1)^{3/2} + \cdots + \frac{m}{2}\left(\frac{2m}{F_{\mathrm{C}}}\right)^{1/2}(-\zeta_1)^{3/2}\, U^{\mathrm{C}}_{1j}[\zeta_j - z_j(\zeta_1)] + \cdots + \frac{m}{2}\,\Omega_{jj'}(0)[\zeta_j - z_j(\zeta_1)][\zeta_{j'} - z_{j'}(\zeta_1)] + \cdots, \tag{3.38}$$

where only terms of leading order are shown explicitly.

Note that $\mathscr{W}^{(1)}$, the first-order correction to the action, does not depend on ζ_j in the small-fluctuation approximation. Hence, its expansion is of a simple form except for one peculiarity which is connected with a logarithmic divergence for $\zeta_1 \to 0$. Therefore, we write

$$\mathscr{W}^{(1)}(\zeta) = \tfrac{1}{4}\ln(-2mF_{\mathrm{C}}\zeta_1)^{1/2} + \cdots + \tfrac{1}{2}\{\mathrm{Tr}\ln\hat{K}_\tau - \ln\dot{r}_1(\tau)\}_{\tau\to 0} + \cdots. \tag{3.39}$$

The unique role of ζ_1 clearly demonstrates that in the small-fluctuation approximation the caustic Σ_{C} is approximated by the hypersurface $\zeta_1 = 0$. It follows from eq. (3.38) that the surfaces $\mathscr{W}(\zeta) = \mathrm{const} < \mathscr{W}_{\mathrm{EP}}(0)$ are of the form $-\zeta_1 \propto (\zeta_\perp - \zeta_0)^{2/3}$, whereas the surface $\mathscr{W}(\zeta) = \mathscr{W}_{\mathrm{EP}}(0)$ is characterized by $-\zeta_1 \propto \zeta_\perp^{4/3}$. See fig. 9 for an illustration.

Concerning the validity of the quasiclassical approximation we take an appropriate generalization of the criterion [eq. (2.25)] which is given by

$$\hbar\left|\frac{\partial\mathscr{W}^{(1)}}{\partial \boldsymbol{R}}\right| \ll \left|\frac{\partial\mathscr{W}}{\partial \boldsymbol{R}}\right|. \tag{3.40}$$

In the present case, no problem arises with this condition for $\boldsymbol{R}$ near the origin, provided that $\hbar$ is sufficiently small. The situation is different near the caustic. Taking the derivatives in eq. (3.40) in the normal direction, we obtain from eqs.

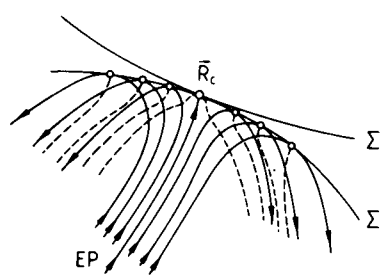

Fig. 9. Paths (→→) in the inverted potential near the escape point. The caustic (——) is marked by Σ_{C}. The surfaces for $\mathscr{W}(\boldsymbol{R}) = \mathrm{const}$ (– – –) are seen to intersect the incoming paths orthogonally.

(3.38) and (3.39), and for $\zeta_\perp = 0$, the condition $|\zeta_1|^3 \gg \hbar^2/mF_C$ which corresponds to one of the inequalities (2.26). Therefore, the quasiclassical approximation breaks down near a caustic the same way it does near a turning point in a one-dimensional problem.

In accordance with the principle of analytical continuation outlined in section 2.3, it is possible to bypass the caustic (which is given by $\zeta_1 = 0$ in the small-fluctuation approximation) if we allow ζ_1 to assume complex values. At this point, we observe that $\mathscr{W}(\zeta)$ of eq. (3.38) is a solution of the eikonal equation (3.3) also for complex ζ_1. A similar statement can also be made with regard to $\mathscr{W}^{(1)}(\zeta)$. Therefore, and in view of eq. (2.31), we are led to substitute

$$-\zeta_1 \to e^{-i\pi}\zeta_1 \tag{3.41}$$

in both expressions (3.38) and (3.39). Let us now mark all quantities obtained by this analytical continuation by a tilde, e.g., $\tilde{\mathscr{W}}(\zeta) = \mathscr{W}[-\zeta_1 \to e^{-i\pi}\zeta_1]$. Then, the transmitted wave may be written as

$$\tilde{\psi}(\zeta) = \exp\left\{-\frac{1}{\hbar}[\tilde{\mathscr{W}}(\zeta) + \hbar\tilde{\mathscr{W}}^{(1)}(\zeta)]\right\}, \tag{3.42}$$

where $\zeta_1 > 0$.

For a general orientation, let us remark that, beyond the caustic, $\tilde{\mathscr{W}}(\zeta)$ is a complex function. This complex function may be represented by surfaces $\mathrm{Re}\,\tilde{\mathscr{W}} = \text{const}$ and $\mathrm{Im}\,\tilde{\mathscr{W}} = \text{const}$. It follows from the eikonal equation (3.3) that at each point, the two normal directions to these surfaces are perpendicular (Landauer 1950). Furthermore, the surface $\mathrm{Im}\,\tilde{\mathscr{W}} = 0$ coincides with the caustic. Schematically, this is shown in fig. 10 for a two-dimensional system.

It is possible to obtain the wave function [eq. (3.42)] directly by modifying appropriately the method of characteristics introduced in section 3.1. For the time being, we are interested only in the small-fluctuation approximation. Then one may proceed as follows.* We allow the parameter τ to assume purely imaginary values. In view of eqs. (3.41) and (3.34) we substitute

$$\tau \to it, \tag{3.43}$$

where t is real and positive. Note that this procedure is justified by the very fact already made use of earlier, namely that $\hat{\Omega}_\tau$ and $\boldsymbol{r}_\tau$ allow a representation as power series in τ and that

$$\dot{\hat{\Omega}}_0 = 0 \quad \text{as well as} \quad \dot{\boldsymbol{r}}_0 = 0.$$

The extension of the escape path $\tilde{\boldsymbol{r}} = \tilde{\boldsymbol{r}}_t$ is defined by

$$m\ddot{\tilde{\boldsymbol{r}}}_t = -\frac{\partial V}{\partial \tilde{\boldsymbol{r}}_t}, \quad \tilde{\boldsymbol{r}}_0 = \boldsymbol{R}_C, \quad \dot{\tilde{\boldsymbol{r}}}_0 = 0, \tag{3.44}$$

where the dot means here differentiation with respect to t. (Note that the t-differentiation operates always on quantities marked by a tilde; therefore, no

* Recall the discussion in the second part in section 2.5 and fig. 5.

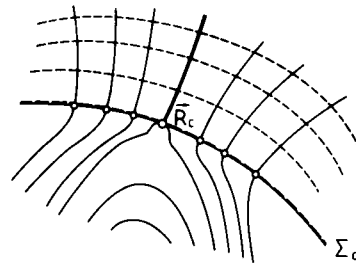

Fig. 10. Surfaces of constant Re $\tilde{\mathscr{W}}(\boldsymbol{R})$ (——) and of constant Im $\tilde{\mathscr{W}}(\boldsymbol{R})$ (– – –) near the caustic. The surfaces of constant $\mathscr{W}(\boldsymbol{R})$ (——) in the classically forbidden region complete the picture in a consistent way.

ambiguity arises with the two types of dot-symbols.*) This path describes a classical motion in the potential V in contrast to eq. (3.10), where the potential is inverted.

In the small-fluctuation approximation, eq. (3.17) is replaced by $\boldsymbol{R} = \tilde{\boldsymbol{r}}(t) + \boldsymbol{\eta}$. Further changes are as follows:

(a) Equation (3.18) is replaced by

$$[\hat{1}\partial_t^2 + \hat{\tilde{U}}_t]\hat{\tilde{K}}_t = 0, \qquad \hat{\tilde{K}}_0 = \hat{K}_0, \qquad -\mathrm{i}\dot{\hat{\tilde{K}}}_0 = \dot{\hat{K}}_0, \tag{3.45}$$

where $\hat{\tilde{U}}_t$ is obtained from eq. (3.13) if $\tilde{\boldsymbol{r}}_t$ is substituted in place of $\boldsymbol{r}_\tau$.

(b) Equations (3.14), (3.22) and (3.23) are replaced by

$$\begin{aligned}
&\tilde{\mathscr{W}}_{\mathrm{EP}}(t) = \mathscr{W}_{\mathrm{EP}}(0) - \mathrm{i}m\int_0^t \mathrm{d}t'\,\dot{\tilde{\boldsymbol{r}}}_{t'}^2,\\
&\tilde{\mathscr{W}}(\boldsymbol{\eta}, t) = \tilde{\mathscr{W}}_{\mathrm{EP}}(t) - \mathrm{i}m\dot{\tilde{\boldsymbol{r}}}_t\cdot\boldsymbol{\eta} + \tfrac{1}{2}m\boldsymbol{\eta}\cdot\hat{\tilde{\Omega}}_t\cdot\boldsymbol{\eta},\\
&\hat{\tilde{\Omega}}_t = -\mathrm{i}\dot{\hat{\tilde{K}}}_t\hat{\tilde{K}}_t^{-1},
\end{aligned} \tag{3.46}$$

(c) Equation (3.28) is replaced by

$$\mathscr{W}^{(1)}(t) = \tfrac{1}{2}\mathrm{Tr}\ln\{\hat{\tilde{K}}_t\cdot\exp(-\mathrm{i}\hat{\omega}_0 t)\}. \tag{3.47}$$

It is advantageous to use again the ζ-coordinate system. Specifically, we put $\tilde{\boldsymbol{z}}_t = \tilde{\boldsymbol{r}}_t - \boldsymbol{R}_C$. In place of eq. (3.34), we now choose $t = t_1$ such that $\tilde{z}_1(t_1) = \zeta_1 > 0$. Considering eq. (3.46), and introducing $\eta_j = \zeta_j - \tilde{z}_j(t_1)$ as an abbreviation, we obtain analogously to eq. (3.35) the expression

$$\tilde{\mathscr{W}}(\zeta) = \tilde{\mathscr{W}}_{\mathrm{EP}}(t_1) - \mathrm{i}m\dot{\tilde{\boldsymbol{r}}}_{t_1}\cdot\boldsymbol{\eta} + \tfrac{1}{2}m\boldsymbol{\eta}\cdot\hat{\tilde{\Omega}}_{t_1}\cdot\boldsymbol{\eta}. \tag{3.48}$$

* Following eq. (3.41), we have introduced the convention to mark the analytically continued quantities with a tilde. Note that this notation might be ambiguous by a factor $\pm$ i for time-integrated or time-differentiated quantities. In what follows, however, the explicit definition applies.

Note specifically

$$|\tilde{\psi}(\eta)|^2 = \exp\left\{-2\mathscr{W}_{\mathrm{EP}}(0)/\hbar - \mathrm{Re}\left[\frac{m}{\hbar}\boldsymbol{\eta}\cdot\hat{\tilde{\Omega}}_{t_1}\cdot\boldsymbol{\eta} + 2\tilde{\mathscr{W}}^{(1)}(t_1)\right]\right\}. \tag{3.49}$$

We assume that $\mathrm{Re}\,\hat{\tilde{\Omega}}_{t_1}$ is a positive definite operator in the subspace perpendicular to $\boldsymbol{n}_{\mathrm{C}}$.

Let us now calculate the probability current through the plane $\zeta_1 = \text{const}$:

$$\begin{aligned} J(\zeta_1) &= \int \mathrm{d}^{N-1}\{\zeta_j\}\,\boldsymbol{n}_{\mathrm{C}}\cdot\frac{\hbar}{m}\,\mathrm{Im}\,\tilde{\psi}^*\frac{\partial\tilde{\psi}}{\partial\boldsymbol{R}} \\ &= \frac{-1}{m}\int \mathrm{d}^{N-1}\{\zeta_j\}\,|\tilde{\psi}|^2\,\mathrm{Im}\left[\frac{\partial\tilde{\mathscr{W}}}{\partial\zeta_1} + \hbar\frac{\partial\tilde{\mathscr{W}}^{(1)}}{\partial\zeta_1}\right]. \end{aligned} \tag{3.50}$$

Calculating $\partial\tilde{\mathscr{W}}/\partial\zeta_1$, we obtain a result similar to eq. (3.36). Due to the form of $|\tilde{\psi}|^2$, the terms linear in η_j vanish upon integration. We also observe that terms contained in $\partial\tilde{\mathscr{W}}/\partial\zeta_1$ which are bilinear in η_j may be neglected since their contribution to J is smaller than the leading term by a factor $\hbar$. Within the same approximation, $\partial\tilde{\mathscr{W}}^{(1)}/\partial\zeta_1$ may also be neglected. Thus, we obtain in leading order

$$J(\zeta_1) = \left[\frac{(\pi\hbar/m)^{N-1}}{\det\mathrm{Re}\,\hat{\tilde{\Omega}}_{\perp}(t)}\right]^{1/2}\left.\frac{\dot{\tilde{r}}_1(t)\exp(-2\mathscr{W}_{\mathrm{EP}}(0)/\hbar)}{\det\mathrm{Re}\{\hat{\tilde{K}}_t\exp(-\mathrm{i}\hat{\omega}t)\}}\right|_{t=t_1}, \tag{3.51}$$

where $\hat{\tilde{\Omega}}_{\perp}$ is the projection of $\hat{\tilde{\Omega}}$ onto the subspace perpendicular to $\boldsymbol{n}_{\mathrm{C}}$.

It is worth noting that the leading contribution to the current as given in eq. (3.51) does not suffer from the small-fluctuation approximation. In fact, systematic corrections to this approximation are of higher order in η_j than the order of the leading terms in $\tilde{\mathscr{W}}$ and $\tilde{\mathscr{W}}^{(1)}$. Consequently, these corrections will give rise to changes in J only by relative order of $\hbar$.

The current $J(\zeta_1)$ as given above does not depend on the value of ζ_1. One way* to understand this is as follows. Since eqs. (3.3) and (3.4) are also valid for $\tilde{\mathscr{W}}$ and $\tilde{\mathscr{W}}^{(1)}$, we find that

$$\frac{\partial j_k}{\partial R_k} = \frac{\hbar}{m}\,\mathrm{Im}\left[\left(\frac{\partial\tilde{\mathscr{W}}^{(1)}}{\partial R_k}\right)^2 - \frac{\partial^2\tilde{\mathscr{W}}^{(1)}}{\partial R_k^2}\right]|\tilde{\psi}|^2. \tag{3.52}$$

Integrating this expression over a volume enclosed between two planes $\zeta_1 = \zeta_1'$ and $\zeta_1 = \zeta_1''$, we find that the difference $J(\zeta_1') - J(\zeta_1'')$ is smaller than, say, $J(\zeta_1')$ by a factor $\hbar$.

*It is also possible to show that this property follows directly from eqs. (3.44–3.47). See also a similar discussion in section 3.5.

3.4. Decay rate in quasiclassical approximation and in instanton technique

In analogy with the procedure leading to eq. (2.27), we integrate $\mathrm{Im}\,\psi^*(\mathscr{H}-E)\psi$ with respect to $\mathrm{d}^N R$ in the region to the "left" side of the plane $\zeta_1 = \text{const}$. Thus, we obtain Γ as the ratio of the current $J(\zeta_1)$ as given in eq. (3.51) and a normalization which is given by eq. (3.32). Since $J(\zeta_1)$ does not depend on ζ_1, one may take as well the limit $\zeta_1 \to 0$, i.e., $t_1 \to 0$.

However, this limit has to be taken with care since both $\dot{\tilde{r}}_1(t)$ as well as $\det \mathrm{Re}\,\tilde{\hat{K}}_t$ vanish for $t \to 0$, although their ratio remains finite. On the other hand, the analytical properties ensure that

$$\left(\frac{\dot{\tilde{r}}_1(t)}{\det \mathrm{Re}\,\tilde{\hat{K}}_t}\right)_{t\to 0} = \left(\frac{\dot{r}_1(\tau)}{\det \hat{K}_\tau}\right)_{\tau\to 0} \tag{3.53a}$$

Thus, we may write $\Gamma = \mathscr{A}\exp(-\mathscr{B})$ in accordance with eq. (2.2) where the exponent is

$$\mathscr{B} = 2\mathscr{W}_{\mathrm{EP}}(0)/\hbar \tag{3.54}$$

and the prefactor is

$$\mathscr{A} = \left[\frac{m \det \hat{\omega}_0}{\pi\hbar \det \hat{\Omega}_\perp(0)}\right]^{1/2}\left(\frac{\dot{r}_1(\tau)}{\det \hat{K}_\tau}\right)_{\tau\to 0}. \tag{3.55a}$$

The restriction of $\hat{\Omega}$ to a subspace in the determinant above is of some inconvenience. It follows from eq. (3.25) that in leading order, $\Omega_{11}(\tau) \to \ddot{r}_1(\tau)/\dot{r}_1(\tau)$ for $\tau \to 0$. Consequently, we have

$$\det \hat{\Omega}_\perp(0) = \left(\frac{\dot{r}_1(\tau)}{\ddot{r}_1(\tau)} \det \hat{\Omega}_\tau\right)_{\tau\to 0}, \tag{3.53b}$$

since the contributions of the finite elements Ω_{1j} and Ω_{j1} to $\det \hat{\Omega}$ drop out in the limit taken above. Using this relation and the definition (3.23), we may write eq. (3.55a) in the form

$$\mathscr{A} = \left[\frac{m\dot{r}_1(\tau)\ddot{r}_1(\tau) \det \hat{\omega}_0}{\pi\hbar \det\{\dot{\hat{K}}_\tau \hat{K}_\tau\}}\right]^{1/2}_{\tau\to 0}. \tag{3.55b}$$

As a technical device, we select a number z, and change the equation of motion (3.18) for the small-fluctuation operator $\hat{K}^z_\tau$ as follows:

$$\begin{aligned}
&[-\hat{1}\partial_\tau^2 + \hat{U}_\tau + z\hat{1}]\hat{K}^z_\tau = 0,\\
&\hat{K}^z_\tau \to \exp\{[\hat{\omega}_0^2 + z\hat{1}]^{1/2}\tau\}, \quad \tau \to -\infty,\\
&\mathrm{Re}[\hat{\omega}_0^2 + z\hat{1}]^{1/2} \geqslant 0.
\end{aligned} \tag{3.56}$$

In the following, we choose $z = \varepsilon$, where ε is a small quantity, $\varepsilon > 0$. Considering eq. (3.30), we introduce

$$\dot{\boldsymbol{r}}_\tau^\varepsilon = \hat{K}_\tau^\varepsilon \cdot \boldsymbol{\chi}_1^- . \tag{3.57a}$$

Note that $\dot{\boldsymbol{r}}_\tau^\varepsilon$ satisfies the small-fluctuation equation (3.12) if $\hat{U}$ is replaced by $\hat{U} + \varepsilon\hat{1}$ and that there exists a Wronskian type of relation

$$\ddot{\boldsymbol{r}}_\tau^\varepsilon \cdot \dot{\boldsymbol{r}}_\tau - \dot{\boldsymbol{r}}_\tau^\varepsilon \cdot \ddot{\boldsymbol{r}}_\tau = \varepsilon \int_{-\infty}^{\tau} \mathrm{d}\tau' \, \dot{\boldsymbol{r}}_{\tau'}^\varepsilon \cdot \dot{\boldsymbol{r}}_{\tau'} . \tag{3.57b}$$

In leading order, one may replace $\dot{\boldsymbol{r}}^\varepsilon$ by $\dot{\boldsymbol{r}}$ on the right-hand side. Therefore,

$$-\dot{\boldsymbol{r}}_0^\varepsilon \cdot \ddot{\boldsymbol{r}}_0 = \frac{\varepsilon}{2m} \mathscr{B}. \tag{3.57c}$$

We proceed now on the assumption that the limits $\tau \to 0$ and $\varepsilon \to 0$ can be interchanged. Then, we obtain from eqs. (3.55b) and (3.57c)

$$\mathscr{A} = \left[\frac{m\dot{r}_1^\varepsilon(0)\ddot{r}_1^\varepsilon(0) \det \hat{\omega}_0}{\pi\hbar \det \hat{K}_0^\varepsilon \dot{\hat{K}}_0^\varepsilon} \right]_{\varepsilon\to 0}^{1/2} = \left[\frac{(-\varepsilon)\mathscr{B} \det \hat{\omega}_0}{2\pi\hbar \det \hat{K}_0^\varepsilon \dot{\hat{K}}_0^\varepsilon} \right]_{\varepsilon\to 0}^{1/2}, \tag{3.58}$$

where we have made use of the fact that $\ddot{\boldsymbol{r}}_0^\varepsilon$ is parallel to $\boldsymbol{n}_C$ up to corrections of $O(\varepsilon)$.

