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# Amplification of small signals via stochastic resonance

Peter Jung and Peter Hänggi
*Institute of Physics, University of Augsburg, D-8900 Augsburg, Germany*
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Stochastic resonance is a cooperative effect of noise and periodic driving in bistable systems. It can be used for the detection and amplification of weak signals embedded within a large noise background. In doing so, the noise triggers the transfer of power to the signal. In this paper we first present general properties of periodically driven Brownian motion, such as the long-time behavior of correlation functions and the existence of a "supersymmetric" partner system. Within the framework of nonstationary stochastic processes, we present a careful numerical study of the stochastic resonance effect, without restrictions on the modulation amplitude and frequency. In particular, in the regime of intermediate driving frequencies which has not yet been covered by theories, we have discovered a secondary resonance at smaller values of the noise strength.

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## I. INTRODUCTION

Stochastic resonance (SR) has been invoked by Benzi and co-workers [1] to explain the more or less periodic occurrence of earth's ice ages. The main effect of SR is to enhance the response of a bistable system to weak periodic driving by the injection of random noise. The slowly varying eccentricity of the earth causes weak periodic modulations of the earth's climatic system. Benzi *et al.* model the climate system as bistable in the sense that there are two metastable states, each characterized by a different typical temperature. The result of the cooperative interaction of climatic fluctuations and the weak periodic modulations is argued to be the periodic occurrence of such crucial climatic events like ice ages.

A more recent experiment has been done by McNamara, Wiesenfeld, and Roy [2] and by Vemuri and Roy [3] with a bidirectional ring laser. The stable states are the clockwise and counterclockwise propagating waves, respectively. Random noise, inherent in every laser system, causes random switching between the two modes. An additional acousto-optic modulator periodically changes the probabilistic weights of the two modes. Vemuri and Roy have taken the power spectrum of the stochastic intensity of one of the modes. The power spectrum exhibits sharp peaks at the driving frequency and at its higher harmonics, where the sharpness of the peaks is solely determined by the experimental facilities, but not by the laser system. The ratio between the area under such a peak and the noise background, i.e., the signal-to-noise ratio, shows a resonancelike curve as a function of the input noise strength. The resonance condition is roughly fulfilled when the unperturbed thermal (Kramers-type [4]) hopping frequency is equal to the frequency of the modulation. Very recently Moss and Bulsara [5] have used the SR effect in a superconducting quantum interference device (SQUID) for the detection of weak magnetic signals embedded in a large noise background.

A theoretical description of the SR effect in a bistable system based on a two-state model has been given by McNamara and Wiesenfeld [6]. They have proposed a master equation for the populations in the two stable states, where the transition probabilities are periodic with the modulation frequency. The expressions for the transition probabilities, however, are only valid for very small frequencies (adiabatic assumption) and small modulation strengths. Within this theory one finds $\delta$ peaks in the power spectrum at the external driving frequency. The signal-to-noise ratio, i.e., the ratio of the weight of the $\delta$ peak and the noise background, shows the SR effect. This theory, however, does not predict higher harmonic peaks in the power spectrum. Since the two-state approximation cannot cover all details of a bistable system, investigations within a continuous description have been put forward by Fox [7] and Presilla, Marchesoni, and Gammaitoni [8]. Both theories are based on a perturbative expansion of the Greens function for a weak periodic modulation into the complete set of functions defined by the spectrum of the unperturbed Fokker-Planck operator. The theory by Presilla, Marchesoni, and Gammaitoni in fact predicts the SR effect. The expansion, however, does not consistently describe the details of the time dependence of the Greens function as it can be seen for the correlation functions, where additional phases occur in comparison with exact results [9]. Stochastic resonance has also been studied in detail by Marchesoni and co-workers [8a–8c] and by Moss and co-workers [10,11] using an analog simulation technique. In [11] SR has been characterized for the first time by the characteristic behavior of peak heights of the escape time distribution.

In [9] and [12] we have given exact answers to spectral and ergodic problems, i.e., we have exactly predicted various $\delta$ peaks in the power spectrum and its selection rules for symmetric potentials, where only spikes at odd superharmonics occur [9]. The goal of this paper is to describe SR within the framework of nonstationary stochastic processes, and to present numerical results without the restriction to small driving amplitudes or frequencies. This paper is organized as follows. In Sec. II our bistable model and some qualitative description of SR are given. In Sec. III we provide the necessary background knowledge of nonstationary processes and present our Floquet-type description of the system. New results for the Floquet spectrum, such as the *supersymmetry property* and *generalized selection rules* for the δ peaks in the power spectrum are also proven here. In addition we show that the weights of the δ peaks in the power spectrum are related to Fourier coefficients of certain time-dependent mean values. We thus derive *a formally exact result for the response of the system to periodic modulation.* This formal result may be evaluated approximately for small modulation frequencies by an adiabatic theory, as well as for small modulation strengths within linear response theory [13] (see also Ref. [14]). In Sec. IV we present explicit numerical evaluations of our formal exact expression for the system response. In contrast to all the papers quoted above, we do not characterize SR by a signal-to-noise ratio. The latter cannot be defined uniquely within a Fokker-Planck description, since after subtracting a δ peak at the driving frequency there is no clear-cut way of identifying a noise background. Instead, we describe the SR amplification device by its amplification characteristics similarly to what is usually done to characterize a micro-electronic device. Here we do not restrict ourselves to small modulation strengths and/or small frequencies only. In this section we also discuss the influence of finite coherence times of the modulation to SR, this being a concession to the real physical world. In Sec. VI, we consider the full power spectrum of the system. It turns out that the results of the two-state approximation in Ref. [6] do not apply to the state-continuous model. The results differ significantly all over the frequency range. Especially the integral of the power spectrum over the frequency, which always equals a constant for two-state models [11], is no longer constant in the state-continuous description.

## II. THE BISTABLE MODEL

Although many of the considerations in the next section are very general, i.e., they do not depend on the specific choice of the system, we introduce a bistable model for the presentation of concrete results. The model we use is the standard double-well system

$$\dot{x}=ax-bx^3+A\sin(\Omega t+\varphi)+\xi(t)\ , \tag{2.1}$$

where $a,b>0$ and $\xi(t)$ is white Gaussian noise with zero mean, i.e.,

$$\begin{aligned}\langle \xi(t)\rangle &=0\ ,\\ \langle \xi(t)\xi(t')\rangle &=2D\delta(t-t')\ ,\end{aligned} \tag{2.2}$$

