Lasers with injected signals: fluctuations and linewidths

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We consider a single mode laser coupled with an external periodic field. The statistical properties of the laser field (photon statistics) are studied under the combined action of fluctuations due to spontaneous emission processes and an external coherent signal. The main focus of our analytical and numerical investigations is on the eigenvalue spectrum, the field correlation function and the line shape function. We find that the linewidth of the single mode laser increases with increasing external field.

1. Introduction

A laser with an injected signal (LIS) is a laser with two partially transmitting mirrors, coupled to an external coherent signal. One of these mirrors serves to couple the external signal to the laser while the other couples out a certain output intensity. The laser operates in a single mode and is homogeneously broadened. Technically such systems appear to be interesting for a number of reasons. Semiconductor lasers, for instance, produce a relatively small output power. To achieve a large output power one has to superpose the outputs of a large number of semiconductor lasers. To optimize the focused power of such an array of lasers, it would be favorable if the phases of the single output laser fields would be locked to a certain value, being the same for all lasers [1]. A LIS having the property that the phase locks to the phase of the external coherent signal (see also [2]) would be a candidate for such a configuration. Another more recently found application is the use of LIS as a superregenerative optical amplifier [3]. Here the LIS is operated as a Q-switched laser. The time, the laser needs to build up a certain intensity, when switched on, depends strongly on the amplitude of the external signal. This time is used as a measure of the amplitude of the external signal, where the gauge of such a device can be obtained from the theory of the decay of an instable state [4, 5]. Controlling the linewidth of a laser or an array of lasers by using electronic modulation instrumentation [6] is a complicate and cumbersome task. LIS allows in a very easy way to control the linewidth of a single laser or an array of lasers by modifying the coupling between the external signal and the laser(s). In this paper we provide a quantitative theory for such a device. Furthermore for laser radar studies the effect of injection locking on the linewidth is of some importance.

In this paper we study the single mode laser equation with inversion and polarization adiabatically eliminated in the presence of an external coherent signal. Our focus is on the coherence of a LIS described by the field correlation function [7] or equivalently its Fourier transform, the lineshape function. It has been argued in [1] that the phase quieting, being a consequence of the phase locking, leads to a significant narrowing of the linewidth. Our main result, however, yields just the opposite: The external signal locks the phase of the laser field, but *increases* the linewidth. This result is based *(i)* on full numerical solution of the laser Fokker-Planck equation without *any* approximations and *(ii)* on careful perturbative expansions for small noise strength and for small external signals. The paper is organized as follows: In Sect. 2 we introduce the model and the basic relations between the linewidth and the spectrum of the laser Fokker-Planck equation. In Sect. 3 we present perturbative results for small external signals I_e compared to the spontaneous emission noise q, i.e. $I_e/q \to 0$ and for the limit of weak noise, i.e. $q/I_e \to 0$. In Sect. 4 we report on full numerical solutions of the laser Fokker-Planck equation, which confirm the analytic results in Sect. 3. In the conclusions we give a general explanation, based on heuristic arguments, why the linewidth should increase with increasing external signals.

2. Basic equations

The laser Langevin equation for a single mode with the complex field $\underset{\sim}{b} = b_1 + i b_2$, with adiabatically eliminated inversion and polarization reads in dimensionless form [8]

$$\dot{\underline{b}} = a\underline{b} - |\underline{b}|^2 \underline{b} + \underline{E} + \sqrt{q}\,\underline{\Gamma}(t), \tag{1a}$$

with white Gaussian noise $\underline{\Gamma}(t) = \Gamma_1 + i\Gamma_2$, i.e.

$$\langle \underline{\Gamma}(t)\,\underline{\Gamma}^*(t') \rangle = 4\delta(t-t'), \tag{1b}$$

$$\langle \Gamma_1(t)\,\Gamma_2(t') \rangle = 0, \tag{1c}$$

the pump parameter a, and the spontaneous emission noise strength q.