We compare now the present result with the one obtained by the instanton technique (Coleman 1979, 1985). The escape path here corresponds to the bounce there if the escape path is supplied by a path of return, which is done by putting $\boldsymbol{r}_{-\tau} = \boldsymbol{r}_\tau$. Clearly, the action of a bounce is then $2\mathscr{W}_{EP}(0) = \hbar\mathscr{B}$.

In the instanton technique, the prefactor contains products of eigenvalues λ of the small-fluctuation equation. These eigenvalues are defined by

$$[-\hat{1}\partial_\tau^2 + \hat{U}_\tau] \cdot \boldsymbol{\varphi}_\tau = \lambda \boldsymbol{\varphi}_\tau, \tag{3.59}$$

where $\boldsymbol{\varphi}_\tau$ is required to remain bounded for $|\tau| \to \infty$. Since $\hat{U}_{-\tau} = \hat{U}_\tau$, the eigenfunctions can be chosen to be either odd or even functions of τ. Therefore, we must have either $\boldsymbol{\varphi}_0 = 0$ or $\dot{\boldsymbol{\varphi}}_0 = 0$.

Consider now an arbitrary complex number z, and let us construct the small-fluctuation operator $\hat{K}_\tau^z$ according to eq. (3.56). Then it is necessary and sufficient for $-z$ to be an eigenvalue λ to an odd eigenfunction that there exists a nontrivial vector $\boldsymbol{c}$ such that

$$\hat{K}_0^z \cdot \boldsymbol{c} = 0. \tag{3.60a}$$

Furthermore, $-z$ is an eigenvalue λ to an even eigenfunction only if

$$\dot{\hat{K}}_0^z \cdot \boldsymbol{d} = 0. \tag{3.60b}$$

The two relations above can be solved for $\boldsymbol{c}$ or for $\boldsymbol{d}$ nontrivially only if the appropriate determinants vanish. Therefore, $-z = \lambda$ only if

$$\det \hat{K}_0^z \dot{\hat{K}}_0^z = 0. \tag{3.60c}$$

Consider now the case where $\hat{U}_\tau$ is replaced by its asymptotic form $\hat{U}^{(0)}$. Correspondingly, we may construct the small-fluctuation operator $\hat{K}_\tau^{(0)z}$. Then we assert that

$$\frac{\det\{-\hat{1}(\partial_\tau^2 - z) + \hat{U}^{(0)}\}}{\det\{-\hat{1}(\partial_\tau^2 - z) + \hat{U}_\tau\}} = \frac{\det \hat{K}_0^{(0)z}\, \dot{\hat{K}}_0^{(0)z}}{\det \hat{K}_0^z\, \dot{\hat{K}}_0^z}, \tag{3.61}$$

where det means a determinant in continuous τ and N-dimensional configuration space.

The proof of the assertion is the same as the one given by Coleman (1979, 1985) for the case $N = 1$; it is based on a comparison of the analytical properties of both sides of eq. (3.61) as functions of z. In particular, one finds that the zeros and the poles agree, as well as the asymptotic behavior for $|z| \to \infty$.

Since $\hat{K}_\tau^{(0)z} = \exp([\hat{\omega}_0^2 + z\hat{1}]^{1/2}\tau)$, we have

$$\det \hat{K}_0^{(0)z}\, \dot{\hat{K}}_0^{(0)z} = \det[\hat{\omega}_0^2 + z\hat{1}]^{1/2}. \tag{3.62}$$

Note that in all cases of interest, $\det([\hat{\omega}_0^2 + \varepsilon\hat{1}]/\hat{\omega}_0^2) \to 1$ for $\varepsilon \to 0$. Therefore, we may write eq. (3.58) as

$$\mathscr{A} = \left(\frac{\mathscr{B}}{2\pi}\right)^{1/2} \left[-\varepsilon \frac{\det\{-\hat{1}\partial_\tau^2 + \hat{U}^{(0)}\}}{\det\{-\hat{1}\partial_\tau^2 + \hat{U}_\tau + \varepsilon\hat{1}\}} \right]_{\varepsilon\to 0}^{1/2}. \tag{3.63}$$

Recall that $\dot{\boldsymbol{r}}_\tau = -\dot{\boldsymbol{r}}_{-\tau}$ is an odd eigenfunction of eq. (3.59) with eigenvalue 0. This means that we may write eq. (3.63) in the form best known in instanton technique (Coleman 1979, 1985), namely,

$$\mathscr{A} = \left(\frac{\mathscr{B}}{2\pi}\right)^{1/2} \left[-\frac{\det\{-\hat{1}\partial_\tau^2 + \hat{U}^{(0)}\}}{\det'\{-\hat{1}\partial_\tau^2 + \hat{U}_\tau\}} \right]^{1/2}, \tag{3.64}$$

where det′ means that the eigenvalue zero has to be omitted. Thus, the prefactor [eq. (3.58)] obtained from our many-dimensional WKB technique is identical to the instanton expression (3.64).

So far we have assumed that there is only one escape path. If there are several such paths, the total decay rate is the sum of contributions of the type (3.54) and (3.64) from each escape path.

3.5. Reflected wave

Essentially, the wave function of a metastable state consists of a principle wave ψ_0, which resembles near the origin the ground state of a set of harmonic oscillators and which extends towards the caustic in a narrow tube surrounding the escape path. There, the wave is partially transmitted and it emerges as the transmitted wave $\tilde{\psi}$ in the classically accessible region. It remains now to calculate the reflected wave ψ_1. It has been shown in the one-dimensional case that ψ_1 can be obtained by analytical continuation. Considering eq. (2.33), we

perform the substitution

$$-\zeta_1 \to -e^{-2\pi i}\zeta_1 \tag{3.65}$$

in eqs. (3.38) and (3.39). Let us mark all quantities obtained by this analytical continuation with a prime, e.g., $\mathscr{W}'(\zeta) = \mathscr{W}([-\zeta_1 \to -e^{-2\pi i}\zeta_1])$. Clearly, $\mathscr{W}'$ and $\mathscr{W}'^{(1)}$ satisfy the eikonal equation (3.3) and the transport equation (3.4) to the same degree of accuracy as the original expressions do. Therefore, we conclude that the reflected wave is given by

$$\psi_1(\zeta) = \frac{1}{2}\exp\left(-\frac{1}{\hbar}[\mathscr{W}' + \mathscr{W}'^{(1)}]\right), \tag{3.66}$$

where we have inserted an additional factor* of $\frac{1}{2}$ in order to obtain from $\psi_0 + \psi_1$ the same probability current density normal to the plane $\zeta_1 \to -0$, as we do from the wave function $\tilde{\psi}(\zeta)$ at $\zeta_1 \to +0$.

Again, it is possible to obtain the wave function [eq. (3.66)] directly; in the present case, this is done by extending the method of characteristics to $\tau > 0$. Thus, the extension of the escape path is just the return path $\boldsymbol{r}_\tau = \boldsymbol{r}_{-\tau}$, and, in the small-fluctuation approximation, most of the relations of section 3.2 apply to the present case as well. For instance, $\mathscr{W}'(\eta, \tau)$ is equal to expression (3.22) for $\mathscr{W}(\eta, \tau)$ in the case $\tau > 0$. However, care has to be taken with expression (3.28) for $\mathscr{W}'^{(1)}(\tau)$ on account of the logarithmic branch point at $\tau = 0$. Specifically, we have for $\tau > 0$

$$\mathscr{W}'^{(1)}(\tau) = \mathrm{Re}\{\tfrac{1}{2}\mathrm{Tr}[\ln\{\hat{K}_\tau \exp(-\hat{\omega}_0\tau)\}]\} + i\pi. \tag{3.67}$$

Consider now the equation of motion (3.18) of the small-fluctuation operator $\hat{K}_\tau$. Since $\hat{U}_\tau = \hat{U}_{-\tau}$, we may derive the following Wronskian-type of relation

$$\dot{\hat{K}}^{\mathrm{T}}_{-\tau}\hat{K}_\tau + \hat{K}^{\mathrm{T}}_{-\tau}\dot{\hat{K}}_\tau = \mathrm{const}, \tag{3.68}$$

which will be found useful in a moment.

Next we aim to calculate the current $J(\tau)$ from $\psi = \psi_0 + \psi_1$ through a plane perpendicular to the escape path, i.e., through a plane perpendicular to $\dot{\boldsymbol{r}}_\tau$ at $\boldsymbol{r}_\tau$. We obtain the result

$$J(\tau) = \left[\frac{(\pi\hbar/m)^{N-1}}{\det\{\frac{1}{2}[\hat{\Omega}_\perp(\tau) + \hat{\Omega}_\perp(-\tau)]\}}\right]^{1/2}\left[\frac{\dot{\boldsymbol{r}}_\tau^2}{\det(\hat{K}_\tau\hat{K}_{-\tau})}\right]^{1/2}\exp(-\tfrac{1}{2}\mathscr{B}), \tag{3.69}$$

which may be compared with $J(\zeta_1)$ of eq. (3.51). Using a local basis with one axis pointing in the direction of $\dot{\boldsymbol{r}}_\tau$ and using eq. (3.68), we find after some lengthy arguments that $J(\tau)$ is independent of τ. In view of eq. (3.53a), we conclude that $J(\tau) = J(\zeta_1 = 0)$.

* See also the comment below eq. (2.33).

With increasing τ, the return path approaches the origin. Again, we assume that there is a sufficiently large environment of the origin where the potential is purely harmonic. Then, for τ large and positive, the small-fluctuation operator assumes the form

$$\hat{K}_\tau = \mathrm{e}^{\hat{\omega}_0 \tau} \hat{A} + \mathrm{e}^{-\hat{\omega}_0 \tau} \hat{B}, \tag{3.70}$$

where $\hat{A}$ and $\hat{B}$ are constant matrices. According to eq. (3.20), they have to obey the relation

$$\hat{A}^{\mathrm{T}} \hat{\omega}_0 \hat{B} = \hat{B}^{\mathrm{T}} \hat{\omega}_0 \hat{A}. \tag{3.71}$$

It has already been remarked in connection with eq. (3.20) that there is a vector χ_1^- such that $\hat{K}_\tau \cdot \chi_1^- = \dot{r}_\tau$. Since $\dot{r}_\tau \to 0$ for $\tau \to \pm\infty$, we find that $\hat{A} \cdot \chi_1^- = 0$. This means that $\det \hat{A} = 0$. On the other hand, there are no arguments which tell us that $\det \hat{B}$ should vanish. Therefore, we assume $\det \hat{B} \neq 0$ and write

$$\hat{\Omega}_\tau = \hat{\omega}_0 - 2\hat{\omega}_0 \mathrm{e}^{-\hat{\omega}_0 \tau} [\hat{C} + \hat{\omega}_0 \mathrm{e}^{-2\hat{\omega}_0 \tau}]^{-1} \hat{\omega}_0 \mathrm{e}^{-\hat{\omega}_0 \tau}, \tag{3.72}$$

where the matrix

$$\hat{C} = \hat{\omega}_0 \hat{A} \hat{B}^{-1} = \hat{C}^{\mathrm{T}} \tag{3.73}$$

is symmetric on account of eq. (3.71). Furthermore, its determinant is zero. Let χ_1^+ be the normalized eigenvector of $\hat{C}$ to the eigenvalue zero. Then, for sufficiently* large τ, we obtain

$$\hat{\Omega}_\tau - \hat{\omega}_0 = 2 \frac{\hat{\omega}_0 \mathrm{e}^{-\hat{\omega}_0 \tau} \hat{P}_1^+ \hat{\omega}_0 \mathrm{e}^{-\hat{\omega}_0 \tau}}{\chi_1^+ \cdot \hat{\omega}_0 \mathrm{e}^{-2\hat{\omega}_0 \tau} \cdot \chi^+}, \tag{3.74}$$

where $\hat{P}_1^+$ is the projector on χ_1^+. We conclude that, in general, $\mathscr{W}'(\eta, \tau)$ cannot be put in a simple form comparable with eq. (3.31).

Consider now the case where the lowest eigenvalue ω_{01} of $\hat{\omega}_0$ is smaller than the remaining ones, ω_{0n}, $n = 2, \ldots, N$, such that $\exp(-\omega_{01}\tau) \gg \exp(-\omega_{0n}\tau)$. Taking the matrix elements of eq. (3.74) with respect to the normal coordinates, we then obtain

$$\Omega_{nn'} = \delta_{nn'} \begin{cases} -\omega_{01}, & n = 1, \\ \omega_{0n}, & n = 2, \ldots, N. \end{cases} \tag{3.75}$$

Clearly, we will find in this case that $\dot{r}_n(\tau) \propto \delta_{n1} \exp(-\omega_{01}\tau)$ which means that the return path (escape path) leads to (leaves) the origin on a straight line in the direction of the low-frequency mode. Furthermore, it follows from eq. (3.26) that $\dot{\mathscr{W}}^{(1)} = -\omega_{01}$, i.e.,

$$\mathscr{W}'^{(1)} = -\omega_{01}\tau + \text{const} = \ln \bar{R}_1 + \text{const}. \tag{3.76}$$

* Precisely, the condition is $\chi_k^+ \cdot \hat{\omega}_0 \exp(-2\hat{\omega}_0\tau) \cdot \chi_k^+ \ll C_k$, where $\hat{C} \cdot \chi_k^+ = C_k \chi_k^+$ for $k = 2, \ldots, N$. Note that eq. (3.74) may also be written as $\hat{\Omega}_\tau = \hat{\omega}_0 - 2\hat{P}_\tau(\ddot{r}_\tau \cdot \dot{r}_\tau)/(\dot{r} \cdot \ddot{r})$, where $\hat{P}_\tau$ is the projector on $\ddot{r}_\tau$.

In writing down the last part of eq. (3.76), we have chosen a coordinate system $(\bar{R}_1, \ldots, \bar{R}_N)$ in the direction of the normal modes, and we have also chosen $\boldsymbol{\eta}$ in eq. (3.17) such that $\boldsymbol{\eta} \cdot \boldsymbol{r}_\tau = 0$. As the final result, we obtain

$$\psi_1(\boldsymbol{R}) = \frac{\mathrm{i}\Gamma}{2\omega_{01}} (2\pi)^{1/2} \frac{a_{01}}{\bar{R}_1} \exp\left(\frac{\bar{R}_1^2}{2a_{01}^2} - \sum_{n=2}^{N} \frac{\bar{R}_n^2}{2a_{0n}^2} \right), \tag{3.77}$$

where $a_{0n} = (\hbar/2m\omega_{0n})^{1/2}$ and Γ is given by eq. (3.54). The overall constant in eq. (3.77) has been obtained from the condition that the probability current calculated from $\psi_0 + \psi_1$ through a plane normal to the $\bar{R}_1$-direction be equal to the result of eq. (3.69).

The reflected wave of eq. (3.77) is an appropriate generalization of the one-dimensional form (2.12b) to N-dimensional systems.* The above presentation suggests that this result can be obtained only if restrictions are imposed on the frequencies of the metastable state. In particular, it appears that one slow mode has to enslave the remaining fast-moving ones. Alternatively, one may require that the escape path leaves the origin on a straight line. At present, we do not know whether there are significant corrections – if any at all – to the decay rate (presumably only to the prefactor $\mathscr{A}$), if this condition is not satisfied.**

At the end of this section, we wish to comment on the extension of the method of characteristics which we have introduced so far only in the small-fluctuation approximation. We recall that this extension has been meant to provide solutions to the eikonal equation and to the transport equation for complex coordinates. We will give arguments which show that this can be done quite generally by solving the equations of motion eqs. (3.7)–(3.9) for complex coordinates.

As an introduction, we consider first the case of a harmonic potential. There, the solution satisfying the boundary conditions of the present problem is

$$\begin{aligned} \boldsymbol{R}_\tau &= \mathrm{e}^{\hat{\omega}_0 \tau} \cdot \boldsymbol{\chi}, \\ \mathscr{W} &= \tfrac{1}{2} m \boldsymbol{\chi} \cdot \hat{\omega}_0 \mathrm{e}^{2\hat{\omega}_0 \tau} \cdot \boldsymbol{\chi}, \end{aligned} \tag{3.78}$$

where the vector $\boldsymbol{\chi}$ represents an initial direction which is arbitrary. Clearly, complex coordinates are obtained if the times τ and the initial direction $\boldsymbol{\chi}$ are chosen to be complex quantities.*** Although the mapping $(\tau; \chi_1, \ldots, \chi_N) \to (R_1, \ldots, R_N)$ is a projection which does not allow inversion, in general, it is possible to eliminate $(\tau; \chi_1, \ldots, \chi_N)$ in $\mathscr{W}$ in favor of $(R_1, \ldots, R_N)$ since there is a relation between the time and the initial directions which expresses energy conservation. For the present case this elimination is trivial on account of the

* The exponent $1 + \nu$ is now replaced by 1 since the imaginary part of the energy has been omitted in the Schrödinger equation.

** One may wonder whether this condition is necessary to guarantee a quasistationary state.

*** Actually, τ could have been chosen to be real. However, one may use the redundancy of the $(\tau, \boldsymbol{\chi})$-representation in the more complicated case as discussed below to ones advantage.

linear nature of this mapping. For instance, we may write $\chi = \exp(-\hat{\omega}_0 \tau) \cdot \boldsymbol{R}$, insert this in the expression for the action and then we obtain $\mathscr{W} = \frac{1}{2} m \boldsymbol{R} \cdot \hat{\omega}_0 \cdot \boldsymbol{R}$.

Consider now the case where the potential energy $V(\boldsymbol{R})$ is an analytical function of the coordinates; e.g., a polynomial of third degree is of sufficient complexity in the problem of quantum decay. The solution of the equations of motion (3.7) and (3.8),

$$R_k = R_k(\tau; \chi_1, \ldots, \chi_N),$$
$$\mathscr{W} = \mathscr{W}(\tau; \chi_1, \ldots, \chi_N), \tag{3.79}$$

is known to be unique if $R_k(\tau)$ is bounded. Therefore, we expect that the expressions on the right-hand side of eq. (3.79) are analytical functions of τ and also of $(\chi_1, \ldots, \chi_N)$, with the possible exception of poles. Clearly, this property allows us to conclude that relation (3.79) does not depend on the path of integration in the complex τ-plane.

The elimination procedure described above may now be applied to $\mathscr{W}$ of eq. (3.79). There, we may consider one variable, e.g., χ_N, to be fixed for reasons discussed above (energy conservation). Note, however, that this inversion fails if the corresponding Jacobian vanishes; this defines a hypersurface in $\{R_k\}$-space which should be identified with the caustic. In the space of complex coordinates, this hypersurface is of dimension $2(N-1)$, i.e., two less than the dimensions of the space itself. Therefore, the complex coordinate space will not be disconnected by the caustic.* We have made use of this property in the analytical continuation procedure where we obtain an expression for $\mathscr{W}(\boldsymbol{R})$ which should agree with eq. (3.79) in the sense explained above. Note that the caustic appears as a branching surface of $\mathscr{W}(\boldsymbol{R})$. Similar considerations apply to $\mathscr{W}^{(1)}(\boldsymbol{R})$.

3.6. The Caldeira–Leggett model

This model consists (Caldeira and Leggett 1981, 1983, 1984) of one particle (which we will call the object in the following) in a metastable potential as shown in fig. 1, which interacts with a dissipative environment. Specifically, the cubic form of eq. (2.10a, b) for a potential is considered. The environment is represented by a set of harmonic oscillators, say, with coordinates x_j, interacting with the particle by linear forces. Specifically, we take from Caldeira and Leggett (1983) the following Lagrangian:

$$L = \frac{1}{2} M \dot{q}^2 - v(q) + \frac{1}{2} \sum_{j=2}^{N} m_j \left[\dot{x}_j^2 - \omega_j^2 \left(x_j - \frac{C_j}{m_j \omega_j^2} q \right)^2 \right]. \tag{3.80}$$

Eventually, one takes the limit $N \to \infty$ such that the frequencies ω_j are distributed continuously. Of importance is the quantity [see eqs. (4.8) and (4.9) of

*See also the discussion on complex orbits in sections 2.5 and 4.1.

Caldeira and Leggett (1983)]

$$J(\omega) = \tfrac{1}{2}\pi \sum_j \frac{C_j^2}{m_j \omega_j}\, \delta(\omega - \omega_j) \to \eta\omega. \tag{3.81}$$

The equation above reveals a redundancy in the description of the environment. Therefore, we may put without loss of generality $M = m_j = m$, $C_j = m\omega_j^2$. Furthermore, we introduce a uniform notation for the coordinates $q = R_1$, $x_j = R_j, j = 2, \ldots, N$. Then eqs. (3.80) and (3.81) assume the form

$$L = \sum_{k=1}^{N} \tfrac{1}{2} m \dot{R}_k^2 - V(\{R_k\}),$$

$$V = v(R_1) + \sum_{j=2}^{N} \tfrac{1}{2} m\omega_j^2 (R_j - R_1)^2,$$

$$J(\omega) = \tfrac{1}{2}\pi m \sum_j \omega_j^3\, \delta(\omega - \omega_j) \to \eta\omega. \tag{3.82}$$

In the above representation, the environment appears to consist of a cloud of springs attached to the particle carrying masses at the other end.