The model (2.1), (2.2) describes the overdamped Brownian motion in the bistable potential $U(x)=bx^4/4-ax^2/2$. The phase $\varphi$ is the initial phase of the modulator. In this paper we assume that the phase is not known, i.e., the phase is equally distributed between 0 and $2\pi$ [15]. This assumption is typical for most of the experiments. For convenience we transform Eqs. (2.1) and (2.2) into a form where all variables and parameters are dimensionless, i.e., $x\rightarrow x\sqrt{b/a}$, $t\rightarrow at$ and $A\rightarrow A(b/a^3)^{1/2}$, $D\rightarrow Db/a^2$, $\Omega\rightarrow\Omega/a$. Equation (2.1) is thus transformed into

$$\dot{x}=x-x^3+A\sin(\Omega t+\varphi)+\xi(t)\ . \tag{2.3}$$

The potential in scaled units has minima at $x_{1,2}=\pm1$ and a relative maximum at $x_3=0$; the barrier height equals $\Delta U=\frac{1}{4}$. As explained in detail in Refs. [1–3] the noise makes it possible to pass over the potential barrier even if the modulation strength $A$ is too small for a deterministic transition. Since the transition rate $k$ out of a metastable state due to random noise obeys for small noise strength $D$ an Arrhenius law, i.e., $k=\nu_0\exp[-\Delta U(A,t)/D]$, transition events are most likely to take place when the barrier height has a minimum. The theory for transition rates in these situations has been put forward in Ref. [16]. The modulation $A\sin\Omega t$ thus clocks the escape process. As a consequence, the escape process also has—apart from its random nature—a coherent component. The coherent component obviously has a maximum if the random noise by itself produces in the statistical average two escape events within one period of the modulation, i.e., $\Omega=\pi\nu_0\exp(-\Delta U/D)$. This is approximately the condition for SR being valid for small driving frequencies $\Omega$. We want to point out, however, that SR *does not* imply a resonance for the escape rate itself. The incoherent part of the rate increases exponentially for increasing noise strength, and thus hides the effect of SR.

The stochastic trajectory $x(t)$, in contrast, is a better candidate for observing SR, since it is convenient to extract coherent parts from its power spectrum. Because the power spectrum is related to the correlation function (in what sense this is true for a nonstationary process will be discussed in the next section), the correlation function, or the power spectrum, presents the appropriate quantity to look at.

## III. ESSENTIALS IN THE THEORY OF NONSTATIONARY STOCHASTIC PROCESSES

The correlation function (2.2) guarantees the Markovian property of the stochastic process (2.1). Due to the explicit time dependence of the deterministic flow, however, the process is nonstationary. The Fokker-Planck equation (FPE) corresponding to (2.1) and (2.2),

$$\frac{\partial P(x,t;\varphi)}{\partial t}=-\frac{\partial}{\partial x}[x-x^3+A\sin(\Omega t+\varphi)]P_\varphi(x,t;\varphi)+D\frac{\partial^2}{\partial x^2}P(x,t;\varphi)\ , \tag{3.1}$$

has a periodic drift coefficient in time with the period $T=2\pi/\Omega$. Thus the FPE (3.1) has no eigenfunctions and eigenvalues; instead, (3.1) possesses the Floquet-type solutions

$$P^{(\mu)}(x,t;\varphi)=\exp(-\mu t)p^{(\mu)}(x,t;\varphi)\ , \tag{3.2}$$

where the functions $p^{(\mu)}(x,t)$ are periodic in $t$, i.e., $p^{(\mu)}(x,t+T)=p^{(\mu)}(x,t)$. The Floquet-type coefficients $\mu$ are in general complex, since the Fokker-Planck operator is non-Hermitian. Expanding the function $p^{(\mu)}(x,t)$ into a Fourier series in time, we observe that the relevant time scales of the Floquet-type solution $P^{(\mu)}(x,t;\varphi)$ are $\lambda_{mn}^{-1}=(\mu_m+in\Omega)^{-1}$, where $n=0,\pm1,\pm2,\ldots$. It has been shown in Refs. [9], and [12] that the inverse time scales $\lambda_{mn}$ are identical with the set of eigenvalues of the FPE,

$$\frac{\partial W(x,\theta,t)}{\partial t}=-\left[\frac{\partial}{\partial x}(x-x^3+A\sin\theta)-\Omega\frac{\partial}{\partial\theta}+D\frac{\partial^2}{\partial x^2}\right]W(x,\theta,t)\ , \tag{3.3}$$

which corresponds to a two-dimensional stationary stochastic process, i.e.,

$$\dot{x}=x-x^3+A\sin\theta+\xi(t)\ ,\qquad \dot{\theta}=\Omega\ . \tag{3.4}$$

Both descriptions, the one-dimensional nonstationary stochastic process (3.1) and the two-dimensional stationary stochastic process (3.3), are fully equivalent when one requires for (3.3) periodic boundary conditions in $\theta$. The Floquet coefficient $\mu=0$ and its corresponding *time-dependent* function $p_0(x,t;\varphi)$ in the one-dimensional description [(3.1),(3.2)] represent the whole class of eigenvalues $\{in\Omega\}$ and corresponding *time-independent* eigenfunctions in the two-dimensional description. There is no additional unphysical information within the two-dimensional description as claimed in recent work [17]. Integrating $W(x,\theta,t)$ over $\theta$ *does not* yield the time-dependent probability density $P(x,t)$ as implied in [17], but rather yields the cycle averaged probability density $\bar{P}(x,t)=(1/2\pi)\int_0^{2\pi}P(x,t;\varphi)d\varphi$ which approaches, with $\varphi$ uniformly distributed, for large times a stationary distribution. For uniformly distributed phases $\varphi$, the two-dimensional FPE (3.3) approaches a stationary density $W_{st}(x,\theta)$ [9] which is connected to the periodic asymptotic solution $P_{as}(x,t;\varphi)$ of (3.1) for large times (which is unique [18] up to the phase $\varphi$) by $P_{as}(x,t;\varphi)=2\pi W_{st}(x,\theta=\Omega t+\varphi)$ [9].

### A. Correlation functions and spectral densities

For large times, the solution of the FPE (3.1) approaches the periodic function $P_{as}(x,t;\varphi)$, which is the Floquet-type solution for the vanishing Floquet coefficient. The analog to the stationary correlation function for stationary processes is the quasistationary correlation function

$$K(t,t';\varphi)=\langle x(t)x(t')\rangle_\varphi=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xyP(x,t|y,t';\varphi)\times P_{as}(y,t';\varphi)dx\,dy\ . \tag{3.5}$$

For large times $\tau=t-t'$ the transition probability density $P(x,t|y,t';\varphi)$ approaches the unique asymptotic solution $P_{as}(x,t;\varphi)$ yielding for the correlation function

$$K_{as}(t,t';\varphi)\rightarrow\langle x(t;\varphi)\rangle_{as}\langle x(t';\varphi)\rangle_{as}\ ,\quad t>t'\rightarrow\infty,\tau\rightarrow\infty\ . \tag{3.6}$$

Note that the correlation function for a fixed phase $\varphi$ is *not a function of the time difference* $\tau$ as for stationary processes, but rather is a symmetric function of $t$ and $t'$ separately. Expanding the periodic mean values in (3.6) in a Fourier series, i.e., with $M_0=0$,

$$\langle x(t);\varphi)\rangle=\sum_{n=-\infty}^{\infty}M_n\exp[in(\Omega t+\varphi)]\ , \tag{3.7}$$

the phase-averaged asymptotic correlation function reads

$$\bar{K}_{as}(t,t')=\bar{K}_{as}(\tau)=\sum_{n=-\infty}^{\infty}|M_n|^2\exp(in\Omega\tau)=2\sum_{n=1}^{\infty}|M_n|^2\cos n\Omega\tau\ . \tag{3.8}$$

