Power Spectrum of a Driven Bistable System.

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Abstract. – A periodically driven bistable system is considered in the region of validity of a two-state hopping approximation. In the limit of weak noise the hopping process becomes discrete not only in space but also in time so that an explicit evaluation of the autocorrelation function becomes possible. This function turns out to be a *non-analytic* function of time leading to a rather peculiar structure of the power spectrum. Results are compared with numerical and experimental data.

A periodically driven bistable system (DBS) serves as an archetype model for a variety of physical processes where non-linearity, noise and deterministic external perturbations are to be accounted for simultaneously. Such processes can include isomerization reactions catalized by light [1], propagation of signals in sensory neurons [2], operation of a ring laser [3], to name only a few. Although the properties of a DBS are presently discussed mainly in the context of a phenomenon named *stochastic resonance* [4], it definitely has much broader application. The attractive feature of a bistable system as a model of a real process is the presence of two essentially different time scales. The first time scale is of dynamical origin and is related to relaxation of the system around the stable minima. The second time scale, being much larger, is due to hopping between the minima and is determined by the intensity of the noise. The two mentioned time scales are well separated only in case the noise is weak. This separation ensures the «universality» of the model: most of its properties become insensitive to the details of the stochastic dynamics (such as the shape of the bistable potential), but depend only on the hopping intensity. In addition, in this case one can hope to obtain *explicit* (often elementary) expressions for the main characteristics of the stochastic process, being an important perspective for a problem that in general case cannot be solved exactly.

Another aspect of the separation of the time scales in the weak-noise limit is the extremely sharp sensitivity of the hopping process to the momentary (time-dependent) shape of the bistable potential. Here one can expect that some results may exhibit discontinuities, cusps and other non-analytic features. This contrasts with other approaches (*e.g.*, the linear-

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response treatment [5]) where a smooth dependence on time and other parameters is usually observed. As a non-trivial example of such non-analytic behaviour, one can mention the decay of a driven metastable state [6]. Here, instead of an intuitively assumed smooth deviation from a quasi-adiabatic regime with the increase of the driving frequency, one observes an abrupt breakaway from this regime when the frequency reaches some threshold value. In the present study of a DBS we will show that the non-analyticity of the autocorrelation function in the limit of weak noise provides an understanding of certain peculiarities (dips or peaks) in the power spectrum. These characteristic features have previously been observed in some of the analogous simulations in the presence of weak noise [2*a*,7] and were described therein as «unexplained» or «strange» [2*a*, 7*b*-*d*].

Consider a Langevin equation with a Gaussian white-noise source

$$\dot{x}(t) = -\partial U^{(0)}(x)/\partial x + A\sin\Omega t + \xi(t), \qquad \langle \xi(t)\xi(t')\rangle = 2D\delta(t-t'). \tag{1}$$

The unmodulated potential $U^{(0)}(x)$ is assumed to be symmetric and to have two minima at $x_\pm^{(0)}$, respectively. The modulation amplitude A is expected to be sufficiently small to ensure the bistability of the time-dependent potential $U(x;t) = U^{(0)}(x) - Ax\sin\Omega t$ possessing minima, respectively, at $x_\pm(t)$. In the limit of weak noise the particle has the best chance to escape to the neighbouring minimum at the instant when the corresponding barrier is the smallest, and otherwise it may be located in the direct vicinity of $x_+(t)$ or $x_-(t)$. Thus, to a very good approximation (we discuss the limitations of such an approximation below) one has a hopping process which is discrete *both* in space and in time, *i.e.*

$$\dot{n}_\pm = W_\mp(t)\,n_\mp - W_\pm(t)\,n_\pm\,, \tag{2}$$

with

$$W_\pm(t) = \alpha\delta\left(t - m \pm \frac{1}{4}\right), \qquad |m| = 0, 1, \ldots \tag{3}$$

