Localization in a Driven Two-Level Dynamics.

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Abstract. – A periodically driven two-state dynamics, being analysed within the Floquet formalism, exhibits localization of the amplitude dynamics in an infinite frequency range, extending from the bare tunnel splitting up to infinity. In contrast, the suppression of tunnelling in a driven symmetric double well is restricted to a limited frequency regime, extending from the bare tunnel splitting up to the first resonance frequency with higher-lying states. With the amplitude dynamics of a periodically driven two-level system not being restricted to describe coherent tunnelling transport only, the localization phenomenon within the infinite frequency range does allow for novel applications for systems in strong laser fields.

The intriguing phenomenon of suppression of coherent tunnelling in a symmetric bistable potential induced by an external periodic force has recently been discovered in a study of time-periodic Schrödinger equations by Großmann *et al.* [1] (possible experimental realizations are discussed in [1*b*]): for a restricted regime of parameter values of external frequency ω and forcing strength S the time-periodic quantum mechanics given by the Hamilton operator (in dimensionless variables)

$$H(x,t) = H_0(x) + H_1(x)\cos(\omega t) \tag{1}$$

with

$$H_0(x) = -\frac{1}{2}\partial_x^2 - \frac{1}{4}x^2 + \frac{1}{64D}x^4 \tag{2}$$

and perturbation

$$H_1(x) = xS \tag{3}$$

exhibits novel coherent tunnelling phenomena. In particular, the time evolution of an initial wave packet prepared in one of the wells essentially *remains localized for all future times t*. The parameter $D = E_B/\hbar\omega_0$ denotes a dimensionless measure of the barrier height.

The physical mechanism underlying this novel localization phenomenon is presently not well understood. If one neglects the *spatial* structure of coherent tunnelling it may be feasible to ask if such a localization which is solely based on a two-level dynamics still occurs. On naive grounds the perturbed two-level system is expected to possess finite-energy splittings only, which would forbid localization features. The main objective of this study is thus the problem of possible localization of the (tunnelling-related) amplitude dynamics in a periodically driven two-level system. In addition we shall focus on a comparison between the suppression of tunnelling as predicted by the full time-periodic tunnelling system in (1)-(3) [1] and the results based on the corresponding two-level dynamics.

The general dynamics of a driven two-state quantum system has been studied a long time ago by Shirley [2] within the Floquet formalism. This driven two-level dynamics recently underwent a renaissance when describing a whole variety of topics [3] such as Autler-Townes doublets, Landau-Zener transitions, multiphoton transitions and quantum chaos in the presence of strong fields, to name but a few. In these previous studies [2,3], however, a possible suppression phenomenon for the driven amplitude dynamics has not been addressed.

Within the two-state approximation one considers the lowest two unperturbed eigenfunctions $|\varphi_1\rangle$, $|\varphi_2\rangle$ of H_0. Due to the inversion symmetry of H_0, the matrix elements of the total Hamiltonian are given by

$$\langle\varphi_1|H_0|\varphi_1\rangle = E_1; \qquad \langle\varphi_2|H_0|\varphi_2\rangle = E_2;$$
$$\langle\varphi_1|H_0|\varphi_2\rangle = \langle\varphi_2|H_0|\varphi_1\rangle = 0; \qquad \langle\varphi_1|H_1|\varphi_1\rangle = \langle\varphi_2|H_0|\varphi_2\rangle = 0;$$
$$\langle\varphi_1|H_1|\varphi_2\rangle = \langle\varphi_2|H_1|\varphi_1\rangle = S\langle\varphi_1|x|\varphi_2\rangle.$$

Denoting the matrices of the unperturbed and the perturbation part of the Hamiltonian in the two-level approximation by $\boldsymbol{H}_0$ and $\boldsymbol{H}_1$, respectively, the infinite-dimensional Floquet matrix has the form [2]

$$\begin{pmatrix} \boldsymbol{H}_0 - 2\omega\mathbf{1} & (1/2)\boldsymbol{H}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ (1/2)\boldsymbol{H}_1 & \boldsymbol{H}_0 - 1\omega\mathbf{1} & (1/2)\boldsymbol{H}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (1/2)\boldsymbol{H}_1 & \boldsymbol{H}_0 & (1/2)\boldsymbol{H}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (1/2)\boldsymbol{H}_1 & \boldsymbol{H}_0 + 1\omega\mathbf{1} & (1/2)\boldsymbol{H}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & (1/2)\boldsymbol{H}_1 & \boldsymbol{H}_0 + 2\omega\mathbf{1} \end{pmatrix}. \tag{4}$$

