Non-Markovian process driven by quadratic noise: Kramers-Moyal expansion and Fokker-Planck modeling

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A linear non-Markovian process driven by additive exponentially correlated Gaussian *quadratic* noise is considered. An exact master equation for a probability distribution of the process is presented. A Kramers-Moyal-type expansion of the master equation is studied. Some limiting cases of the expansion are investigated. It is demonstrated that any finite truncation of the expansion fails. An alternative and correct Fokker-Planck modeling is constructed.

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I. INTRODUCTION

A modeling of dynamics of noise-driven systems by stochastic differential equations has been the subject of intensive studies in the past decades (it is impossible to quote all papers; for recent surveys see Refs. [1,2]). For one-dimensional systems, it is given by an equation of the form

$$\frac{dx}{dt}=F(x,t;\xi(t)) , \qquad (1)$$

where $x=x(t)$ is a "relevant" variable characterizing the system and $\xi(t)$ is noise (a random perturbation). If $\xi(t)$ is taken to be *white noise*, then the function F in (1) has to be linear in $\xi(t)$ [otherwise Eq. (1) makes no sense]. Then $x(t)$ is a Markovian process and its dynamics is determined by two quantities: its initial probability distribution $p(x,0)$ and its conditional distribution $p(x,t|x_0,s)$, $t\geq s$. If $\xi(t)$ enters nonlinearly in (1), then $\xi(t)$ must not be white noise, implying that $x(t)$ is a non-Markovian process. Its dynamics is determined not only by $p(x,t|x_0,s)$ and $p(x,0)$, but also by all finite-dimensional distributions.

Nonwhite- (colored-) noise-driven processes are described by equations that cannot be analyzed analytically for general cases. Therefore a variety of approximation methods have been proposed [3–5], most of which are based on an expansion of some expressions and a truncation of the series obtained. When dynamics is described by differential equations with a linear noise term, then the white-noise limit exists and the equations simplify. The white-noise limit is the simplest test of correctness of the approximations imposed. If the noise term is a nonlinear function of noise, then the white-noise limit does not exist in general and a simple criterion of testing the validity of applied approximations is not yet available.

In most of papers on colored-noise-driven systems, a linear noise case has been investigated [1,3–5]. Nonlinear noises have been rather seldom treated. In [6,7], limit theorems on convergence of the process (1) to a diffusion process were presented. This asymptotic theory of stochastic differential equations [6,7] is founded on some general properties of processes such as, e.g., an asymptotic independence property (strong mixing) and it has found a great number of applications in different branches of science. In [8–11], processes driven by quadratic noises were treated, but not with full mathematical rigor. In the paper we consider one of the simplest differential equation with a nonlinear noise term, namely, a first-order differential equation with a nonlinear deterministic part and an additive quadratic noise part composed of colored noise in the form of an exponentially correlated Gaussian process (an Ornstein-Uhlenbeck process) [8–10,12]. By use of this model one can discuss all significant problems concerning, among others, (i) a Kramers-Moyal expansion [13,14] of a master equation for $x(t)$ and (ii) the existence of limiting cases and correctness of Markovian as well as diffusion approximations.

The paper is organized as follows. In Sec. II we present a model and an exact master equation for $p(x,t|x_0,0)$ with its explicit solution. The master equation has the same structure as a space-nonlocal diffusion equation [15]. In Sec. III a Kramers-Moyal expansion of the master equation is derived and a few coefficients of it are depicted explicitly. The limits of a weak intensity and a short correlation time of noise is considered in Sec. IV. Section V is devoted to the problem of a diffusion approximation of the process. It is demonstrated that a truncation of the Kramers-Moyal expansion yields an incorrect description. The correct construction of the Fokker-Planck approximation is presented in Sec. VI. Final re-

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marks, conclusions, and a summary are contained in Sec. VII.

