Can Colored Noise Improve Stochastic Resonance?

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The phenomenon of stochastic resonance is studied in the presence of colored noise. Several sources of colored noise are introduced and the consequences for the asymptotic time-periodic probability and the (phase-averaged) power spectrum are discussed. Based on space-time symmetry considerations, selection rules for the occurrence of δ-spikes in the power spectrum are derived. The effect of colored noise on the amplification of small periodic signals is studied in terms of effective, time-periodic Fokker–Planck equations: In overdamped systems driven by colored noise, we find that SR is suppressed with increasing noise color. In contrast, for colored noise induced by inertia (as well as for asymmetric dichotomic noise), one obtains an enhancement of SR. This latter result is obtained by studying the Kramers equation perturbed by a small periodic force.

1. INTRODUCTION AND CONCLUSIONS

The study of time-dependent stochastic systems recently underwent a renaissance in the context of phenomena like "stochastic resonance" (SR) (see refs. 1 and 2 for recent reviews) and "resonance activation."[3] SR is a cooperative effect of noise and periodic forcing in a bistable system. It is characterized by a noise-induced large response to a weak periodic signal. Thus, the SR effect can be used to amplify a weak (trial) signal by subjecting it to noise of external or internal character. Roughly speaking, the signal-to-noise ratio exhibits as a function of noise intensity a bell-shaped curve, i.e., increasing noise can—in a counterintuitive manner—enhance the signal amplification. In order for SR to occur in moderate-to-overdamped

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systems it is essential that the unperturbed system possesses a frequency scale (lowest eigenvalue λ_1) which is decreasing exponentially with decreasing noise intensity. This in turn implies that the system is required to exhibit some sort of metastability which is usually introduced by a multi- or bistable potential field. Likewise, such a bistability can also be induced differently either by a nonlinear friction mechanism which implies two basins of attraction, such as, e.g., two-stable limit cycles[3] (see also ref. 4), or can be induced dynamically via the well-known effect of "resonance hysteresis"[(5)] in periodically driven nonlinear oscillatory systems. SR might not necessarily occur with noise-induced bistability, due to the possible absence of a softening of the first nonvanishing eigenvalue.

The application of the periodic force of period T_e alternately raises and lowers each potential well with respect to the barrier separating the metastable states. Now with the forward and backward hopping times denoted by T^+ and T^-, the system optimally follows the external modulation when $T_e \approx T^+ + T^-$. With T^+ and T^- related to the lowest frequency scale $\lambda_{\min}$ in the unperturbed system, i.e., $\lambda_{\min} = 1/T^+ + 1/T^-$, the condition for SR thus reads for symmetric escape times $T^+ = T^-$

$$\Omega \approx \frac{\pi}{2} \lambda_{\min} \tag{1.1}$$

Here, $\Omega = 2\pi/T_e$ denotes the (angular) frequency of the coherent external small signal.

In this work our focus will be on the role of realistic noise for SR. In many situations the time scale of random perturbations is very much shorter than that of the characteristic time scale of the system. It is then a good working assumption to use uncorrelated (i.e., δ-correlated) random forces. This assumption considerably simplifies the problem, because it allows one to treat the dynamics within the notion of Markov processes. In the physical world, however, this idealization is never exactly realized. In order to understand the importance of corrections to white noise, and more generally, in order to investigate the role of noise correlations of arbitrary strength, it is thus necessary to study also nonwhite noise, i.e., "noisecolor" of small to moderate-to-large correlation strength. Strong noise color is not unrealistic for many physical applications. Usually a strongly correlated noise emerges as the result of a coarse graining over a hidden set of slow variables, or it simply is applied by the experimenter externally. Historically, colored noise of arbitrary long correlation times was introduced by

[3] A nonconservative flow processing two stable limit cycles (and one intermediate unstable limit cycle) is given by $\ddot{x} + a\sin(\dot{x}) + x = 0$.

Kubo in his cornerstone papers on the theory of the transition of Gaussian-like line shapes toward Lorentzian-like ones in nuclear magnetic resonance[6] ("motional narrowing"), or for paramagnetic resonance[7] ("exchange narrowing").

In practice, one wishes to monitor only a few, and preferably just *one* physical variable. With the objective to obtain a tractable description, the theorist, too, prefers a low, preferably a one-dimensional description. There is a price to be paid, however, for such a simplification. This is so because a low-dimensional flow implies a loss of the Markov property of the original higher-dimensional system. In recent years, a lot of sweat has been invested by a large number of theoretical practitioners in developing such efficient (non-Markovian) colored noise approaches (note the various review articles—and references therein—in ref. 8). Our principal objective is to use and to expand upon this colored noise work for the study of nonstationary processes such as the phenomenon of SR.

The paper is organized as follows: The next section introduces several classes of Gaussian colored noise sources characterized by several time scales. In particular, we consider oscillatorlike noise, which assumes correlation forms other than the commonly used simple exponential decay typical for Gaussian Ornstein–Uhlenbeck noise. In Section 3 we elaborate on general properties, such as the selection rules for δ-function peaks in the power spectra of periodically driven stochastic processes. In Section 4 we lay the groundwork for colored noise and SR by setting up the relevant approximation schemes. Section 5 is devoted to the study of noise color for SR in overdamped systems at small and moderate-to-large noise correlation times. The last section treats SR for the bistable Kramers equation.

The main conclusions are here summarized:

(i) The phase-averaged power spectrum of periodically driven stochastic systems subjected to colored noise sources exhibits δ-peaks at multiples of the external driving frequency. With a symmetric bistable flow, Gaussian stationary colored noise sources composed of one to many time scales yield selection rules for the weights of corresponding δ-peaks. For example, for additive noise and additive driving only δ-peaks at odd-numbered multiples of the driving frequency occur.

(ii) The asymptotic long-time probability $p_t(x; \varphi)$ for SR in the presence of colored noise is time-periodic.

(iii) The effect of noise color for SR can be dealt with by an effective time-periodic Fokker–Planck operator, both in the limit $\tau \to 0$ (small-τ SR approximation) and also for $\tau \gg 1$ (unified colored-noise SR approximation).

(iv) SR in overdamped systems driven by additive exponentially correlated colored Gaussian noise $\xi(t)$ is always *reduced* as compared to the case with white noise ($\tau = 0$) of the same strength D. The amplification $\eta(\Omega, A)$ is well approximatea by use of linear response theory. The peak for SR is shifted to larger noise intensities due to the fact that colored noise suppresses (exponentially) the hopping rate with increasing noise color.

(v) The result of SR for the driven Kramers equation predicts an *enhancement* of SR within the regime of moderate to large friction γ for increasing noise color as measured by the time scale γ^{-1}.