The phase-averaged correlation function thus is a function of the time differences $\tau$ [19]. Furthermore it does not decay in time, but performs ever present periodic cosine oscillations. Systems having this property are termed *not strongly mixing* [13]. In our two-dimensional description, the periodic oscillations are due to the branch of purely imaginary eigenvalues. The asymptotic spectral density $\bar{S}_{as}(\omega)$, which is the Fourier transform of the asymptotic time-homogeneous correlation function $\bar{K}_{as}(\tau)$, is given by

$$\bar{S}_{as}(\omega)=\int_{-\infty}^{\infty}K_{as}(\tau)\exp(-i\omega\tau)d\omega=2\pi\sum_{n=-\infty}^{\infty}|M_n|^2\delta(\omega-n\Omega)\ . \tag{3.9}$$

It exhibits $\delta$ spikes at multiples of the driving frequency $\Omega$ with weights given by the Fourier coefficients of the time-dependent mean value $\langle x(t)\rangle_{as}$. The full spectral density, obtained from the full correlation function valid for arbitrary times, has in addition to the $\delta$ spikes a Lorenzian-like "background." The existence of $\delta$ spikes in the power spectrum is very general for periodically driven stochastic systems. They are due to the fact that the corresponding probability distribution always approaches a *periodic* asymptotic distribution [20]. This is also true in the case when the deterministic dynamics is chaotic [12,21]. The same results [(3.8), (3.9)] could have been obtained also from the two-dimensional stationary stochastic process [(3.3), (3.4)] (see the Appendix). The fact that $S(\omega)$ is positive is thus a consequence of the validity of Wiener-Khintchin's theorem for the two-dimensional *stationary* stochastic process [(3.3), (3.4)]. As a result we conclude that the Wiener-Khintchin theorem also holds for our nonstationary process after phase averaging over the uniformly distributed initial phase of the modulation.

Next we mention that, depending on the symmetry of the problem, not all the weights of the $\delta$ spikes in the spectral density are different from zero. There are selection rules for the occurrence of $\delta$ spikes, which are now derived for the more general situation

$$\dot{x}=h(x)+g(x)\sin(\Omega t+\varphi)+\xi(t) \ . \tag{3.10}$$

For symmetric potentials, i.e., $h(x)=-h(-x)$ and symmetric functions $g(x)=g(-x)$, the mean value $\langle x(t);\varphi)\rangle$ possesses only odd Fourier coefficients. This is a consequence of the fact that the asymptotic probability must be invariant under the parity symmetry transformation $\mathcal{T}_1$: $x\to -x$, $t\to t+T/2$. Expanding $P_{\rm as}(x,t;\varphi)$ in a Fourier series with respect to time, the Fourier coefficients obey the symmetry relation $c_n(x)=(-1)^n c_n(-x)$. Carrying through the integration for the mean value $\langle x(t);\varphi\rangle$ only the odd coefficients $c_{2n+1}(x)$ do contribute. Thus the δ spikes at *even multiples* of the driving frequency assume *zero weight.* For symmetric potentials and antisymmetric function $g(x)=-g(-x)$, the mean value $\langle x(t)\rangle$ is identical zero. This is also a consequence of the invariance of the asymptotic probability with respect to the parity symmetry transformation which reads here $\mathcal{T}_2$: $x\to -x$. This parity implies that $P_{\rm as}(x,t;\varphi)$ is symmetric at all times $t$, i.e., $P_{\rm as}(x,t;\varphi)=P_{\rm as}(-x,t;\varphi)$. Thus the mean $\langle x(t);\varphi\rangle$ is identical to zero. As a consequence there are *no δ spikes at all* in the spectral density. Asymmetric potentials possess δ spikes at even and odd multiples of the driving frequency. All those selection rules have already been observed by simulations [11] and by actual experiments [3].

The "power" [22] in the $\nu$th frequency component is obtained by integrating the spectral density $S(\omega)$ over the δ peaks at $\nu\Omega$ and $-\nu\Omega$, i.e.,

$$P_\nu=4\pi|M_\nu|^2 \ . \tag{3.11}$$

With the total input power contained in the modulation, given by

$$P_{\rm in}=\pi A^2 \ , \tag{3.12}$$

the power amplification at the frequency $\omega=\Omega$ is given by

$$\eta(A,\Omega)=\frac{P_1}{P_{\rm in}}=4\left[\frac{|M_1|}{A}\right]^2 . \tag{3.13}$$

The quantity $\eta(A,\Omega)$ in Eq. (3.13) presents our measure for stochastic resonance.

Another interesting interpretation of the physical meaning of $M_1$ is the following [23]: The equal-time correlation between the stochastic output $x(t)$ and the modulation $A\sin(\Omega t+\varphi)$ averaged over the time is given by

$$C(A,\Omega,D)=\langle\langle x(t)\sin(\Omega t+\varphi)\rangle_\xi\rangle_t \ . \tag{3.14}$$

From (3.7) we find

$$\mathrm{Im}M_1=\frac{1}{T}\int_0^T\langle x(t;\varphi)\rangle\sin(\Omega t+\varphi)=C(A,\Omega,D) \ . \tag{3.15}$$

The imaginary part of the first Fourier coefficient of the mean value $\langle x(t;\varphi)\rangle$ thus describes also the equal-time correlations explained above. Finally we want to mention that within the two-dimensional description in $x$ and $\theta$ the Fourier coefficients $M_n$ and the correlation $C$ are given by

$$M_n=\int_{-\infty}^{\infty}dx\int_0^{2\pi}d\theta\, x W_{\rm st}(x,\theta)\exp(in\theta) \tag{3.16a}$$

and

$$C=\langle x\sin\theta\rangle \ . \tag{3.16b}$$

These relations allow us to extend the quantitative description of stochastic resonance to more realistic systems, where the modulation is not perfectly coherent (see Sec. IV C).

### B. Inverted potentials and supersymmetry

It has been shown that for overdamped one-dimensional Brownian motion in confining potentials $f(x)$, the Fokker-Planck operator is isospectral (except for the eigenvalue $\lambda=0$) with that of Brownian motion in the inverted potential $-f(x)$ [24,25]. In this section we show that this remains true for the Floquet spectrum of periodically modulated overdamped stochastic systems. This concept is very similar to what others have termed "supersymmetry" in quantum mechanics and field theory [26,27]. The major difference is that our Fokker-Planck operator is non-Hermitian and cannot be transformed into a Hermitian operator. The starting point is the two-dimensional Fokker-Planck equation (3.3) with the general binding force field $h(x)$, i.e.,

$$\begin{aligned}\frac{\partial W(x,\theta,t)}{\partial t}&=\left[-\frac{\partial}{\partial x}[h(x)+A\sin\theta]\right.\\&\qquad\left.-\Omega\frac{\partial}{\partial\theta}+D\frac{\partial^2}{\partial x^2}\right]W(x,\theta,t)\\&=L_{\rm FP}W(x,\theta,t) \ .\end{aligned} \tag{3.17}$$

The transformed operator

$$L=\phi_0^{-1}(x)\mathbf{L}_{\rm FP}\phi_0(x) \ , \tag{3.18a}$$

with $\phi_0(x)$ given by

$$\phi_0(x)=\exp\left[\frac{1}{2D}\int_0^x h(y)dy\right] \tag{3.18b}$$

is obtained after some calculations to read

$$\begin{aligned}L=&-\tfrac{1}{2}h'(x)-\frac{1}{4D}h^2(x)+D\frac{\partial^2}{\partial x^2}-\frac{A}{2D}h(x)\sin\theta\\&-A\sin\theta\frac{\partial}{\partial x}-\Omega\frac{\partial}{\partial\theta} \ .\end{aligned} \tag{3.19}$$