Throughout this paper we restrict ourselves to a positive pump parameter a. The external coherent complex valued field $\underline{E} = E_1 + iE_2$ is in resonance with the relevant atomic transition frequency and the cavity (no detuning). Note that (1a, b, c) are valid within rotating wave and slowly varying amplitude approximation, i.e. the fast time dependence of the external signal is not visible any more on the time scale of the envelope function, i.e. $\underline{E} = \underline{\text{const.}}$ Transforming (1a, b) into equations for the field intensities $I = \underline{b}^*\underline{b}$, $I_e = \underline{E}^*\underline{E}$, and phases $\varphi = \arctan(b_2/b_1)$, $\varphi_e = \arctan(E_2/E_1)$,

$$\dot{I} = 2(a-I)\,I + 2\sqrt{II_e}\cos(\varphi_e - \varphi) + 2q + 2\sqrt{qI}\,\Gamma_1, \tag{2a}$$

$$\dot{\varphi} = \sqrt{\frac{I_e}{I}}\sin(\varphi_e - \varphi) + \sqrt{\frac{q}{I}}\,\Gamma_2, \tag{2b}$$

we observe that the external field couples the time dependence of the intensity to that of the phase and breaks the phase translation symmetry of the unperturbed system ($I_e = 0$). The absolute phase φ_e, however, is unimportant, since only the phase difference $\phi = \varphi_e - \varphi$ enters in (2a, b). The second equation (2b) has, if one assumes the intensity I to be constant, the form of a "locking equation". The phase φ tends to "lock" into the phase of the external signal φ_e. Thus, the external field confines the motion of the phase of the laser field [2]. The coherence of the laser is characterized by the amplitude correlation function $K(t-t') = \langle \underline{b}^*(t)\,\underline{b}(t') \rangle - \langle \underline{b}^*(t) \rangle \langle \underline{b}(t') \rangle$, or equivalently by its Fourier transform, the spectral line shape function. Note that we subtract nonvanishing mean values in the definition of $K(t-t')$ to avoid the occurrence of trivial δ-function contributions in the spectral density at zero frequency. Clearly the linewidth is not affected by this procedure. We also like to point out that with an incoherent external field the mean value $\langle \underline{b}(t) \rangle$ will again be zero, i.e. the corresponding δ-function contribution assumes a zero weight. From this point of view a δ-function contribution due to nonvanishing mean values carries little physical information. Since the Langevin equations (1a, b) are nonlinear, the best access to correlation functions can be achieved from the statistically equivalent laser Fokker-Planck equation for the real and imaginary parts of the complex field $\underline{b}$, i.e.

$$\begin{aligned}\frac{\partial P(b_1,b_2,t)}{\partial t} &= -\frac{\partial}{\partial b_1}\left[a b_1 - (b_1^2+b_2^2)\,b_1 + E_1\right] P(b_1,b_2,t)\\ &+ q\frac{\partial^2}{\partial b_1^2} P(b_1,b_2,t) - \frac{\partial}{\partial b_2}\left[a b_2 - (b_1^2+b_2^2)\,b_2 + E_2\right]\\ &\cdot P(b_1,b_2,t) + q\frac{\partial^2}{\partial b_2^2} P(b_1,b_2,t) \equiv \mathbf{L}_{\mathrm{FP}}\, P(b_1,b_2,t).\end{aligned} \tag{3}$$

The boundary conditions for $P(b_1, b_2, t)$ are chosen to be $P(b_1, b_2, t) \to 0$ as $b_{1,2} \to \infty$.

The Fokker-Planck equation in (3) obeys strict detailed balance (with an incoherent external field this property is lost). The corresponding generalized potential $V_e(b_1, b_2)$, which determines the stationary probability as $P_{\mathrm{st}} = Z^{-1}\exp[-V_e(b_1,b_2)/q]$, thus reads

$$\begin{aligned}V_e(b_1,b_2) = &-\tfrac{1}{2}a(b_1^2+b_2^2)\\ &+\tfrac{1}{4}(b_1^2+b_2^2)^2 - E_1 b_1 - E_2 b_2.\end{aligned} \tag{4}$$

The stationary correlation function $K(t-t')$ can be expressed in terms of eigenvalues λ_{nm} and eigenfunctions Ψ_{nm} of the Fokker-Planck operator, defined by $\mathbf{L}_{\mathrm{FP}}\,\Psi_{nm} = -\lambda_{nm}\Psi_{nm}$, i.e.

$$K(t-t') = \sum_{n,m \neq 0,0} g_{nm}\,\mathrm{e}^{-\lambda_{nm}\tau}, \tag{5}$$

where g_{nm} is determined by the left and right eigenfunctions of the Fokker-Planck operator $\mathbf{L}_{\mathrm{FP}}$ [9] and $\tau = |t-t'|$. The eigenvalues $\{\lambda_{nm}\}$ are all real valued due to the strict detailed balance symmetry inherent in (3).