(time is measured in the units of the modulation period). Here $n_\pm(t)$ denote the probability to occupy the corresponding site, while $W_\pm(t)$ are the escape rates which can be obtained as the Kramers rate for the instantaneous configuration of the potential $U(x;t)$ [8]. In this approximation, also known as the quasi-adiabatic approximation, the values of α in eq. (3) can be readily evaluated [9,10]. Introducing the *maximal* Kramers frequency $\omega_{\mathrm{K}}^{\max}$ corresponding to the most shallow well and neglecting the slight modulations of the equilibrium positions, one estimates α to be $(D/Ax_+^{(0)})^{1/2}\omega_{\mathrm{K}}^{\max}/\Omega$. Mostly, we will be interested in $\Omega > \omega_{\mathrm{K}}^{\max}$, in which case α is small and has the meaning of the hopping probability. Nevertheless, the results are expected to be applicable for $\alpha \sim 1$ as well. The breakdown of the approach is anticipated only for exponentially large values of α when the driving frequency is comparable with the *unmodulated* Kramers frequency corresponding to $A = 0$. In such cases there exists a finite probability that the particle will hop to another minimum during the intermediate stages of modulation, not «waiting» for the shallowing of the corresponding well. For such small frequencies—which will not be discussed in the present study—eq. (3) is to be replaced by a smooth (analytic) function of time which would bring the problem closer to the treatment of McNamara and Wiesenfeld [5*a*]. The opposite, large-frequency limitation of our treatment comes from the restricted validity of the quasi-adiabatic approximation in eq. (2). The frequencies considered should be small compared with the intrawell frequencies, $\sim \partial^2 U/\partial x^2$ at $x = x_\pm^{(0)}$. Nevertheless, we note that the region of validity of the forthcoming consideration is *asymptotically large* in the weak-noise limit. In practice it covers several orders of magnitude above the Kramers frequency.

Having stated the expected region of validity of eqs. (2) and (3), we proceed to our main concern—the autocorrelation function

$$S(t, \tau) \equiv \langle [x(t+\tau) - \langle x(t+\tau)\rangle][x(t) - \langle x(t)\rangle]\rangle . \tag{4}$$

From this definition and eqs. (2), (3) one obtains

$$S(t, \tau) = [x_+(t) - x_-(t)][x_+(t+\tau) - x_-(t+\tau)][1 + \exp[-\alpha]]^{-2} \exp[-2\alpha(\tau + 1/2)] \cdot$$
$$\cdot \exp\left[\alpha\left(\left\{t + \frac{1}{4}\right\} - \left\{t + \tau + \frac{1}{4}\right\} + \left\{t - \frac{1}{4}\right\} - \left\{t + \tau - \frac{1}{4}\right\}\right)\right], \tag{5}$$

where $\{...\}$ denotes the fractional part of a real-value number. One can see that as a function of τ the autocorrelation function has factorised into an exponentially decaying part, and a periodic part. In such cases one would normally expect the corresponding spectral density to consist of a set of Lorentzian-type peaks and a monotonously decaying background. The non-trivial point, however, is that the periodic part containing the fractional parts is a *non-analytic* function of τ, *i.e.* it cannot be represented by any *finite* number of Fourier harmonics. Thus, a rather peculiar structure of the spectrum is anticipated. To demonstrate this more explicitly we, for the moment, replace the analytic part of the periodic factor, *i.e.* the product of the distances between the equilibrium positions in eq. (5), by a constant $(x_+ - x_-)^2$. The contribution of the time dependence of this factor which becomes important at large frequencies will be discussed below later. Upon averaging the resulting expression of $S(t, \tau)$ over time t, one ends up with

$$\bar{S}(\tau) = \frac{\exp[-\alpha(2\tau + 1)]}{(1 + \exp[-\alpha])^2} (x_+ - x_-)^2 \exp[\alpha\{2\tau\}](1 - \{2\tau\}(1 - \exp[-\alpha])) . \tag{6}$$

Note that, although the averaging removed the discontinuities of eq. (5), eq. (6) is still «weakly non-analytic», having a cusp at each integer value of 2τ. This is clearly seen from

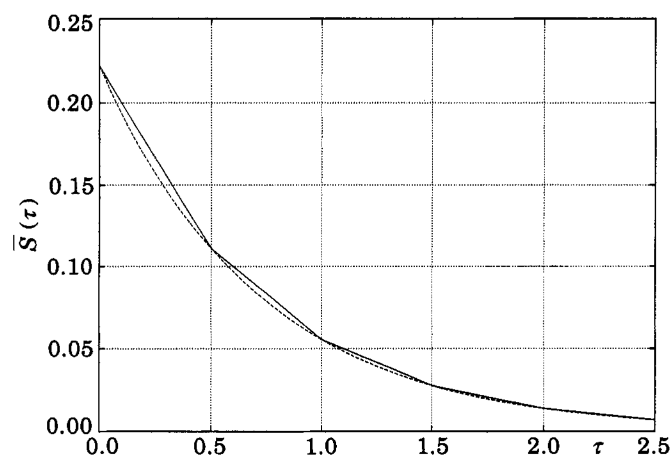

Fig. 1. – The «non-analytic» shape of the phase-averaged autocorrelation function (solid line), eq. (6), with $x_+ - x_- = 1$ and $\alpha = \ln 2$. The smooth dashed line is a guide for the eye.