Its eigenvalues $\varepsilon = \varepsilon(S, \omega)$ are termed quasi-energies (¹). These quasi-energies determine the behaviour of a solution $|\Psi(t)\rangle$ of the Schrödinger equation according to the Floquet theorem [2, 4, 5]; *i.e.*

$$|\Psi_\varepsilon(t)\rangle = |\Phi_\varepsilon(t)\rangle \exp[-i\varepsilon t], \tag{5}$$

with the time-periodic Floquet function

$$|\Phi_\varepsilon(t)\rangle = |\Phi_\varepsilon(t + T)\rangle, \tag{6}$$

where $T = 2\pi/\omega$ denotes the period of the external force. Note the specific structure inherent in (4): the Floquet matrix is real-valued and symmetric and only the main diagonals of $\boldsymbol{H}_0$ and the off-diagonals of $\boldsymbol{H}_1$ are different from zero. From these specific properties we find, following the same reasoning put forward originally by von Neumann and Wigner in [6], that the variation of *one parameter only* is sufficient to create a degeneracy of two quasi-energies. Therefore the exact crossing of the two quasi-energies $\varepsilon_{1,2}(\omega, S)$ occurs along a *one-dimensional manifold* $\mathscr{M}$ in the parameter space (ω, S). In passing, we like to point out that the feature of a one-dimensional manifold of exact crossings is lost for the case of an asymmetry for the double-well potential and/or a perturbation $H_1(x)$ which contains in addition elements of even symmetry in x. In this latter case, due to the loss of generalized

(¹) The set $\{\varepsilon(S,\omega)\}$ can be arranged into two classes $\{\varepsilon_1(S,\omega) + k\omega\}$, $\{\varepsilon_2(S,\omega) + k\omega\}$ with $k = 0, \pm 1, \pm 2, \ldots$, obeying $\varepsilon_1 + \varepsilon_2 = E_1 + E_2$.

parity $x \to -2$, $t \to t + \pi/\omega$, the system does—if at all—exhibit isolated exact crossings only.

The exact crossing of the first two tunnelling-related quasi-energies yields a necessary—but not sufficient-criterion for suppression of coherent tunnelling: a crossing of these tunnelling-related first two quasi-energies $\varepsilon_1, \varepsilon_2$ does not imply a time-independent behaviour for a wave packet made up of the two unperturbed eigenfunctions $\Psi_L(x, t=0) = (1/\sqrt{2})[\langle x|\varphi_1\rangle - \langle x|\varphi_2\rangle]$ (L: left well). This is so because of the Floquet theorem in (5), *i.e.* the presence of the explicit *periodic time dependence* of the Floquet functions $\Phi_\varepsilon(x,t)$. Thus, to check whether a suppression of driven coherent tunnelling actually occurs at all times the result of stroboscopic suppression monitored at multiples of the external period, *i.e.* $t_n = n(2\pi/\omega)$, is not sufficient to claim suppression at all times t. It is thus necessary to check suppression also between multiple periods.

Next let us compare the one-dimensional manifold describing the exact crossings of the first two quasi-energies of the full tunnelling problem in (1) gainst the one-dimensional manifold generated by the driven two-level modelling in (4) (see fig. 1). With $E_2 - E_1 \equiv \Delta$ being the unperturbed tunnel splitting we note that both the exact manifold $\mathcal{M}_{\text{d.w.}}$ (double well) and the manifold $\mathcal{M}_{\text{t.l.}}$ (two level) of the two-level dynamics start at $S = 0$ and external driving frequency $\omega = \Delta/2$. Most importantly, we note that the symmetric double well *yields exact crossing only within a limited-frequency regime* $\Delta/2 \leq \omega \leq \omega_{\text{res}} + \Delta$, where $\omega_{\text{res}} \equiv E_3 - E_2$ denotes the first fundamental resonance angular frequency between the 3rd eigenvalue and the 2nd eigenvalue of H_0. In contrast, the periodically driven two-level dynamics yields exact crossings for all frequencies $\Delta/2 \leq \omega < \infty$! As can be understood readily any two-level modelling thus falls short in describing the influence generated by higher quasi-energies $\varepsilon_3, \varepsilon_4, \ldots$ of the full problem.

