Comment on “Bistability driven by weakly colored Gaussian noise: the Fokker-Planck boundary layer and mean first-passage times”

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Comment on "Bistability Driven By Weakly Colored Gaussian Noise: The Fokker-Planck Boundary Layer and Mean First-Passage Times"

In a recent Letter, Doering, Hagen, and Levermore[1] made an attempt to evaluate the mean first-passage time (MFPT) for the archetypal bistability: $\dot{x}=x-x^3+u$, $\dot{u}=-\epsilon^{-2}u+\sigma\epsilon^{-2}\xi(t)$, with $\xi(t)$ being white Gaussian noise, i.e., $\langle\xi(t)\xi(s)\rangle=2\delta(t-s)$. This flow implies colored noise $\langle u(t)u(0)\rangle=(\sigma/\epsilon)^2\exp(-|t|/\epsilon^2)$. The authors of Ref. 1 make use of a scaling $z(t)=\epsilon u(t)/\sigma$ and proceed to evaluate at *weak noise* ($\sigma^2\ll 1$) "the MFPT T of the variable $x(t)$ from the well at $x=1$ to the unstable point $x=0$." They do not use a MFPT approach based on the adjoint operator, but use instead a method based on a steady-state density per unit flux, $G(x,z)$. For $G(x,z)$ they correctly use the boundary condition $G(0,z)=0$, $z>0$ (dash-dotted line in Fig. 1). We wish to point out, however, that this boundary condition is not suitable to evaluate the activation rate to leave the *domain of attraction* or, equivalently, the smallest nonvanishing eigenvalue of the two-dimensional Fokker-Planck dynamics. Figure 1 depicts the flow for the deterministic equation $[\xi(t)=0]$ with $\epsilon^2=0.2$. The flow exhibits an inversion symmetry. Note that the separatrix (dotted line) has a tilt and does not coincide with the line $x=0$. Particles starting at $(1,z)$ will reach ($x=0$, $u>0$) only after having crossed the separatrix [MFPT ($x=1\rightarrow$ separatrix) T_0] at *negative u* values. After crossing the separatrix they will settle near $x=-1$ [MFPT ($x=1\rightarrow x=-1$) $2T_0$] before crossing the line $x=0$, $u>0$. Doering, Hagen, and Levermore[1] now integrate $G(x,z)$ over the positive half plane $x>0$ only. For $x>0$, one has regions of integration with a MFPT [$x=1\rightarrow x\cong 0$, $u=O(\sigma)$] $T_1\gtrsim T_0$, and a MFPT ($x=1\rightarrow$ hatched region in Fig. 1) $T_1=2T_0$, respectively. Near $x=0$, $u\lesssim 0$ (critical region), the boundary layer for the MFPT T_0, of width $O(\sigma)$, cuts the line $x=0$, thereby yielding a *mixed contribution* to T in Eq. (8) of Ref. 1, i.e., $T_0\leq T<2T_0$. The final result in Eq. (23) of Ref. 1 predicts for T a growth proportional to ϵ.

We have performed precise numerical calculations (error $<0.1\%$) for the smallest eigenvalue $\lambda(\epsilon)$ $[=T_0^{-1}(\epsilon),\sigma^2\ll 1]$ of the two-dimensional Fokker-Planck process. In the inset of Fig. 1 this eigenvalue is compared with the inverse MFPT $T(\epsilon)$ of Doering, Hagen, and Levermore to reach the line $x=0$, i.e., for the sake of illustration we plot $\lambda_D(\epsilon)\equiv T^{-1}(\epsilon)$ for small values of the correlation time, ϵ^2, of the noise.

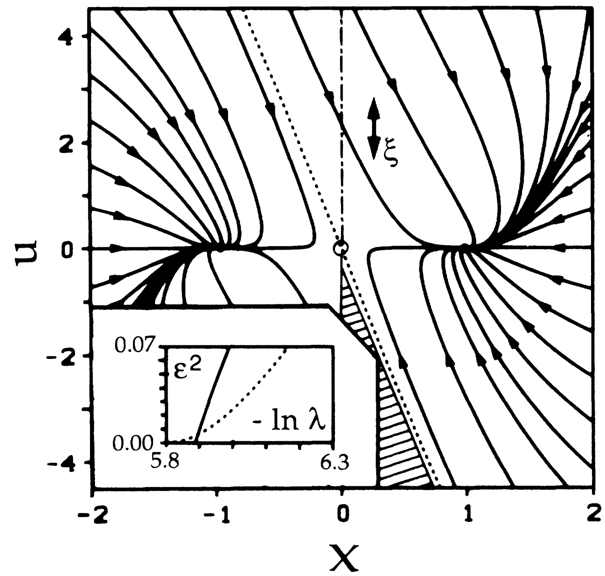

FIG. 1. Bistable flow diagram $\dot{x}=x-x^3+u$, $\dot{u}=-\epsilon^{-2}u$ with $\epsilon^2=0.2$. Inset: Smallest nonvanishing eigenvalue of the two-dimensional bistable flow for $\sigma^2=0.05$. Solid line, exact result; dotted line, $T^{-1}(\epsilon^2)$ from Eq. (23) in Ref. 1.

In agreement with the above argumentation, this MFPT $T(\epsilon)$ in Eq. (23) of Ref. 1 does not compare with the inverse rate of escape or the MFPT T_0 to reach the separatrix. As exhibited in the inset of Fig. 1, the eigenvalue $\lambda(\epsilon)$ does not show a growth proportional to ϵ, i.e., $\lambda(\epsilon)\neq\lambda_D=\lambda_0(1-\text{const}\times\epsilon)$.

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[1] C. R. Doering, P. S. Hagen, and C. D. Levermore, Phys. Rev. Lett. **59**, 2129 (1987).