By now it is clear that the Hamiltonian of the Caldeira–Leggett model is just a special case of eq. (3.1). In order to make progress in the calculation of the decay rate of the metastable state in the well at $\boldsymbol{R} \sim 0$, we have to find $\boldsymbol{r}_\tau$ of the escape path. According to eqs. (3.10) and (3.82), it obeys the equation

$$m\ddot{r}_1 = v'(r_1) - \sum_{j=2}^{N} m\omega_j^2 (r_j - r_1), \tag{3.83a}$$

$$m\ddot{r}_j = m\omega_j^2 (r_j - r_1). \tag{3.83b}$$

As already found previously, it is advantageous to supplement a return path by putting $\boldsymbol{r}_{-\tau} = \boldsymbol{r}_\tau$, $\boldsymbol{r}_{\pm\infty} = 0$. Then, one may introduce Fourier transforms,

$$\boldsymbol{r}_\tau = \int \frac{d\omega}{2\pi}\, e^{-i\omega\tau} \boldsymbol{r}_\omega, \tag{3.84}$$

and eq. (3.83b) is solved by

$$r_j(\omega) = \frac{\omega_j^2}{\omega^2 + \omega_j^2}\, r_1(\omega). \tag{3.85}$$

Substituting this result in eq. (3.83a), one obtains the following equation of motion for the coordinate r_1 of the object

$$-m\ddot{r}_1(\tau) + H * r_1 + v'(r_1(\tau)) = 0,$$

$$H * r_1 = \int d\tau'\, H(\tau - \tau') r_1(\tau'), \tag{3.86}$$

where the linear operator H is defined most conveniently by its Fourier transform

$$H_\omega = m \sum_{j=2}^{N} \frac{\omega_j^2 \omega^2}{\omega^2 + \omega_j^2} \to \eta|\omega|. \tag{3.87}$$

In the limit indicated above, H represents a friction linear in the velocity.

Caldeira and Leggett (1981, 1983, 1984) have succeeded in solving the equation $H * r_1 + v' = 0$, which is eq. (3.86) without acceleration term. It can be considered as the heavy damping limit of eq. (3.86), which is realized when $\gamma = \eta/m \gg \omega_0$. The solution is given by

$$r_1^{\mathrm{CL}}(\tau) = \frac{4R_C}{3} \frac{1}{\tau^2 \omega_B^2 + 1}, \quad \omega_B = \frac{2\pi}{\tau_B} = \frac{m\omega_0^2}{\eta} = \frac{\omega_0^2}{\gamma},$$

$$r_1^{\mathrm{CL}}(\omega) = \frac{4\pi R_C}{3\omega_B} \exp(-|\omega|/\omega_B). \tag{3.88}$$

The quantities $r_j^{\mathrm{CL}}(\omega)$ follow from eq. (3.85). We wish to add that numerical calculations (Chang and Chakravarty 1984) have shown that the above heavy damping limit is approached very smoothly.

Interesting properties of the escape path (3.88) can be calculated as follows. We define

$$\begin{aligned} \Delta(\boldsymbol{r}_\tau \cdot \boldsymbol{r}_{\tau'}) &= \boldsymbol{r}_\tau \cdot \boldsymbol{r}_{\tau'} - r_1(\tau) r_1(\tau') \\ &= \int \frac{\mathrm{d}\omega\, \mathrm{d}\omega'}{(2\pi)^2} \mathrm{e}^{-\omega\tau - \mathrm{i}\omega\tau'} \sum_{j=2}^{N} \frac{\omega_j^4}{(\omega^2 + \omega_j^2)(\omega'^2 + \omega_j^2)} r_1^{\mathrm{CL}}(\omega) r_1^{\mathrm{CL}}(\omega') \\ &= \frac{4\gamma R_C^2}{9\omega_B^2} \int \mathrm{d}\omega\, \mathrm{d}\omega' \frac{\exp[-\mathrm{i}\omega\tau - \mathrm{i}\omega'\tau' - (|\omega| + |\omega'|)/\omega_B]}{|\omega| + |\omega'|}, \end{aligned} \tag{3.89}$$

where we have obtained the last line of this equation by making use of eq. (3.88). One recognizes that eq. (3.89) can also be written as

$$\begin{aligned} \Delta(\boldsymbol{r}_\tau \cdot \boldsymbol{r}_{\tau'}) &= \frac{4\gamma R_C^2}{9\omega_B^2} \int_0^{\omega_B} \mathrm{d}\omega \frac{4}{(\tau^2\omega^2 + 1)(\tau'^2\omega^2 + 1)} \\ &= \frac{4\gamma R_C^2}{9\omega_B^2} \begin{cases} \omega_B[4 - 3\omega_B^2(\tau^2 + \tau'^2)], & |\omega_B\tau|, |\omega_B\tau'| \ll 1, \\ 2\pi/(|\tau| + |\tau'|), & |\omega_B\tau|, |\omega_B\tau'| \gg 1. \end{cases} \end{aligned} \tag{3.90}$$

In particular, we have near the origin, $\Delta(r_\tau^2) \gg r_1^2(\tau)$; this means that the escape path starts perpendicular to the R_1-axis.

Using eq. (3.90), we may also calculate the curvature κ of the escape path

$$\kappa_\tau^2 = \frac{(\dot{r}_\tau)^2(\ddot{r}_\tau)^2 - (\dot{r}_\tau \cdot \ddot{r}_\tau)^2}{[(\dot{r}_\tau)^2]^3} = \begin{cases} \text{const}, & |\omega_B\tau| \ll 1, \\ |9\tau\omega_B^2|/(8\pi R_C^2\gamma), & |\omega_B\tau| \gg 1. \end{cases} \tag{3.91}$$

We recognize that this path leaves the origin with infinite curvature.

Very schematically, the escape path of fig. 8 may be thought of describing the present situation if we identify $-r_2(\tau)$ there with $[\Delta(r_\tau^2)]^{1/2}$ here. It is interesting to note that at the escape point* where $R_{C1} = r_1^{CL}(\tau = 0) = 4R_C/3 > R_C$, the particle has lost the potential energy $v(4R_C/3) = -4v_B$; this energy is now stored in the springs.

Considering definitions (3.14) and (3.54), we calculate

$$\hbar\mathscr{B}^{CL} = 2\int_{-\infty}^{0} d\tau\, m(\dot{r}_\tau^{CL})^2 = \int \frac{d\omega}{2\pi}\, m\omega^2 |r_\omega^{CL}|^2 = \frac{2\pi\eta R_C^2}{9} + \frac{4\pi m\omega_B R_C^2}{9}. \quad (3.92)$$

In the limit $m \to 0$, this agrees with the result of Caldeira and Leggett (1981, 1983, 1984).

Concerning the prefactor $\mathscr{A}$, note that the square root of the ratio of determinants in eq. (3.63) can be expressed as the ratio of two Gaussian functional integrals with respect to N-dimensional paths.** Observe now the simple structure of $\hat{U}_\tau$ in the Caldeira–Leggett model, where all matrix elements but one are time-independent. Therefore, it is possible to perform the functional integrals with respect to $N-1$ components of the paths ("integrating out the coordinates of the environment"). The result of this operation is expressed by the relation

$$\frac{\det\{-\hat{1}\partial_\tau^2 + \hat{U}^{(0)}\}}{\det\{-\hat{1}(\partial_\tau^2 - \varepsilon) + \hat{U}_\tau\}} = \frac{\det\{-m\partial_\tau^2 + H + m\omega_0^2\}}{\det\{-m(\partial_\tau^2 - \varepsilon) + H^\varepsilon + v''[r_1(\tau)]\}}, \quad (3.93)$$

where the operator H is defined by eq. (3.86) and H^ε is obtained by replacing $\omega^2 \to \omega^2 + \varepsilon$ in eq. (3.87). Therefore,

$$H_\omega^\varepsilon = H_\omega + \varepsilon \sum_j \frac{m\omega_j^4}{(\omega^2 + \omega_j^2)^2} + O(\varepsilon^2). \quad (3.94)$$

Concerning the operator $-m(\partial_\tau^2 - \varepsilon) + H^\varepsilon + v''$, note that in the case of $\varepsilon = 0$, it has an eigenvalue zero which belongs to the eigenfunction $\dot{r}_1(\tau)$. Therefore, we may calculate for finite ε, the eigenvalue λ^ε closest to zero by perturbation theory:

$$\lambda^\varepsilon = \frac{\varepsilon}{\mathscr{B}_0}\int \frac{d\omega}{2\pi} m\left[1 + \sum_j \frac{\omega_j^4}{(\omega^2 + \omega_j^2)^2}\right]\omega^2 |r_1(\omega)|^2 = \varepsilon\frac{\mathscr{B}}{\mathscr{B}_0}, \quad (3.95)$$

where $\hbar\mathscr{B} = 2\mathscr{W}_{EP}(0)$ and the normalization is given by

$$\hbar\mathscr{B}_0 = \hbar\mathscr{B}_0\{[r_1(\tau)]\} = 2m\int_{-\infty}^{0} d\tau[\dot{r}_1(\tau)]^2. \quad (3.96)$$

* Note that in the present context, R_C of eq. (2.10) is just a parameter which has nothing to do with the one coordinate $R_{C1} = 4R_C/3$ of the escape point.

** The problem of a negative eigenvalue in the denominator determinant can be handled by deforming the contour of integration properly.

Thus, we conclude that the prefactor can be expressed alternatively as

$$\mathscr{A} = \left(\frac{\mathscr{B}_0}{2\pi m}\right)^{1/2} \left[-\frac{\det\{-m\partial_\tau^2 + H + m\omega_0^2\}}{\det'\{-m\partial_\tau^2 + H + v''\}} \right]^{1/2}. \tag{3.97}$$

This form is well-established in the literature (Caldeira and Leggett 1981, 1983, 1984, Chang and Chakravarty 1984, Grabert and Weiss 1984).

Although the present formalism of calculating decay rates does not raise any problems, we should observe that the Caldeira–Leggett model does not conform so well with the concept of a quasistationary decaying state on account of the presence of arbitrarily small frequency modes. For the same reasons, the escape path leaves the origin with infinite curvature. This seems to leave some questions open with respect to the concept of a quasistationary state.

4. *Statistical matrix and multidimensional decay at finite temperatures*

4.1. *Decay at finite temperatures in N dimensions*

In the discussion of section 2.5 we have argued that our theory requires closed extremal orbits which connect the metastable region (A) with the outside region (C) and which are traversed in time $\tau_1 = \hbar/kT$. We have also pointed out that only an extension to complex extremal paths provides sufficient flexibility to meet these requirements in the multidimensional problem.

An attempt to describe the properties of such a complex orbit in a multidimensional space should be guided by the experience we have gained in the zero temperature ($\tau_1 = \infty$) case of section 3. There, an essential part of the theory is the escape path which connects the bottom of the well with the outside region at the escape point $\boldsymbol{R}_C$ (fig. 8). A closed orbit (bounce) is obtained by adding a return path which retraces the escape path. Obviously, the total time needed to complete such an orbit is $\tau_1 = \infty$. We have also learned in section 3.3 that the escape path can be extended in region C by letting $\tau \to it$; see, e.g., eq. (3.43). This procedure reminds us of the discussion in the second part of section 2.5 which has been illustrated by fig. 5.

It is fairly obvious that at finite temperatures $kT = \hbar/\tau_1$, we need an escape path (see fig. 11) which connects in the time $\tau_1/2$ the turning points on two surfaces of the same potential energy, but surfaces pertaining to the well region A and outside region C, respectively. Again, this escape path together with its return path forms a closed orbit which will be completed in time τ_1. We may also argue that two external legs can be attached at $\boldsymbol{R}_{2,3}$ by letting $\tau \to it$. The corresponding diagram in space and complex time is shown in fig. 12a, b; it should be compared with fig. 5. Evidently, for $\boldsymbol{R}_1 = \boldsymbol{R}$ we have $\boldsymbol{P}_1 = \boldsymbol{P}$; the

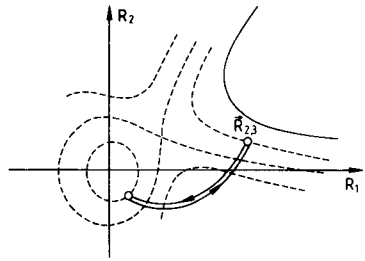

Fig. 11. Periodic orbit under the barrier completed in time τ_1. The endpoints are turning points on surfaces of equal potential energy.

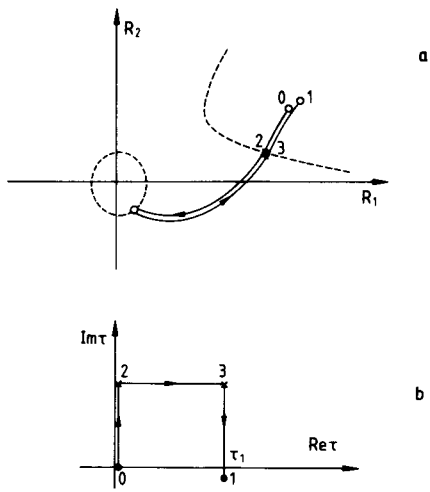

Fig. 12. (a) Real path under the barrier ($2 \to 3$) in time τ_1 and real paths in the outside world ($0 \to 2$, $3 \to 1$) for imaginary τ. (b) The corresponding contour in the complex τ-plane. (For $\boldsymbol{R}_1 = \boldsymbol{R}_0 \equiv \boldsymbol{R}$, the endpoint of the contour is on the real axis.) Note the corners at the classical turning points.

trajectory is closed and also periodic in time τ_1. Now, we assume that this combination of escape and return paths augmented by external legs can be deformed continuously in the multidimensional complex planes $\boldsymbol{R} = \{R_k\}$ such that for real τ and $0 \leqslant \tau \leqslant \tau_1$ it is the solution of the equation*

$$m\ddot{\boldsymbol{r}}_\tau = \frac{\partial V}{\partial \boldsymbol{r}_\tau}. \tag{4.1}$$

* It differs from eq. (3.10) by different boundary conditions.

We expect that this solution is an analytic function of τ; moreover, we expect that such an orbit exists for any choice of the starting point $\boldsymbol{R}$ and that the action $\mathscr{S}([\boldsymbol{r}_\tau]; \tau_1)$ depends only on the topology of the orbit. We will give some arguments in favor of this assumption at the end of Appendix C; moreover, it can be verified for the Caldeira–Leggett model by analytical (see section 4.3) and numerical (Ludviksson 1989) calculations.

With only a moderate increase in the action $\mathscr{S}$, there are also closed extremal paths $\bar{\boldsymbol{r}}_\tau$ in the multidimensional vicinity of the periodic orbit which contribute also to the outgoing current. To evaluate their contribution, we put

$$\bar{\boldsymbol{r}}_\tau = \boldsymbol{r}_\tau + \boldsymbol{x}_\tau, \quad \boldsymbol{x}_0 = \boldsymbol{x}_{\tau_1} = \boldsymbol{\rho}, \tag{4.2}$$

where $\boldsymbol{x}_\tau$ is small. We now expand [cf. eq. (3.15) and below]

$$V(\bar{\boldsymbol{r}}_\tau) = V(\boldsymbol{r}_\tau) + (\boldsymbol{x}_\tau \cdot \partial_{\boldsymbol{r}_\tau}) V(\boldsymbol{r}_\tau) + \tfrac{1}{2}(\boldsymbol{x}_\tau \cdot \partial_{\boldsymbol{r}_\tau})^2 V(\boldsymbol{r}_\tau) + \cdots, \tag{4.3}$$

and obtain the action through second order in $\boldsymbol{x}_\tau$:

$$\mathscr{S}([\bar{\boldsymbol{r}}_\tau]; \tau_1) = \mathscr{S}([\boldsymbol{r}_\tau]; \tau_1) + \int_0^{\tau_1} \mathrm{d}\tau \{ m\dot{\boldsymbol{r}}_\tau \cdot \dot{\boldsymbol{x}}_\tau + \tfrac{1}{2} m\dot{\boldsymbol{x}}_\tau^2 + (\boldsymbol{x}_\tau \cdot \partial_{\boldsymbol{r}_\tau}) V(\boldsymbol{r}_\tau) + \tfrac{1}{2}(\boldsymbol{x}_\tau \cdot \partial_{\boldsymbol{r}_\tau})^2 V(\boldsymbol{r}_\tau) \}. \tag{4.4}$$

Taking into account the periodicity of the orbit $\boldsymbol{r}_\tau$ as well as its equation of motion (4.1), we obtain after integration by parts

$$\mathscr{S}([\bar{\boldsymbol{r}}_\tau]; \tau_1) - \mathscr{S}([\boldsymbol{r}_\tau]; \tau_1) = \tfrac{1}{2} m \int_0^{\tau_1} \mathrm{d}\tau (\dot{\boldsymbol{v}}_\tau^2 + \boldsymbol{v}_\tau \cdot \hat{U}_\tau \cdot \boldsymbol{v}_\tau), \tag{4.5}$$

where $\boldsymbol{v}_\tau$ is the first-order approximation to $\boldsymbol{x}_\tau$, which satisfies the small-fluctuation equation

$$[-\hat{1}\partial_\tau^2 + \hat{U}_\tau] \cdot \boldsymbol{v}_\tau = 0 \tag{4.6}$$

and the boundary condition

$$\boldsymbol{v}_0 = \boldsymbol{v}_{\tau_1} = \boldsymbol{\rho}. \tag{4.7}$$

Above, we have introduced the operator $\hat{U}_\tau$ which is defined analogous to eq. (3.13)

$$(\hat{U}_\tau)_{kk'} = \frac{1}{m} \frac{\partial^2 V}{\partial R_k \partial R_{k'}}\bigg|_{\boldsymbol{R}=\boldsymbol{r}_\tau} \tag{4.8}$$

but for a different extremal path.

Let us choose a linearly independent set of solutions to the small-fluctuation equation as follows:

$$[-\hat{1}\partial_\tau^2 + \hat{U}_\tau]\begin{Bmatrix}\hat{K}_\tau \\ \hat{M}_\tau\end{Bmatrix} = 0, \tag{4.9}$$

where $\hat{K}_\tau$ and $\hat{M}_\tau$ satisfy the initial conditions

$$\hat{K}_0 = 0, \qquad \dot{\hat{K}}_0 = \hat{1}, \qquad \hat{M}_0 = \hat{1}, \qquad \dot{\hat{M}}_0 = 0. \tag{4.10}$$

In terms of $\hat{K}_\tau$ and $\hat{M}_\tau$ we may write $\boldsymbol{v}_\tau$ [cf. eq. (3.21)] as follows:

$$\boldsymbol{v}_\tau = \{\hat{K}_\tau \hat{K}_1^{-1}[\hat{1} - \hat{M}_1] + \hat{M}_\tau\}\cdot\boldsymbol{\rho}. \tag{4.11}$$

Above, we have introduced

$$\hat{K}_1 = \hat{K}_{\tau_1}, \qquad \hat{M}_1 = \hat{M}_{\tau_1} \tag{4.12}$$

and have assumed that $\det \hat{K}_1 \neq 0$.