Given the two amplitudes

$$a_{1,2}(t) = \langle \varphi_{1,2} | \Psi_L(t) \rangle, \tag{7}$$

the dynamics of $\{a_1(t), a_2(t)\}$ is within the two-level approximation determined by

$$i\dot{a}_1(t) = E_1 a_1(t) + b a_2(t) \cos(\omega t), \qquad a_1(0) = \frac{1}{\sqrt{2}}, \tag{8}$$

$$i\dot{a}_2(t) = E_2 a_2(t) + b a_1(t) \cos(\omega t), \qquad a_2(0) = -\frac{1}{\sqrt{2}}, \tag{9}$$

where $b \equiv S x_{1,2} = S\langle \varphi_1 | x | \varphi_2 \rangle$ denotes the dipole matrix element. Starting with an initial

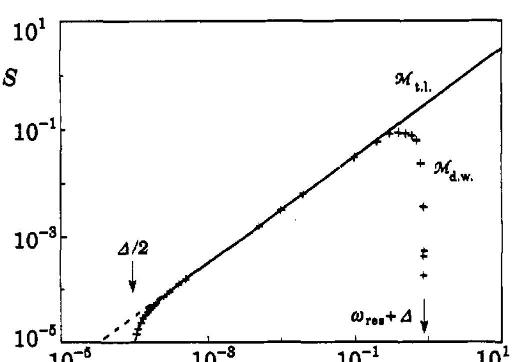

Fig. 1. – Double-logarithmic plot depicting the one-dimensional manifolds along which the first two quasi-energies cross for the first time: $\mathcal{M}_{\text{d.w.}}$ (double well) and $\mathcal{M}_{\text{t.l.}}$ (two level). The dashed line has been evaluated from the first zero of the Bessel function $J_0(2b/\omega)$, see (17). The D-parameter used is $D = 2$, *i.e.* $\Delta = 1.895 \cdot 10^{-4}$.

localization we readily observe that (8), (9) cannot preserve mathematically such an exact localization for all future times. The suppression of tunnelling has been monitored in [1] by the probability to stay, *i.e.* $P(t) \equiv \left| \int_{-\infty}^{\infty} dx\, \Psi_L^*(x,t)\, \Psi_L(x,0) \right|^2$, where $|\Psi_L(0)\rangle$ is the wave packet initially centred in the left well (see above). For the two-level dynamics this becomes

$$P(t) = |a_1^*(t)\, a_1(0) + a_2^*(t)\, a_2(0)|^2 , \tag{10}$$

where with $a_1(0) = -a_2(0) = 1/\sqrt{2}$, $P(t=0) = 1$. The dynamics of the probability to stay over the first period of the driving force is depicted for the full double-well problem and the two-state approximation in fig. 2. There, we have chosen parameter values (S, ω) from the one-dimensional manifolds $\mathcal{M}_{\text{d.w.}}$ and $\mathcal{M}_{\text{t.l.}}$, respectively. It can be seen that in the vicinity of $\omega \geqslant \Delta/2$ (see fig. 2*a*)) the behaviour resembles the one known from the undriven case, *i.e.* we find a behaviour of the form

$$P(t) \approx \cos^2\left(\frac{\Delta}{2} t\right). \tag{11}$$

Thus the probability to stay is zero for times $t_n = nT/4$, $n = 1, 3, 5 \ldots$. With increasing values of frequency and force on the one-dimensional manifold $\mathcal{M}$ the oscillation behaviour within the driving period T flattens out; *i.e.* its amplitude decreases, see fig. 2*b*)-*d*). At $\omega = \Delta$, the behaviour of the Floquet functions yields for $P(t)$ the approximation [7]

$$P(t) = \left| \int_{-\infty}^{\infty} \Psi_L^*(x,t)\, \Psi_L(x,0)\, dx \right|^2 \simeq \frac{1}{4}\{3 + \cos[2\omega t]\}. \tag{12}$$

Put differently, at $\omega = \Delta$ coherent tunnelling is not yet effectively suppressed, yielding $P(t = nT/4) = 1/2$, $n = 1, 3, 5, \ldots$. This behaviour is approximatively confirmed by our numerics