(vi) With overdamped SR being characterized mainly by the behavior of the hopping rate, we conclude that SR (within the linear response approximation) is reduced also for symmetric (additive as well as multiplicative) non-Gaussian, dichotomic noise $\xi_{\mathrm{DM}}(t)$.[36–38] This is so because symmetric dichotomic noise always leads to an exponential decrease for the rate with increasing noise color.[36]

(vii) For asymmetric dichotomic noise, however, an increase of the noise color generally does not yield a lower rate.[39] In this case, therefore, SR can be enhanced or suppressed, depending on the specific structure of the asymmetric dichotomic noise.[39]

2. CLASSIFICATION OF COLORED NOISE SOURCES

We next discuss various classes of Gaussian colored noise sources.

2.1. Exponentially Correlated Gaussian Noise (Ornstein–Uhlenbeck Noise)

The archetypal source for colored noise consists of an exponentially correlated process given by a Gauss–Markov process $\xi(t)$,

$$\dot{\xi}(t) = -\frac{1}{\tau}\xi + \frac{\sqrt{D}}{\tau}\xi_w(t) \tag{2.1}$$

where $\xi_w(t)$ denotes Gaussian white noise of vanishing mean and with correlation $\langle \xi_w(t)\, \xi_w(s) \rangle = 2\delta(t-s)$. It then follows from (2.1) that the stationary correlation of $\xi(t)$ is given by

$$\langle \xi(t)\, \xi(s) \rangle = (D/\tau) \exp(|t-s|/\tau) \equiv S(t-s) \tag{2.2}$$

Its spectrum thus assumes a Lorentzian form, i.e.,

$$S(\omega) = \int_{-\infty}^{\infty} S(t) \exp(i\omega t)\, dt = S(-\omega) = \frac{2D}{1 + \tau^2\omega^2} \tag{2.3}$$

such that $S(\omega = 0) = 2D$.

2.2. Harmonic Noise

For colored noise possessing structural features, such as, e.g., resonancelike spectral shapes, a noisy damped oscillator presents an ideal archetype. It is represented by a two-dimensional Gauss–Markov process of the form[9, 10]

$$\dot{\xi} = v \tag{2.4a}$$

$$\ddot{\xi} = \dot{v} = -\gamma v - \omega_0^2 \xi + \sqrt{D}\, \xi_w(t) \tag{2.4b}$$

The stationary correlation is then given by ($\gamma^2 < 4\omega_0^2$):

$$S(t) = \frac{D\tau}{\gamma\omega_0^2} \exp\left(-\frac{1}{2}\gamma\, |t|\right)\left[\cos(\omega_1 t) + \frac{\gamma}{2\omega_1} \sin(\omega_1 t)\right] \tag{2.5}$$

with $\omega_1^2 = \omega_0^2 - \gamma^2/4$. This yields for the spectrum

$$S(\omega) = \frac{2D}{\omega^2\gamma^2 + (\omega^2 - \omega_0^2)^2} \tag{2.6}$$

This very harmonic noise has recently been studied in the context of colored-noise-driven bistability by Schimansky-Geier and Zülicke.[10]

2.3. Multiscale Colored Noise

The harmonic noise in Section 2.2 can be generalized to include memory damping composed of many time scales. Following the generalized Langevin treatment in refs. 11–13, we set

$$\dot{\xi} = v \tag{2.7a}$$

$$\ddot{\xi} = \dot{v} = -\omega_0^2 \xi - \int_0^t \phi(t-s)\, \dot{\xi}(s)\, ds + \eta(t) \tag{2.7b}$$

with $\eta(t)$ a non-Markovian stationary Gaussian noise of zero mean and correlation

$$\langle \eta(t)\, \eta(s) \rangle = D\phi(t-s) \tag{2.7c}$$

The colored noise $\xi(t)$ can be recast as an $(n+2)$-dimensional Gauss–Markov process by writing[13]

$$\begin{aligned}
\dot{\xi} &= v \\
\ddot{\xi} &= \dot{v} = -\omega_0^2 \xi + \eta_1 \\
\dot{\eta}_1 &= -c_1 v - \gamma_1 \eta_1 + \eta_2 + \xi_w^{(1)}(t) \\
\dot{\eta}_2 &= -c_2 \eta_1 - \gamma_2 \eta_2 + \eta_3 + \xi_w^{(2)}(t) \\
&\vdots \\
\dot{\eta}_n &= -c_n \eta_{n-1} - \gamma_n \eta_n + 0 + \xi_w^{(n)}(t)
\end{aligned} \tag{2.8}$$

where

$$\gamma_i \geqslant 0, \qquad c_i > 0, \qquad i = 1,..., n; \qquad \gamma_n > 0$$

and

$$\langle \xi_w^{(i)}(t)\, \xi_w^{(j)}(s) \rangle = 2\delta_{ij} D \gamma_i \left[\prod_{m=1}^{i} c_m \right] \delta(t-s) \tag{2.9}$$

The Laplace transform $\phi(z)$ of the memory damping assumes the continued-fraction expansion

$$\phi(z) = \frac{c_1}{z+\gamma_1+} \, \frac{c_2}{z+\gamma_2+} \cdots \frac{c_n}{z+\gamma_n} \tag{2.10}$$

Clearly, this form allows both for oscillatory decaying and multi-exponentially-decaying memory damping functions $\phi(t)$. In the following section we shall investigate some general properties of these various colored noise sources for stochastic resonance in a symmetric bistable potential.

3. SELECTION RULES FOR STOCHASTIC RESONANCE

In the context of stochastic resonance we consider a symmetric double-well system driven by a deterministic periodic modulation and noise $\xi(t)$. Using the bistable nonequilibrium Ginzburg–Landau flow dynamics, we set

$$\dot{x} = ax - bx^3 + g(x) \mathscr{A} \sin(\Omega t + \varphi) + \xi(t) \tag{3.1}$$

where $\xi(t)$ is either white Gaussian noise or one of the colored Gaussian noise sources introduced in the previous section. With $\xi(t)$ white noise the process $x(t)$ in (3.1) constitutes a *nonstationary Markov process* with a time-periodic Fokker–Planck operator. For colored noise $\xi(t)$ of Gaussian character the previous three noise sources all can be represented as Gauss–Markov processes of rank 1, 2, and $n+2$, respectively. Following the general theory in ref. 14, it then follows that the composed process made up by the Gauss–Markov process and the state variable $x(t)$ constitutes a nonlinear nonstationary Markov process of rank 2, 3, and $n+3$, respectively, with a (multidimensional) *time-periodic* Fokker–Planck operator. Integrating over the components of the Gauss–Markov noise process then yields at large times a *periodic* asymptotic probability solution for the periodically driven (non-Markovian) process $x(t)$, i.e.,

$$P_{as}(x, t; \varphi) = P_{as}(x, t + 2\pi/\Omega; \varphi) \tag{3.2}$$

Likewise, the periodicity of the multidimensional Fokker–Planck operator implies periodic, phase-averaged asymptotic correlations. In particular, the non-Markovian asymptotic correlation becomes, with $t - t' \equiv \tau$, $t > t'$ (see refs. 15, 16), *after phase averaging*,

$$S_{as}(\tau) \equiv \lim_{t > t' \to \infty, \tau \to \infty} \langle x(t)\, x(t') \rangle_{\varphi} = 2 \sum_{n=1}^{\infty} |M_n|^2 \cos(n\Omega\tau) \tag{3.3}$$

with the set $\{M_n\}$ determined from the asymptotic, periodic mean value, i.e.,

$$\langle x(t); \varphi \rangle_{as} = \sum_{n=-\infty}^{\infty} M_n \exp[in(\Omega t + \varphi)], \qquad M_0 = 0 \tag{3.4}$$