In eq. (4.5) we integrate by parts again and observe that $\boldsymbol{v}_\tau$ satisfies eqs. (4.6) and (4.7). Then, we obtain

$$\mathscr{S}([\bar{\boldsymbol{r}}_\tau]; \tau_1) - \mathscr{S}([\boldsymbol{r}_\tau]; \tau_1) = \tfrac{1}{2} m \boldsymbol{v}_0 \cdot (\dot{\boldsymbol{v}}_{\tau_1} - \dot{\boldsymbol{v}}_0). \tag{4.13}$$

Considering eqs. (4.7) and (4.11), we may write

$$\mathscr{S}([\bar{\boldsymbol{r}}_\tau]; \tau_1) = \mathscr{S}([\boldsymbol{r}_\tau]; \tau_1) + \tfrac{1}{2} m \boldsymbol{\rho}\cdot\hat{\Omega}\cdot\boldsymbol{\rho}, \tag{4.14}$$

where

$$\hat{\Omega} = -[\hat{1} - \dot{\hat{K}}_1]\hat{K}_1^{-1}[\hat{1} - \hat{M}_1] + \dot{\hat{M}}_1. \tag{4.15}$$

Obviously, $\hat{\Omega}$ is a symmetric matrix. Furthermore, since $\dot{\boldsymbol{r}}_\tau$ is a periodic solution of the small-fluctuation equation (4.6), the vector $\dot{\boldsymbol{r}}_0$ $(= \dot{\boldsymbol{r}}_{\tau_1})$ is an eigenvector of $\hat{\Omega}$ with eigenvalue zero. For convenience, let us choose a representation where the first basis vector $\boldsymbol{e}_1$ is parallel to $\dot{\boldsymbol{r}}_0$. Then

$$\hat{\Omega}\cdot\boldsymbol{e}_1 = 0, \quad \boldsymbol{e}_1 \parallel \dot{\boldsymbol{r}}_0. \tag{4.16}$$

This means that we have

$$\Omega_{1k} = \Omega_{k1} = 0, \quad k = 1, 2, \ldots, N. \tag{4.17}$$

4.2. Gaussian fluctuations about periodic orbits

To complete the quasiclassical approximation to the statistical matrix $\rho(\boldsymbol{R} + \boldsymbol{\rho}, \boldsymbol{R} + \boldsymbol{\rho}; \tau_1)$ we must calculate the prefactor due to Gaussian fluctuations about $\bar{\boldsymbol{r}}_\tau = \boldsymbol{r}_\tau + \boldsymbol{x}_\tau$. In view of the approximation $\boldsymbol{x}_\tau \to \boldsymbol{v}_\tau$, this prefactor will be, to lowest order, the same as for the periodic orbit $\boldsymbol{r}_\tau$.

Since $\boldsymbol{r}_\tau$ and, hence, the operator $\hat{U}_\tau$ is complex-valued, the calculation of the Gaussian fluctuations follows somewhat unusual lines. A complete calculation

has been carried through for the one-dimensional case in Appendix C. The outcome is fairly simple: the final result is the same as for real-valued operators $\hat{U}_\tau$ provided that we replace the determinant det, which means the product of all eigenvalues, by $\det^+$, which now includes only positive eigenvalues as explained by eqs. (C.47), and (C.48).

For the sake of simplicity, therefore, we will consider $\hat{U}_\tau$ to be real in the following discussions, which allows us to disregard the details mentioned above. Generalization to complex $\hat{U}_\tau$ follows the arguments of Appendix C and is straightforward though lengthy.

We introduce $\hat{K}_\tau^z$ and $\hat{M}_\tau^z$ that are solutions of

$$\{-\hat{1}\partial_\tau^2 + \hat{U}_\tau + z\hat{1}\}\begin{Bmatrix}\hat{K}_\tau^z\\ \hat{M}_\tau^z\end{Bmatrix} = 0 \tag{4.18}$$

subject to the initial conditions

$$\hat{K}_0^z = 0, \qquad \dot{\hat{K}}_0^z = \hat{1}, \qquad \hat{M}_0^z = \hat{1}, \qquad \dot{\hat{M}}_0^z = 0. \tag{4.19}$$

First let us consider the eigenvalue problem

$$[-\hat{1}\partial_\tau^2 + \hat{U}_\tau]\boldsymbol{\varphi}_\tau = \lambda\boldsymbol{\varphi}_\tau, \tag{4.20}$$

where the eigenfunction $\boldsymbol{\varphi}_\tau$ satisfies zero boundary conditions

$$\boldsymbol{\varphi}_0 = \boldsymbol{\varphi}_{\tau_1} = 0. \tag{4.21}$$

For $\lambda = -z$, we may write any solution of the differential equation (4.20) which vanishes at $\tau = 0$ as follows:

$$\boldsymbol{\varphi}_\tau = \hat{K}_\tau^z \cdot \boldsymbol{a}. \tag{4.22}$$

For $-z$ to be an eigenvalue λ we must demand ($\hat{K}_1^z = \hat{K}_{\tau_1}^z$, etc.)

$$\hat{K}_1^z \cdot \boldsymbol{a} = 0. \tag{4.23}$$

This equation has a nontrivial solution only if

$$\det \hat{K}_1^z = 0. \tag{4.24}$$

Similar relations may be derived in the case where $\hat{U}_\tau$ is replaced by $\hat{U}^{(0)}$ as defined by [see also eq. (3.2)]

$$\hat{U}_{kk'}^{(0)} = \frac{1}{m}\frac{\partial^2 V}{\partial R_k \partial R_{k'}}\bigg|_{R=0} \tag{4.25}$$

and where the matrices $\hat{K}_\tau^{(0)}$ and $\hat{M}_\tau^{(0)}$ are constructed according to eqs. (4.9) and (4.10) with $\hat{U}^{(0)}$ replacing $\hat{U}_\tau$.

We assert that

$$\frac{\prod_n(\lambda_n^{(0)} + z)}{\prod_n(\lambda_n + z)} = \frac{\det\{-\hat{1}\partial_\tau^2 + \hat{U}^{(0)} + \hat{1}z\}}{\det\{-\hat{1}\partial_\tau^2 + \hat{U}_\tau + \hat{1}z\}} = \frac{\det \hat{K}_1^{(0)z}}{\det \hat{K}_1^z}. \tag{4.26}$$

The proof of the assertion is based on the fact that both sides have the same zeros and poles and approach unity as $|z| \to \infty$. Thus, we have for zero boundary (ZB) conditions, taking $z = 0$ in the above equation,

$$R_{\mathrm{ZB}} = \frac{\det\{-\hat{1}\partial_\tau^2 + \hat{U}^{(0)}\}}{\det\{-\hat{1}\partial_\tau^2 + \hat{U}_\tau\}}\bigg|_{\mathrm{ZB}} = \frac{\det \hat{K}_1^{(0)}}{\det K_1}. \tag{4.27}$$

Next we consider the same eigenvalue problem [eq. (4.20)] but this time with periodic boundary conditions. Although the eigenvalues and eigenfunctions are different now, we retain the labeling λ and $\boldsymbol{\varphi}$.

The eigenfunctions now satisfy

$$\boldsymbol{\varphi}_0 = \boldsymbol{\varphi}_{\tau_1}, \qquad \dot{\boldsymbol{\varphi}}_0 = \dot{\boldsymbol{\varphi}}_{\tau_1}. \tag{4.28}$$

Now we must write an arbitrary solution of eq. (4.20) as follows:

$$\boldsymbol{\varphi}_\tau = \hat{K}_\tau^z \cdot \boldsymbol{c} + \hat{M}_\tau^z \cdot \boldsymbol{d}. \tag{4.29}$$

For $-z$ to be an eigenvalue λ we must require that the boundary conditions (4.28) are satisfied, i.e.,

$$\begin{aligned} \hat{K}_1^z \cdot \boldsymbol{c} + \hat{M}_1^z \cdot \boldsymbol{d} &= \boldsymbol{d}, \\ \dot{\hat{K}}_1^z \cdot \boldsymbol{c} + \dot{\hat{M}}_1^z \cdot \boldsymbol{d} &= \boldsymbol{c}, \end{aligned} \tag{4.30}$$

which is equivalent to

$$\det\begin{pmatrix} \hat{1} - \hat{M}_1^z & -\hat{K}_1^z \\ -\dot{\hat{M}}_1^z & \hat{1} - \dot{\hat{K}}_1^z \end{pmatrix} = 0. \tag{4.31}$$

We deduce by the usual argument that for periodic boundary conditions

$$\frac{\det\{-\hat{1}\partial_\tau^2 + \hat{U}^{(0)} + \hat{1}z\}}{\det\{-\hat{1}\partial_\tau^2 + \hat{U}_\tau + \hat{1}\hat{z}\}}\bigg|_{\mathrm{PB}} = \frac{\det\begin{pmatrix} \hat{1} - \hat{M}_1^{(0)z} & -\hat{K}_1^{(0)z} \\ -\dot{\hat{M}}_1^{(0)z} & \hat{1} - \dot{\hat{K}}_1^{(0)z} \end{pmatrix}}{\det\begin{pmatrix} \hat{1} - \hat{M}_1^z & -\hat{K}_1^z \\ -\dot{\hat{M}}_1^z & \hat{1} - \dot{\hat{K}}_1^z \end{pmatrix}}. \tag{4.32}$$

Let us now assume that $\hat{K}_1^z$ is nonsingular. In that case the system (4.30) is equivalent to the existence of a nontrivial solution of

$$\{-[\hat{1} - \dot{\hat{K}}_1^z][\hat{K}_1^z]^{-1}[\hat{1} - \hat{M}_1^z] + \dot{\hat{M}}_1^z\} \cdot \boldsymbol{d} = 0 \tag{4.33}$$

Consider now the function of the complex variable w

$$\frac{(\det \hat{K}_1^z)\det\{[w\hat{1} - \dot{\hat{K}}_1^z][\hat{K}_1^z]^{-1}[w\hat{1} - \hat{M}_1^z] - \hat{M}_1^z\}}{\det\begin{pmatrix} w\hat{1} - \hat{M}_1^z & -\hat{K}_1^z \\ -\dot{\hat{M}}_1^z & w\hat{1} - \dot{\hat{K}}_1^z \end{pmatrix}}. \tag{4.34}$$

The zeros of numerator and denominator coincide and for $|w| \to \infty$ their ratio approaches unity. Hence, it is equal to one everywhere and we may write

$$\left.\frac{\det\{-\hat{1}\partial_\tau^2 + \hat{U}^{(0)} + z\hat{1}\}}{\det\{-\hat{1}\partial_\tau^2 + \hat{U}_\tau + z\hat{1}\}}\right|_{\mathrm{PB}} = \frac{(\det \hat{K}_1^{(0)z})\det \hat{\Omega}^{(0)z}}{(\det \hat{K}_1^z)\det \hat{\Omega}^z}, \tag{4.35}$$

where, in accordance with eq. (4.15) we have put

$$\hat{\Omega}^z = -[\hat{1} - \dot{\hat{K}}_1^z][\hat{K}_1^z]^{-1}[\hat{1} - \hat{M}_1^z] + \dot{\hat{M}}_1^z. \tag{4.36}$$

We know that the small-fluctuation-operator with periodic boundary conditions has an eigenvalue zero (corresponding to the eigenfunction $\dot{\boldsymbol{r}}_\tau$), which has to be omitted. To take account of this, we consider

$$R_{\mathrm{PB}} = \left.\frac{\det\{-\hat{1}\partial_\tau^2 + \hat{U}^{(0)}\}}{\det'\{-\hat{1}\partial_\tau^2 + \hat{U}_\tau\}}\right|_{\mathrm{PB}} = \lim_{\varepsilon\to 0}\left.\frac{\varepsilon\det\{-\hat{1}\partial_\tau^2 + \hat{U}^{(0)}\}}{\det\{-\hat{1}\partial_\tau^2 + \hat{U}_\tau + \varepsilon\hat{1}\}}\right|_{\mathrm{PB}}, \tag{4.37}$$

where, again, the determinants are calculated for periodic boundary conditions and the prime denotes omission of the zero eigenvalue.

Due to the property (4.16) we may write

$$R_{\mathrm{PB}} = \frac{(\det \hat{K}_1^{(0)})\det \hat{\Omega}^{(0)}}{\alpha(\det \hat{K}_1)\det_\perp \hat{\Omega}}, \tag{4.38}$$

where the subscript $\perp$ denotes restriction to the subspace perpendicular to $\dot{\boldsymbol{r}}_0$. Furthermore,

$$\alpha = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{-[\hat{1} - \dot{\hat{K}}_1^\varepsilon][\hat{K}_1^\varepsilon]^{-1}[\hat{1} - \hat{M}_1^\varepsilon] + \dot{\hat{M}}_1^\varepsilon\}_{11}. \tag{4.39}$$

To evaluate α, we consider the vector

$$\boldsymbol{v}_\tau^\varepsilon = \{\hat{K}_\tau^\varepsilon[\hat{K}_1^\varepsilon]^{-1}[\hat{1} - \hat{M}_1^\varepsilon] + \hat{M}_\tau^\varepsilon\}\cdot \boldsymbol{e}_1, \tag{4.40}$$

which, according to eqs. (4.6) and (4.11) is a closed solution of

$$[-\partial_\tau^2\hat{1} + \hat{U}_\tau + \varepsilon\hat{1}]\cdot \boldsymbol{v}_\tau^\varepsilon = 0. \tag{4.41}$$

In terms of $\boldsymbol{v}_\tau^\varepsilon$ we may express α as follows:

$$\alpha = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\boldsymbol{e}_1\cdot[\dot{\boldsymbol{v}}_{\tau_1}^\varepsilon - \dot{\boldsymbol{v}}_0^\varepsilon]. \tag{4.42}$$

On the other hand, we have

$$\partial_\tau\{\boldsymbol{v}_\tau^\varepsilon\cdot\dot{\boldsymbol{v}}_\tau - \dot{\boldsymbol{v}}_\tau^\varepsilon\cdot\boldsymbol{v}_\tau\} = -\varepsilon\boldsymbol{v}_\tau^\varepsilon\cdot\boldsymbol{v}_\tau, \tag{4.43}$$

where $\boldsymbol{v}_\tau$ is defined as in eq. (4.40) with $\varepsilon = 0$. Integrating eq. (4.43) and observing that $\boldsymbol{v}_0 = \boldsymbol{v}_{\tau_1}$, we find to lowest order

$$(\dot{\boldsymbol{v}}_{\tau_1}^\varepsilon - \dot{\boldsymbol{v}}_0^\varepsilon)\cdot\boldsymbol{v}_0 = \varepsilon\int_0^{\tau_1} \mathrm{d}\tau\, \boldsymbol{v}_\tau\cdot\boldsymbol{v}_\tau. \tag{4.44}$$

Eventually, we conclude that

$$\alpha = \int_0^{\tau_1} \mathrm{d}\tau (\boldsymbol{v}_\tau \cdot \boldsymbol{v}_\tau) = \frac{1}{|\dot{\boldsymbol{r}}_0|^2} \int_0^{\tau_1} \mathrm{d}\tau \, |\dot{\boldsymbol{r}}_\tau|^2. \tag{4.45}$$

We are now in a position to write down the quasiclassical approximation to the statistical matrix at $\boldsymbol{R}_1 = \boldsymbol{R} + \boldsymbol{\rho}$, $\boldsymbol{R} = \boldsymbol{r}_0 = \boldsymbol{r}_1$:

$$\rho(\boldsymbol{R} + \boldsymbol{\rho}, \boldsymbol{R} + \boldsymbol{\rho}; \tau_1) = \left(\det \frac{m\hat{\omega}_0}{2\pi\hbar \sinh \hat{\omega}_0 \tau_1} \right)^{1/2} \left[\frac{\det \hat{K}_1^{(0)}}{\det \hat{K}_1} \right]^{1/2} \times \exp\left(-\frac{1}{\hbar} \mathscr{S}([\boldsymbol{r}_\tau]; \tau_1) - \frac{m}{2\hbar} \boldsymbol{\rho} \cdot \hat{\Omega} \cdot \boldsymbol{\rho} \right), \tag{4.46}$$

where again $\hat{\omega}_0 = (\hat{U}^{(0)})^{1/2}$. We have chosen a representation where a ratio of determinants appears since it is easier to handle. That the overall constant is chosen properly can be seen by inserting the trivial orbit $\boldsymbol{r}_\tau = 0$, where $\hat{U}_\tau = \hat{U}^{(0)}$. Then one obtains $\rho(0, 0; \tau_1) = [\det(m\hat{\omega}_0 / 2\pi\hbar \sinh \hat{\omega}_0 \tau_1)]^{1/2}$, which is known to be valid for harmonic oscillators.

Eventually, we need the total current flowing through a plane perpendicular to $\boldsymbol{e}_1$ which is defined in eq. (4.16). Since $\hat{\Omega} \cdot \boldsymbol{e}_1 = 0$, we need to take the derivative in the exponent only with respect to $\mathscr{S}([\boldsymbol{r}_\tau]; \tau_1)$. This operation produces the factor

$$(\partial_{\boldsymbol{R}} - \partial_{\boldsymbol{R}_1}) \mathscr{S}([\boldsymbol{r}_\tau]; \tau_1)|_{\boldsymbol{R}_1 = \boldsymbol{R}} = 2\boldsymbol{P} = 2m\dot{\boldsymbol{r}}_0. \tag{4.47}$$

Thus, we obtain the decay rate

$$\begin{aligned} \Gamma &= \frac{1}{Z_0} \int \mathrm{d}^{N-1} \rho_\perp (\boldsymbol{e}_1 \cdot \boldsymbol{j}) \\ &= \frac{1}{Z_0} \left[\frac{m}{2\pi\hbar} \det \left(\frac{\hat{\omega}_0}{\sinh \hat{\omega}_0 \tau_1} \right) \right]^{1/2} \left[\frac{\det \hat{K}_1^{(0)}}{\det \hat{K}_1} \right]^{1/2} \frac{|\dot{\boldsymbol{r}}_0|}{(\det{}_\perp \hat{\Omega})^{1/2}} \\ &\quad \times \exp\left(-\frac{1}{\hbar} \mathscr{S}[(\boldsymbol{r}_\tau]; \tau_1) \right). \end{aligned} \tag{4.48}$$

Here, in analogy with the one-dimensional case, Z_0 is the partition function corresponding to the harmonic approximation of $V(\boldsymbol{R})$ at the origin. Hence,

$$Z_0^{-1} = \det(2 \sinh \hat{\omega}_0 \tau_1 / 2). \tag{4.49}$$

Furthermore, we note that

$$\hat{K}_\tau^{(0)} = \frac{1}{\hat{\omega}_0} \sinh \hat{\omega}_0 \tau_1, \qquad \hat{M}_\tau^{(0)} = \cosh \omega_0 \tau_1 . \tag{4.50}$$

Considering eq. (4.36), we obtain

$$\hat{\Omega}^{(0)} = 2\hat{\omega}_0 \tanh \hat{\omega}_0 \tau_1 / 2 . \tag{4.51}$$

Combining suitably eqs. (4.37), (4.38), (4.45), (4.49) and (4.51) and inserting it into eq. (4.48), we obtain a decay rate which can be written in the standard form $\bar{\Gamma} = \mathscr{A} e^{-\mathscr{B}}$ of eq. (2.2), where the exponent is given by

$$\mathscr{B} = \frac{1}{\hbar} \mathscr{S}([\boldsymbol{r}_\tau]; \tau_1), \tag{4.52}$$

whereas the prefactor

$$\mathscr{A} = \left(\frac{\mathscr{S}_0([\boldsymbol{r}_\tau]; \tau_1)}{2\pi\hbar} \right)^{1/2} \left| \frac{\det(-\partial_\tau^2 \hat{1} + \hat{U}^{(0)})}{\det'(-\partial_\tau^2 \hat{1} + \hat{U}_\tau)} \right|_{\mathrm{PB}}^{1/2}. \tag{4.53}$$

In addition, we have introduced the quantity

$$\mathscr{S}_0([\boldsymbol{R}_\tau]; \tau_1) = m \int_0^{\tau_1} \mathrm{d}\tau |\dot{\boldsymbol{R}}_\tau|^2. \tag{4.54}$$

Note one peculiarity of this result. At first, we had to calculate the Gaussian fluctuations for zero boundary condition (ZB). Now, it has been found that the contribution to the current from the closed paths in the vicinity of the periodic orbit has, in effect, changed the requirement of zero boundary condition to the one of periodic boundary condition (PB). This allows us to interpret the above quantities as being part of a contribution $Z_1 = (i\tau_1/2) Z_0 \mathscr{A} \exp(-\mathscr{B})$ to the partition function that leads to an imaginary part of the free energy $\mathscr{F}$ and from there – see eq. (2.47) – to the decay rate $\bar{\Gamma}$.

Thus, the relations above confirm and generalize Affleck's theory and interpretation to the multidimensional case. Specifically, they mean a generalization of eqs. (2.51) and (2.53). In the limit $T \to 0$ ($\tau_1 \to \infty$, $\mathscr{S}_0 \to \mathscr{S} \to 2\mathscr{W}_{\mathrm{EP}}(0)$), this relation agrees also with eqs. (3.54) and (3.64). It is this form (4.53) which emerges naturally (although less rigorously) from instanton-type calculations and forms the starting-point of many finite-temperature calculations in the literature (see, e.g., Larkin and Ovchinnikov 1992).

4.3. The Caldeira–Leggett model at finite temperatures

In what follows, we wish to apply the theory developed in sections 2.5, 2.6, 4.1 and 4.2 to the Caldeira–Leggett model as presented in section 3.6. Most of the motivation for such an application originates from our desire to demonstrate the multiple orbit theory of section 2.6 for a nontrivial model.