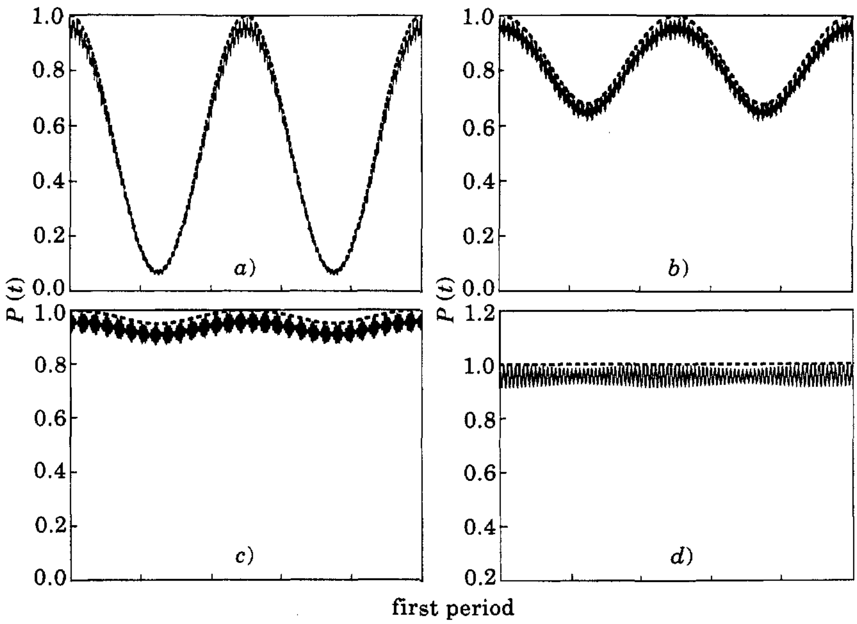

Fig. 2. – The probability to stay $P(t)$ over the first period of the external force for: *a*) $\omega = 0.53\Delta$, $S = 1.03 \cdot 10^{-5}$; *b*) $\omega = \Delta$, $S = 5.23 \cdot 10^{-5}$; *c*) $\omega = 2.64\Delta$, $S = 1.56 \cdot 10^{-4}$; *d*) $\omega = 52.8\Delta$, $S = 3.17 \cdot 10^{-3}$. Solid line: double-well potential ($D = 2$, $\Delta = 1.895 \cdot 10^{-4}$); dashed line: two-level system.

depicted in fig. 2*b*). Note that in contrast to the undriven case, *i.e.* $P(t) = \cos^2((\Delta/2)t)$, the amplitude of $P(t)$ is already suppressed, however, by ca. 60%.

At values $\Delta \ll \omega \ll \omega_{\text{res}}$ of the driving frequency and with S on $\mathscr{M}_{\text{d.w.}}$ we find numerically that the Floquet functions are closely given by

$$\Phi_{\varepsilon_1}(x,t) \approx \varphi_2(x)|\sin(\omega t)| - i\varphi_1(x)\cos(\omega t), \tag{13}$$

$$\Phi_{\varepsilon_2}(x,t) \approx \varphi_1(x)|\sin(\omega t)| - i\varphi_2(x)\cos(\omega t). \tag{14}$$

The wave packet initially centred in the left well thus yields by virtue of (13) and (14) for the probability to stay

$$P(t) = \left| \int_{-\infty}^{\infty} \Psi_{\text{L}}^*(x,t)\,\Psi_{\text{L}}(x,0)\,\mathrm{d}x \right|^2 = \cos^2(\omega t) + \sin^2(\omega t) = 1, \tag{15}$$

indicating a localization of the particle (cf. fig. 2*d*)). For the frequency $\omega = \omega_{\text{res}}$ the Floquet functions contributing to the dynamics of a wave packet centred in the left well will also have dominant admixtures from the higher-lying Floquet states: even for very small values of the forcing strength S one finds numerically that at least three Floquet functions will contribute significantly to the dynamics of $\Psi_{\text{L}}(x,t)$ [1]. This implies that the suppression predicted by the two-state approximation no longer takes place for the physical tunnelling system in (1)-(3).