fig. 1. We next consider the spectral density $\bar{S}(\omega)$. When performing the Fourier transformation of eq. (6) one can integrate within a period with subsequent summation of a geometric progression. One thus obtains after some manipulations

$$\bar{S}(\omega) = (x_+ - x_-)^2 \frac{\Omega \exp[-\alpha] \operatorname{tgh}(\alpha/2)}{\pi\omega^2} \frac{1 - \cos(\pi\omega/\Omega)}{1 - 2\exp[-\alpha]\cos(\pi\omega/\Omega) + \exp[-2\alpha]}, \quad (7)$$

where we explicitly restored the driving frequency Ω, instead of the previously used value $\Omega = 2\pi$. This result is general and is valid for any system whose dynamics can be reduced to eqs. (1)-(3) with a negligible time dependence for the difference $x_\pm(t)$. Specific properties of a system are contained solely in the parameter α. For small α, on average the spectral density decays proportional to ω^{-2}, with the values of the constant being independent of the driving frequency Ω. In the extreme situation, $\alpha \to 0$, eq. (7) describes, accurate to a factor, a delta-function of ω. Evidently, when hopping is completely forbidden, the system never equilibrates. In this case, in neglect of a slight modulation of the equilibrium positions, the autocorrelation function remains constant—see eq. (6)—leading to the afore-mentioned delta-shaped spectral density. Alternatively, in case of a fast equilibration (large α) one has $\bar{S}(\omega) \sim \exp[-\alpha](1 - \cos(\pi\omega/\Omega))/\omega^2$. The system completely equilibrates after the first hopping, *i.e.* after one-half of the modulation period the autocorrelation function has practically relaxed to zero already. This sharp cut-off explains the dependence on ω. Non-zero contributions to $S(t, \tau)$ as defined by eq. (5) come from those initial conditions at $\tau = 0$ when the particle is placed in the «wrong» (shallow) well—otherwise the system is in equilibrium from the start. The probability of such a wrong placement is, however, exponentially small, which explains the factor $\exp[-\alpha]$ in $\bar{S}(\omega)$. For *arbitrary* values of α a general property of a driven bistable system, which is given explicitly by the main result (7), is that the spectral density has zero values at even multiples of the driving frequency. These

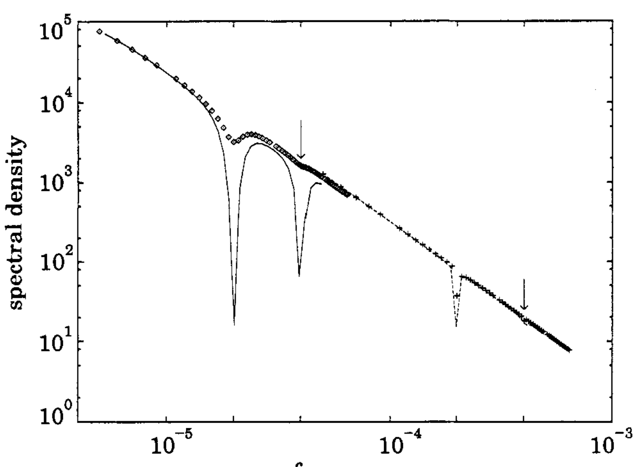

Fig. 2. – Spectral density $\bar{S}(\omega)$ for two different values of the driving frequency Ω and the driving amplitude $A = 0.121$. The noise intensity D equals 1/70. Solid line: eq. (7) for $\Omega = 10^{-5}$, dashed line: eq. (7) for $\Omega = 10^{-4}$; symbols: numerically exact data for a modulated quartic potential. Arrows indicate small dips at $\omega = 4\Omega$. Note the overlap of numerical data for $\Omega = 10^{-5}$ (diamonds) and $\Omega = 10^{-4}$ (crosses) around $\omega = 5 \cdot 10^{-5}$, where, in agreement with the theory, the data do not depend on the driving frequency. The finite size of the dips, which from eq. (7) are infinitely deep, is a plotting artifact.

zeros imply unlimitedly deep *dips* in the logarithmic scale. This property is a direct consequence of the mentioned non-analyticity of the autocorrelation function. Below we will assess the applicability of our conclusion to realistic situations with small but finite noise. At the moment we wish to compare eq. (7)—which is surprisingly simple taking into account the general complexity of the problem—with numerically exact data for $\bar{S}(\omega)$.