First of all, note that the equation of motion [eqs. (3.83a, b)] is still valid, but now it has to be solved for a periodic orbit. Consequently, we introduce the discrete frequencies

$$\omega_n = \omega(n) = \frac{2\pi}{\tau_1} n \tag{4.55}$$

and define Fourier transforms as follows:

$$r_\tau = \frac{1}{\tau_1} \sum_n e^{-i\omega_n \tau} r_{\omega_n},$$

$$r_{\omega_n} = \int_0^{\tau_1} d\tau\, e^{i\omega_n \tau} r_\tau. \tag{4.56}$$

Elimination of the environmental coordinates r_k, $k = 2, \ldots, N$, leads to the same type [eq. (3.86)] of equation of motion for the coordinate r_1 of the object except for the redefinitions (4.55) and (4.56) of the Fourier transform; specifically, the dissipative part [cf. eqs. (3.86) and (3.87)] is given by

$$(H * r_1)_{\omega_n} = \eta |\omega_n| r_1(\omega_n). \tag{4.57}$$

Therefore,

$$\{m\omega_n^2 + \eta|\omega_n| + m\omega_0^2\}\, \xi_n - \tfrac{3}{2} m\omega_0^2 \frac{1}{\tau_1} \sum_{n'} \xi_{n-n'} \xi_{n'} = 0, \tag{4.58}$$

where the coordinate of the object has been rescaled according to

$$\xi_n = \frac{1}{R_C} r_1(\omega_n). \tag{4.59}$$

In the limit of heavy damping

$$\gamma = \frac{\eta}{m} \gg \omega_0, \tag{4.60}$$

the acceleration term $m\omega_n^2$ in the curly brackets of eq. (4.58) can be neglected; Larkin and Ovchinnikov (1983a, b, 1984) have shown how to solve eq. (4.58) in this limit. Specifically, for an orbit of multiplicity p, where the primitive period is

$$\tau_p = \frac{\tau_1}{p}, \tag{4.61}$$

we have

$$\xi_{pn} = \tfrac{2}{3} \tau_1 (\tanh b) \exp[-b|n| + i\omega_{pn}\tau_a],$$

$$\xi_\tau = \tfrac{2}{3} (\tanh b) \sum_n \exp[-b|n| - i\omega_{pn}(\tau - \tau_a)], \tag{4.62}$$

where

$$\tanh b = \frac{\tau_B}{\tau_p}, \qquad \tau_B = \frac{2\pi\gamma}{\omega_0^2}. \tag{4.63}$$

Note that $T_B = \hbar / k\tau_B$ is the crossover temperature in the above mentioned limit. Also, we wish to draw attention to the fact that $r_1(\tau = 0) = R_C \xi_{\tau=0}$ can be given

any value by an appropriate choice of the integration constant τ_a (e.g., $\mathrm{Im}\,\tau_a \neq 0$). Note also that we have orbits for $\tau_p < \tau_B$; in this case $\mathrm{Im}\,b = \pi/2$.

Calculating the action for a single orbit, we find that

$$\mathscr{S}^{(1)} = \mathscr{S}([\boldsymbol{r}_\tau]; \tau_1) = \tau_B v_B \left[\frac{3}{2} - \frac{1}{2}\left(\frac{\tau_B}{\tau_1}\right)^2 \right], \tag{4.64}$$

and that its Legendre transform is given by

$$\mathscr{W}^{(1)}(E) = \tfrac{3}{2}\tau_B v_B \left[1 - \left(\frac{E}{v_B}\right)^{2/3} \right], \tag{4.65}$$

which should be compared with fig. 6.

The corresponding quantities for an orbit of multiplicity p are

$$\mathscr{S}^{(p)} = p\mathscr{S}([\boldsymbol{r}_\tau]; \tau_p) = p\tau_B v_B \left[\frac{3}{2} - \frac{1}{2}\left(\frac{\tau_B}{\tau_p}\right)^2 \right],$$

$$\mathscr{W}^{(p)} = p\mathscr{W}^{(1)}(E). \tag{4.66}$$

Of interest will also be the quantity

$$\frac{\partial^2 \mathscr{S}^{(p)}}{\partial \tau_1^2} = -3\frac{p^3 \tau_B^3 v_B}{\tau_1^4}. \tag{4.67}$$

The computational procedure for the fluctuation determinants which appear in the expression (4.53) for the prefactor, follows the same reasoning as in section 3.6, that has led us to eq. (3.97). Accordingly, we have

$$\mathscr{A} = \left(\frac{\mathscr{S}_0([\boldsymbol{r}_1(\tau)]; \tau_1)}{2\pi\hbar m} \right)^{1/2} \left| \frac{\det(-m\partial_\tau^2 + H + m\omega_0^2)}{\det'(-m\partial_\tau^2 + H + v''(\boldsymbol{r}_1(\tau)))} \right|^{1/2}. \tag{4.68}$$

The calculation of the prefactor is carried out in Appendix D. There it is shown that it depends on the multiplicity p in a nontrivial way. Accordingly, $\mathscr{A} \to \mathscr{A}^{(p)}$, which is given by

$$\mathscr{A}^{(p)}(\tau_1) = \left(\frac{9\pi v_B}{\hbar \tau_B} \right)^{1/2} \left(\frac{\tau_B}{\tau_1} \right)^2 \frac{p^{1/2}}{\Gamma^2(p)} \frac{\Gamma(2p + 2\sigma)}{\Gamma(2\sigma)}, \tag{4.69}$$

where σ is defined by

$$\sigma = \frac{1}{2}\left[1 + \frac{\tau_1}{\tau_B}\left(\frac{\gamma^2}{\omega_0^2} - 1 \right) \right]. \tag{4.70}$$

According to what has been said in section 2.6, we should sum the contributions of all orbits with arbitrary multiplicity p. Hence,

$$\bar{\Gamma} = \sum_{p=1}^{\infty} (-1)^{p+1} \mathscr{A}^{(p)}(\tau_1) \exp\left(-\frac{1}{\hbar} \mathscr{S}^{(p)} \right). \tag{4.71}$$

As in the dissipation-free case, we solve the summation problem by a Laplace transform, evaluated by the method of steepest descent. Thus, we obtain*

$$\bar{\Gamma} = \sum_{p=1}^{\infty} (-1)^{p+1} \int dE\, \tilde{\mathscr{A}}^{(p)} \exp\left\{ -\frac{1}{\hbar}[\mathscr{W}^{(p)}(E) + E\tau_1] \right\}, \tag{4.72}$$

where, by virtue of eq. (4.67),

$$\tilde{\mathscr{A}}^{(p)} = \mathscr{A}^{(p)} \left| \frac{1}{2\pi\hbar} \frac{\partial^2 \mathscr{S}^{(p)}}{\partial \tau_1^2} \right|^{-1/2} = \frac{1}{\hbar} \frac{\Gamma(2p + 2\sigma)}{p\Gamma^2(p)\Gamma(2\sigma)}. \tag{4.73}$$

Interchanging the order of summation and integration in eq. (4.72), the decay rate can finally be put into the form

$$\bar{\Gamma} = \frac{1}{\hbar} \int_0^{\infty} dE\, e^{-E/kT} \phi(z), \quad z = e^{-\mathscr{W}(E)/\hbar}, \tag{4.74}$$

where $\phi(z)$ can be expressed in terms of the Gauss hypergeometric function (Abramowitz and Stegun 1986) as

$$\begin{aligned} \phi(z) &= \sum_{p=1}^{\infty} (-1)^{p+1} \frac{p\Gamma(2p + 2\sigma)}{\Gamma^2(p + 1)\Gamma(2\sigma)} z^p \\ &= 4z\sigma(\sigma + \tfrac{1}{2}) F(\sigma + 1, \sigma + \tfrac{3}{2}; 2; -4z). \end{aligned} \tag{4.75}$$

At this point we would like to mention that in Larkin and Ovchinnikov (1984), and subsequently also in Grabert and Weiss (1984) and Grabert et al. (1987), a calculation of $\bar{\Gamma}$ close to the crossover temperature T_B has been carried through according to Affleck's prescription (2.76) listed in (iii) of section 2.6:

$$\bar{\Gamma}^{A} = \int_{-\infty}^{v_B} dE\, \tilde{\mathscr{A}}^{(1)}(\tau_1) \exp\left\{ -\frac{1}{\hbar}[\mathscr{W}_2^{(1)}(E) + E\tau_1] \right\}, \tag{4.76}$$

where $\mathscr{W}_2^{(1)}$ is the second-order expansion of $\mathscr{W}^{(1)} = \mathscr{W}$ in $E - v_B$ as shown in eq. (2.77). Some arguments concerning this prescription are found in Appendix E, where also the explicit form of $\bar{\Gamma}^{A}$ is given in eq. (E.11).

Ludviksson (1989) has calculated the decay rate [eq. (4.72)] numerically. (For details see also Appendix E.) In Table 1, we have listed some of his results for the case $\gamma = 4\omega_0$, $v_B = \hbar\omega_0$ in the form of the ratio $Q = \bar{\Gamma}/\Gamma_R(T)$, where

$$\Gamma_R(T) = \begin{cases} \dfrac{\omega_0^2}{2\pi\gamma} e^{-v_B/kT}, & T > T_B, \\ \dfrac{\omega_0^2}{2\pi\gamma} e^{-v_B/kT_B}, & T < T_B. \end{cases} \tag{4.77}$$

* See also Appendix E.

Table 1
Decay rate divided by $\Gamma_R(T)$, which is the classical Kramers rate truncated for $T < T_B$. The parameters are $\gamma = 4\omega_0$ and $v_B = \hbar\omega_0$. Present theory: Q; Affleck's prescription: Q^A

T/T_B	Q	Q^A
0.4	0.0161	0.0177
0.5	0.0545	0.0585
0.6	0.241	0.253
0.7	1.35	1.43
0.8	8.58	10.5
0.9	52.0	101
1.0	268	1010
2.0	6.13	9.28
4.0	1.70	2.03
6.0	1.24	1.39
8.0	1.09	1.18
10.0	1.02	1.09

Note that for $T > T_B$, $\Gamma_R(T)$ is the classical Kramers rate in the heavy damping limit. For the sake of comparison, we have also listed numerical data $Q^A = \bar{\Gamma}^A/\Gamma_R(T)$, which have been calculated in Grabert et al. (1987) according to Affleck's prescription. In view of the very different analytical expressions, one may call the agreement reasonable* above and below the crossover temperature, with some reservation in the crossover region. Fair agreement is obtained also for other values of γ/ω_0 and v_B, with a tendency of improvement for large values of $v_B/\hbar\omega_0$.

The agreement for temperatures far above the crossover temperatures is not a coincidence. In fact, as we have shown in Appendix E, our theory is, in the high-temperature limit, equivalent to the ansatz of a quadratic expansion of $V(\boldsymbol{R})$ at $\boldsymbol{R}_B$. Such an expression has been discussed by Affleck (1981) and others.**

For temperatures far below the crossover temperature, only small energies contribute to the decay rate [eq. (4.72)]. Consequently, only the simple orbit $p = 1$ is of importance and we agree with the standard theory.

* The agreement here relates only to the comparison of different methods. Arguments for the validity of semiclassical approximations, even for the quoted parameter values, are given in Larkin and Ovchinnikov (1992).

** See, e.g., the paper by Wolynes (1981).

5. Conclusion

This chapter has been concerned with the decay of a metastable state which may exist in a system where the potential energy $V(\boldsymbol{R})$ of an object features a relative minimum at some point of the multidimensional configuration space $\boldsymbol{R}$. In the present paper, such a minimum has been assumed to be at $\boldsymbol{R} = 0$, where we have put $V(0) = 0$. Clearly, there must be an environment of $\boldsymbol{R} = 0$ where $V(\boldsymbol{R}) \geqslant 0$, i.e., which is classically inaccessible; but there must also be an extensive outside, i.e., an accessible region $V(\boldsymbol{R}) < 0$, where the object is found in the decayed state.

There are two main parts: one corresponding to the decay at zero temperature, where a wave function ψ suffices for a description of the state, and the other corresponding to the decay at finite temperatures, where a statistical matrix ρ is required.

Let us first recall the case $T = 0$, where the wave function ψ is calculated in quasiclassical accuracy. Specifically, this calculation is based entirely on the standard quasiclassical ansatz, where the wave function is put equal to $\exp(-[\text{action}]/\hbar)$ and the (Euclidean) action is expanded in powers of $\hbar$. Generally, the action is a complex function of position. In case of a metastable state, however, it is real in a very large part of the inaccessible region.

This ansatz substituted in the Schrödinger equation leads to a coupled set of nonlinear first-order partial differential equations (eikonal equation and transport equation) which can be solved by the methods of characteristics. These characteristics are trajectories in a phase space (configuration and momentum space), which obey classical equations of motion.

Of central importance is one single trajectory called the escape path. This is a classical trajectory of zero energy in the "inverted" potential $-V(\boldsymbol{R})$, which connects the metastable minimum at $\boldsymbol{R} = 0$ with an escape point $\boldsymbol{R}_C$ on the border of the outside region. There are also other trajectories in the inverted potential which start with zero energy at $\boldsymbol{R} = 0$ and which follow the escape path for some distance. Eventually, however, they will be reflected off at more or less close distances to the boundary of the accessible region. There is an envelope to these paths which may be called a caustic, and which lies entirely in the inaccessible region except for the escape point.

One can show that the wave function falls off rapidly at distances which scale with $\hbar^{1/2}$ from the escape path. Therefore, it is possible to introduce the small-fluctuation approximation, where only paths which are sufficiently close to the escape path are taken into consideration.

The problem now arises how the wave function can be obtained in the region beyond the caustic, which is by and large classically accessible. It is shown that a process of analytical continuation in the configuration space extended to complex coordinates resolves the problem. Alternatively, it is possible to extend the trajectories (essentially, as they touch the caustic) to complex phase space and to complex times. Note as a special case that the extension of the escape

path, which starts at $\boldsymbol{R}_C$ into the accessible region, is a real classical trajectory in the true potential $V(\boldsymbol{R})$.

We should keep in mind that this complexity merely reflects the fact that in multidimensional systems cases are rarely found where the action is either purely real or purely imaginary.

Concerning the wave function, the following picture emerges. Essentially, it consists of a principal wave ψ_0 $(= \psi_0^*)$ which resembles, near the origin, the ground state of a set of harmonic oscillators and which extends toward the caustic in a narrow tube surrounding the escape path. There, it is partially transmitted and reflected. The transmitted wave $\tilde{\psi}$ $(\neq \tilde{\psi}^*)$ represents a wave propagating in the direction of the extension of the escape path. The reflected wave ψ_1 $(= -\psi_1^*)$ returns to the origin.

By general arguments, the decay rate Γ of the metastable state is equal to the probability current which penetrates into the outside region. It has been shown that the decay rate thus obtained is exactly the same as that calculated in the instanton technique. This equivalence can partially be understood by noting that the escape path here corresponds to the bounce there.

At finite temperatures, we represent the statistical matrix in the form of a Feynman path integral. In quasiclassical approximation, the integration with respect to the paths is dominated by extremal paths, that are paths for which the action is stationary. Two types of such paths are of importance. There are some that contribute to the normalization and others that lead to a finite probability current in the outside, i.e., the classically accessible region. The extremal paths are solutions of classical equations of motion in the inverted potential. Of importance are periodic orbits completed in time $\tau_1 = \hbar/kT$ and which connect the metastable well with the outside region. One can easily see that for $\tau_1 \to \infty$ $(T \to 0)$ this periodic orbit degenerates into the bounce discussed above.

For the same reasons that has led us above to the conclusion that, in general, the action is a complex quantity, we have to extend the class of extremal orbits to include also complex orbits. Also, we have found it necessary to include multiple orbits (see, e.g., Hänggi and Hontscha 1988), where a primitive orbit completed in time $\tau_p = \tau_1/p$ is traversed p times. This generalization constitutes the main message of our theory as compared with the one put forward, e.g., by Affleck.

In the vicinity of the periodic orbit there are closed extremal paths which also contribute to the outgoing current. These neighboring paths can be taken into account by a technique similar to what we have referred to above as the small-fluctuation approximation.

One result is worthy of note. As is quite generally the case in the quasiclassical approximation to path integrals, the Gaussian fluctuations have to be calculated for zero boundary conditions. Now, it has been found that the integration of the contribution to the current from those neighboring paths mentioned above changes the expression for Gaussian fluctuations at zero boundary condition

exactly to an expression for Gaussian fluctuations at periodic boundary conditions. This allows us to interpret the overall current contribution as a contribution to the partition function where the paths are known to obey periodic boundary conditions. Thus, we have been able to confirm strictly the connection between the decay rate and the imaginary part of the free energy.

However, our theory means a generalization since it requires the inclusion of complex periodic orbit of arbitrary multiplicity. The summation with respect to the multiplicity is simple in a one-dimensional system but it might become a formidable task in the multidimensional case. We have carried through such a program for the Caldeira–Leggett model in the case of heavy damping and found that, numerically, the agreement between our theory and the standard one may be called reasonable in view of the fact that the corresponding analytical expressions are quite different. On the other hand, we have been able to show that for temperatures far above and far below the crossover temperature, our theory does lead to the standard limiting expression.

Although progress has been made, we have not found quantitative criteria for the assumption on the quasistationarity of the decaying state.

Appendix A. Transmission through a smooth barrier in quasiclassical approximation

For an illustration of the quasiclassical expression for the transmission coefficient [eq. (2.68)] let us compare the quasiclassical result with the analytical one for a barrier of the shape

$$v(R) = \frac{v_B}{\cosh^2 \alpha R}. \tag{A.1}$$

Let us calculate the abbreviated action $\mathcal{W}(E)$ defined by eq. (2.37) for a closed orbit and for $E < v_B$. Then ($v(R_0) = E$)

$$\begin{aligned}\mathcal{W}(E) &= 4\int_0^{R_0} \mathrm{d}R\, \sqrt{2m(v(R) - E)} \\ &= \frac{\sqrt{2mv_B}}{\alpha}(1-\varepsilon)\int_0^1 (1-s)^{1/2}(1-(1-\varepsilon)s)^{-1}s^{-1/2}\,\mathrm{d}s,\end{aligned} \tag{A.2}$$

where we have made the substitution

$$\varepsilon = \frac{E}{v_B}, \quad s = \frac{1}{1-\varepsilon}\tanh^2 \alpha R. \tag{A.3}$$

After some calculations, we obtain

$$\mathcal{W}(E) = \frac{2\pi}{\alpha}(1-\varepsilon^{1/2})(2mv_B)^{1/2}. \tag{A.4}$$

Although we initially assumed $E < V_B (\varepsilon < 1)$, the above result holds for all values of E (> 0). We thus obtain within quasiclassical accuracy

$$\frac{1}{D(E)} = 1 + \exp\left[\frac{2\pi}{\hbar\alpha}(2mv_B)^{1/2}(1-\varepsilon^{1/2})\right], \tag{A.5}$$

whereas the exact expression (Landau and Lifschitz 1975) is

$$\frac{1}{D(E)} = \frac{\sinh^2(\pi k/\alpha) + \cosh^2[\pi(2mv_B/\hbar^2\alpha^2 - \frac{1}{4})^{1/2}]}{\sinh^2(\pi k/\alpha)}, \tag{A.6}$$

where $\hbar k = (2mE)^{1/2}$. We realize that eqs. (A.5) and (A.6) agree in the quasiclassical limit $\hbar \to 0$, i.e., in the range

$$(\hbar\alpha)^2/2m \ll E, v_B. \tag{A.7}$$

In contrast, the expression for the transmission coefficient in harmonic approximation, i.e., replacing the barrier by a parabolic one with the same curvature at the top, is given by

$$\frac{1}{D(E)} = 1 + \exp\left[\frac{\pi}{\hbar\alpha}(2mv_B)^{1/2}(1-\varepsilon)\right], \tag{A.8}$$

which differs from eq. (A.5). Thus, the agreement of expressions (A.5) and (A.6) in the quasiclassical limit is not related to the fact that the smooth potential (A.1) can be approximated by a parabola close to its maximum.

Appendix B. Some results for the one-dimensional decay

In the case of the cubic potential (2.10) one may calculate $\mathscr{W}(E)$ explicitly for all E in terms of the Gauss hypergeometric function (Abramowitz and Stegun 1968), with the result

$$\mathscr{W}(E) = m\omega_0 R_C^2 \frac{1}{\sqrt{2}}(-3c)^{5/2}\frac{\pi}{4} z F\left(-\frac{1}{4}, \frac{1}{4}; 2, z\right). \tag{B.1}$$

The quantities in the expression above are explained as follows: c is the smallest root of

$$c^3 - \frac{1}{3}c = \frac{2}{27}\left(1 - \frac{2E}{v_B}\right), \tag{B.2}$$

and

$$z = \frac{1}{3}\left[\left(\frac{2}{3c}\right)^2 - 1\right]. \tag{B.3}$$

Note that $-\infty < c < -\frac{1}{3}$ and $-\frac{1}{3} < z < 1$ for $\infty > E/v_B > 0$. Consequently, $\mathscr{W}(E)$ is an analytical function for $\mathrm{Re}\, E > 0$.