II. MODEL

The model we are considering is a linear dynamically stable flow given by the equation [9,10,12]

$$\frac{dx}{dt}=-ax+\xi^2(t), \quad x(0)=x_0\in R^1 , \tag{2}$$

where $x\in \mathrm{R}^1$, $a>0$, and $\xi(t)$ is an exponentially correlated Gaussian process (an Ornstein-Uhlenbeck process) with the properties

$$\langle \xi(t)\rangle=0, \quad \langle \xi(t)\xi(0)\rangle=\alpha D\exp(-\alpha t) , \quad \alpha, D>0 . \tag{3}$$

The parameter $\tau_c=1/\alpha$ is a correlation time of the noise and D denotes its intensity.

In view of possible applications of Eq. (2), let us mention input-output systems [8] as they occur in engineering sciences, thermoelectrical instruments [9], turbulent fluid flows [16], or quantum optics [17]. The stochastic process $x(t)$ in (2) is non-Markovian, but the pair $\{x(t),\xi(t)\}$ constitutes a Markovian process with a degenerate diffusion tensor. The statistics of the pair process is determined by a two-dimensional Fokker-Planck equation, which is, however, not tractable. An evolution equation for the conditional probability distribution $p(x,t|x_0 0)$ of the process $x(t)$ alone has been derived in a previous paper [18]. However, its explicit form was presented for the case $a=4\alpha$ only. Thus we could not carry out a systematic analysis for the general case. Recently, however, using the same methods as in [18], we have succeeded in obtaining an evolution equation for $p(x,t|x_0,0)$, valid for all values of parameters a, α, and D. It has the explicit form

$$\frac{\partial}{\partial t}p(x,t|x_0,0) =\frac{\partial}{\partial x}(ax-\alpha D)p(x,t|x_0,0) +\alpha D\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}H(x-y,t)p(y,t|x_0,0)dy , \tag{4}$$

where

$$H(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{H}(k,t)e^{-ikx}dk , \tag{5}$$

$$\hat{H}(k,t)=\frac{1}{ik}\,\frac{J_{\nu-1}(s_k)J_{-\nu+1}(s_k e^{-at/2})-J_{-\nu+1}(s_k)J_{\nu-1}(s_k e^{-at/2})}{J_{\nu+1}(s_k)J_{-\nu+1}(s_k e^{-at/2})-J_{-\nu-1}(s_k)J_{\nu-1}(s_k e^{-at/2})} , \tag{6}$$

$$s_k=2\nu\sqrt{iDk} , \quad \nu=-2\alpha/a , \tag{7}$$

and $J_\nu(z)$ stands for the ordinary Bessel function.

Equation (4) is exact. Its interesting structure can be interpreted as follows. The first term of the right-hand side of (4) is the drift with a noise-induced part αD. This noise-induced part is solely due to the nonvanishing second mean [cf. (2)] with $\langle \xi^2(t)\rangle=\alpha D$. Thus the conditional average in (2) readily yields the same drift. If the kernel $H(x,t)$ were proportional to the Dirac δ function, i.e., $H(x,t)\sim\delta(x)$, then the second term would be purely diffusional. Hence $H(x,t)$ can be termed a space-nonlocal diffusion function and Eq. (4) could be interpreted as *a space-nonlocal diffusion equation* [15]. With the diffusion operator being nonlocal one expects at the same time a retarded (time-nonlocal) evolution over time [19]. Note, however, that the operator $\hat{H}(k,t)$ intrinsically corresponds to a time-convolutionless (but not memoryless) form of the evolution operator [20].