Introducing the operator $a$ and its Hermitian adjoint $a^\dagger$

$$\begin{aligned}a&=\sqrt{D}\frac{\partial}{\partial x}-\frac{1}{2\sqrt{D}}h(x) \ ,\\a^\dagger&=-\sqrt{D}\frac{\partial}{\partial x}-\frac{1}{2\sqrt{D}}h(x) \ ,\end{aligned} \tag{3.20}$$

the transformed operator $L$ takes on the simple form

$$L=-a^\dagger a+\frac{A}{\sqrt{D}}(\sin\theta)a^\dagger-\Omega\frac{\partial}{\partial\theta} \ . \tag{3.21}$$

Repeating the steps above with the inverted potential, i.e., $\tilde{h}(x) = -h(x)$, we end up with a transformed operator

$$\tilde{L} = -aa^\dagger - \frac{A}{\sqrt{D}}(\sin\theta)a - \Omega\frac{\partial}{\partial\theta} . \quad (3.22)$$

From (3.21) and (3.22) we can derive the following relation:

$$aL^\dagger(x,-\theta) = \tilde{L}(x,\theta)a , \quad (3.23)$$

where $L^\dagger(x,\theta)$ is the adjoint operator of $L$, i.e.,

$$L^\dagger(x,\theta) = -a^\dagger a + \frac{A}{\sqrt{D}}(\sin\theta)a + \Omega\frac{\partial}{\partial\theta} . \quad (3.24)$$

The eigenvalues of $L$ and $L^\dagger$ are denoted by $\lambda$, while the eigenfunctions are denoted by $\Sigma(x,\theta)$ and $\Lambda(x,\theta)$, respectively, i.e.,

$$L(x,\theta)\Sigma(x,\theta) = \lambda\Sigma(x,\theta) ,$$
$$L^\dagger(x,\theta)\Lambda(x,\theta) = \lambda\Lambda(x,\theta) . \quad (3.25)$$

Since the spectrum of the operator $L$ does not depend on the sign of $\theta$, we find

$$aL^\dagger(x,-\theta)\Lambda(x,-\theta) = \tilde{L}(x,\theta)a\Lambda(x,-\theta)$$
$$= \lambda a\Lambda(x,-\theta) , \quad (3.26)$$

which means that $a\Lambda(x,-\theta)$ is an eigenfunction of $\tilde{L}(x,\theta)$ with the eigenvalue $\lambda$, which is also an eigenvalue of $L$. Thereby we conclude that the Fokker-Planck operator with the inverted potential $-V(x)$ is isospectral with the original potential $V(x)$. Since the operators L and $\tilde{L}$ are not of Schrödinger type we *cannot* define supersymmetric partner potentials as in quantum mechanics. The isospectral property, however, is very important for practical calculations. For instance, we can conclude that the escape rate out of the bistable potential $V(x) = -x^2/2 + x^4/4$, which is given by the smallest real Floquet coefficient [16,28], equals that of the inverted metastable potential $-V(x)$.

## IV. SIGNAL AMPLIFICATION CHARACTERISTICS

In this section we present explicit results for the signal power amplification (3.13) in our double-well system introduced in Sec. II. To obtain the time-dependent mean values and its Fourier coefficients we utilize the matrix-continued-fraction (MCF) technique [29]. The technical details for the application of the MCF technique in our system are explained in Ref. [16]. We shall present only results here. In Fig. 1, the signal power amplification is plotted as a function of the noise strength $D$ at $A = 0.2$ for various values of the driving frequency $\Omega$. For large $\Omega$, the curve is rather flat and the power amplification is smaller than 1, i.e., the signal is damped and not amplified. For decreasing $\Omega$, the curve develops a peak, where the power amplification is larger than 1, i.e., the signal is amplified by the action of the noise. Note that the power amplification always decreases to a finite value, when the noise strength tends to zero (below the resonance point). This limit value can be obtained from a linearization around the potential minima and is given by $1/(4+\Omega^2)$. In the adiabatic limit $\Omega \ll \exp(-\Delta U/D)$ (see Sec. IV A), the amplification increases for decreasing noise strength and approaches a maximum at $D = 0$. This limit, however, is never reached for a *finite* frequency $\Omega$, since the adiabatic approach breaks down for values of the noise strength $D$, at which the adiabatic condition $D \gg \Delta U/|\ln(\Omega)|$ is not valid any more. In this regime of $D$ values, the system approaches the linear regime upon decreasing $D$.

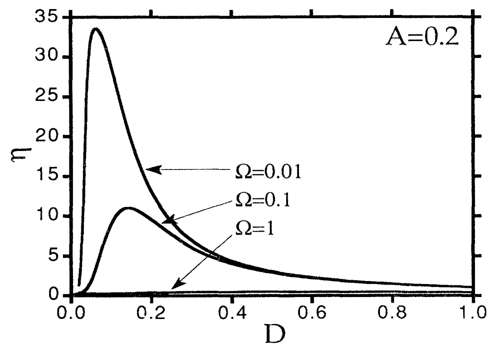


FIG. 1. The signal amplification factor $\eta$ (3.13) is shown as a function of the noise strength $D$ at $A = 0.2$ for three different driving frequencies $\Omega$.

In Fig. 2, the power amplification is shown as a function of the noise strength $D$ at $\Omega = 0.1$ for increasing values of the modulation strength $A$. For $A > A_c(\Omega)$, with $A_c(\Omega \to 0) = \sqrt{4/27}$, the resonance structure of the power amplification disappears and we observe a monotonous decrease with increasing $D$. This transition corresponds in the deterministic system to a transition from nonswitching to switching with hysteresis. The transition value $A_c(\Omega)$ scales in the deterministic system for small frequencies like $A_c(\Omega) \propto \Omega^{2/3}$ [30]. The maximum power amplification shows a monotonous decreasing behavior for increasing modulation strength $A$. This important characteristic of SR is neither described by a linear

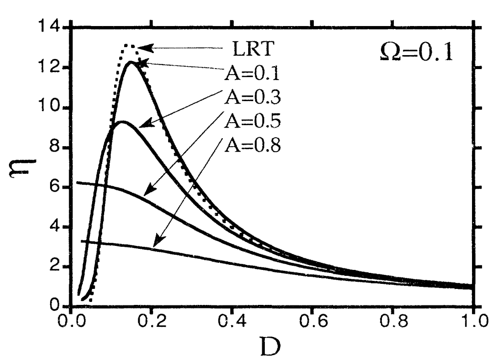


FIG. 2. The signal amplification factor $\eta$ (3.13) is depicted as a function of the noise strength $D$ at $\Omega = 0.1$ for increasing values of the driving amplitude $A$. The result of LRT (4.15) is plotted as the dotted curve.

response theory [14], nor by the two-state approximation in Ref. [6].

Another effect, which has been overlooked so far, is the existence of an additional peak in the power amplification, and thus also in the Fourier coefficient $M_1(D)$. This peak is best visible when the frequency $\Omega$ is not too small, i.e., much larger than the thermal hopping frequency. In Fig. 3 the Fourier coefficient $|M_1|$ is shown as a function of $D$ at $\Omega=2$ and $A=0.2$. We clearly recognize an additional peak at $D\approx 0.05$. The shape of this small "secondary" peak looks very much like the shape of the primary peak. This similarity of the peaks implies the following explanation for this secondary peak. The primary peak has been explained by the clocking of the escape mechanism by the external forcing [see below Eq. (2.3)]. This argument, however can be generalized as explained in the following.