Without the external field, the Fokker-Planck operator is symmetric with respect to inversion, i.e. $b_1, b_2 \to -b_1, -b_2$. Therefore, the eigenfunctions can be classified into odd and even eigenfunctions with respect of changing the sign with inversion or not. Only the odd eigenfunctions give nonvanishing weights g_{nm} in (5). Since even and odd eigenfunctions occur alternately with increasing eigenvalues there is a large "gap" between the smallest non-vanishing eigenvalue and the next odd eigenvalue [10]. Thus, the first non-vanishing eigenvalue mainly determines the time scale on which field fluctuations decay

$$K(\tau) \approx \sigma^2\,\mathrm{e}^{-\lambda_{\min}\tau}, \tag{6}$$

where $\sigma^2 = \langle \underline{b}^*\underline{b} \rangle - \langle \underline{b}^* \rangle \langle \underline{b} \rangle$, and the line shape

$$S(\omega) = \int_{-\infty}^{\infty} K(\tau)\,\mathrm{e}^{-i\omega\tau}\,\mathrm{d}\tau = \sigma^2\,\frac{2\lambda_{\min}}{\lambda_{\min}^2 + \omega^2}, \tag{7}$$

with the linewidth

$$\alpha = \lambda_{\min}. \tag{8}$$

This scheme, however, does not apply in the presence of a symmetry breaking external field. In particular, many more eigenvalues and eigenfunctions contribute to the correlation function in (5) (depending on the strength of the symmetry breaking perturbation). Thus, in this case it is more appropriate to characterize the decay of fluctuations by an effective eigenvalue [11] defined as a Lorenzian approximation of the non-Lorenzian line shape having the same "area" under the correlation function, i.e.

$$S_{\mathrm{Lor}}(\omega=0) \overset{!}{=} S(\omega=0).$$

The effective eigenvalue reads in terms of the eigenvalues

$$\lambda_{\text{eff}} = \frac{\sigma^2}{S(\omega=0)} = \sigma^2 \left(\sum_{n,m \neq 0,0} \frac{g_{nm}}{\lambda_{nm}} \right)^{-1}. \tag{9}$$

Numerical results in Sect. 4 in fact show that λ_{eff} differs significantly from the smallest non-vanishing eigenvalue $\lambda_{\min}$ in the relevant regime of parameters.

3. Approximative results

In this section we derive approximate results for the eigenvalues and the linewidth in the weak-noise limit q, $q/I_e \to 0$ and in the limit of weak-fields $I_e/q \to 0$. To carry through a perturbative analysis in the weak-noise limit, we move the origin of the coordinate frame into the deterministic ($q=0$) stationary state $\underline{b}^{\text{st}}$ and scale the deviations by the noise strength, i.e.

$$\underline{\tilde{b}} = (\underline{b} - \underline{b}^{\text{st}})/\sqrt{q}, \tag{10}$$

with

$$b_1^{\text{st}} = 2\sqrt{a/3} \cos\left(\frac{1}{3} \arccos \frac{E_1}{\hat{E}_1}\right)$$

if $E_1 < \hat{E}_1 = \sqrt{\frac{4}{27} a^3}$, and

$$b_1^{\text{st}} = \left(\frac{1}{2} E_1 + \frac{1}{2}\sqrt{E_1^2 - \hat{E}_1^2}\right)^{\frac{1}{3}} + \frac{a}{3}\left(\frac{1}{2} E_1 + \frac{1}{2}\sqrt{E_1^2 - \hat{E}_1^2}\right)^{-\frac{1}{3}} \tag{11}$$