After neglect of transients (i.e., $\tau \to \infty$), the phase-averaged correlation is not strongly mixing;[15, 16] the weights $|M_n|^2$ give rise to δ-spikes at multiples of the driving frequency Ω, i.e.,

$$S_{as}(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |M_n|^2 \, \delta(\omega - n\Omega) \tag{3.5}$$

With $\xi(t)$ being white noise and the bistable flow $h(x) = ax - bx^3 = -h(-x)$ being antisymmetric, we previously found the following selection rules[15, 16]:

1. $g(x) = g(-x)$ symmetric $\Rightarrow |M_{2n}|^2 = 0$
   i.e., only δ-spikes at odd multiples of the driving frequency occur (3.6)

2. $g(x) = -g(-x)$ asymmetric $\Rightarrow |M_n| = 0$ for all n
   i.e., no δ-spikes at all occur (3.7)

These selection rules for SR can now be generalized for colored noise by observing the generalized parity symmetry obeyed by the corresponding *multidimensional Fokker–Planck operator.*

First, let us discuss the case where $g(x) = -g(-x)$ is antisymmetric. The corresponding multidimensional Fokker–Planck operator then implies the parity symmetry $\mathscr{P}$:

Ornstein–Uhlenbeck noise:

$$\begin{aligned} x &\to -x \\ \xi &\to -\xi \end{aligned} \tag{3.8}$$

Harmonic noise:

$$\begin{aligned} x &\to -x \\ \xi &\to -\xi \\ v &\to -v \end{aligned} \tag{3.9}$$

Memory damping:

$$\begin{aligned} x &\to -x \\ \xi &\to -\xi \\ v &\to -v \\ \eta_i &\to -\eta_i, \qquad i = 1,\dots, n \end{aligned} \tag{3.10}$$

These generalized parity symmetries thus imply that $P_{\mathrm{as}}(x, t; \varphi)$ is *symmetric in x at all times t.* Therefore, $\langle x(t); \varphi \rangle_{\mathrm{as}} = 0$, which again implies *no δ-spikes at all* for the asymptotic long-time spectrum, i.e.,

$$g(x) = -g(-x) \quad \text{asymmetric} \Rightarrow |M_n| = 0 \qquad \text{for all } n \tag{3.11}$$

With a *symmetric* modulation shape function $g(x) = g(-x)$, e.g., $g(x) = 1$, the multidimensional periodic Fokker–Planck operator still obeys a generalized parity, but now it also involves the time variable. With the period of the external driving given by $T_e = 2\pi/\Omega$ the new generalized parity $\mathscr{P}$ reads for the state variables the same as before, but the time variable is transformed according to

$$t \to t + \tfrac{1}{2} T_e \tag{3.12a}$$

Expanding the periodic multidimensional, asymptotic probability $p_{\mathrm{as}}(x, \xi, v,\dots, \eta_i,\dots, t; \varphi)$ into a Fourier series and integrating over the state variables of the noise, one ends up with a periodic, non-Markovian asymptotic probability $P_{as}(x, t; \varphi)$ composed of Fourier components $\{c_n(x)\}$ obeying $c_n(x) = (-1)^n c_n(-x)$. The periodic, asymptotic mean $\langle x(t); \varphi \rangle$ thus involves only *odd* Fourier components. This in turn implies the selection rule

$$g(x) = g(-x) \quad \text{symmetric} \Rightarrow \text{only} \quad M_{2n+1} \neq 0 \tag{3.12b}$$

i.e., for white noise $\xi_w(t)$, as well as for all stationary Gaussian colored noise sources introduced in Section 2, the selection rules for the δ-spikes in the asymptotic spectral density *do not change*!

Moreover, with a symmetric flow $h(x) = h(-x)$ and antisymmetric shape function $g(x) = -g(-x)$ a reasoning just as before would imply the selection rule $M_{2n+1} = 0$, $M_{2n} \neq 0$; i.e., only even-numbered δ-spikes survive the asymptotic spectrum $S_{as}(\omega)$.

4. APPROXIMATION SCHEMES FOR NONSTATIONARY COLORED NOISE PROCESSES

In the previous section we dealt with colored noise by embedding the correlated noise into a higher-dimensional Markov process. In the presence of nonlinear flows, however, these many-dimensional Markovian schemes do not prove to be very effective in obtaining analytical (or even numerical) results. The exception is the linear flow driven by colored noise, which can be solved in closed form in any finite dimension (see Appendix A). As a consequence, a great many statistical physicists engaged in research aimed at obtaining tractable approximation schemes which are low-dimensional (mostly one-dimensional) in character.[8] Generally, the approximation schemes become useful only if they reduce to an (effective) Fokker–Planck form. Such a procedure, however, is not without limitation: The approximation has a limited range of validity, addressing mostly asymptotic regimes of small or very large noise correlation time. For the sake of simplicity and clarity only, we shall in the following address exponentially decaying colored noise characterized by a unique time scale τ; see (2.2).