Equation (4) seems to be complicated. Nevertheless, it can be solved. A Fourier transform of $p(x,t|x_0,0)$ (i.e., its characteristic function) obeys a first-order partial differential equation of two variables and it can be solved by the usual method of characteristics. But here we need not do this because the explicit form of $p(x,t|x_0,0)$ is known [12,18] (it has been obtained by use of the "curtailed" functional method) and full one-dimensional dynamics of the system can be determined. Hence we can test various previous approximative methods put forward in the literature [8–10]. To make the paper self-contained, let us present the solution of Eq. (4). It explicitly reads

$$p(x,t|x_0,0)=\frac{\nu}{\pi}[\sin(\pi\nu)/\pi\nu]^{1/2} \times\int_{-\infty}^{\infty}dk\,\exp[-ik(x-x_0e^{-at})+(2\alpha+a)t/4] \times s_k^{-1}[J_{\nu+1}(s_k)J_{-\nu+1}(s_k e^{-\alpha t/2})-J_{-\nu-1}(s_k)J_{\nu-1}(s_k e^{-at/2})]^{-1/2} . \tag{8}$$

Representative cases of Eq. (8) are illustrated in [18].

III. KRAMERS-MOYAL EXPANSION

A Kramers-Moyal expansion [13,14] of the master equation (4) can be obtained by use of the property of commutativity of convolution in (4) and utilizing the identity

$$f(x-y)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}y^n\frac{d^n}{dx^n}f(x) . \tag{9}$$

Then Eq. (4) can be rewritten as

$$\frac{\partial}{\partial t}p(x,t|x_0,0)=\frac{\partial}{\partial x}(ax-\alpha D)p(x,t|x_0,0)+\alpha D\sum_{n=2}^{\infty}K_n(t)\frac{\partial^n}{\partial x^n}p(x,t|x_0,0)\ , \qquad (10)$$

where

$$K_n(t)=\frac{i^{n-2}}{(n-2)!}\frac{\partial^n}{\partial k^n}\hat{H}(k,t)|_{k=0}\ . \qquad (11)$$

Although the expansion coefficients $K_n(t)$ are given formally upon combining (11) and (6), the calculation of their explicit form presents a hopeless task. To gain insight into their structure, here we present the first three higher-order generalized diffusive terms, i.e.,

$$K_2(t)=2\alpha D\frac{1-e^{-(a+2\alpha)t}}{a+2\alpha}\ . \qquad (12)$$

$$K_3(t)=-4(\alpha D)^2\frac{a-2(a+\alpha)e^{-(a+2\alpha)t}+(a+2\alpha)e^{-2(a+\alpha)t}}{a(a+\alpha)(a+2\alpha)}\ , \qquad (13)$$

$$K_4(t)=8(\alpha D)^3(a^2/C)F(t)\ , \qquad (14)$$

where

$$C=2(a+2\alpha)^2(a+\alpha)(3a+2\alpha)(2\alpha-a)\ , \qquad (15)$$

$$F(t)=(5\nu-6)(\nu+1)-(\nu^2-5\nu+6)e^{-2a(1-\nu)t}-(2\nu^4-11\nu^3+14\nu^2+9\nu-18)e^{-a(1-\nu)t}+(4\nu^4+16\nu^3-8\nu^2-16\nu+12)e^{-a(2-\nu)t}-(2\nu^4-5\nu^3-2\nu^2+11\nu-6)e^{-a(3-\nu)t} \qquad (16)$$

with ν defined in (7).

The higher-order terms are very complex. With our interest being the long-time asymptotics of (10), we ask the challenging question, is it possible to rescale Eq. (10) in such a way that any nontrivial limit exists which reduces Eq. (10) to a simpler form? With a nontrivial limit we mean that not only the drift term in (10) survives, but also a diffusive term so that K_2 does not vanish.

IV. LIMITING BEHAVIORS

In the long-time limit, i.e., $K_n(t)\to\bar{K}_n$ as $t\to\infty$, we obtain

$$\bar{K}_2=\frac{2\alpha D}{a+2\alpha}\ , \qquad (17)$$

$$\bar{K}_3=-\frac{4(\alpha D)^2}{(a+\alpha)(a+2\alpha)}\ , \qquad (18)$$

$$\bar{K}_4=8(\alpha D)^3\frac{3a+5\alpha}{(a+\alpha)(3a+2\alpha)(a+2\alpha)^2}\ . \qquad (19)$$

From the general expression in (11) one can infer that $\bar{K}_n\sim(-\nu D)^{n-1}$ with ν in (7).