if $E_1 > \hat{E}_1$. Since the absolute phase of the external signal is irrelevant we have chosen for convenience $E_2 = 0$, i.e. $b_2^{\text{st}} = 0$. Using $\sqrt{q}$ as the small parameter the Fokker-Planck operator can be written in the form

$$\mathbf{L}_{\text{FP}} = \mathbf{L}_{\text{FP}}^{(0)} + \sqrt{q}\, \mathbf{L}_{\text{FP}}^{(1)} + q\, \mathbf{L}_{\text{FP}}^{(2)},$$

where

$$\mathbf{L}_{\text{FP}}^{(0)} = -\frac{\partial}{\partial \tilde{b}_1}(a\tilde{b}_1 - 3 b_1^{\text{st}} \tilde{b}_1 - 2 b_1^{\text{st}} b_2^{\text{st}} \tilde{b}_2 - (b_2^{\text{st}})^2 \tilde{b}_1) + \frac{\partial^2}{\partial \tilde{b}_1^2}$$
$$- \frac{\partial}{\partial \tilde{b}_2}(a\tilde{b}_2 - 3 b_2^{\text{st}} \tilde{b}_2 - 2 b_1^{\text{st}} b_2^{\text{st}} \tilde{b}_1 - (b_1^{\text{st}})^2 \tilde{b}_2) + \frac{\partial^2}{\partial \tilde{b}_2^2},$$

$$\mathbf{L}_{\text{FP}}^{(1)} = \frac{\partial}{\partial \tilde{b}_1}(3 b_1^{\text{st}} (\tilde{b}_1)^2 + 2 b_2^{\text{st}} \tilde{b}_1 \tilde{b}_2 + b_1^{\text{st}} (\tilde{b}_2)^2)$$
$$+ \frac{\partial}{\partial \tilde{b}_2}(3 b_2^{\text{st}} (\tilde{b}_2)^2 + 2 b_1^{\text{st}} \tilde{b}_1 \tilde{b}_2 + b_2^{\text{st}} (\tilde{b}_1)^2),$$

$$\mathbf{L}_{\text{FP}}^{(2)} = \frac{\partial}{\partial \tilde{b}_1}((\tilde{b}_1)^3 + \tilde{b}_1 (\tilde{b}_2)^2) + \frac{\partial}{\partial \tilde{b}_1}((\tilde{b}_1)^3 + \tilde{b}_1 (\tilde{b}_2)^2). \tag{12}$$

Accordingly, the eigenvalues are denoted by

$$\lambda_{nm} = \lambda_{nm}^{(0)} + \sqrt{q}\, \lambda_{nm}^{(1)} + q\, \lambda_{nm}^{(2)}. \tag{13}$$

In leading order, $\sqrt{q}^0$, the drift term of $\mathbf{L}_{\text{FP}}^{(0)}$ is linear and the eigenvalues are obtained from the two-dimensional Ornstein-Uhlenbeck process

$$\lambda_{nm}^{(0)} = n\Lambda_1 + m\Lambda_2, \tag{14}$$

with

$$\Lambda_1 = -a + 3(b_1^{\text{st}})^2, \quad \Lambda_2 = -a + (b_1^{\text{st}})^2.$$

The smallest eigenvalue $\lambda_{\min} = \lambda_{01} = -a + (b_1^{\text{st}})^2$ vanishes for a vanishing external signal since $b_1^{\text{st}}(E_1 = 0) = \sqrt{a}$. This zero mode for vanishing injected signal corresponds to the dynamics along the valley of the "Mexican hat" potential $V_0(b_1, b_2)$ defined by the stationary solution of the Fokker-Planck equation without external signal, i.e. $V_0(b_1, b_2) = V_e(b_1, b_2; \underline{E} = \underline{0})$.