Most importantly, we thus observe that the periodically driven two-level dynamics proceeds to describe essentially complete suppression of coherent tunnelling on $\mathscr{M}_{\text{t.l.}}$ for all frequencies $\omega \gtrsim \omega_{\text{res}}$—extending up to infinity—while the exact dynamics of coherent driven tunnelling in (1)-(3) no longer exhibits «coherent destruction» (because of a mixing with higher-lying quasi-energy states).

The one-dimensional manifold $\mathscr{M}_{\text{t.l.}}$ can be determined approximatively from (4) following the reasoning by Shirley. By use of the approximation derived in eq. (27) in [2], and observing that the quasi-energies can be defined only modulo ω, the crossing of ε_1 and ε_2 is approximatively determined by the first zero of the Bessel function $J_0(2b/\omega) = 0$ (²). With the first zero located at $y_1 = 2.404\,82\ldots$ we thus find for the manifold $\mathscr{M}_{\text{t.l.}}$ the approximation

$$S = \frac{2.404\,82\ldots}{2\langle\varphi_1|x|\varphi_2\rangle}\,\omega, \qquad \omega > 0; \qquad \textit{i.e.} \tag{16}$$

$$S \approx 0.3172\,\omega, \qquad \text{for } D = 2. \tag{17}$$

This result is depicted in fig. 1 with the dashed line. The approximation in (16) fits very well the linear regime of the one-dimensional manifolds $\mathscr{M}_{\text{d.w.}}$ and $\mathscr{M}_{\text{t.l.}}$, respectively. The approximation fails, however, for low frequencies, $\omega \lesssim 2\Delta$. This approximation in (16) can also be derived directly from an approximate treatment of the amplitude dynamics (8), (9) [8, 9]. Of interest is also the behaviour as b (or S) becomes very large: from (8), (9) one readily finds that $\lim_{b\to\infty} P(t) = 1$, for all fixed ω-values, *i.e.* the driven two-level amplitude dynamics always exhibits a localization (³) for very large driving strength (this result is consistent with: $\varepsilon_1(\infty,\omega) = \varepsilon_2(\infty,\omega) = E_1 + (1/2)\Delta$).

In conclusion, the periodically driven two-level system in (8), (9) does predict almost complete localization of the amplitude dynamics on an infinite frequency range in the linear

(²) Higher-order roots of the Bessel function $J_0(y)$ correspond to additional crossings between ε_1 and ε_2 at fixed ω and higher S-values. These additional crossings are not considered herein.

(³) Also note the role of initial preparation: a preparation in the *ground state*, *i.e.* $a_1(0) = 1 (a_2(0) = 0)$ yields instead $\lim_{b\to\infty} P(t) = \cos^2((b/\omega)\sin\omega t)$, being rapidly oscillating.

regime of the exact manifold $\mathcal{M}_{\text{t.l.}}$. In clear contrast, coherent destruction of tunnelling is limited to a restricted frequency interval within the linear regime on $\mathcal{M}_{\text{d.w.}}$ This latter frequency range is very accurately modelled by the two-level dynamics, cf. fig. 1. The different behaviour can be elucidated as follows: at $\omega > \omega_{\text{res}}$ the corresponding forcing strength S increases according to (16) proportional to the driving frequency ω. The large amplitude S makes it increasingly more difficult to localize an initial wave packet in a symmetric double well, due to the ever-present, with increasing S enhanced mixing with higher-lying quasi-energy states. Likewise, for small barrier heights (*i.e.* decreasing D-values, or increasing bare tunnel splittings Δ) the amplitude needed to suppress tunnelling necessarily increases. This fact, therefore, implies that the good agreement depicted in fig. 1 between the driven two-level approximation and the exact dynamics in (1)-(3) worsens also with decreasing barrier height.

The result of an essentially complete localization for the periodically driven two-level dynamics (see fig. 2) for frequencies $\omega \geqslant \Delta$ on $\mathcal{M}_{\text{t.l.}}$ carries further interesting consequences. The complex-valued amplitude dynamics near suppression in (8), (9) could be utilized to the effect of freezing the polarization modes of two coupled modes, propagating in an optical ring resonator, see ref. [3*g*]. Generally, the exact crossing of two quasi-energies modifies any interference behaviour of transition probabilities of two-level atoms in strong laser fields. For example, within the approximation in (16) the first zero of the Bessel function leads to drastic changes (*i.e.* infinite widths, see fig. 14 in [10]) for the absorption spectrum of the Hanle effect in a transverse, oscillating magnetic field.

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