The escape process can also be clocked in a "subharmonic" way, i.e., the period of the thermal hopping $T_H=2/k$ can be a odd multiple of the period $T$ of the driving frequency [transition rate $k=\nu_0\exp(-\Delta U/D)$]. This is motivated by possible "wait loops" of duration $nT$ $(n=0,1,2,\dots)$ in the potential wells. The complete period, including the wait loops, is thus given by $(2n+1)T$. The generalized stochastic resonance condition $2/k=(2n+1)T$ yields resonance at a sequence of values for the noise strength $D_n=\Delta U/|\ln[\nu_0 T(2n+1)/2]|$, with $D_{n+1}<D_n$. The primary peak is located at $D_0$, while the secondary peak is at a smaller value $D_1$. For a quantitative description at driving frequencies of the order of the well frequency, one should merely use directly the eigenvalue [16,28] (depending on frequency $\Omega$ and amplitude $A$) instead of an Arrhenius-type expressions. Otherwise, the resulting values $D_n$ are not compatible with the weak noise assumption for the Arrhenius law. Using nevertheless an Arrhenius law, the prediction for $D_1$ is in qualitative agreement with the position of our second peak.

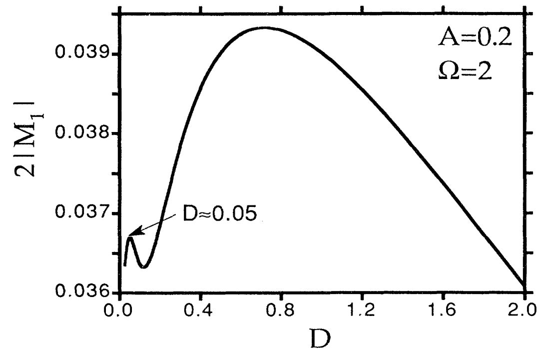


FIG. 3. The absolute value of the Fourier coefficient $M_1$ is shown as a function of the noise strength $D$ at $\Omega=2$ and $A=0.2$. The secondary resonance at $D\approx 0.05$ is indicated by an arrow. Note the self-similar shape of this secondary peak as compared with the main peak.

### A. Adiabatic approximation

In this section we present the adiabatic theory of our system which is valid when the frequency of the external signal $\Omega$ is small compared to all other typical frequencies of the system. For small noise strengths $D$, the signal frequency $\Omega$ has to be small compared to the thermal hopping frequency, i.e., $\Omega \ll \exp(-\Delta U/D)$. From this condition it is obvious that the adiabatic limit is not reached in the limit $D\to 0$ at finite frequencies. The asymptotic $(t\to\infty)$ probability distribution is given by [16]

$$P_{as}(x,t;\varphi)=\frac{1}{Z_0}\frac{\exp\left[-\frac{V_0(x)}{D}+\frac{A}{D}x\cos(\Omega t+\varphi)\right]}{\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{A^2\cos^2(\Omega t+\varphi)\sqrt{2D}}{4D^2}\right]^n\mathcal{D}_{-n-1/2}(-1/\sqrt{2D})}\,, \tag{4.1}$$

where $\mathcal{D}_{-n}(x)$ denote parabolic cylinder functions and $Z_0$ is a normalization constant, i.e.,

$$Z_0=\exp\left[\frac{1}{8D}\right](2D)^{1/4}\sqrt{\pi}\,. \tag{4.2}$$

For ease of notation we have used a cos-type modulation instead of a sin-type modulation. The asymptotic mean value $\langle x(t;\varphi)\rangle$ can be obtained from (4.1) via an integration i.e.,

$$\langle x(t;\varphi)\rangle=\frac{A\cos(\Omega t+\varphi)}{\sqrt{2D}}\frac{\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{A^2\cos^2(\Omega t+\varphi)\sqrt{2D}}{4D^2}\right]^n\mathcal{D}_{-n-3/2}(-1/\sqrt{2D})}{\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{A^2\cos^2(\Omega t+\varphi)\sqrt{2D}}{4D^2}\right]^n\mathcal{D}_{-n-1/2}(-1/\sqrt{2D})}\,. \tag{4.3}$$

Using the asymptotic expansion for the parabolic cylinder functions [31]

$$\mathcal{D}_{-n-1/2}(-1/\sqrt{2D}\,)\approx\frac{\sqrt{2\pi}}{\Gamma(n+\frac{1}{2})}\exp\left[\frac{1}{8D}\right](2D)^{1/4-n/2} \quad \text{for } D\to 0\ , \tag{4.4}$$

we find for the asymptotic mean value in the weak noise limit

$$\langle x(t;\varphi)\rangle=\tanh\left[\frac{A}{D}\cos(\Omega t+\varphi)\right]\ . \tag{4.5}$$

The first Fourier coefficient $M_1$ an be found by rewriting $\tanh(x)$ in terms of exponentials and expanding the exponentials into modified Bessel functions $I_n(x)$ [31], i.e.,

$$|M_1|=\frac{I_1(A/D)}{I_0(A/D)}=1-\frac{D}{2A}+\mathcal{O}(D^2)\ . \tag{4.6}$$

Thus, in the adiabatic limit, the signal amplification $\eta$ approaches a maximum value $\eta_{\max}=4/A^2$ in the limit $D\to 0$. This value is more generally given by $d^2/A^2$, where $d$ is the distance between the minima of the potential. This can be understood in terms of the time-dependent probability distribution. For weak noise, the time-dependent probability distribution is—even for small modulation amplitudes $A\sin(\Omega t+\varphi)$—exponentially concentrated in the right or in the left well, respectively. Therefore the mean value performs an oscillation between the positions of the two potential minima. The influence of finite noise is then to decrease the amplitude of the oscillation of the mean value, i.e., to reduce the amplification of the signal.

For a large noise strength $D$, we obtain from (4.3) by expanding into powers of $A/D$ the approximate result

$$\langle x(t;\varphi)\rangle=\frac{A}{D}\langle x^2\rangle_{\rm st}\sin(\Omega t+\varphi)+\mathcal{O}\left[\frac{A^2}{D^2}\right]\ , \tag{4.7}$$

where $\langle x^2\rangle_{\rm st}$ denotes a stationary mean value of the unperturbed system. The absolute value of the Fourier coefficient $M_1$ is thus given by

$$|M_1|\approx\frac{1}{2}\frac{A}{\sqrt{2D}}\frac{\mathcal{D}_{-3/2}(-1/\sqrt{2D}\,)}{\mathcal{D}_{-1/2}(-1/\sqrt{2D}\,)} \approx\frac{A}{4\sqrt{D}}\frac{|\Gamma(-\frac{1}{4})|}{\Gamma(\frac{1}{4})}\propto 1/\sqrt{D}\ . \tag{4.8}$$