The injected signal generates a ladder of finite (for small injected signal very close lying) eigenvalues, which all contribute to the linewidth. At this point it becomes evident that the identification of the smallest non-vanishing eigenvalue with the linewidth breaks down in the presence of injected signals. On the other hand, using (9) the effective eigenvalue λ_{eff} being a reliable measure of the linewidth can be obtained directly from the Ornstein-Uhlenbeck process generated by $\mathbf{L}_{\text{FP}}^{(0)}$

$$\lambda_{\text{eff}}^{-1} = \frac{\Lambda_1 \Lambda_2}{\Lambda_1 + \Lambda_2}\left(\frac{1}{\Lambda_1^2} + \frac{1}{\Lambda_2^2}\right) > \lambda_{\text{eff}}(E_1 = 0) = 0. \tag{15}$$

Most important, all eigenvalues $\lambda_{nm}^{(0)}$ and λ_{eff} being an approximation for the linewidth *increase* with increasing amplitude of the injected signal. The next non-vanishing contributions to the eigenvalues are of the order q, i.e. $\lambda_{nm}^{(1)} = 0$. Second order standard peturbation theory yields after some lengthy calculations

$$\lambda_{nm}^{(2)} = \frac{2(b_1^{\text{st}})^2}{\Lambda_2 + 2a} \frac{1}{\Lambda_2} [(m+2)(m+1)n - m(m-1)(n+1)]$$
$$- \frac{2(b_1^{\text{st}})^2}{\Lambda_1}\left(\frac{27n^2 + 3m + 2m^2 + 18nm}{\Lambda_1} + \frac{6nm + 3n + m + 2m^2}{\Lambda_2}\right)$$
$$+ \frac{3n^2 + 2nm + m}{\Lambda_1} + \frac{3m^2 + 2nm + n}{\Lambda_2}. \tag{16}$$

The comparison of the approximate eigenvalues with numerical results (next section) in Fig. 1 confirms our perturbative approach. Also the effective eigenvalue approaches the analytic prediction (15) for small noise very well.

To obtain results in the limit of weak external signals we utilize a time scale separation of the phase-dynamics and intensity-dynamics in (2). For large pump parameters, the mean intensity becomes also large $I_{\text{st}} \approx a$. Thus, the phase motion $\dot{\varphi} \sim 1/\sqrt{I}$ slows down in comparison to the relaxation rate of the intensity. In other words, the intensity approaches its stationary value while the phase does not change appreciably. In order that this argument applies we have to require that the noise strength and the external field is small compared to the pump parameter. Consistent decoupling of the phase dynamics yields the Langevin equation for the phase difference $\varphi - \varphi_e = \phi$

$$\dot{\phi} = -\varepsilon \sin \phi + \sqrt{D}\, \Gamma_1(t), \tag{17}$$

where $\varepsilon = \sqrt{I_e/a}$ and $D = q/a$.

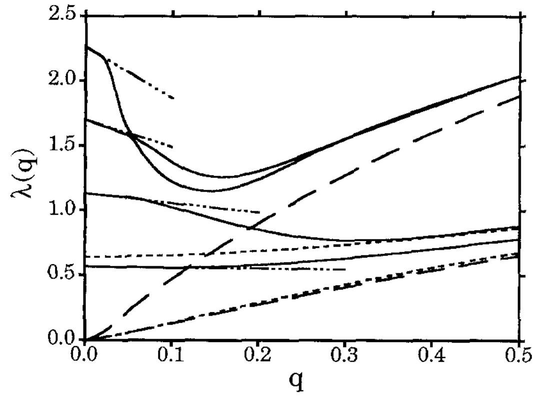

Fig. 1. The smallest few eigenvalues are shown as a function of the noise strength q and $a=1$. The long-dashed lines are the first two eigenvalues without injected signal. The four solid lines correspond to the two doublets of eigenvalues with an external signal $E_1=E_2=0.5$. The dashed-dotted curves are the weak-noise approximations for the eigenvalues (13), (14), (16). The short-dashed curves represent the effective eigenvalues without injected signal (above the lower lying long-dashed curve) and the effective eigenvalue with a finite injected signal (well above the smallest non-vanishing eigenvalue)

It seems tempting to linearize (17) in order to obtain an analytic estimate for the eigenvalues $\lambda_m(\varepsilon)$. Such a procedure, however, leads to incorrect results, due to the violation of periodic boundary conditions and the fact that the periodic potential with finite barrier heights is poorly approximated by a parabolic potential for $D>\varepsilon^2$. In order to find an estimate we perform a perturbation theory to second order (see Appendix) and obtain for ε small (i.e. $\varepsilon^2 \ll D$) the result

$$\lambda_m(\varepsilon)=m^2 D+\frac{\varepsilon^2}{2D}\frac{m^2}{4m^2-1}>\lambda_m(\varepsilon=0);$$
$$m=0, \pm 1, \pm 2, \ldots \tag{18}$$