A. Limit of weak noise

If D is sufficiently small, i.e., $D\ll 1$, then from (10) we get an asymptotic expansion [21] in powers of D, that is,

$$\frac{\partial p}{\partial t}\sim\frac{\partial}{\partial x}(ax-\alpha D)p+\sum_{n>1}D^nf_n(\alpha,a)\frac{\partial^n p}{\partial x^n}\ , \qquad (20)$$

where $p\equiv p(x,t|x_0,0)$ and $f_n(\alpha,a)$ are rational functions of α and a [cf. Eqs. (17)–(19)].

B. Limit of short correlation time

For this case, when $\alpha\to\infty$ (or $\tau_c\to 0$), one finds that

$$\frac{\partial p}{\partial t}\sim\frac{\partial}{\partial x}(ax-\alpha D)p+\alpha\sum_{n>1}g_nD^n\frac{\partial^n p}{\partial x^n}\ , \qquad (21)$$

where g_n are whole numbers. The absolute values of $\{g_n\}$ increase as n increases. In particular $g_2=1, g_3=-2$, and $g_4=5$. Note that *all higher-order Kramers-Moyal diffusive terms are of first order in* α.

C. Weak-intensity short correlation time of noise

In the limit $D\to 0$ and $\alpha\to\infty$ so that $\alpha D=\gamma=$const, Eq. (10) reduces to a pure drift equation, that is,

$$\frac{\partial p}{\partial t}=\frac{\partial}{\partial x}(ax-\gamma)p\ . \qquad (22)$$

We note that in agreement with the remarks stated in the Introduction, the limit in (IV B) does not exist for $\alpha\to\infty$, except for the case when a is proportional to α, $a\sim\alpha$. Indeed, rescaling time $t\to at$ yields an equation of a form similar to (21). In this case, the Kramers-Moyal expansion does not reduce to a simpler form. It contains an infinite number of terms and it is not even an asymptotic expansion. Therefore the only candidate for further consideration is the case in (IV A) and Eq. (20). Because of the Pawula theorem [22], one may consider at most three cases: (i) a truncated version of (20) with a drift term only, (ii) a truncated version of (20) with a drift and K_2 terms, and (iii) the full expansion (20) with infinite number of terms. The first case corresponds to the Dirac δ distribution of the deterministic process $y(t)=\langle x(t)\rangle$,

$$\frac{dy(t)}{dt}=-ay(t)+\alpha D\ . \qquad (23)$$

We are not interested here in the third case because we know the exact solution (8) of Eq. (10). Thus we consider the usual Kramers-Moyal-Fokker-Planck truncation at second order only.

V. DIFFUSION-TYPE TRUNCATION

For the second case (ii), a probability density $P \equiv P(x,t|x_0,0)$ is determined by the equation

$$\frac{\partial}{\partial t}P = \frac{\partial}{\partial x}(ax - \alpha D)P + \alpha D K_2 \frac{\partial^2}{\partial x^2}P \ . \tag{24}$$

We ask whether (24) is an approximate form of Eq. (10) and whether (24) describes approximately the original process $x(t)$ in (2). To answer this question, let us consider a solution of Eq. (24) with appropriate boundary conditions. To construct them correctly, let us notice that from (2) it follows that for any realization of the noise,

$$x(t) = x_0 e^{-at} + \int_0^t e^{-a(t-s)} \xi^2(s) ds \ . \tag{25}$$

The integrand in (25) is non-negative. Hence

$$x(t) \ge x_0 e^{-at} \quad \text{for } t \ge 0 \ . \tag{26}$$

Therefore for any probability distribution characterizing $x(t)$, the relation

$$P(x,t|x_0,0) = 0 \quad \text{for } x < x_0 e^{-at} \tag{27}$$

should hold. Of course it is not possible to solve (24) with the boundary condition (27). Instead, let us focus on its stationary solution. In this case

$$\frac{d}{dx}(ax - \alpha D)P_{\rm st}(x) + \alpha D K_2 \frac{d^2}{dx^2} P_{\rm st}(x) = 0 \ ,$$
$$P_{\rm st}(x) = 0 \quad \text{for } x < 0 \ . \tag{28}$$