### B. Linear response theory for stochastic resonance

In this section we shall discuss the application of linear response theory (LRT) [13] to periodically driven stochastic systems. The perturbation of the Fokker-Planck operator $L_0$ without periodic driving is of the gradient type, i.e.,

$$L_{\rm ext}=-\frac{\partial}{\partial x}A\sin(\Omega t+\varphi)\ . \tag{4.9}$$

The response function

$$R(t)=\int_{-\infty}^{\infty}x\exp(L_0t)L_{\rm ext}P_{\rm st}(x)dx\ , \tag{4.10}$$

where $P_{\rm st}(x)$ is the stationary probability of the *unperturbed* system, is connected with the correlation function $K_{xx}(\tau=t-t')=\langle x(t)x(t')\rangle$ of the *unperturbed* system by [13]

$$R(t)=-\frac{1}{D}\dot{K}_{xx}(t)\ . \tag{4.11}$$

For small noise strength $D$, we can approximate $K_{xx}(t)$ by a single exponential term, i.e.,

$$K_{xx}(t)\approx\langle x^2\rangle\exp(-\lambda_{\min}t)\ , \tag{4.12}$$

where $\lambda_{\min}$ is the smallest nonvanishing eigenvalue of the operator $L_0$. For the one-sided Fourier transform $X(\omega)$ of the response function $R(t)$, we find

$$X(\omega)=X'(\omega)-iX''(\omega) =\frac{1}{D}\langle x^2\rangle\left[\frac{\lambda_{\min}^2}{\lambda_{\min}^2+\omega^2}-i\omega\frac{\lambda_{\min}}{\lambda_{\min}^2+\omega^2}\right]\ . \tag{4.13}$$

The mean value $\langle x(t;\varphi)\rangle$ is obtained from the convolution integral of the response function with the external perturbation $A\sin(\Omega t+\varphi)$. The absolute value of the first Fourier coefficient, i.e., $|M_1|$, is then evaluated to read

$$|M_1|=\frac{A}{2}|X(\Omega)|=\frac{A}{2D}\langle x^2\rangle\frac{\lambda_{\min}}{(\lambda_{\min}^2+\Omega^2)^{1/2}}\ . \tag{4.14}$$

Substituting (4.14) into (3.13), and using the approximate expression for the eigenvalue $\lambda_{\min}\approx\sqrt{2}/\pi\exp[-1/(4D)]$, valid at weak noise, we obtain for the signal power amplification in linear response

$$\begin{aligned}\eta(\Omega)&=\left[\frac{\langle x^2\rangle}{D}\right]^2\frac{1}{1+(\pi^2/2)\Omega^2\exp(1/2D)}\\ &=\frac{1}{2D}\left[\frac{\mathcal{D}_{-3/2}(-1/\sqrt{2D}\,)}{\mathcal{D}_{-1/2}(-1/\sqrt{2D}\,)}\right]^2 \times\frac{1}{1+(\Omega^2\pi^2/2)\exp(1/2D)}\\ &\approx\frac{1}{D^2}\frac{1}{1+(\Omega^2\pi^2/2)\exp(1/2D)}\ .\end{aligned} \tag{4.15}$$

Clearly, this linear response theory *does not* involve a dependence on the driving amplitude $A$. This expression shows qualitatively the correct behavior as a function of the noise strength $D$ (cf. Fig. 2). For $D\to 0$ the exponential term in (4.15) forces $\eta$ to approach zero while for large $D$, where (4.15) is, strictly speaking, no longer valid, the $D^{-2}$ term would force $\eta$ to approach zero. In the adiabatic limit ($\Omega=0$), the expression (4.15) would approach infinity for $D\to 0$; i.e., for (i) $\Omega\to 0$ and (ii) $D\to 0$. It is, however, obvious from (4.14) that the mean value $\langle x(t;\varphi)\rangle$ also would diverge in the adiabatic limit for $D\to 0$. Thus this limit is not consistent with the assumptions of linear response theory. But also for finite frequencies, (4.15) does not predict the correct behavior for very small values of $D$, being a consequence of the single exponent approximation in (4.12). Equation (4.15) predicts in contrast to the numerical results an exponential decrease of $\eta$ for $D \rightarrow 0$. Furthermore, the additional small peak, for instance, is not reproduced by LRT. A similar expression for the signal-over-noise ratio has been derived in Ref. [14] on the basis of a linear response theory. They have compared their result with simulations of the stochastic differential equation. They have obtained reasonable agreement within their accuracy.

In conclusion, LRT produces qualitatively the correct picture for small values of the modulation; other important properties such as the $A$ dependence of the signal amplification and the additional secondary resonance are clearly beyond the regime of validity of LRT.

### C. The influence of finite coherence times

In the real world where periodic modulations are produced, for instance, by lasers these modulations have a finite coherence time. The destruction of the phase relation within a certain coherence time $\tau$ can be included in our model by adding Wiener noise to the phase of the modulator. The extended Langevin equations then read [16]

$$\dot{x} = ax - bx^3 + A \sin\theta + \xi(t) \,, \quad \dot{\theta} = \Omega + \xi_p(t) \,, \tag{4.16}$$

where

$$\langle \xi_p(t) \xi_p(t') \rangle = 2\frac{1}{\tau}\delta(t-t') \,. \tag{4.17}$$

The one-dimensional description does not lead to a Fokker-Planck equation, since the phase noise results in a non-Gaussian noise term. In contrast, the two-dimensional description yields the two-dimensional Fokker-Planck equation

$$\frac{\partial W(x,\theta,t)}{\partial t} = \left[ -\frac{\partial}{\partial x}(x - x^3 + A \sin\theta) - \Omega \frac{\partial}{\partial \theta} + D\frac{\partial^2}{\partial x^2} + \frac{1}{\tau}\frac{\partial^2}{\partial \theta^2} \right] W(x,\theta,t) \,. \tag{4.18}$$

This Fokker-Planck equation does not possess a set of purely imaginary eigenvalues [16]. Instead, one branch of eigenvalues is given by

$$\lambda_{0n} = in\Omega - n^2\frac{1}{\tau} \,. \tag{4.19}$$

As a consequence the correlation function $\langle x(t)x(0) \rangle$ decays for long times with the decay constant $\tau^{-1}$. The $\delta$ spikes in the spectral density at frequencies $n\Omega$ thus are now broadened and have a finite width proportional to $n^2/\tau$. Another consequence is that the time-dependent mean value $\langle x(t) \rangle$ now exhibits *damped* oscillations. In accordance with the arguments at the end of Sec. III A, we measure stochastic resonance now in terms of the coefficient $M_1 = \langle x \exp i\theta \rangle$, see (3.16a). In Fig. 4, the Fourier coefficient $|M_1|$ is shown as a function of the noise strength $D$ for $\Omega=0.1$, $A=0.2$, and $\tau=100$. In Fig. 4 we observe that the SR peak decreases with decreasing coherence times. Thus if one wishes to use SR for the detection of small signals (whose coherence times $\tau$ are *a priori* unknown) embedded in a noise background, one has to take into account that SR works best when the coherence times are long enough, i.e., $\tau \gg 1$. This crucially restricts the technical applicability of SR.

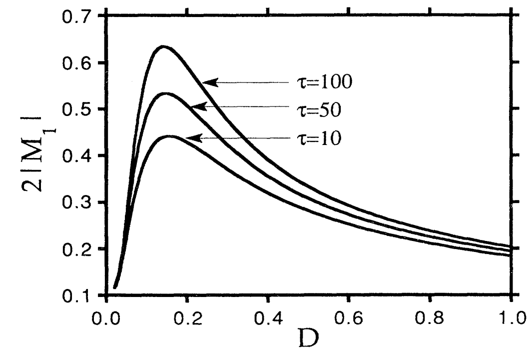


FIG. 4. The absolute value of the Fourier coefficient $M_1$ is plotted as a function of the noise strength $D$ at $\Omega=0.1$ and $A=0.2$ for decreasing values of the coherence time $\tau$.