4.1. Small-Correlation-Time Approximation

Given (3.1), we use for $\xi(t)$ Ornstein–Uhlenbeck noise [see (2.1)] and set for the shape function $g(x)$ a constant, i.e., $g(x) \equiv 1$. The nonstationary bistable flow thus reads

$$\dot{x} = ax - bx^3 + \mathcal{A} \sin(\Omega t + \varphi) + \xi(t) \tag{4.1a}$$

with

$$\langle \xi(t)\, \xi(0) \rangle = (D/\tau) \exp(-|t|/\tau) \tag{4.1b}$$

In the following we shall work with dimensionless variables, i.e., we rescale the state variables and the time according to (the direction of the arrow refers to the dimensionless variables):

$$\begin{aligned} at &\to t \\ a\tau &\to \tau \\ \Omega/a &\to \Omega \\ (ba^{-1})^{1/2}\, x &\to x \\ (ba^{-3})^{1/2}\, \varepsilon &\to \varepsilon \\ (ba^{-3})^{1/2}\, \mathscr{A} &\to \mathscr{A} \\ \tfrac{1}{4} D/\Delta U &\to D \end{aligned} \tag{4.2}$$

where $\Delta U = a^2/4b$ denotes the barrier height of the bistable, symmetric potential. Equation (4.1) is then recast in dimensionless quantities as

$$\dot{x} = x - x^3 + \mathscr{A} \sin(\Omega t + \varphi) + \xi(t) \tag{4.3}$$

with $\langle \xi(t)\, \xi(0) \rangle = (D/\tau) \exp(-|t|/\tau)$. Because the colored noise source is not affected by the periodic driving force, the effective Fokker–Planck approximation at small noise color follows by using the same reasoning as put forward for time-homogeneous processes.[8, 17–20] The result thus reads

$$\begin{aligned} \dot{p}_t(x;\tau) = &-\frac{\partial}{\partial x} \{ [x - x^3 + \mathscr{A} \sin(\Omega t + \varphi)]\, p_t(x;\tau) \} \\ &+ D \frac{\partial^2}{\partial x^2} \{ [1 + \tau(1 - 3x^2)]\, p_t(x;\tau) \} \end{aligned} \tag{4.4}$$

In this context we emphasize again[20] the regime of validity for (4.4): The result in (4.4) holds good for small $\tau \to 0$, with the (dimensionless) ratio $\tau/D \ll 1$ staying small. The Fokker–Planck equation (4.4) serves as our starting point to evaluate the effect of noise color on SR in Section 5. In contrast to the white noise case ($\tau = 0$), it involves a state-dependent diffusion coefficient. In passing, we point out that the decoupling approximation,[20, 21] although somewhat crude, would describe the effect of noise color solely within a renormalization of the noise intensity, i.e.,

$$D \to D/[1 - \tau(1 - 3\langle x^2 \rangle)] \approx D/(1 + 2\tau) \tag{4.5}$$

4.2. Unified-Colored-Noise Approximation (UCNA)

Following ref. 22, we consider next an approximation scheme aimed at eliminating adiabatically the velocity $\dot{x}$.

Upon a differentiation of (4.3) we obtain

$$\ddot{x} = \dot{x}(1-3x^2) + \mathscr{A}\Omega \cos(\Omega t + \varphi) + \dot{\xi} \tag{4.6}$$

With a substitution of $\dot{\xi}$ [see (2.1)] the flow in (4.3) is equivalently represented as

$$\begin{aligned}\ddot{x} &= \dot{x}(1-3x^2) + \mathscr{A}\Omega \cos(\Omega t + \varphi) \\ &\quad + \frac{1}{\tau}[x - x^3 + \mathscr{A} \sin(\Omega t + \varphi) - \dot{x}] + \frac{D^{1/2}}{\tau}\xi_w(t)\end{aligned} \tag{4.7}$$

By use of a new time scale, i.e., $\hat{t} \equiv t = \tau^{-1/2}t$, this is recast as

$$\begin{aligned}\ddot{x} + \dot{x}\gamma(x, \tau) - (x - x^3) - \mathscr{A}\tau\left(\frac{1}{\tau^2} + \Omega^2\right)^{1/2} \\ \times \sin(\Omega\tau^{1/2}t + \hat{\varphi}) = D^{1/2}\xi_w(\tau^{1/2}t)\end{aligned} \tag{4.8}$$

where $\gamma(x, \tau) \equiv [\tau^{-1/2} + \tau^{1/2}(3x^2 - 1)]$, and $\hat{\varphi} = \varphi + \alpha$ with $\tan\alpha = \Omega\tau$. Note that $|\gamma(x, \tau)|$ approaches infinity both for $\tau \to 0$ and $\tau \to \infty$. With $\gamma(x, \tau) \to \infty$, we may set $\ddot{x}$ equal to zero, yielding the UCNA approximation for SR, i.e.,

$$\begin{aligned}\dot{x} = \gamma^{-1}(x, \tau)\{(x - x^3) + \mathscr{A}[1 + \Omega^2\tau^2]^{1/2} \\ \times \sin(\Omega\tau^{1/2}t + \hat{\varphi}) + D^{1/2}\tau^{1/4}\xi_w(t)\}\end{aligned} \tag{4.9}$$

With $\tau \to 0$, this adiabatic approximation is well justified. In contrast, with $\tau \to \infty$ the variation of $\dot{x}$ is with $\lim_{\tau\to\infty} \tau \sin(\Omega\tau^{1/2}t)$ being rapidly oscillating, not slowly varying. The approximation in (4.9) nevertheless still holds good for $\tau \to \infty$ if we note that the best Fokker–Planck approximation to the Langrangian $\mathscr{L}$ of the path integral solution[23–26] for the nonstationary colored noise process $x(t)$, i.e.,

$$\begin{aligned}\mathscr{L}[x(s)] = (4D)^{-1}\{\tau\ddot{x} + \dot{x}[1 - \tau(1-3x^2)] - \mathscr{A}\Omega\tau \cos(\Omega t + \varphi) \\ - [x - x^3 + \mathscr{A} \sin(\Omega t + \varphi)]\}^2\end{aligned} \tag{4.10}$$

is obtained by setting $\ddot{x} = 0$; see also refs. 23–26.[4] This in turn is (within the prepoint discretization) consistent with the white noise Langevin equation

[4] In writing (4.10), we have neglected boundary terms; for details see ref. 25.

(4.9)! The corresponding Fokker–Planck equation possesses a strictly positive diffusion coefficient, independent of the size of the correlation time τ. The UCNA for time-homogeneous processes has been proven to yield good results for the invariant stationary probability. We thus expect that (4.9) also yields good results for the time-periodic asymptotic probability $p_{\mathrm{as}}(x, t; \varphi; \tau)$ for both $\tau \to 0$ and $\tau \to \infty$.