Consider now the expressions (2.68) and (2.69) for $D(E)$ and $\bar{\Gamma}$, respectively. According to the concluding comments of section 2.2, the approximation (i), where $D(E) = \exp[-(1/\hbar)\mathscr{W}(E)]$, may be used for low temperatures. There, we require $\mathscr{W}(E)$ only for small values of E. An appropriate expansion* of eq. (B.1) leads to

$$\frac{1}{\hbar}\mathscr{W}(E) = \frac{1}{\hbar\omega_0}\left[\frac{36v_B}{5} - E\ln\frac{432v_B}{E} - E\right]. \tag{B.4}$$

Thus, we may write

$$\bar{\Gamma} = \frac{1}{2\pi\hbar Z_0}\int_0^\infty dE \exp\left(-\frac{E}{kT} - \frac{1}{\hbar\omega_0}\left[\frac{36v_B}{5} - E\ln\frac{432v_B}{E} - E\right]\right). \tag{B.5}$$

If v_B is large compared to kT, $\hbar\omega_0$, we may evaluate the integral by steepest descent. In the limit $T \to 0$, we recover indeed the decay rate as given by eqs. (2.17) and (2.24). For the sake of completeness, we remark that at finite temperatures, we obtain nominally an exponent

$$\mathscr{B}(T) = \frac{v_B}{\hbar\omega_0}\left[\frac{36}{5} - 432\exp\left(-\frac{\hbar\omega_0}{kT}\right)\right]. \tag{B.6}$$

However, the finite temperature correction is irrelevant, particularly in comparison with the correction of the normalization $Z_0(\tau_1)$.

For large temperatures, we approximate $\mathscr{W}(E) \approx \mathscr{W}_1(E)$ according to eq. (2.71) whence we find that

$$kT_B = \hbar\omega_0/2\pi. \tag{B.7}$$

Thus, eq. (2.72) may be written in the form

$$\bar{\Gamma} = \frac{\omega_0}{2\pi}\frac{1}{2Z_0\sin(\hbar\omega_0/2kT)}e^{-v_B/kT}. \tag{B.8}$$

Note that $2Z_0\sin(\hbar\omega_0/2kT) \to 1$ for $kT \gg \hbar\omega_0$; this confirms eqs. (2.2) and (2.3).

The divergence of eq. (B.8) for $T \to T_B +$ is a consequence of grossly overestimating the small energy contribution in the replacement $\mathscr{W} \to \mathscr{W}_1$ (see fig. 6). Therefore, it seems reasonable to make use of the following approximation

$$\bar{\Gamma} = \frac{1}{2\pi\hbar Z_0}\int_0^\infty dE\, e^{-E/kT}\{[1 + e^{(1/\hbar)\mathscr{W}_1(E)}]^{-1} + \theta(v_B - E)[e^{-(1/\hbar)\mathscr{W}_2(E)} - e^{-(1/\hbar)\mathscr{W}_1(E)}]\}, \tag{B.9}$$

*One should make use of the appropriate linear transformation formulas (Abramowitz and Stegun 1968).

where $\mathscr{W}_2(E)$ is given by eq. (2.77). For the present case, we have

$$\frac{1}{kT_2} = \left(\frac{5}{18\,kT_B v_B}\right)^{1/2}. \tag{B.10}$$

In the form (B.9), we may extend the lower limit of integration to $-\infty$. Thus, we obtain

$$\bar{\Gamma} = \frac{kT_B}{2\pi\hbar Z_0}\,\mathrm{e}^{-v_B/kT}\left\{\frac{\pi}{\sin(\pi T_B/T)} - \frac{T}{T-T_B} + \frac{T_2}{T_B}\,\pi^{1/2}\,\mathrm{erfc}\left(\frac{T_2}{T_B} - \frac{T_2}{T}\right)\exp\left(\frac{T_2}{T_B} - \frac{T_2}{T}\right)^2\right\}. \tag{B.11}$$

Clearly, in the region of the crossover temperature $T \sim T_B$, the first two terms in the curly brackets vanish and one recovers Affleck's prescription (iii), which has led us to eq. (2.79). It appears that we have to require that $T_2/T_B \sim (v_B/\hbar\omega_0)^{1/2} \gg 1$ in order to have this approximation to be meaningful.

In the limit of high temperatures, we obtain the classical result including quantum correction:

$$\bar{\Gamma} = \frac{\omega_0}{2\pi}\,\mathrm{e}^{-v_B/kT}\left[1 + \frac{1}{12}\left(\frac{\hbar\omega_0}{kT}\right)^2 - \frac{5\pi^2}{9}\,\frac{(\hbar\omega_0)^2}{kTv_B}\right]. \tag{B.12}$$

The second term in the brackets represents quantum corrections already found by Wigner (1932). The third term is a much smaller correction in the limit $v_B \gg kT$.

Appendix C. Gaussian fluctuations about complex paths

In a quasiclassical approximation to the path integral (2.55) it is necessary to calculate the contributions of Gaussian fluctuations about the extremal path $\boldsymbol{r}_\tau$, which is presently a complex path.

In this section, we are concerned with a one-dimensional system. Accordingly, we have

$$R_\tau = r_\tau + z_\tau, \quad z = x + \mathrm{i}y,$$

$$W_\tau = \frac{1}{m}\,v''(r_\tau) = U_\tau + \mathrm{i}V_\tau, \tag{C.1}$$

the second-order contribution to the action is given by

$$\mathscr{S}_2 = \tfrac{1}{2}m\int_0^{\tau_1} \mathrm{d}\tau\{\dot{z}_\tau^2 + W_\tau z_\tau^2\} = \mathscr{S}'_2 + \mathrm{i}\mathscr{S}''_2. \tag{C.2}$$

Since Gaussian fluctuations have to be calculated for zero boundary (ZB) conditions $z_{\tau_1} = z_0$, we obtain by partial integrations

$$\mathscr{S}'_2 = \tfrac{1}{2}m\int_0^{\tau_1} \mathrm{d}\tau\,\{x_\tau(-\ddot{x}_\tau + U_\tau x_\tau - V_\tau y_\tau) + y_\tau(\ddot{y}_\tau - U_\tau y_\tau - V_\tau x_\tau)\}$$

$$\mathscr{S}''_2 = \tfrac{1}{2}m\int_0^{\tau_1} \mathrm{d}\tau\,\{x_\tau(-\ddot{y}_\tau + U_\tau y_\tau - V_\tau x_\tau) + y_\tau(-\ddot{x}_\tau + U_\tau x_\tau - V_\tau y_\tau)\}. \quad \text{(C.3)}$$

Next, we introduce the two-component set of functions (x_n, y_n) which satisfies the equations

$$\left\{(-\partial_\tau^2 + U_\tau)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - V_\tau\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\}\begin{pmatrix}x_n\\ y_n\end{pmatrix} = \lambda_n\begin{pmatrix}x_n\\ y_n\end{pmatrix},$$

$$\int_0^{\tau_1} \mathrm{d}\tau(x_n x_{n'} + y_n y_{n'}) = \delta_{nn'}. \quad \text{(C.4)}$$

One may convince oneself easily that $(x_n, y_n) = (y_{\bar{n}}, -x_{\bar{n}})$ is also an eigenfunction and that its eigenvalue is $\lambda_n = -\lambda_{\bar{n}}$. It is important to note that this set with positive and negative eigenvalues is complete. Therefore, we may expand

$$\begin{pmatrix}x\\ y\end{pmatrix} = \sum_n c_n\begin{pmatrix}x_n\\ y_n\end{pmatrix}, \quad \text{(C.5)}$$

and inserting eq. (C.5) into eq. (C.3), we arrive at

$$\mathscr{S}'_2 = \tfrac{1}{2}m\sum_n \lambda_n c_n^2,$$

$$\mathscr{S}''_2 = -\tfrac{1}{2}m\sum_n c_{\bar{n}} c_n \lambda_n, \quad \text{(C.6)}$$

where $\bar{n}$ is so defined that $\lambda_{\bar{n}} = -\lambda_n$.

The path integral in eq. (2.55) is now replaced by Gaussian integrals with respect to the expansion coefficients c_n. For these integrals to converge we must restrict the expansion (C.5) to such values of n for which $\lambda_n > 0$. In that case we have $\mathscr{S}''_2 = 0$. We may interpret the restriction as follows: For each degree of freedom n there is only a one-dimensional integration in the complex c_n-plane; furthermore, the integration contour is in the direction of steepest descent from the saddle point;* the direction of steepest ascent – corresponding to a rotation by $\frac{1}{2}\pi$ in the complex plane – is excluded.

Let us introduce the Jacobi fields

$$z_j(\tau;\xi) = x_j(\tau;\xi) + \mathrm{i}y_j(\tau;\xi), \quad j = 1, 2, \quad \text{(C.7)}$$

* Of course, the sign problem ("which way to cross the mountain pass") mentioned in section 2.6 [eq. (2.65)] remains.

which are solutions of the equation

$$\left\{(-\partial_\tau^2 + U)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - V\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - \xi\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}\begin{pmatrix}x_j\\ y_j\end{pmatrix} = 0 \tag{C.8}$$

subject to the initial conditions

$$z_j(0;\xi) = 0, \qquad \dot{z}_1(0;\xi) = 1, \qquad \dot{z}_2(0;\xi) = -i. \tag{C.9}$$

Observe, that for $\xi = 0$ we have

$$(-\partial_\tau^2 + U_\tau + iV_\tau)z_j(\tau;0) = 0; \tag{C.10}$$

whence it follows that

$$z_2(\tau;0) = -iz_1(\tau;0). \tag{C.11}$$

We now construct a linear combination (with real coefficients) of $z_1(\tau;\xi)$ and $z_2(\tau;\xi)$ that satisfies the boundary con litions

$$a_1 z_1(\tau_1;\xi) + a_2 z_2(\tau_1;\xi) = 0. \tag{C.12}$$

Separating the real and imaginary parts of eq. (C.12) leads to a homogeneous system of equations that has a nontrivial solution only if

$$\mathrm{Im}(z_1(\tau_1;\xi)[z_2(\tau_1;\xi)]^*) = x_1(\tau_1;\xi)y_2(\tau_1;\xi) - y_1(\tau_1;\xi)x_2(\tau_1;\xi) = 0. \tag{C.13}$$

Next we compare the eigenvalues above with those derived from

$$W^{(0)} = \frac{1}{m}v''(0) = U^{(0)} + iV^{(0)},$$

$$U^{(0)} = \omega_0^2, \qquad V^{(0)} = 0, \tag{C.14}$$

and define the ratio (denoting zero boundary conditions by the subscript ZB)

$$R_{\mathrm{ZB}}^2 = \left(\frac{\prod_n^+ \lambda_n^0}{\prod_n^+ \lambda_n}\right)_{\mathrm{ZB}}^2 = \left|\frac{\prod_n \lambda_n^0}{\prod_n \lambda_n}\right|_{\mathrm{ZB}}, \tag{C.15}$$

where the superscript $+$ in the product above denotes restriction to positive eigenvalues λ_n. For further progress, we compare the expressions

$$\left.\frac{\prod_n(\lambda_n^{(0)} - \xi)}{\prod_n(\lambda_n - \xi)}\right|_{\mathrm{ZB}}$$

$$= \left.\frac{\det\left\{(-\partial_\tau^2 + U^{(0)})\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - V^{(0)}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - \xi\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}}{\det\left\{(-\partial_\tau^2 + U_\tau)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - V_\tau\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - \xi\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}}\right|_{\mathrm{ZB}} \tag{C.16}$$

and

$$\frac{\mathrm{Im}\,(z_1^{(0)}(\tau_1;\xi)[z_2^{(0)}(\tau_1;\xi)]^*)}{\mathrm{Im}\,(z_1(\tau_1;\xi)[z_2(\tau_1;\xi)]^*)}, \tag{C.17}$$

where $z_1^{(0)}$ and $z_2^{(0)}$ are defined analogous to eqs. (C.8) and (C.9), with W_τ replaced by $W^{(0)}$. Evidently, both expressions (C.16) and (C.17) have simple poles for $\xi = \lambda_n$ and zeros at $\xi = \lambda_n^{(0)}$ and approach 1 as $|\xi| \to \infty$.

Therefore, we conclude that both expressions coincide in the entire ξ-plane. Putting $\xi = 0$ and using eq. (C.11) we conclude that

$$R_{\mathrm{ZB}} = \left| \frac{z_1^{(0)}(\tau_1;0)}{z_1(\tau_1;0)} \right|. \tag{C.18}$$

By definition, the time derivative of the classical path $\dot{r}_\tau$ obeys the equation

$$(-\partial_\tau^2 + U + \mathrm{i}V)\dot{r}_\tau = 0; \tag{C.19}$$

whence it follows that

$$z_1(\tau;0) = \dot{r}_0 \dot{r}_\tau \int_0^\tau \mathrm{d}\tau' \dot{r}_{\tau'}^{-2}. \tag{C.20}$$

Furthermore,

$$z_1^{(0)}(\tau;0) = \frac{1}{\omega_0} \sinh \omega_0 \tau, \tag{C.21}$$

and we find that

$$|\dot{r}_0|^2 R_{\mathrm{ZB}} = \frac{\sinh \omega_0 \tau_1}{\omega_0 |\int_0^{\tau_1} \mathrm{d}\tau\, \dot{r}_\tau^{-2}|}. \tag{C.22}$$

We observe that for an analytical potential

$$\int_0^{\tau_1} \mathrm{d}\tau\, \dot{r}_\tau^{-2} = \oint \mathrm{d}r\, \dot{r}^{-3} \tag{C.23}$$

does not depend on the particular form of the orbit as long as it encloses the two turning points of classical motion (cf. fig. 4). From the definition of τ_1, we find that

$$\oint \mathrm{d}r\, \dot{r}_\tau^{-3} = -m \frac{\partial \tau_1}{\partial E} = -m \frac{\partial^2 \mathscr{W}(E)}{\partial E^2}. \tag{C.24}$$

Let us now consider fluctuations with periodic boundary conditions. Although the eigenvalues and eigenfunctions are different from those for zero boundary conditions, we will for the sake of simplicity, use the same labeling.

We define

$$R_{\mathrm{PB}}^2 = \left(\frac{\prod_n^+ \lambda_n^{(0)}}{\prod_n^+ \lambda_n} \right)_{\mathrm{PB}}^2 = \left| \frac{\prod_n \lambda_n^{(0)}}{\prod_n' \lambda_n} \right|_{\mathrm{PB}}, \tag{C.25}$$

where $\prod_n^+$ means restriction to positive eigenvalues and $\prod_n'$ means that the eigenvalue zero (corresponding to the eigenfunction $\propto \dot{r}_\tau$) is to be omitted.

In order to control the zero eigenvalue, let us first define a modified ratio

$$\begin{aligned}(R_{\mathrm{PB}}^\varepsilon)^2 &= \left| \prod_n \frac{\lambda_n^{(0)}}{\lambda_n^\varepsilon} \right|_{\mathrm{PB}} \\ &= \left| \frac{\det\left\{ (-\partial_\tau^2 + U^{(0)}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - V^{(0)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}}{\det\left\{ (-\partial_\tau^2 + U_\tau^\varepsilon) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - V_\tau \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}} \right|_{\mathrm{PB}},\end{aligned} \tag{C.26}$$

where $U_\tau^\varepsilon = U_\tau + \varepsilon$.

We now introduce four Jacobi fields $z_j^\varepsilon(\tau; \xi)$ that satisfy eq. (C.10), with U_τ replaced by U_τ^ε, subject to the initial conditions

$$\begin{aligned} &z_{1,2}^\varepsilon(0; \xi) = 0, \qquad \dot{z}_1^\varepsilon(0; \xi) = 1, \qquad \dot{z}_2^\varepsilon(0; \xi) = -\mathrm{i}, \\ &\dot{z}_{3,4}^\varepsilon(0; \xi) = 0, \qquad z_3^\varepsilon(0; \xi) = 1, \qquad z_4^\varepsilon(0; \xi) = -\mathrm{i}. \end{aligned} \tag{C.27}$$

For $\xi = 0$ we have relations similar to eqs. (C.11) and (C.12). In particular,

$$z_2^\varepsilon(\tau; 0) = -\mathrm{i} z_1^\varepsilon(\tau; 0), \qquad z_4^\varepsilon(\tau; 0) = -\mathrm{i} z_3^\varepsilon(\tau; 0). \tag{C.28}$$

We now form linear combinations of $z_j^\varepsilon(\tau; \xi)$ with real coefficients that satisfy the periodic boundary conditions. This can be done in two steps. First, we form two independent solutions

$$\begin{aligned} &Z_1^\varepsilon(\tau; \xi) = z_1^\varepsilon(\tau; \xi)[1 - z_3^\varepsilon(\tau_1; \xi)] + z_3^\varepsilon(\tau; \xi) z_1^\varepsilon(\tau_1; \xi), \\ &Z_2^\varepsilon(\tau; \xi) = (1, 3) \leftrightarrow (2, 4). \end{aligned} \tag{C.29}$$

For $\xi = 0$ we have, on account of eq. (C.28),

$$Z_2^\varepsilon(\tau; 0) = -\mathrm{i} Z_1^\varepsilon(\tau; 0). \tag{C.30}$$

Secondly, we construct a linear combination of $Z_l^\varepsilon(\tau; \xi)$ that satisfies

$$\sum_{l=1,2} \alpha_l \dot{Z}_l^\varepsilon(0; \xi) = \sum_{l=1,2} \alpha_l \dot{Z}_l^\varepsilon(\tau_1; \xi). \tag{C.31}$$

The condition for this system to have a nontrivial solution is

$$\mathrm{Im}\{[\dot{Z}_1^\varepsilon(\tau_1; \xi) - \dot{Z}_1^\varepsilon(0; \xi)][\dot{Z}_2^\varepsilon(\tau_1; \xi) - \dot{Z}_2^\varepsilon(0; \xi)]^*\} = 0. \tag{C.32}$$

By reasoning similar to that leading to eq. (C.18), we conclude that

$$(R_{\mathrm{PB}}^{\varepsilon})^2 = \left| \frac{\dot{Z}_1^{(0)}(\tau_1;0) - \dot{Z}_1^{(0)}(0;0)}{\dot{Z}_1^{\varepsilon}(\tau_1;0) - \dot{Z}_1^{\varepsilon}(0;0)} \right|^2, \tag{C.33}$$

where we have made use of eq. (C.30).