### V. SPECTRAL DENSITIES

In the preceding section we have discussed SR in terms of the weight of the $\delta$ spikes in the spectral density $S(\omega)$. In this section we discuss the full behavior $S(\omega)$. For the calculations of $S(\omega)$ we follow a matrix-continued-fraction technique [29] described in detail in Refs. [32] and [33]. The technical details are presented elsewhere. We want to point out, however, that the error of our numerical numbers can be estimated to be much smaller than the line thickness in our figures. In Fig. 5, the real part of the one-sided Fourier transform of the autocorrelation function $K(t) = \langle x(t)x(0) \rangle$, i.e.,

$$K(\omega) = \int_0^\infty \langle x(\tau)x(0) \rangle \exp(-i\omega\tau) d\tau \,, \tag{5.1}$$

which is half the spectral density, i.e., $S(\omega) = 2\mathrm{Re}[K(\omega)]$, is plotted for increasing values of the modulation strength $A$ at the driving frequency $\Omega=0.1$ and the coherence time $\tau=100$. For vanishing $A$ the spectral density shows the usual Lorenzian-like shape [13]. For $A=0.2$, the predicted peak builds up at the driving frequency $\Omega$. We want to emphasize that the line shape is modified not only such that the spectral density becomes somewhat smaller at all frequencies different from the odd multiples of the driving frequency, but rather changes it shape completely. At small frequencies $\omega$, the value of $S(\omega)$ is strongly reduced, since $S(0)$ is proportional to the inverse escape rate $k(A)$, which is strongly enhanced with increasing modulation strength $A$. This becomes quite evident at $A=0.5$. Here, the Lorenzian-like background structure around $\omega=0$ no longer exists. For infinite coherence times, the shapes of the curves for

$\mathrm{Re}[K(\omega)]$ are qualitatively the same, but the width of the peak at the driving frequency would be zero ($\delta$ spike). The imaginary part of $K(\omega)$ shows a typical resonance structure at the driving frequency $\Omega$. For infinite coherence times, one would observe a characteristic pole at the driving frequency.

Another interesting quantity is the *total power*, defined by

$$P_{\mathrm{tot}} = \int_{-\infty}^{\infty} S(\omega) d\omega \ . \tag{5.2}$$

As already mentioned in the Introduction, the total power in a two-state model does not depend on the driving amplitude and frequency. Here, within the state-continuous bistable model this does not hold true any more. The total power $P_{\mathrm{tot}}$ being identical with the second moment $2\pi\langle x^2 \rangle$ (the brackets denote averaging over the realizations of the noise and over the initial phase of the modulator) is shown in Fig. 6. The unexpected result is that for larger frequencies the total power first decreases and then increases for increasing modulation strength $A$. This behavior can be understood from the shape of the phase-averaged asymptotic probability distribution [16]: For large frequencies, the peaks are shifted towards smaller values of $x$ at small values of the modulation strength $A$, i.e., the variance is reduced. On further increasing the modulation strength $A$, however,

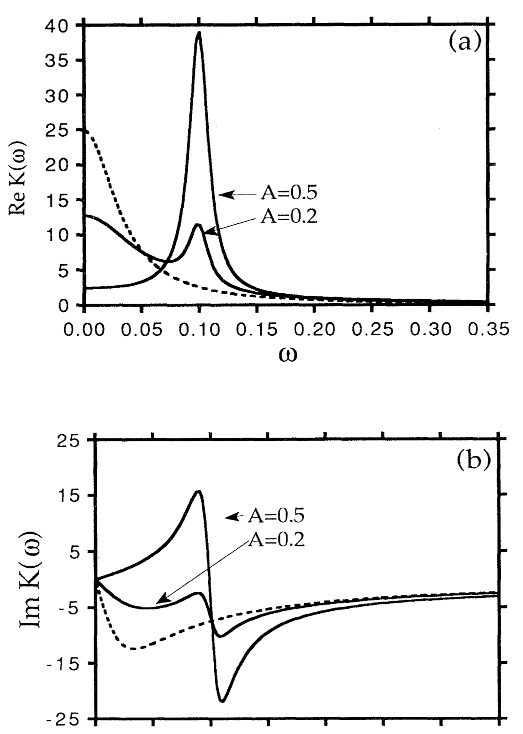


FIG. 5. The real part (a) and the imaginary part (b) of the one-sided Fourier transform of the correlation function $K(\omega)$ (5.1) are shown at $\Omega=0.1$ zero driving amplitude (dotted lines) and for $A=0.2$ and $0.5$.

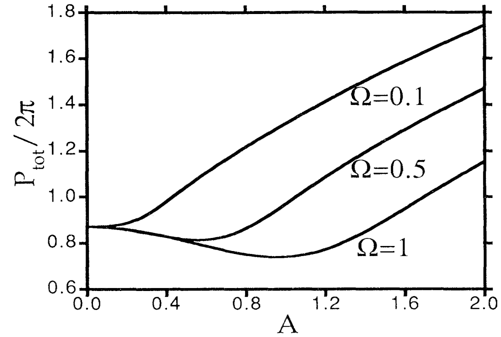


FIG. 6. The total "power" $P_{\mathrm{tot}}$, i.e., the integral over the spectral density (4.21), is depicted as a function of the driving amplitude $A$ at $D=0.1$ for three different driving frequencies $\Omega$.

the probability distributions broaden and the variance increases again. In contrast, for small driving frequencies $\Omega$, the peaks of the probability distribution are shifted toward larger values of $x$ even for small modulation strength, and thus the variance increases monotonously for increasing modulation strength. To understand this point, we have to carefully study the energy-flow diagram for a periodically driven stochastic system.

The input powers are the power of the periodic modulation $P_{\mathrm{sig}}$ plus the power of the noise $P_{\mathrm{noise}}$. The output power is the total power of the stochastic output $x(t)$, defined in (5.2), and in addition a very large dissipation $P_{\mathrm{diss}}$ going to a second reservoir. Utilizing the white noise assumption as is done throughout this paper, the input noise has an infinite total power as indicated by the infinite variance of the white noise process. Since the total output power contained in the process $x(t)$, (5.2), is finite, the system, acting as a nonlinear filter, has to dissipate infinite power. This is even true in the weak-noise limit, $D \to 0$, since here also the variance of the noise is infinite. The limit of zero noise is a discontinuous limit from infinite power to zero power. In real systems, the white noise assumption does not hold and the system dissipates finite energy into the environment via viscosity [we remind the reader that the equation of motion (2.1) is valid in an overdamped regime]. The periodic driving also adds up to the input power, but has in addition the function of a distributor of power. It regulates the transfer of input power into the output power in $x(t)$ and into the second reservoir. For large frequencies, as seen in Fig. 6, the power in $x(t)$ decreases for small modulations, i.e., the modulation distributes more power from the input noise to the second reservoir than to the stochastic output $x(t)$. For small driving frequencies, in contrast, the power of the stochastic output increases with increasing modulation strength.