5. STOCHASTIC RESONANCE FOR COLORED NOISE

As mentioned in the Introduction, the SR describes the amplification of small signals embedded in a noisy background. It occurs when the time scale of the external driving frequency $T_e = 2\pi/\Omega$ is of the order of the escape time from one of the metastable wells. Given *weak noise*, the noise strength $D \equiv D_{\mathrm{SR}} = \Delta U/[\ln(T_e \omega_0/4\pi)]$, with ω_0 denoting the angular well frequency, results in a hopping dynamics between the two metastable states most highly correlated with the external driving signal. Following ref. 15, the SR is measured by the ratio between the "power" at the (angular) frequency Ω of $S_{\mathrm{as}}(\omega)$ and the input power $P_{\mathrm{in}} = \pi A^2$. The signal amplification $\eta(\mathscr{A}, \Omega)$ is then given by[15]

$$\eta(\mathscr{A}, \Omega) = 4\,\frac{|M_1|^2}{\mathscr{A}^2} \tag{5.1}$$

with M_1 the first Fourier component of the asymptotic, time-periodic mean value $\langle x(t); \varphi \rangle_{\mathrm{as}}$. The $|M_1|$ can be evaluated approximately by use of linear response theory.[15, 27, 28] With $\chi(\omega) = \int_0^\infty \exp(i\omega t)\, x(t)\, dt$ the (one-sided) Fourier transform of the response function $\chi(t)$, which describes the perturbation in $x(t)$, one finds[15]

$$|M_1| \approx \frac{\mathscr{A}}{2}\, |\chi(\Omega)| \tag{5.2}$$

5.1. SR for Small Noise Color

Given the small-correlation-time approximation in (4.4), the stochastic operator describing the perturbation reads

$$\Gamma_{\mathrm{ext}}(t) = -\mathscr{A} \sin(\Omega t + \varphi)\, \frac{\partial}{\partial x} \equiv \mathscr{A} \sin(\Omega t + \varphi)\, \Gamma_{\mathrm{ext}} \tag{5.3}$$

i.e., $\Gamma_{\mathrm{ext}} \equiv -\partial/\partial x$. In linear response the expectation, i.e.,

$$\langle x(t) \rangle = \int_{-\infty}^{t} \chi(t-s)\, \mathscr{A} \sin(\Omega s + \varphi)\, ds \tag{5.4}$$

gives for the response function the correlation function result[27]

$$\chi(t) = \theta(t)\langle x(t)\,\phi[x(0)]\rangle \tag{5.5}$$

where $\phi[x] = -(\partial/\partial x)[\ln p_{\mathrm{st}}(x, \tau)]$ is a fluctuation of vanishing mean and $\theta(t)$ denotes the step function. Here, $p_{\mathrm{st}}(x; \tau)$ denotes the stationary, unperturbed probability. From (4.4) it is given explicitly by

$$p_{\mathrm{st}}(x, \tau) = \frac{Z^{-1}}{|[1+\tau(1-3x^2)]|} \exp -\left\{\frac{-\frac{1}{2}x^2 + \frac{1}{4}x^4}{D}\left[1 - \frac{1}{2}\tau(x - x^3)^2\right]\right\} \tag{5.6}$$

where Z denotes the normalization constant. Thus, the fluctuation $\phi[x, \tau]$ is evaluated to read

$$\phi[x, \tau] = D^{-1}[1 + \tau(1-3x^2)]^{-1}\,(x - x^3 + 6D\tau x) \tag{5.7}$$

An alternative form for the response function can be obtained following ref. 27: If we denote by Γ the unperturbed ($\mathscr{A} = 0$) Fokker–Planck operator in (4.4), we find, with $\psi(x)$ defined by

$$\phi[x, \tau]\, p_{\mathrm{st}}(x) = [\Gamma_{\mathrm{ext}}\, p_{\mathrm{st}}](x) \equiv -[\Gamma\psi p_{\mathrm{st}}](x) \tag{5.8}$$

the result

$$\chi(t) = -\theta(t)\frac{d}{dt}\langle x(t)\,\psi[x(0)]\rangle \tag{5.9}$$

From (5.8) one finds with τ small

$$\frac{d\psi}{dx} = \{D[1 + \tau(1-3x^2)]\}^{-1} \approx D^{-1}[1 - \tau(1-3x^2)] \tag{5.10}$$

yielding

$$\psi(x) = D^{-1}[x - \tau(x - x^3)] \tag{5.11}$$

In conclusion, the response function $\chi(t)$ can advantageously be recast as the time derivative of the correlation

$$\begin{aligned}\chi(t) &= -\theta(t)D^{-1}\frac{d}{dt}\langle x(t)\{x(0) - \tau[x(0) - x^3(0)]\}\rangle \\ &\equiv -\theta(t)D^{-1}\frac{d}{dt}C_{x\psi}\end{aligned} \tag{5.12}$$

In the context of SR, we set for the correlation $\langle x(t)\,\psi[x(0);\tau]\rangle$ the long-time approximation[5]

$$C_{x\psi}(t) \approx \langle x\psi(x)\rangle \exp(-\lambda_{\min} t) + O(D) \tag{5.13}$$

$$= (\langle x^2\rangle + \tau D) \exp(-\lambda_{\min} t) + O(D) \tag{5.14}$$

To obtain (5.14), we eliminated $\langle x^4\rangle$ by observing that $(d/dt)\langle x^2\rangle = 0 = \langle \Gamma^+ x^2\rangle$, where Γ^+ is the adjoint operator of Γ. Upon noticing that $|M_1| = \frac{1}{2}\mathscr{A}\,|\chi(\Omega)|$, we thus obtain for the amplification $\eta(\mathscr{A}, \Omega; \tau; D)$ in linear response approximation, i.e., $\eta(\mathscr{A}, \Omega) \approx \eta(\Omega)$, from (5.1), (5.2) the result

$$\eta(\Omega) = |\chi(\Omega)|^2 = \left(\frac{\langle x^2\rangle + \tau D}{D}\right)^2 \frac{1}{1 + (\Omega/\lambda_{min})^2} \tag{5.15}$$

In the limit of small noise color and $\tau/D \ll 1$, $\lambda_{\min}$ has been evaluated in previous works[19, 20, 29] to read

$$\lambda_{\min}(\tau) = \frac{\sqrt{2}}{\pi}\left(1 - \frac{3}{2}\tau\right) \exp\left(-\frac{1}{4D}\right) \tag{5.16}$$

With $\langle x^2\rangle \approx 1$, we obtain therefore (see footnote 5)

$$\eta(\Omega) \approx \frac{1}{D^2}(1 + 2\tau D)\frac{1}{1 + (1 + 3\tau)(\Omega^2\pi^2/2)\exp(1/2D)} \tag{5.17}$$

At stochastic resonance, i.e., $\Omega = (\pi/2)\lambda_{\min}$, the peak height becomes

$$\max \eta(\Omega) \equiv \eta_{\max} \approx \frac{1 + 2\tau D_{\mathrm{SR}}(\tau)}{D_{\mathrm{SR}}^2(\tau)}\left(\frac{1}{1 + \pi^2/4}\right) \tag{5.18}$$

with

$$D_{\mathrm{SR}}(\tau) = \frac{1}{4}\left[\ln\left(\frac{1 - \frac{3}{2}\tau}{\sqrt{2}\,\Omega}\right)\right]^{-1} > D_{\mathrm{SR}}(\tau = 0) \tag{5.19}$$

Put differently, with colored noise present, the peak in SR is shifted toward higher D values. Moreover, with $\eta_{\max} \propto 1/D_{\mathrm{SR}}^2$, the peak value is, with $D_{\mathrm{SR}}(\tau) > D_{\mathrm{SR}}(\tau = 0)$, reduced compared to SR at $\tau = 0$. These characteristic colored noise features become more pronounced with increasing noise