We consider a specific example of a quartic potential with $U^{(0)}(x) = -x^2/2 + x^4/4$, yielding $x_{\pm}^{(0)} = \pm 1$. The corresponding Fokker-Planck equation was solved using the matrix continued-fraction method [11]. Numerical results are plotted against the analytical expression in fig. 2. The agreement is good both on the qualitative and the quantitative levels, except for the fact that the dips are finite and are observed only for the two lowest harmonics of the driving frequency. To understand this deviation recall that the infinite dips arise due to the non-analyticity of $\bar{S}(\tau)$ which in turn is due to the discrete nature of the hopping process in the limit of weak noise. Increasing of noise destroys the non-analyticity as the δ-peaks of the transition probability in eq. (3) acquire a finite width $\delta t \sim (A/D)^{-1/2}\Omega^{-1}$. Nevertheless, the «fingerprints» of this non-analyticity—the rapid change of the derivative of $\bar{S}(\tau)$ and hence the (finite) dips in $\bar{S}(\omega)$—will still be detectable, provided one considers the low-frequency part of the spectrum with $\omega \ll (\delta t)^{-1}$. On the other hand, for higher frequencies the resolution of the spectral analysis will be too fine to notice the «non-analyticity» of the autocorrelation function so that no dips are expected. An elementary estimation [12] gives the condition of existence of the m-th dip: $A/D \gg (2m)^2$. In a realistic experimental or computational situation one typically has $A/D \sim 10^1$, or less; therefore not more than two dips at $\omega = 2\Omega$ and $\omega = 4\Omega$ may be observed. This finding is in accord with experimental studies [2*a*] and [7*b*-*d*]. Moreover, to observe at least one dip, one needs a sufficiently large ratio of A/D, which explains why the dips *vanish* with increasing of noise. This behaviour was observed experimentally and was often considered as an indicator that the dips are an artifact of the measurement technique [13].

We now briefly consider the modulation of the equilibrium positions. One can show that after averaging this effect becomes very small, being proportional only to the fourth (!) power of the driving amplitude [12]. Nevertheless, this modulation leads to a qualitatively new effect—*peaks* at even multiples of the driving frequency. The major peak at $\omega = 2\Omega$ has a width $\delta\omega \sim \omega_K^{max}(D/A)^{1/2}$ and a height $\sim A^4/\delta\omega$. This peak will prevail over the corresponding dip in the direct vicinity of 2Ω in case the latter is sufficiently large, *i.e.* $\Omega \gg \omega_K^{max} D^{1/2} A^{-3/2}$, but otherwise eq. (7) remains applicable. Note that the width of the peak is extremely narrow since it is determined by the Arrhenius factor. Thus, in practice they may be easily mistaken for the delta-peaks which, however, are *forbidden* by symmetry rules [5*c*].

The numerical solution of the Fokker-Planck equation for a modulated quartic potential discussed above exhibits peaks in the spectral density at the second multiple of the driving frequency in complete accord with the analytical prediction. Peaks at even harmonics were observed experimentally in ref. [7*d*], although these were associated exclusively with the coloured noise therein. From the above treatment one can see that, in principle at least, peaks may be present in the case of white noise as well. An explanation of why similar peaks were not observed in other analogous simulations [2*a*, 7*b*,*c*] may be contained in the fact that in contrast to ref. [7*d*] those studies employed the two-stage filtering technique [13]. Most likely, two-stage filtering which records only the hopping events destroys the peaks which arise due to a delicate correlation of hopping with the modulation of the equilibrium positions.

In the present work we have shown that for a periodically driven bistable system the weak-noise consideration provides an accurate expression for the spectral density of the autocorrelation function which in practice ranges over several decades of magnitude of the driving frequency—see fig. 2. The weak-noise treatment also explained the «strange» dips and peaks at even harmonics, previously observed in experimental studies. The *dips* emerge due to the non-analyticity of the autocorrelation function in the weak-noise limit—see fig. 1. Traces of this non-analyticity can be detected even with increasing noise making the dips at the lowest even harmonics rather realistic. However, there exists a threshold value for the ratio of the noise intensity to the driving amplitude above which no dips are possible. On the other hand, *peaks* occur due to correlation of the modulation in the equilibrium positions with the hopping process. Thus, they represent a more «delicate» phenomenon compared to dips which emerge exclusively due to hopping. In this sense, peaks are less universal than dips, and are more vulnerable to the peculiarities of the measurement technique, such as the two-stage filtering.

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