## VI. CONCLUSIONS

In this paper we have systematically studied the effect of stochastic resonance. A description of SR within the theory of nonstationary stochastic processes has been put forward. The parameter dependence of SR has been described in a wide range of values for the modulation amplitude $A$, frequency $\Omega$, and noise strength $D$.

Finally we would like to give some comments on the issue (raised in Ref. [6]) of physical interpretation of SR involving a transfer of power from the noise to the signal. As already mentioned in the Introduction, the integral over the spectral density is always a constant in a two-state model. The integral has been interpreted as the total output power of the stochastic system. Increasing the coherent power thus automatically reduces the noise power of the total output. An increase in the signal-to-noise ratio thus has been interpreted as the transfer of power from the noise background to the signal part of the output. This, however, is not true for a continuous model as discussed in Sec. V. Thus stochastic resonance is generally not a simple transfer of noise power from the noise background to the power of the coherent output. Having in mind the discussion of the energy flow in the preceding section, we know that there is an energy flow from the noise input to the stochastic output $x(t)$ and another one to a second reservoir. Energy conservation is only valid with respect to the sum of all involved powers, i.e., the signal, the input noise, the stochastic finite output $x(t)$, and the dissipation into the second reservoir. From these considerations it becomes evident that the conservation of power with changing modulation strength $A$ in $x(t)$ alone in the two-state approximation is only an accidental and not a generic property. The periodic modulation, acting as a "power distributor," regulates the pumping of power from the input noise to the process $x(t)$ and to the second reservoir. Furthermore it regulates the distribution of power to the coherent (the weight of the $\delta$ spike in the spectral density) and incoherent parts of $x(t)$. In conclusion, the generic mechanism of SR is connected with an involved transfer of power, contained in the input noise, into the stochastic output $x(t)$ and into a second reservoir.

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## APPENDIX: TWO-DIMENSIONAL DESCRIPTION

In the main parts of this paper we have used the one-dimensional description in terms of the time-dependent Fokker-Planck operator [(3.1), (3.2)] for the nonstationary stochastic process. We have already mentioned the equivalence with the description in terms of the two-dimensional Fokker-Planck equation (3.3). In this appendix we briefly describe the derivation of the results for the long-time correlation function (3.8) and the spectral density (3.9) within this two-dimensional theory.

To this end, we introduce in addition to the Fokker-Planck operator of the two-dimensional stochastic process (3.3), i.e.,

$$L_{\mathrm{FP}} = -\frac{\partial}{\partial x}[h(x) + A\sin\theta] - \Omega\frac{\partial}{\partial\theta} + D\frac{\partial^2}{\partial x^2} \quad \text{(A1)}$$

its Hermitian adjoint operator, i.e.,

$$L_{\mathrm{FP}}^{\dagger} = [h(x) + A\sin\theta]\frac{\partial}{\partial x} + \Omega\frac{\partial}{\partial\theta} + D\frac{\partial^2}{\partial x^2} \ . \quad \text{(A2)}$$

The operator $L_{\mathrm{FP}}^{\dagger}$ has the same eigenvalues $\lambda_{nm}$ as $L_{\mathrm{FP}}$ [defined by $\mathbf{L}_{\mathrm{FP}}\psi_{mn}(x,\theta) = -\lambda_{mn}\psi_{mn}(x,\theta)$], but other eigenfunctions $\varphi_{mn}(x,\theta)$. The following theorems are readily verified.

(a) If $\lambda_{m0}$ is an eigenvalue of $L_{\mathrm{FP}}$ with the right eigenfunction $\psi_{m0}(x,\theta)$, then $\lambda_{mn} = \lambda_{m0} + in\Omega$ are also eigenvalues of $L_{\mathrm{FP}}$ with the eigenfunctions $\psi_{mn}(x,\theta) = \psi_{m0}(x,\theta)\exp(in\theta)$ and $\varphi_{mn}(x,\theta) = \varphi_{m0}(x,\theta)\exp(-in\theta)$.

(b) From (a) it follows that there exists a branch of pure imaginary eigenvalues $\lambda_{0n} = in\Omega$. The corresponding left eigenfunctions are obtained from (A2) and read $\varphi_{0n}(x,\theta) = \exp(-in\theta)$, while the corresponding right eigenfunctions are given by $W_{\mathrm{st}}(x,\theta)\exp(in\theta)$. Here $W_{\mathrm{st}}(x,\theta)$ is defined by $L_{\mathrm{FP}}W_{\mathrm{st}}(x,\theta) = 0$.

The existence of the purely imaginary branch of eigenvalues is no artifact of the extended two-dimensional description as claimed in Refs. [17], but rather is the *necessary consequence* of the periodicity of the asymptotic probability density $p_{\mathrm{as}}(x,t;\varphi)$, defined in Sec. III. Within our two-dimensional description, the time is contained in the variables $t$ and $\theta$. Thus $\theta$ is not an independent stochastic variable. This leads to some peculiarities mentioned in the beginning of Sec. III. Calculating mean values in the two-dimensional description, i.e., performing integrations with respect to $x$ and $\theta$, is equivalent to computing mean values in the one-dimensional description by averaging over the noise *and over the uniformly distributed initial phase* $\varphi$ *of the modulator.* The time-homogeneous, phase-averaged correlation function $K(\tau) = \langle x(\tau)x(0)\rangle$ (where the phase has been assumed to be uniformly distributed between 0 and $2\pi$) is thus given by

$$K(\tau) = \sum_{n,m} g_{mn}\exp(-\lambda_{mn}\tau) \ , \quad \text{(A3)}$$

where the weights $g_{nm}$ are given in terms of the left and right eigenfunctions, i.e.,

$$g_{mn} = \int_{-\infty}^{\infty}\int_{0}^{2\pi} x\,\psi_{mn}(x,\theta)dx\,d\theta \times \int_{-\infty}^{\infty}\int_{0}^{2\pi} x' W_{\mathrm{st}}(x',\theta') \times \varphi_{mn}(x',\theta')dx'd\theta' \ . \quad \text{(A4)}$$

For large times $\tau$, the correlation function $K(\tau)$ is governed by the terms with purely imaginary eigenvalues $\lambda_{0n}=in\Omega$. Using the spectral theorems above, the corresponding weights $g_{0n}$ are given by

$$g_{0n}=\left|\int_{-\infty}^{\infty}\int_{0}^{2\pi}xW_{st}(x,\theta)\exp(-in\theta)dx\,d\theta\right|^2 . \quad (A5)$$

With the relation $P_{as}(x,t;\varphi)=2\pi W_{st}(x,\theta=\Omega t+\varphi)$, and substituting $\theta$ by $\Omega t+\varphi$, i.e., $t\in[0,\ldots,2\pi/\Omega=T]$, Eq. (A5) is rewritten as

$$g_{0n}=\frac{1}{T^2}\left|\int_{-\infty}^{\infty}\int_{0}^{T}xP_{as}(x,s;\varphi)\times\exp[-in(\Omega s+\varphi)]dx\,ds\right|^2 =|M_n|^2 \quad (A6)$$

where $M_n$ are the Fourier coefficients of the time-dependent mean value $\langle x(t;\varphi)\rangle$, see (3.7). Inserting (A6) into (A3), we thus reobtain the general results in Eqs. (3.8) and (3.9).


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