[5] At weak noise, the correction of $O(D)$ can be approximated by a single exponential, see Eq. (6.3.46) in ref. 27, which in our case reads $(D/2)\exp(-2t) + O(\tau D^2)$. For $D \to 0$, this implies for (5.17) the finite limiting value $\eta(\Omega, D \to 0) = (4 + \Omega^2)^{-1}$; see also ref. 2.

color τ. Indeed, with τ increasing, $\lambda_{\min}(\tau)$ becomes exponentially reduced in comparison to $\lambda_{\min}(\tau=0)$[6]; see ref. 30. With Ω fixed, we therefore must *increase* D to match the SR condition $\Omega \approx (\pi/2)\lambda_{\min}$. This results in a drastic *shift* of the SR peak toward higher D values, and a corresponding *reduction* of the peak height; see Section 5.2. In Fig. 1 we depict the result of the theory in (5.17). Indeed, we find that small noise color suppresses the effect of SR.

5.2. SR within UCNA

In order to treat both small noise color and large noise color on the same basis, we consider the UCNA in (4.9). To compare with Section 5.1, we use the original time scale, i.e., $t = \tau^{1/2}\hat{t}$. The stochastic operator describing the time-periodic perturbation then reads

$$\begin{aligned} \Gamma_{\text{ext}}(t) &= -\mathscr{A}(1+\Omega^2\tau^2)^{1/2}\sin(\Omega t+\hat{\varphi})\left\{\frac{\partial}{\partial x}\frac{1}{[1-\tau(1-3x^2)]}\right\} \\ &\equiv \mathscr{A}(1+\Omega^2\tau^2)^{1/2}\sin(\Omega t+\hat{\varphi})\Gamma_{\text{ext}} \end{aligned} \tag{5.20}$$

with

$$\Gamma_{\text{ext}} = -\frac{\partial}{\partial x}\left(\frac{1}{[1-\tau(1-3x^2)]}\right)$$

[6] A rough estimate based on the decoupling theory in (4.5) yields $\lambda_{\min}(\tau) \propto \exp[-(1+2\tau)/4D]$; numerically[30] one finds instead $\lambda_{\min}(\tau) \propto \exp[-(1+0.4\tau)/4D]$.

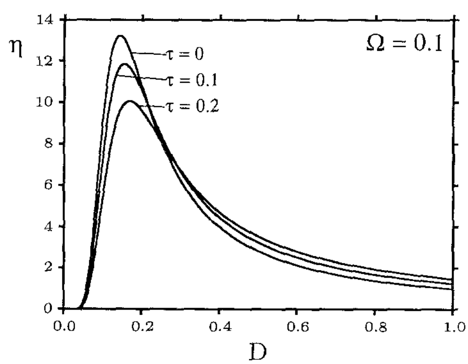

Fig. 1. The signal amplification η, evaluated within linear response approximation [see Eq. (5.15)], depicted as a function of the noise strength D at $\Omega = 0.1$ for increasing values of the noise correlation time τ.

Proceeding as before, we set

$$[\Gamma_{\mathrm{ext}}\, p_{\mathrm{st}}^{\mathrm{UCNA}}](x) = -[\Gamma\psi p_{\mathrm{st}}^{\mathrm{UCNA}}](x) \tag{5.21}$$

which yields for the fluctuation $\psi(x)$

$$\frac{d}{dx}\psi^{\mathrm{UCNA}}(x,\tau) = D^{-1}[1-\tau(1-3x^2)]$$

This yields

$$\psi^{\mathrm{UCNA}}(x,\tau) = D^{-1}[x-\tau(x-x^3)] \tag{5.22}$$

Note that $\psi^{\mathrm{UCNA}}(x,\tau)$ coincides precisely with the small-τ result in (5.11). In contrast to (5.11), however, we can use (5.22) for arbitrary τ values, $0 \leqslant \tau < \infty$. The response function $\chi^{\mathrm{UCNA}}(t)$ thus reads

$$\chi^{\mathrm{UCNA}}(t) = -\theta(t)D^{-1}\frac{d}{dt}\langle x(t)\{x(0)-\tau[x(0)-x^3(0)]\}\rangle \tag{5.23}$$

with the long-time (low-frequency) approximation

$$\approx \theta(t)D^{-1}\lambda_{\mathrm{min}}(\tau, D)\langle x\psi^{\mathrm{UCNA}}(x,\tau)\rangle \exp(-\lambda_{\mathrm{min}} t) \tag{5.24}$$

By the use of previous works[20, 23–26, 29, 30] we obtain within UCNA and path-integral methods for λ_{min} the crossover approximation[30]

$$\lambda_{\mathrm{min}}^{\mathrm{UCNA}}(\tau, D) = \frac{\sqrt{2}}{\pi}(1+3\tau)^{-1/2}\exp\left[\frac{-1}{4D}\left(\frac{1+(27/16)\tau+\frac{1}{2}\tau^2}{1+(27/16)\tau}\right)\right] \tag{5.25}$$

The corresponding result for the amplification thus reads

$$\begin{aligned}\eta^{\mathrm{UCNA}}(\Omega) &= |\chi^{\mathrm{UCNA}}(\omega=\Omega)|^2 \\ &= \left(\frac{(1-\tau)\langle x^2\rangle + \tau\langle x^4\rangle}{D}\right)^2 \frac{1}{1+[\Omega/\lambda_{\mathrm{min}}^{\mathrm{UCNA}}(\tau, D)]^2}\end{aligned} \tag{5.26}$$

where the mean values are evaluated with the stationary probability

$$\begin{aligned}p_{\mathrm{st}}^{\mathrm{UCNA}} &= Z^{-1}\,|1-\tau(1-3x^2)|\exp-\left[\frac{\tau}{2D}(x-x^3)^2\right] \\ &\quad\times\exp-\left(\frac{\frac{1}{4}x^4-\frac{1}{2}x^2}{D}\right)\end{aligned} \tag{5.27}$$

which has a support over all x values for $\tau < 1$; but $p_{\mathrm{st}}^{\mathrm{UCNA}}(x,\tau) \equiv 0$, for $|x| < [(\tau-1)/3\tau]^{1/2}$, when $\tau > 1$. The result in Eq. (5.26) is depicted in

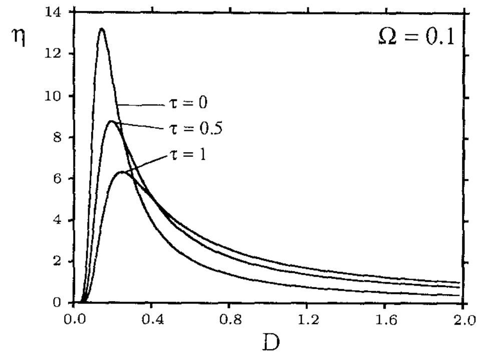

Fig. 2. The amplification factor $\eta(\Omega)$ at $\Omega = 0.1$ as predicted by the UCNA theory in Eqs. (5.25)–(5.27) plotted versus the noise strength D at moderate to large noise color τ.