For the derivation of (18), we used periodic boundary conditions since phases φ and $\varphi+2\pi$ cannot be distinguished. For the eigenfunctions we do not use the standard classification into odd and even eigenfunctions. Instead we use complex eigenfunctions and their complex conjugate partners. The complex right and left eigenfunctions read in order ε

$$\Psi_m(\phi)=\frac{1}{2\pi}\left[\left(1+m^2\frac{\varepsilon^2}{2D^2}\frac{4m^2-3}{(4m^2-1)^2}\right)e^{im\phi}\right.$$
$$\left.+\frac{\varepsilon}{2D}\frac{m+1}{2m+1}e^{i(m+1)\phi}+\frac{\varepsilon}{2D}\frac{m-1}{2m-1}e^{i(m-1)\phi}\right], \tag{19a}$$

$$\Psi_m^+(\phi)=e^{im\phi}-\frac{\varepsilon}{2D}\frac{m}{2m+1}e^{i(m+1)\phi}-\frac{\varepsilon}{2D}\frac{m}{2m-1}e^{i(m-1)\phi}. \tag{19b}$$

The ε^2-term in (19a) has been introduced to guarantee a correct normalization of the eigenfunctions in order ε. Other terms of order ε^2 do not contribute to the correlation functions below. With

$$P_{st}(\phi)=\frac{1}{I_0(\varepsilon/D)}\exp\left(\frac{\varepsilon}{D}\cos\phi\right)=1+\frac{\varepsilon}{D}\cos\phi$$
$$+\frac{\varepsilon^2}{2D^2}\cos^2\phi+O(\varepsilon^3), \tag{20}$$

where $I_0(x)$ is a modified Bessel function, the stationary correlation function expressed in terms of the eigenvalues $\lambda_m(\varepsilon)$ and eigenfunctions $\Psi_m(\phi)$, i.e.

$$K(\tau)=a^2\langle e^{i\phi(t)}e^{-i\phi(t')}\rangle-a^2\langle e^{i\phi(t)}\rangle_{st}\langle e^{-i\phi(t')}\rangle_{st}$$
$$=\sum_m e^{-\lambda_m\tau}\int d\phi\, e^{i\phi}\Psi_m(\phi)\int d\phi\, e^{-i\phi}\Psi_m^+(\phi)\,P_{st}(\phi), \tag{21}$$

is found as

$$K(\tau)=a\left(1-\frac{5}{18}\frac{\varepsilon^2}{D^2}\right)\exp\left(-\left(1+\frac{\varepsilon^2}{6D^2}\right)D\tau\right)$$
$$+\frac{a}{36}\frac{\varepsilon^2}{D^2}\exp\left(-\left(1+\frac{1}{30}\frac{\varepsilon^2}{D^2}\right)4D\tau\right). \tag{22}$$

The lineshape function is a sum of two Lorenzians and can be characterized by the effective eigenvalue (9)

$$\lambda_{eff}=D\left(1+\frac{7}{16}\frac{\varepsilon^2}{D^2}\right), \quad \frac{\varepsilon^2}{D}\ll 1. \tag{23}$$

The result is a line broadening with increasing external signal as in the weak-noise limit.

4. Numerical analysis

In order to obtain the linewidth of a LIS for arbitrary pump parameters and field strengths (in the range of validity of the model (1 a, b)) we have to solve (3) numerically.

This is done by expanding the probability distribution $P(b_1, b_2, t)$ in complete sets of Hermite functions with respect to b_1 and b_2. This leads us to a tridiagonal vectorial system of differential equations for properly chosen vectors of expansion coefficients. The matrices have block structure with 16 blocks per matrix.

Such systems of differential equations can be solved in terms of matrix continued fractions for eigenvalues and eigenfunctions [12] or directly for the correlation function and its Fourier transform [13].