Now we eliminate the two eigenvalues close to zero. For $\varepsilon = 0$, let the eigenfunctions to the double-degenerate eigenvalue zero be $(x_0(\tau), y_0(\tau))$ and $(y_0(\tau), -x_0(\tau))$, respectively. To first order in ε, the matrix elements of the perturbation

$$\varepsilon \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in the two-dimensional subspace are

$$\varepsilon \int_0^{\tau_1} \mathrm{d}\tau \begin{pmatrix} x_0^2 - y_0^2 & 2x_0 y_0 \\ 2x_0 y_0 & y_0^2 - x_0^2 \end{pmatrix}. \tag{C.34}$$

The product of the two eigenvalues is then

$$-\varepsilon^2 \left\{ \left[\int_0^{\tau_1} \mathrm{d}\tau (x_0^2 - y_0^2) \right]^2 + \left[2 \int_0^{\tau_1} \mathrm{d}\tau\, x_0 y_0 \right]^2 \right\} = -\varepsilon^2 \left| \int_0^{\tau_1} \mathrm{d}\tau\, z_0^2 \right|^2, \tag{C.35}$$

which leads to

$$R_{\mathrm{PB}} = \left| \int_0^{\tau_1} \mathrm{d}\tau\, z_0^2 \right| \left| \lim_{\varepsilon \to 0} \varepsilon \frac{\dot{Z}_1^{(0)}(\tau_1;0) - \dot{Z}_1^{(0)}(0;0)}{\dot{Z}_1^{\varepsilon}(\tau_1;0) - \dot{Z}_1^{\varepsilon}(0;0)} \right|. \tag{C.36}$$

We now make use of the fact that

$$\partial_\tau \{ \dot{Z}_1^{\varepsilon}(\tau;0) z_0(\tau) - Z_1^{\varepsilon}(\tau;0) \dot{z}_0(\tau) \} = \varepsilon Z_1^{\varepsilon}(\tau;0) z_0(\tau). \tag{C.37}$$

Furthermore, since $z_0(\tau)$ solves the equation (C.9) with $\xi = 0$, we may write

$$z_0(\tau) = \dot{z}_0(0)\, z_1(\tau;0) + z_0(0) z_3(\tau;0). \tag{C.38}$$

Since $z_0(\tau)$ is periodic ($z_0(\tau) = \dot{r}_\tau$), we may use the condition $z_0(\tau_1) = z_0(0)$ to deduce from eq. (C.29) that to zeroth order we have

$$Z_1^{\varepsilon}(\tau;0) = \frac{z_1(\tau_1;0)}{z_0(0)} z_0(\tau). \tag{C.39}$$

From eq. (C.37) we obtain by integration

$$\dot{Z}_1^{\varepsilon}(\tau_1;0) - \dot{Z}_1^{\varepsilon}(0;0) = \frac{\varepsilon}{z_0(0)} \int_0^{\tau_1} \mathrm{d}\tau\, Z_1^{\varepsilon}(\tau;0) z_0(\tau), \tag{C.40}$$

which, by virtue of eq. (C.39) leads to

$$\dot{Z}_1^{\varepsilon}(\tau_1; 0) - \dot{Z}_1^{\varepsilon}(0; 0) = \varepsilon \frac{z_1(\tau_1; 0)}{z_0^2(0)} \int_0^{\tau_1} \mathrm{d}\tau\, z_0^2(\tau). \tag{C.41}$$

Since z_0 is normalized to unity, we have

$$|z_0(0)|^2 = |\dot{r}_0|^2 \Big/ \int_0^{\tau_1} \mathrm{d}\tau\, |\dot{r}_\tau^2|. \tag{C.42}$$

Inserting eqs. (C.20), (C.41) and (C.42) into eq. (C.36), we finally arrive at the following expression:

$$R_{\mathrm{PB}} \int_0^{\tau_1} \mathrm{d}\tau\, |\dot{r}_\tau|^2 = \frac{|\dot{Z}_1^{(0)}(\tau_1; 0) - \dot{Z}_1^{(0)}(0; 0)|}{|\int_0^{\tau_1} \mathrm{d}\tau\, \dot{r}_\tau^{-2}|}. \tag{C.43}$$

We observe now that

$$z_1^{(0)}(\tau; 0) = \frac{1}{\omega_0} \sinh \omega_0 \tau, \quad z_3^{(0)}(\tau; 0) = \cosh \omega_0 \tau,$$

$$\dot{Z}_1(\tau_1; 0) - \dot{Z}_1(0; 0) = 4 \sinh^2 \frac{\omega_0 \tau_1}{2}. \tag{C.44}$$

A comparison of eqs. (C.21) and (C.43) together with eq. (C.44) allows us to express R_{ZB} in terms of R_{PB} as follows:

$$R_{\mathrm{ZB}} |\dot{r}_0^2| = \frac{1}{2m\omega_0} \coth \frac{\omega_0 \tau_1}{2}\, \mathscr{S}_0(\tau_1) R_{\mathrm{PB}}, \tag{C.45}$$

where we have introduced

$$\mathscr{S}_0(\tau_1) = \mathscr{S}_0([r_\tau]; \tau_1) = m \int_0^{\tau_1} \mathrm{d}\tau\, |\dot{r}_\tau|^2. \tag{C.46}$$

At this point, we find it convenient to introduce the notation

$$R_{\mathrm{XB}} = \left. \frac{\det^+ \{-\partial_\tau^2 + W_\tau^{(0)}\}}{\det^+ \{-\partial_\tau^2 + W_\tau\}} \right|_{\mathrm{XB}}. \tag{C.47}$$

Evidently, for $W_\tau \to U_\tau$ real, we have

$$\frac{\det^+ \{-\partial_\tau^2 + W_\tau^{(0)}\}}{\det^+ \{-\partial_\tau^2 + W_\tau\}} = \left| \frac{\det \{-\partial_\tau^2 + U^{(0)}\}}{\det' \{-\partial_\tau^2 + U_\tau\}} \right|. \tag{C.48}$$

Using the above definitions and relations, we may express the statistical matrix ρ in quasiclassical approximation as follows:

$$\rho(R, R; \tau_1) = \left(\frac{m\omega_0}{2\pi\hbar \sinh \omega_0 \tau_1}\right)^{1/2} \left|\frac{\det^+(-\partial_\tau^2 + W^{(0)})}{\det^+(-\partial_\tau^2 + W_\tau)}\right|_{ZB}^{1/2} \times \exp\left(-\frac{1}{\hbar}\mathscr{S}([r_\tau]; \tau_1)\right). \tag{C.49}$$

We have chosen a representation where a ratio of determinants appears since it is easier to handle. That the overall constant is chosen properly can be seen by inserting the trivial path $r_\tau = 0$, where one expects the result $\rho(0, 0; \tau_1) = (m\omega_0/2\pi\hbar \sinh \omega_0\tau_1)^{1/2}$, which is known to be valid for the harmonic oscillator. Using eq. (C.45) together with $Z_0^{-1} = 2 \sinh \omega_0 \tau_1/2$, we may rewrite eq. (C.49) as follows:

$$\rho(R, R; \tau_1) = Z_0^{-1}\left(\frac{\mathscr{S}_0(\tau_1)}{2\pi\hbar|\dot{r}_0|^2}\right)^{1/2} R_{PB}^{1/2} \exp\left(-\frac{1}{\hbar}\mathscr{S}([r_\tau])\right). \tag{C.50}$$

Calculating the decay current $J(R)$ and the decay rate Γ according to eqs. (2.57) and (2.60), we obtain the result given previously by eqs. (2.44), (2.51) and (2.53), namely

$$\bar{\Gamma} = \left(\frac{\mathscr{S}_0(\tau_1)}{2\pi\hbar}\right)^{1/2} \left|\frac{\det^+(-\partial_\tau^2 + W^{(0)})}{\det^+\{-\partial_\tau^2 + W_\tau\}}\right|_{PB}^{1/2} \exp\left(-\frac{1}{\hbar}\mathscr{S}([r_\tau]; \tau_1)\right). \tag{C.51}$$

Alternatively, we may combine eqs. (C.22), (C.24) and (C.45) to obtain

$$\bar{\Gamma} = Z_0^{-1}\left[2\pi\hbar \frac{\partial^2 \mathscr{W}(E)}{\partial E^2}\right]^{-1/2} \exp\left(-\frac{1}{\hbar}\mathscr{S}([r_\tau]; \tau_1)\right), \tag{C.52}$$

which agrees with eqs. (2.44)–(2.46).

Eventually, we wish to discuss the possibility of deforming the periodic orbit r_τ in the complex r-plane. Evidently, the replacement

$$r_\tau \to r_\tau + \delta \cdot \dot{r}_\tau \tag{C.53}$$

does not change the orbit to $O(\delta)$, and also it leaves the action $\mathscr{S}$ unchanged. In a formal way, we may consider this to be a consequence of the fact that $(\mathrm{Re}\,\dot{r}_\tau, \mathrm{Im}\,\dot{r}_\tau)$ is a solution of the eigenvalue equation (C.4) to the eigenvalue zero.

However, we have also learned that $(\mathrm{Im}\,\dot{r}_\tau, -\mathrm{Re}\,\dot{r}_\tau)$ is also an eigenfunction with eigenvalue zero. On the other hand, $-\mathrm{i}\dot{r}_\tau = \mathrm{Im}\,\dot{r}_\tau - \mathrm{iRe}\,\dot{r}_\tau$ is, in the complex plane, perpendicular to $\dot{r}_\tau$ and, therefore,

$$r_\tau \to r_\tau - \mathrm{i}\delta \cdot \dot{r}_\tau \tag{C.54}$$

means a real deformation of the orbit which leaves the action unchanged.

Clearly, by successive application of this infinitesimal deformation, we may induce global changes in the orbit. Thus, we may say that it is possible to construct an orbit which passes through any preselected point R.

With some restrictions, the above argument can also be extended to the multidimensional case.

Appendix D. Prefactor for the heavily damped object

We consider the fluctuation operator in the denominator of eq. (4.68). Its eigenvalues Λ and eigenfunctions $v_\tau = v(\tau)$, $v_n = v(\omega_n)$ can be found by solving the equation

$$\{m\omega_n^2 + \eta|\omega_n| + m\omega_0^2\}v_n - 3m\omega_0^2[\xi_\tau v_\tau]_n = \Lambda v_n, \tag{D.1}$$

which is written down in terms of Fourier transforms [cf. eqs. (4.55) and (4.56)].

In the limit of heavy damping, the extremal path $r_1(\tau) = R_C\xi_\tau$ is given by eq. (4.62). (For convenience, we substitute $\tau - \tau_a$ by τ in the following.) Considering an orbit of multiplicity p, we choose the ansatz

$$v_\tau = \sum_n C_n^k \exp(i\omega_{pn-k}\tau), \tag{D.2}$$

where the integer k, $0 \leqslant k < p$, plays the role of a "Bloch index". Inserting this expression into eq. (D.1), we obtain*

$$\left[\left(\frac{\omega_1}{\omega_0}\right)^2 (pn + k)^2 + \frac{1}{p}|pn + k|\tanh b + 1\right]C_n^k - 2(\tanh b)\sum_{n'} C_{n'}^k e^{-b|n-n'|}$$
$$= \frac{\Lambda}{m\omega_0^2} C_n^k, \tag{D.3}$$

where $\omega_1 = 2\pi/\tau_1$.

The case $k = 0$ was discussed by Larkin and Ovchinnikov (1984). We will now briefly discuss their procedure. In the limit $\gamma \gg \omega_0$ we are considering here, and for the lowest eigenvalues, we may neglect the acceleration which is the first term in the brackets of eq. (D.3) and we will call this form the truncated eigenvalue equation.

For $k = 0$, one may also choose the eigenfunctions to have a definite parity. In the even case we start from the ansatz

$$C_n^e = (|n| + C)e^{-b|n|}. \tag{D.4}$$

* Note that $\tanh b = \tau_B/\tau_p \propto p$ depends on p.

We insert this in the truncated eigenvalue equation (D.3) and arrive at

$$-\left\{C(\tanh b)|n| + C + \tanh b \frac{1}{\sinh^2 b}\right\} = \frac{\Lambda^e}{m\omega_0^2}(|n| + C), \tag{D.5}$$

which is satisfied for all n provided that

$$-C\tanh b = \frac{\Lambda^e}{m\omega_0^2}, \qquad -C - \tanh b \frac{1}{\sinh^2 b} = \frac{\Lambda^e}{m\omega_0^2} C. \tag{D.6}$$

This leads to a second-order equation for C, which is solved to give the eigenvalues

$$\frac{\Lambda^e_{0,1}}{m\omega_0^2} = -\frac{1}{2}\left[1 \pm \left(1 + \frac{4}{\cosh^2 b}\right)^{1/2}\right]. \tag{D.7}$$

The remaining eigenfunctions of even parity, corresponding to the eigenvalues Λ^e_{2+N}, $N \geqslant 0$, are of the type

$$C^e_n = (|n| + C)e^{-b|n|} + d_n, \tag{D.8}$$

where

$$d_n = d_{-n}, \qquad d_N \neq 0, \qquad d_n = 0 \quad \text{for } |n| > N. \tag{D.9}$$

This leads to the following system of equations:

$$-\tanh b\left\{|n|C + (\coth b)C + \frac{1}{\sinh b}\right\} + \{|n|\tanh b + 1\} d_n e^{b|n|}$$

$$-2(\tanh b)e^{b|n|}\sum_{n'=-N}^{N} d_{n'} e^{-b|n-n'|} = \frac{\Lambda^e_{2+N}}{m\omega_0^2}\{|n| + C + e^{b|n|} d_n\}. \tag{D.10}$$

Assuming that the equations are valid for $|n| > N$, we conclude that for $n = N$

$$[N\tanh b + 1] d_N e^{bN} = \frac{\Lambda^e_{2+N}}{m\omega_0^2} d_N e^{bN}. \tag{D.11}$$

Since $d_N \neq 0$, we must have

$$\frac{\Lambda^e_{2+N}}{m\omega_0^2} = 1 + N\tanh b. \tag{D.12}$$

For the odd eigenfunctions we make the ansatz

$$C^o_n = (n + \bar{C}\,\mathrm{sgn}\, n)e^{-b|n|} + g_n, \tag{D.13}$$

where

$$g_n = -g_{-n}, \qquad g_M \neq 0, \qquad g_n = 0 \quad \text{for } n > M \tag{D.14}$$

and

$$\operatorname{sgn} n = \begin{cases} 1, & n > 0, \\ 0, & n = 0, \\ -1, & n < 0. \end{cases} \tag{D.15}$$

Thus, we arrive at the system of equations

$$\bar{C}\operatorname{sgn} n - \bar{C}(\tanh b)n + [|n|\tanh b + 1]g_n e^{b|n|} - 2(\tanh b)e^{b|n|}\sum_{n'=-M}^{M} g_{n'} e^{-b|n-n'|} = \frac{\Lambda_M^0}{m\omega_0^2}[n + \bar{C}\operatorname{sgn} n + g_n e^{b|n|}]. \tag{D.16}$$

Assuming that the equations are valid for $|n| > M$, we conclude that for $n = M$

$$(M\tanh b + 1)g_M e^{bM} = \frac{\Lambda_M^0}{m\omega_0^2} g_M e^{bM}. \tag{D.17}$$

Since $g_M \neq 0$, we must have

$$\frac{\Lambda_M^0}{m\omega_0^2} = 1 + M\tanh b \quad (M \neq 0). \tag{D.18}$$

For $M = 0$ we obtain

$$\bar{C}(\operatorname{sgn} n - n\tanh b) = \frac{\Lambda_0^0}{m\omega_0^2}(n + \bar{C}\operatorname{sgn} n), \tag{D.19}$$

which holds for all n if

$$\bar{C} = \frac{\Lambda_0^0}{m\omega_0^2}\bar{C}, \qquad -\bar{C}\tanh b = \frac{\Lambda_0^0}{m\omega_0^2}. \tag{D.20}$$

Either $\bar{C} = 0$, corresponding to $\Lambda_0^0 = 0$ or $\bar{C} \neq 0$ in which case $\Lambda_0^0/m\omega_0^2 = 1$.

We have now obtained all eigenvalues of the truncated eigenvalue equation. We introduce the following labeling, which includes all even and odd eigenvalues:

$$\tilde{\Lambda}_0 = 0,$$

$$\tilde{\Lambda}_{\pm 1} = -\frac{1}{2}m\omega_0^2\left[1 \pm \left(1 + \frac{4}{\cosh^2 b}\right)^{1/2}\right],$$

$$\tilde{\Lambda}_\nu = m\omega_0^2[(|\nu| - 2)\tanh b + 1], \quad |\nu| \geqslant 2. \tag{D.21}$$

For states with large quantum numbers, the acceleration term in eq. (D.3), which we have neglected so far (truncation), becomes important. In that case we neglect the last term on the left-hand side of eq. (D.3). Then, the eigenfunction

corresponding to the eigenvalue

$$\lambda_\nu^{(0)} = m\omega_0^2\left[p^2\nu^2\left(\frac{\omega_1}{\omega_0}\right)^2 + |\nu|\tanh b + 1 \right] \tag{D.22}$$

is simply

$$C_n = \delta_{n\nu}.$$

Including the off-diagonal term by lowest-order perturbation theory, we obtain the eigenvalues

$$\lambda_\nu = m\omega_0^2\left[p^2\nu^2\left(\frac{\omega_1}{\omega_0}\right)^2 + (|\nu| - 2)\tanh b + 1 \right]. \tag{D.23}$$

Comparing eqs. (D.21) and (D.23), we recognize that the result (D.21) can be improved if we put

$$\tilde{\Lambda}_\nu = \lambda_\nu, \quad |\nu| \geqslant 2. \tag{D.24}$$

For the prefactor we need also to calculate

$$\begin{aligned}\mathscr{S}_0([r_1(\tau)];\tau_p) &= m\int_0^{\tau_p} \mathrm{d}\tau(\dot{r}_1(\tau))^2 = \frac{2\pi^2}{\tau_B}\frac{mR_B^2}{\cosh^2 b} \\ &= \frac{2\pi^2 mR_B^2}{\tau_B\omega_0^4}\left|\frac{\tilde{\Lambda}_1}{m}\;\frac{\Lambda_{-1}}{m}\right|.\end{aligned} \tag{D.25}$$

As an exercise, we calculate the prefactor $\mathscr{A}_0^{(1)}$ of the decay rate for one primitive orbit completed in time τ_p. There

$$\begin{aligned}\mathscr{A}_0^{(1)} &= \left(\frac{\mathscr{S}_0}{2\pi\hbar}\right)^{1/2}\left\{\frac{\prod_\nu[\omega_{p\nu}^2 + \gamma|\omega_{p\nu}| + \omega_0^2]}{\prod_{\nu\neq 0}\tilde{\Lambda}_\nu/m}\right\}^{1/2} \\ &= \left(\frac{m\omega_0 R_B^2}{2\hbar}\frac{1}{\gamma\omega_0}\right)^{1/2}\frac{\prod_{\nu=1}^{\infty}[\omega_{p\nu}^2 + \gamma|\omega_{p\nu}| + \omega_0^2]}{\prod_{\nu=2}^{\infty}[\omega_{p\nu}^2 + \gamma|\omega_{p(\nu-2)}| + \omega_0^2]}.\end{aligned} \tag{D.26}$$

We now write the above product in the following form

$$\omega_p^2\frac{\prod_{\nu=1}^{\infty}(\nu - p\nu_1)(\nu - p\nu_2)}{\prod_{\nu=2}^{\infty}(\nu - p\mu_1)(\nu - p\mu_2)}, \tag{D.27}$$

where we have introduced the dimensionless quantities

$$\begin{aligned}\nu_{1,2} &= (\omega_1)^{-1}[-\gamma/2 \pm \sqrt{\gamma^2/4 - \omega_0^2}], \\ \mu_{1,2} &= (\omega_1)^{-1}[-\gamma/2 \pm \sqrt{\gamma^2/4 - \omega_0^2 + 2\gamma\omega_p}].\end{aligned} \tag{D.28}$$

Using the relation

$$\prod_{r=0}^{\infty}\frac{(r+v_1)(r+v_2)}{(r+w_1)(r+w_2)} = \frac{\Gamma(w_1)\Gamma(w_2)}{\Gamma(v_1)\Gamma(v_2)}, \quad v_1 + v_2 = w_1 + w_2, \tag{D.29}$$

the product can be written in terms of Γ-functions:

$$\omega_p^2 \frac{\Gamma(2-p\mu_1)\Gamma(2-p\mu_2)}{\Gamma(1-pv_1)\Gamma(1-pv_2)}. \tag{D.30}$$

For very heavy damping, this expression is proportional to γ^4/ω_0^2, which leads to the following limiting form of the prefactor:*

$$\mathscr{A}_0^{(1)} = \left(\frac{m\omega_0 R_B^2}{2\hbar}\right)^{1/2} \left(\frac{\gamma}{\omega_0}\right)^{7/2} \omega_0. \tag{D.31}$$

For $k \neq 0$ parity is not conserved and we need to generalize the ansatz for the eigenfunctions

$$C_n^k = (\alpha|n| + C)e^{-b|n|} + d_n + (\bar{\alpha}n + \bar{C}\operatorname{sgn} n)e^{-b|n|} + g_n, \tag{D.32}$$

where d_n and g_n have the same properties as before. We now insert this ansatz into the truncated form of eq. (D.3). Observing that

$$\left|n + \frac{k}{p}\right| = |n| + \operatorname{sgn} n \frac{k}{p} + \frac{k}{p}\delta_{n,0}, \tag{D.33}$$

we obtain

$$\begin{aligned}
&-\{|n|C\tanh b + C + \alpha \tanh b/\sinh^2 b\} - \bar{C}\{n\tanh b - \operatorname{sgn} n\} \\
&\quad + \frac{k}{p}\{\alpha n + C\operatorname{sgn} n + \bar{\alpha}|n| + \bar{C}(1-\delta_{n,0}) + \delta_{n,0}C\}\tanh b \\
&\quad + \left[\left(|n| + \frac{k}{p}\operatorname{sgn} n + \frac{k}{p}\delta_{n,0}\right)\tanh b + 1\right]\{d_n + g_n\}e^{b|n|} \\
&\quad - 2(\tanh b)\sum_{n'}(d_{n'} + g_{n'})e^{b|n|-b|n-n'|} \\
&= \frac{\Lambda}{m\omega_0^2}\{\alpha|n| + C + \bar{\alpha}n + \bar{C}\operatorname{sgn} n + (d_n + g_n)e^{b|n|}\}.
\end{aligned} \tag{D.34}$$

For $|n| > N$, the above equation reduces to

$$\begin{aligned}
&-\{|n|C\tanh b + C + \alpha\tanh b/\sinh^2 b\} - \bar{C}\{n\tanh b - \operatorname{sgn} n\} \\
&\quad + \frac{k}{p}\{\alpha n + C\operatorname{sgn} n + \bar{\alpha}|n| + \bar{C}(1-\delta_{n,0}) + \delta_{n,0}C\}\tanh b \\
&\quad - 2(\tanh b)\sum_{n'}\{d_{n'} + \operatorname{sgn} n)g_{n'}\}e^{bn'} \\
&= \frac{\Lambda}{m\omega_0^2}\{\alpha|n| + C + \bar{\alpha}n + \bar{C}\operatorname{sgn} n\}.
\end{aligned} \tag{D.35}$$

* See also eq. (33) of Larkin and Ovchinnikov (1984).