Fig. 2. We note the consistent shift toward higher D values induced by the exponential decrease of $\lambda_{\min}$ with increasing noise color τ. Related with this shift is the corresponding reduction of the SR peak value; see Fig. 2. With the moderate-to-large noise-color approximation[30]

$$\lambda_{\min} = \frac{\sqrt{2}}{\pi} \exp -\left(\frac{1+\alpha\tau}{4D}\right) \tag{5.28}$$

where $\alpha = 0.1$ for moderate τ, and $\alpha = 8/27$ as $\tau \to \infty$, we can evaluate the condition for SR in (1.1). Fixed $\Omega = (\pi/2)\lambda_{\min}$ then yields for the noise intensity D_{SR} at the SR-peak location the result

$$D_{SR} \approx \frac{1}{4}(1+\alpha\tau)\left[\ln \frac{1}{(2\Omega)^{1/2}}\right]^{-1} \tag{5.29}$$

D_{SR} thus increases almost linearly with increasing τ. The characteristic behavior depicted in Fig. 2 for SR in the presence of Ornstein–Uhlenbeck noise is in good agreement with the simulation results by Gammaitoni *et al.*[31] for the signal-to-noise ratio [this quantity is not quite identical with $\eta(\Omega)$] at moderate noise color $0.2 < \tau < 1.36$.

6. STOCHASTIC RESONANCE FOR THE KRAMERS EQUATION

The systems considered thus far were restricted to the overdamped case with colored noise and periodic forcing. Here, we discuss thermal equilibrium systems subject to periodic driving. The archetypal situation is given by the stochastic motion of a Brownian particle in a symmetric bistable potential field $U(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$ driven by white Gaussian noise $\xi_w(t)$ which satisfies the fluctuation-dissipation theorem of the second kind, i.e., $\langle \xi_w(t)\, \xi_w(0) \rangle = 2D\gamma\delta(t)$. The Fokker–Planck equation is thus the driven Klein–Kramers equation[32]

$$\dot{p}_t(x, v, t; \varphi) = \left\{ -\frac{\partial}{\partial x} v + \gamma \frac{\partial}{\partial v} v - (x - x^3) \frac{\partial}{\partial v} - \mathscr{A} \sin(\Omega t + \varphi) \frac{\partial}{\partial v} + \gamma D \frac{\partial^2}{\partial v^2} \right\}$$
$$\times p_t(x, v, t; \varphi) \tag{6.1}$$

where $D \equiv kT$ characterizes the noise intensity and γ is the damping coefficient. As before, all variables are assumed to be scaled dimensionless; see (4.2). In the presence of colored noise $\xi(t)$, the corresponding Langevin equation involves a memory friction. For weakly colored noise we give in Appendix B the result for the small-correlation-time approximation for colored-noise-driven thermal equilibrium systems. If we just add colored noise of the Ornstein–Uhlenbeck type to the deterministic flow, the noise does not obey the second fluctuation-dissipation theorem, i.e., the flow so obtained describes a nonequilibrium situation. For this latter nonequilibrium case, analog simulation results for SR have been studied in ref. 31.

We point out that the (x, v) dynamics contracted over the velocity would yield a stochastic flow perturbed by colored noise. In the limit $\gamma \to \infty$ we would recover the Smoluchowski dynamics perturbed by white Gaussian noise. In this sense γ^{-1} is a measure for the amount of noise color present in (6.1).

We now turn to the evaluation of SR. The response function for the variable $x(t)$ is given, as it must be, by the classical fluctuation-dissipation theorem[27, 33] (of the first kind), i.e., with $kT \equiv D$,

$$\chi(t) = -\theta(t) D^{-1} \frac{d}{dt} \langle x(t)\, x(0) \rangle \equiv -\theta(t) D^{-1} \frac{d}{dt} C_{xx}(t) \tag{6.2}$$

The relevant long-time approximation for $\langle x(t)\, x(0) \rangle$ reads

$$C_{xx}(t) \approx \langle x^2 \rangle \exp(-\lambda_K t) + O(D) \tag{6.3}$$

where λ_K denotes the lowest eigenvalue given by the celebrated Kramers rate Γ_K; i.e., $\lambda_K = 2\Gamma_K$. The (one-sided) Fourier transform $\chi(\omega)$ is thus approximated by

$$\chi(\omega) \approx D^{-1} \lambda_K \langle x^2 \rangle \int_0^\infty \exp[-t(i\omega + \lambda_K)]\, dt = \frac{\lambda_K}{D} \frac{\langle x^2 \rangle}{i\omega + \lambda_K} \tag{6.4}$$

From Eqs. (5.1), (5.2) we thus readily find for the amplification in linear response approximation

$$\eta(\Omega) = \frac{\langle x^2 \rangle^2}{D^2} \frac{1}{1 + (\Omega/\lambda_K)^2} \tag{6.5}$$

The essential input into (6.5) is thus given by the Kramers eigenvalue λ_K, which depends both on the noise intensity D and the friction γ. As is well known, the eigenvalue $\lambda_K(D, \gamma)/\lambda_{\mathrm{TST}}$, with $\lambda_{\mathrm{TST}} \equiv (\sqrt{2}/\pi) \exp(-1/4D)$ denoting the transition-state estimate, exhibits as a function of increasing γ a bell-shaped dependence, commonly known as "Kramers turnover."[34] Whereas the detailed theory of this turnover is rather complex,[35] we use here a poor-man's (multiplicative) bridging expression given by Eq. (6.4) in ref. 34. With (angular) well frequency and (angular) barrier frequency given by $\omega_0^2 = 2$ and $\omega_b^2 = 1$, respectively, barrier height $\Delta U = (4D)^{-1}$, and the abbreviated action at energy $E = \Delta U$ evaluated to be $I_b = 4/3$, this bridging expression reads explicitly

$$\lambda_K = \frac{4\gamma}{3D} \left[\frac{(1 + 3D\gamma^{-1})^{1/2} - 1}{(1 + 3D\gamma^{-1})^{1/2} + 1} \right] \frac{\sqrt{2}}{\pi} \left[\left(1 + \frac{\gamma^2}{4} \right)^{1/2} - \frac{\gamma}{2} \right] \exp\left(-\frac{1}{4D} \right) \tag{6.6}$$