The results for the problem under investigation [14] are as follows: The first few small eigenvalues are shown in Fig. 1 as a function of the noise strength. Without injected signal they approach zero for vanishing noise. For finite injected signals they split into pairs and approach finite values for $q\to 0$, given by the eigenvalues of the linearized system (14). Thus, in the weak-noise limit the relevant time scale for the phase dynamics is not given by the diffusion constant any more, but rather by the shape of the symmetry broken potential. This, however, means linewidth broadening, since the phase becomes "faster". In the regime where the eigenvalues increase with increasing noise, the phase dynamics is noise-controlled, but still faster due to the asymmetry.

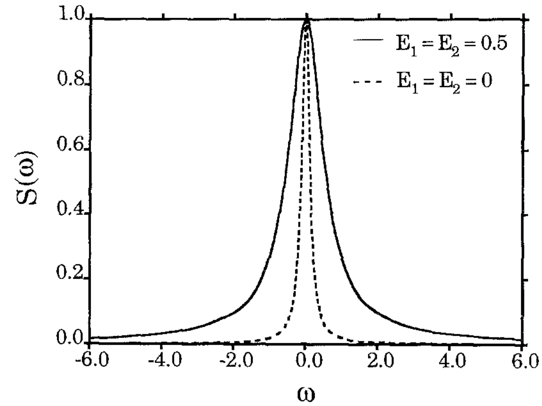

Fig. 2. The numerically evaluated line shape function $S(\omega)$ for $q=0.1$ and $a=1$ without external signal (dashed line) and with finite injected signal (solid line)

As already mentioned above, the smallest eigenvalue is not a reliable measure for the linewidth for small noise strength. The effective eigenvalue behaves even qualitatively different as compared to the smallest non-vanishing eigenvalue. In contrast to the smallest eigenvalue the effective eigenvalue increases monotonously for increasing noise strength (see Fig. 1). Even more importantly, λ_{eff} is always *larger than without injected signal.*

In Fig. 2 this result is again confirmed by the explicit numerical calculation of the Fourier transform of the correlation function. For increasing injected signals the line shape clearly broadens.

5. Conclusions

In this paper we have shown that *the spectral linewidth of a single mode laser with an injected coherent signal always broadens.* At first glance this seems awkward, since the phase motion is, due to the broken rotational symmetry, more concentrated at the potential minimum. On a second glance it becomes clear however, that it is *not the range of values* for the phase of the laser field, but it is rather the *speed* of the phase motion which determines the linewidth. In other words, it does not depend on the absolute value of the phase but rather on its rate of change which increases with increasing external signal.

Appendix: Perturbation scheme for Eq. 17

The Fokker-Planck equation corresponding to (17) reads

$$\frac{\partial P(\phi,t)}{\partial t}=\left(\varepsilon\frac{\partial}{\partial\phi}\sin\phi+D\frac{\partial^2}{\partial\phi^2}\right)P(\phi,t)\equiv \mathbf{L}_{\text{FP}}\,P(\phi,t), \quad \text{(A1)}$$

with the eigenvalues and eigenfunctions defined by

$$\mathbf{L}_{\text{FP}}\,\Psi_m=-\lambda_m\,\Psi_m. \quad \text{(A2)}$$

One expands the eigenfunctions Ψ_m into the set of eigenfunctions of the unperturbed operator, i.e.

$$\Psi_m=\frac{1}{2\pi}\sum_n c_n^m\exp(in\phi). \quad \text{(A3)}$$

This yields the tridiagonal system of equations

$$(\lambda_m-n^2D)\,c_n^m+\tfrac{1}{2}\varepsilon n(c_{n-1}^m-c_{n+1}^m)=0. \quad \text{(A4)}$$

Expanding the eigenvalues λ_n and the coefficients c_n into a series in the small parameter ε

$$\lambda_m=m^2D+\varepsilon\lambda_m^{(1)}+\varepsilon^2\lambda_m^{(2)},$$
$$c_n^m=\delta_{nm}\,c_n^{m(0)}+\varepsilon c_n^{m(1)}+\varepsilon^2 c_n^{m(2)}, \quad \text{(A5)}$$

one solves the recurrence relation (Eq. A4) up to second order in ε, to yield the result given in (18).

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