Comparing terms with the same n-dependence, we must require that

$$-C\tanh b+\frac{k}{p}(\tanh b)\bar{\alpha}=\frac{\Lambda}{m\omega_0^2}\alpha, \tag{D.36a}$$

$$-\bar{C}\tanh b+\frac{k}{p}(\tanh b)\alpha=\frac{\Lambda}{m\omega_0^2}\bar{\alpha}, \tag{D.36b}$$

$$\bar{C}+\frac{k}{p}(\tanh b)C-2(\tanh b)\sum_{n'}g_{n'}\mathrm{e}^{bn'}=\frac{\Lambda}{m\omega_0^2}\bar{C}, \tag{D.36c}$$

$$-C-\alpha(\tanh b)/\sinh^2 b+\frac{k}{p}(\tanh b)\bar{C}-2(\tanh b)\sum_{n'}d_{n'}\mathrm{e}^{bn'}=\frac{\Lambda}{m\omega_0^2}C. \tag{D.36d}$$

For $n=0$ we have the following condition:

$$-C-\alpha\tanh b/\sinh^2 b+\frac{k}{p}(\tanh b)C+\left(\frac{k}{p}\tanh b+1\right)d_0$$

$$-2(\tanh b)\sum_{n'}(d_{n'}+g_{n'})\mathrm{e}^{-b|n'|}=\frac{\Lambda}{m\omega_0^2}(C+d_0). \tag{D.36e}$$

We obtain the five eigenvalues $\tilde{\Lambda}_\nu^k$ with $|\nu|\leqslant 2$ from the above equations by putting $d_n=0$ for $n\neq 0$ and $g_n=0$ for all n. This leads to the following eigenvalue equation [after subtracting eq. (D.36d) from eq. (D.36e)]:

$$(M^k-\Lambda/m\omega_0^2)\begin{pmatrix}\alpha\\ C\\ d_0\\ \bar{\alpha}\\ \bar{C}\end{pmatrix}=0, \tag{D.37}$$

where the matrix M^k (with $q^k=(k/m)\tanh b$) is given by

$$M^k=\begin{pmatrix}0 & -\tanh b & 0 & q^k & 0\\ -\tanh b/\sinh^2 b & -1 & -2\tanh b & 0 & q^k\\ 0 & q^k & 1+q^k & 0 & -q^k\\ q^k & 0 & 0 & 0 & -\tanh b\\ 0 & q^k & 0 & 0 & 1\end{pmatrix}. \tag{D.38}$$

Obviously, for $k=0$ the odd subspace decouples from the even one and the product of the three lowest even eigenvalues is $(\tilde{\Lambda}_0\tilde{\Lambda}_1\tilde{\Lambda}_2/m^3\omega_0^6)=-1/\cosh^2 b$, as derived above.

For $k \neq 0$ the eigenvalue $\tilde{\Lambda}_0^k$ is nonzero and we obtain

$$\tilde{\Lambda}_{-2}^k \tilde{\Lambda}_{-1}^k \tilde{\Lambda}_0^k \tilde{\Lambda}_1^k \tilde{\Lambda}_2^k / (m\omega_0^2)^5 = \det M^k$$
$$= (q^k)^2 (1 + q^k)(q^k - \tanh b)^2. \tag{D.39}$$

We next consider the case $|\nu| - 2 = N \geqslant 1$. By virtue of eqs. (D.36a–d), the equations for $n = \pm N$ reduce to

$$\left[\left(N + \frac{k}{p}\right)\tanh b + 1\right](d_N + g_N) = \frac{\Lambda}{m\omega_0^2}(d_N + g_N),$$
$$\left[\left(N - \frac{k}{p}\right)\tanh b + 1\right](d_N - g_N) = \frac{\Lambda}{m\omega_0^2}(d_N - g_N). \tag{D.40}$$

We conclude that the eigenvalues are

$$\tilde{\Lambda}_\nu^k = m\omega_0^2\left[\left(\left|\nu + \frac{k}{p}\right| - 2\right)\tanh b + 1\right]. \tag{D.41}$$

As for $k = 0$ we take account of the finite mass by putting

$$\tilde{\Lambda}_\nu^k = m\tilde{L}(|\omega_{p\nu+k}|), \tag{D.42}$$

where

$$\tilde{L}(\omega) = \omega^2 + \gamma(\omega - 2\omega_p) + \omega_0^2. \tag{D.43}$$

We also define

$$L^0(\omega) = \omega^2 + \gamma\omega + \omega_0^2. \tag{D.44}$$

We conclude that the prefactor of an orbit of multiplicity p is similar to eq. (D.26) with an extra product with respect to the Bloch index k and $\mathscr{S}_0$ replaced by $p\mathscr{S}_0$. Thus,

$$\mathscr{A}^{(p)} = \left(\frac{p\mathscr{S}_0([r_1(\tau)]; \tau_p)}{2\pi\hbar}\right)^{1/2} \left|\frac{\prod_\nu \prod_{k=0}^{p-1} L^0(|\omega_{p\nu+k}|)}{\prod'_\nu \prod_{k=0}^{p-1} \tilde{L}(|\omega_{p\nu+k}|)}\right|^{1/2}, \tag{D.45}$$

where the prime means that in the product of the denominator the eigenvalue $\tilde{\Lambda}_0^{k=0}$ should be omitted.

Consider now the product

$$\prod_\nu \prod_{k=0}^{p-1} L(|\omega_{p\nu+k}|) = \prod_r L(|\omega_r|) = \frac{1}{L(0)}\left\{\prod_{r=0}^{\infty} L(|\omega_r|)\right\}^2. \tag{D.46}$$

We then write the prefactor in the form

$$\mathscr{A}^{(p)} = \mathscr{A}_1^{(p)} \mathscr{A}_2^{(p)}, \tag{D.47}$$

where

$$\mathscr{A}_1^{(p)} = \left|\frac{\tilde{L}(0)}{L^0(0)}\right|^{1/2} \left|\prod_{r=0}^{\infty} \frac{L^0(|\omega_r|)}{\tilde{L}(|\omega_r|)}\right|, \tag{D.48}$$

and

$$\mathscr{A}_2^{(p)} = \left(\frac{p\mathscr{S}_0(\tau_p)m^4}{2\pi\hbar\Lambda_{-2}^0\Lambda_{-1}^0\Lambda_1^0\Lambda_2^0}\right)^{1/2} \frac{|\prod_{k=0}^{p-1}\prod_{\nu=-2}^{2}\tilde{L}(|\omega_{p\nu+k}|)|^{1/2}}{\prod_{k=1}^{p-1}q^k|q^k - \tanh b|(1+q^k)^{1/2}\omega_0^5}. \tag{D.49}$$

We can, by introducing $\mu_{1,2}$ and $\nu_{1,2}$ as defined in eq. (D.28), write $\mathscr{A}_1^{(p)}$ as follows:

$$\mathscr{A}_1^{(p)} = [1 - 2\tanh b]^{1/2} \left|\frac{\Gamma(-\mu_1)\Gamma(-\mu_2)}{\Gamma(-\nu_1)\Gamma(-\nu_2)}\right|. \tag{D.50}$$

Using the results (D.25) and (D.41) for $\mathscr{S}_0$ and for $\tilde{\Lambda}_\nu^0$, we may write $\mathscr{A}_2^{(p)}$ as

$$\mathscr{A}_2^{(p)} = \left(\frac{mpR_B^2}{2\hbar\gamma}\right)^{1/2} \frac{\omega_1^{5p}}{\omega_0^{5p-2}} \frac{|\prod_{k=0}^{p-1}\prod_{\nu=-2}^{2}[|p\nu+k| - \mu_1][|p\nu+k| - \mu_2]|^{1/2}}{\prod_{k=1}^{p-1}p^{-5/2}(\tanh^2 b)(p-k)k(k\tanh b + p)^{1/2}}. \tag{D.51}$$

Since the five smallest eigenvalues $\tilde{\Lambda}_\nu^k$ were calculated for zero mass, we must, for the sake of consistency, in the expression for $\mathscr{A}_2^{(p)}$ consider $-\mu_2 \to \gamma/\omega_1$ to be very large, leading to

$$\prod_{k=0}^{p-1}\prod_{\nu=-2}^{2}[|p\nu+k| - \mu_2] \Rightarrow (\gamma/\omega_1)^{5p/2}. \tag{D.52}$$

In the same limit we have

$$\mu_1 = 2p - \frac{\omega_0^2}{\gamma\omega_1} = p(2 - \coth b). \tag{D.53}$$

After some algebraic manipulations, we obtain

$$\mathscr{A}_2^{(p)} = \left(\frac{mp}{2\hbar\gamma}\right)^{1/2} R_B \frac{\omega_0^2 \tanh b}{p^2\Gamma^2(p)} \left|\frac{\Gamma(2p - p\coth b)\Gamma(p\coth b)}{\Gamma(-p\coth b)\Gamma(-2p + p\coth b)}\right|^{1/2}. \tag{D.54}$$

Let us now introduce the following abbreviations:

$$y = T_B/T = p\coth b, \qquad \rho = \gamma^2/\omega_0^2. \tag{D.55}$$

In the limit of strong damping,

$$\mu_1 = 2p - y, \qquad -\mu_2 = 2p + (\rho - 1)y,$$
$$\nu_1 = -y, \qquad -\nu_2 = (\rho - 1)y; \tag{D.56}$$

after some manipulations, we obtain for the prefactor

$$\mathscr{A}^{(p)} = \left(\frac{m}{2\hbar\gamma}\right)^{1/2} \omega_0^2 R_B \frac{p^{1/2}}{y^2 \Gamma^2(p)} \left| \frac{\Gamma(1 + \rho y + 2p - y)}{\Gamma(1 + \rho y - y)} \right|. \tag{D.57}$$

Eventually, we introduce

$$2\sigma = 1 + \rho y - y = 1 + \frac{T_B}{T}\left(\frac{\gamma^2}{\omega_0^2} - 1\right); \tag{D.58}$$

whence we may write

$$\mathscr{A}^{(p)}(\tau_1) = \left(\frac{9\pi v_B}{\hbar \tau_B}\right)^{1/2} \left(\frac{\tau_B}{\tau_1}\right)^2 \frac{p^{1/2}}{\Gamma^2(p)} \frac{\Gamma(2p + 2\sigma)}{\Gamma(2\sigma)}, \tag{D.59}$$

as quoted in eq. (4.69) of the text.

Appendix E. Calculations of the decay rate of a heavily damped object

For orientation, we first review some characteristic features of the multiple orbit summation problem already considered in section 2.6 and later in section 4.3. We have

$$\bar{\Gamma} = \sum_{p=1}^{\infty} (-1)^{p+1} A^{(p)}(\tau_1) \exp\left[-\frac{1}{\hbar} \mathscr{S}^{(p)}(\tau_1) \right]. \tag{E.1}$$

However, we do not wish to define $\bar{\Gamma}_p(E)$ directly as done in eq. (2.66). Rather, we prefer to consider in addition to eq. (E.1) a representation of the form

$$\bar{\Gamma} = \sum_{p=1}^{\infty} (-1)^{p+1} \int_0^{\infty} \frac{dE}{2\pi} \tilde{A}^{(p)}(E; \tau_1) \exp\left\{ -\frac{1}{\hbar} [E\tau_1 + \mathscr{W}^{(p)}(E)] \right\}, \tag{E.2}$$

where $\tilde{A}^{(p)}(E; \tau_1)$ has to be so chosen that a steepest descent evaluation of the energy integral leads to a term-by-term agreement of both series (E.1) and (E.2). We will see that there is some arbitrariness in the energy and time dependence of $\tilde{A}^{(p)}(E; \tau_1)$, and we may exploit this ambiguity in order to obtain an expression for $\tilde{A}^{(p)}$ which is only weakly dependent on energy. This seems to be a reasonable choice in a steepest descent approximation, where only E-values near the minimum of $[E\tau_1 + \mathscr{W}^{(p)}(E)]$ are supposed to contribute.

For orientation, we first note that

$$\mathscr{S}^{(p)}(\tau_1) = p\mathscr{S}(\tau_p), \qquad \tau_p = \tau_1/p, \qquad \mathscr{W}^{(p)}(E) = p\mathscr{W}(E) \tag{E.3}$$

are pairs of Legendre transforms

$$\frac{\partial \mathscr{S}(\tau)}{\partial \tau} = E(\tau), \qquad \frac{\partial \mathscr{W}}{\partial E} = -\tau(E) \equiv -\tau_E. \tag{E.4}$$

For illustration, we have for a heavily damped object [see eqs. (4.64) and (4.65)]

$$\begin{aligned} &\mathscr{S}(\tau) = v_B \tau_B [\tfrac{3}{2} - (\tau_B/\tau)^2], \\ &\mathscr{W}(E) = \tfrac{3}{2} v_B \tau_B [1 - (E/v_B)^{2/3}], \\ &\tau_E = \tau_B (E/v_B)^{-1/3}. \end{aligned} \tag{E.5}$$

Evaluating the integral in eq. (E.2) by steepest descent, we find that the energy is implicity given by

$$\tau_p = \tau_1/p = \tau_E \tag{E.6}$$

and that

$$\tilde{A}^{(p)}(E;\tau_1) = \left(-2\pi\hbar p \frac{\partial^2 \mathscr{W}}{\partial E^2}\right)^{1/2} A^{(p)}(\tau_1). \tag{E.7}$$

It is possible to eliminate the energy dependence on the right-hand side of the relation above by means of eqs. (E.5) and (E.6). Thus, one arrives at the expression (4.73).

In a heuristic way, let us now eliminate all the p-dependence in eq. (E.7). Thus, we put $p = \tau_1/\tau_E$ and obtain ($\rho = \gamma^2/\omega_0^2$)

$$\tilde{\tilde{A}}^{(p)}(E) = \frac{\tau_E}{\tau_1 \Gamma^2(\tau_1/\tau_E)} \frac{\Gamma(1 + 2\tau_1/\tau_E + (\rho - 1)\tau_1/\tau_B)}{\Gamma(1 + (\rho - 1)\tau_1/\tau_B)}. \tag{E.8}$$

For temperatures close to the crossover temperature $\tau_1 \sim \tau_B$, we expect that energies $E \sim v_B$ are of importance. Thus, we put $\tau_E = \tau_1 = \tau_B$ and obtain

$$\tilde{\tilde{A}}^{(p)}(E \approx v_B) = \frac{\Gamma(2 + \rho)}{\Gamma(\rho)}. \tag{E.9}$$

Now, the series (E.2) can easily be summed over. For the sake of consistency, we also replace $\mathscr{W}(E)$ by the quadratic approximation $\mathscr{W}_2(E)$ as shown in eq. (2.77), where for the present case

$$\left(\frac{1}{kT_2}\right)^2 = \frac{2}{3}\frac{\tau_B}{\hbar v_B}. \tag{E.10}$$

Proceeding further according to Affleck's prescription of eq. (2.76), we arrive at eq. (4.76); whence we obtain [cf. eq. (2.79)]

$$\bar{\Gamma}^A = \frac{kT_2}{2\pi\hbar} \frac{\Gamma(2 + \rho)}{\Gamma(\rho)} \pi^{1/2} \operatorname{erfc}\left(\frac{T_2}{T_B} - \frac{T_2}{T}\right) \exp\left[-\frac{v_B}{T} + \frac{1}{2}\left(\frac{T_2}{T_B} - \frac{T_2}{T}\right)^2\right], \tag{E.11}$$

which agrees essentially with eq. (43) of Larkin and Ovchinnikov (1984), and eq. (46) of Larkin and Ovchinnikov (1992). Note, however, that the strong energy dependence in eq. (E.8) casts some doubts on the above procedure.

Therefore, we wish to draw attention to the fact that there is a direct way to calculate $\tilde{A}^{(p)}$ for $E \sim v_B$. In order to understand the reasoning, we should recall that the presentation in eq. (E.1) requires different primitive traversal times τ_p for the different periodic orbits of multiplicity p, and that such a flexibility is given only in the case of a nonlinear potential. On the other hand, the presentation in eq. (E.2), requires only data of periodic orbits at a given energy, which can also be obtained in the limit of a harmonic potential. Thus, for $E \sim v_B$ and $\boldsymbol{R} \sim \boldsymbol{R}_B$, we may represent $V(\boldsymbol{R})$ by a quadratic form which is based on the matrix

$$U_{kk'}^{(B)} = \frac{1}{m} \frac{\partial^2 V(\boldsymbol{R})}{\partial R_k \partial R_{k'}}\bigg|_{\boldsymbol{R}=\boldsymbol{R}_B}. \tag{E.12}$$

Then the problem is separable, and putting $\hat{\omega}_B^2 = \hat{U}^{(B)}$, we may single out the unstable mode $\omega_{B_1}^2 < 0$ from the remaining ones. Then we obtain for the relevant part of the action

$$\frac{1}{\hbar} \mathscr{W}(E) = -\frac{1}{kT_B}(E - v_B), \tag{E.13}$$

and for the prefactor

$$\tilde{A}(\tau_1) = \frac{1}{2\pi\hbar} \sinh(\omega_{B_1}\tau_1/2) \frac{\det(\sinh \hat{\omega}_0 \tau_1/2)}{\det(\sinh \hat{\omega}_B \tau_1/2)}. \tag{E.14}$$

Turning now our attention to the Caldeira–Leggett model (1981, 1983, 1984), we find $\omega_{B_1} = i\lambda_1\omega_1$ ($\omega_1 = 2\pi kT/\hbar$), where

$$\lambda_{1,2} = (\omega_1)^{-1}(-\tfrac{1}{2}\gamma \pm [\tfrac{1}{4}\gamma^2 + \omega_0^2]). \tag{E.15}$$

Concerning the stable modes, we need

$$\Lambda_{1,2} = (\omega_1)^{-1}(-\tfrac{1}{2}\gamma \pm [\tfrac{1}{4}\gamma^2 - \omega_0^2]^{1/2}). \tag{E.16}$$

Thus, we obtain

$$\begin{aligned}\tilde{A}(\tau_1) &= \frac{\sin\lambda_1}{2\pi\hbar}\left|\prod_{n=-\infty}^{\infty} \frac{[1-\Lambda_1/n][1-\Lambda_2/n]}{[1-\lambda_1/n][1-\lambda_2/n]}\right| \\ &= \frac{\sin\lambda_1}{2\pi\hbar} \frac{\Gamma(1-\lambda_1)\Gamma(1-\lambda_2)}{\Gamma(1-\Lambda_1)\Gamma(1-\Lambda_2)}.\end{aligned} \tag{E.17}$$

We observe that (i) $\tilde{A}$ depends only on τ_1, and neither on the multiplicity nor on the energy, as to be expected for a harmonic potential, (ii) in the heavy damping limit, $\tilde{A}(\tau_B) \to \tilde{\tilde{A}}(v_B)$, as given by eq. (E.8) and (iii) that $\tilde{A}(\tau_1)$ together with

$\mathscr{W}_1(E)$ leads to the well-known expression

$$\bar{\Gamma} = \frac{kT_B}{\hbar} \frac{\Gamma(1-\lambda_1)\Gamma(1-\lambda_2)}{\Gamma(1-\Lambda_1)\Gamma(1-\Lambda_2)} e^{-v_B/kT}. \tag{E.18}$$

This expression agrees with eq. (47) of Larkin and Ovchinnikov (1984).

Note also that $\bar{\Gamma}$ of eq. (E.18) can be obtained from the imaginary part of the free energy calculated from the trivial orbit $\boldsymbol{R} = \boldsymbol{R}_B$ if an additional factor T_B/T is inserted as shown in eq. (2.74).

Since the Γ-functions depend very sensitively on the arguments, the approximations in the transition from eq. (E.8) to eq. (E.9) may not be so good. Thus, we have introduced in section 4.3 the function $\phi(z)$ according to eq. (4.75), which, in terms of the variable

$$t = (1 + 4z)^{-1/2}, \tag{E.19}$$

can be expressed through the Legendre function (Abramowitz and Stegun 1968) as follows:

$$\phi(z) = -\sigma \frac{2\sigma + 1}{4\sigma + 1} t^{2\sigma}[P_{2\sigma+1}(t) - P_{2\sigma-1}(t)]. \tag{E.20}$$

Using standard numerical routines for the Legendre function, Ludviksson (1989) has integrated eq. (4.74) numerically, with some of the results given in Table 1.

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