It obeys the well-known limiting forms

$$\lambda_K/\lambda_{\mathrm{TST}} \Rightarrow \{(1 + \gamma^2/4)^{1/2} - \gamma/2\} \qquad \text{as} \quad \gamma \geqslant \omega_b = 1 \tag{6.7a}$$

and

$$\lambda_K/\lambda_{\mathrm{TST}} \to \tfrac{4}{3} D^{-1} \gamma \qquad \text{as} \quad \gamma \to 0 \tag{6.7b}$$

The result in (6.7a) holds within the steepest descent approximation, while (6.7b) holds within a harmonic-well approximation and large Arrhenius factors $\Delta U/D \gg 1$. The result for $\eta(\Omega)$ in (6.5), evaluated by use of (6.6), is depicted in Fig. 3. We note that SR is *enhanced* for decreasing damping strength, i.e., SR is enhanced for effective noise color ($\propto \gamma^{-1}$) which is increasing. The *increase* of the SR peak height and the *shift* of the maximum toward smaller D values with *decreasing* γ follow from the fact that in (6.7a) the rate is decreasing for increasing γ. Put differently, increasing γ yields a smaller λ_K, which with fixed Ω at the SR condition $\Omega = (\pi/2)\lambda_K$ must be compensated for by a higher value for the noise intensity D_{SR}. In Fig. 3 we have with $\Omega = 0.01$ chosen not to plot SR curves for very small friction values $\gamma \ll 1$. This is so because the weak-noise estimate in (6.6), (6.7b) for $\gamma \to 0$ is reached for very high barriers only, i.e., $\Delta U/D \gg 1$. This in turn would imply very small Ω values for the SR condition to be obeyed within the weak-noise estimate for λ_K in (6.6).

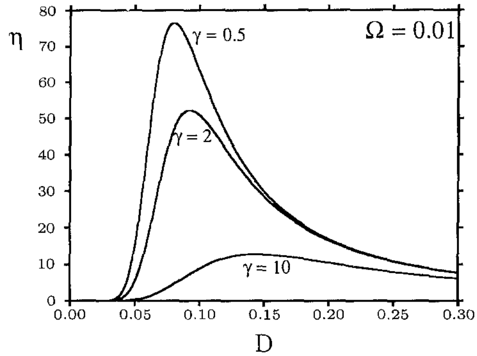

Fig. 3. The signal amplification $\eta(\Omega=0.01)$ for the Kramers equation [Eqs. (6.5) and (6.6)] as a function of the noise strength $D \equiv kT$ for three different damping strengths γ.

APPENDIX A

The linear flow perturbed by Ornstein–Uhlenbeck noise and periodic driving, i.e.,

$$\begin{aligned} \dot{x} &= -\alpha x + \varepsilon - \frac{\mathscr{A}}{\Omega}\cos(\Omega t + \varphi) \\ \dot{\varepsilon} &= -\frac{\varepsilon}{\tau} + \frac{1}{\tau} D^{1/2}\xi_w(t) \end{aligned} \tag{A.1}$$

is exactly solvable. By differentiating $\dot{x} \to \ddot{x}$ we find a periodically driven stochastic oscillator, i.e.,

$$\ddot{x} = -\gamma\dot{x} - \omega^2 x + \mathscr{A}\sin(\Omega t + \varphi) + (kT\gamma)^{1/2}\,\xi_w(t) \tag{A.2}$$

with

$$\gamma = \alpha + (1/\tau) \tag{A.3a}$$

$$\omega^2 = \alpha/\tau \tag{A.3b}$$

$$kT = (D/\tau^2)(\alpha + 1/\tau)^{-1} \tag{A.3c}$$

With these substitutions, the Floquet eigenvalues, Floquet functions, and asymptotic probabilities can be readily read off from the explicit results given for (A.2) in ref. 14. For example, the Floquet eigenvalues are given by

$$\lambda_{nm} = n\alpha + m(1/\tau) \tag{A.4}$$

$n, m = 0, 1, \ldots$. The time-periodic, asymptotic probability $p_{as}(x, t; \varphi)$ is a Gaussian given by

$$p_{as}(x, t; \varphi) = \left(\frac{\alpha(\alpha\tau+1)}{2\pi D}\right)^{1/2} \exp\left\{-\frac{\alpha(\alpha\tau+1)}{2D} \times [x - \mathscr{A}(\omega_0)\sin(\Omega t + \varphi + \hat{\varphi})]^2\right\} \tag{A.5}$$

with

$$\mathscr{A}(\Omega) = \frac{\mathscr{A}}{[(\alpha/\tau - \Omega^2)^2 + \Omega^2(\alpha + 1/\tau)^2]^{1/2}}; \qquad \tan\hat{\varphi} = -\frac{\Omega(\alpha\tau+1)}{\alpha - \Omega^2\tau} \tag{A.6}$$

APPENDIX B

For systems with internal exponentially correlated noise, i.e., a system which is in equilibrium with a heat bath in the absence of a periodic force, the Langevin equation reads

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\gamma \int_0^t \phi(t-t')\, v(t')\, dt' - \frac{dU}{dx} + \mathscr{A}\sin(\Omega t + \varphi) + (\gamma kT)^{1/2}\,\xi(t) \end{aligned} \tag{B.1}$$

where $\langle \xi(t) \rangle = 0$, and

$$\langle \xi(t)\,\xi(t') \rangle = \frac{1}{\tau}\exp\left(-\frac{1}{\tau}|t-t'|\right) \tag{B.2}$$

In order to perform a small-correlation time analysis, we first embed (B.1), (B.2) in the three-dimensional stochastic process

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\gamma z - \frac{dU}{dx} + \mathscr{A}\sin(\Omega t + \varphi) \\ \dot{z} &= -\frac{1}{\tau} z + \frac{1}{\tau} v - \frac{1}{\tau}\left(\frac{kT}{\gamma}\right)^{1/2} \xi_w(t) \end{aligned} \tag{B.3}$$

Expanding the solution of the corresponding Fokker–Planck equation into the complete set of Hermite functions and neglecting all terms which are of order τ^n, $n > 1$, as well as transient effects, we obtain for the Fokker–Planck-type equation for the reduced probability $p(x, v, t) = \int_{-\infty}^{\infty} dz\, p(x, v, z, t)$ the result

$$\dot{p}_t = \left\{ -v \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial v} v(1 + \gamma\tau) + \frac{\partial}{\partial v} \left[\frac{dU}{dx} - \mathscr{A} \sin(\Omega t + \varphi) \right] \right.$$
$$\left. + \gamma kT(1 + \gamma t) \frac{\partial^2}{\partial v^2} + \gamma\tau kT \frac{\partial^2}{\partial x\, \partial v} \right\} p_t \tag{B.4}$$

The stationary solution for $\mathscr{A} = 0$ is given by the canonical form

$$p_{\mathrm{st}}(x, v) = Z^{-1} \exp\{ -[U(x) + \tfrac{1}{2}v^2]/kT \}$$

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