A solution of the problem (28) reads

$$P_{\rm st}(x) = N e^{-B(x)} \int_0^x e^{B(y)} dy, \quad x \ge 0 \ , \tag{29}$$

where

$$B(x) = \frac{(ax - \alpha D)^2}{2a\alpha D K_2} \ . \tag{30}$$

The constant N is a normalization factor. Unfortunately, the solution (29) is non-normalizable and therefore should be ruled out in the case considered. So, the problem (24) with the boundary condition (27) does not have a solution in the class of normalizable distributions. One can try to save the truncation (24) by use of natural boundary conditions on the interval $x \in (-\infty, \infty)$, but then the solution is a Gaussian function on this interval [18] and the probability that $x(t) < x_0 \exp(-at)$ is nonzero, though from (27) it follows that it must be zero. Therefore (24) is unacceptable and such a truncation of a full Kramers-Moyal expansion is incorrect.

VI. FOKKER-PLANCK MODELING

The shortcomings of the above Fokker-Planck approximation obtained by the truncation of the Kramers-Moyal expansion of the master equation are serious. A correct approximation to the master equation (4) by a diffusion parabolic equation can be constructed following Refs. [23,24]. Instead of (24), let us consider an equation of the form

$$\frac{\partial}{\partial t}\tilde{p}(x,t|x_0,0) = \frac{\partial}{\partial x}(ax - \alpha D)\tilde{p}(x,t|x_0,0) + \frac{\partial^2}{\partial x^2} K(x,t)\tilde{p}(x,t|x_0,0) \ , \tag{31}$$

with the diffusion function $K(x,t)$.

A general theory of how to obtain $K(x,t)$ from equations such as (2) or from corresponding two-dimensional Fokker-Planck equations, the form of which is known, has not been elaborated. In our particular case (2), one can observe that Eq. (4) may be rewritten in a form similar to (31) if one defines

$$K(x,t) \equiv K(x,t;x_0,0) = \alpha D \frac{\int_{-\infty}^{\infty} H(x-y,t) p(y,t|x_0,0) dy}{p(x,t|x_0,0)} \ . \tag{32}$$

Because $p(x,t|x_0,0)$ depends on x_0 and the initial time, $K(x,t)$ therefore depends on x_0 and the initial time as well. It may not be named a diffusion function in the usual sense and (31) with (32) is not a Fokker-Planck equation because its coefficients depend on an initial state of the system. This is because the process (2) is non-Markovian. For long times, in states close to stationary, one can approximate the process (2) by a diffusion process, taking a stationary value of the function (32) as a diffusion function,

$$K(x) = \lim_{t \to \infty} K(x,t;x_0,0) \ . \tag{33}$$

Its equivalent form can be obtained from a stationary version of Eq. (31). It reads

$$K(x) = p^{-1}(x) \int_0^x (\alpha D - ay) p(y) dy \ , \tag{34}$$

where

$$p(x) = \lim_{t \to \infty} p(x,t|x_0,0) \tag{35}$$

is a stationary distribution of the process (2), which does not depend on an initial state of the system.

From (34) it follows that

$$K(x) \to 0 \quad \text{as } x \to 0+ \tag{36}$$

and

$$K(x) \to \infty \quad \text{as } x \to \infty \ . \tag{37}$$

The first property is obvious and results from, e.g., the mean value theorem of integral calculus [25]. This property also follows from the observation that the noise should not drive the system in (2) towards negative state values. The property (37) is a consequence of the fact that

$$\lim_{x \to \infty} \int_0^x (\alpha D - ay) p(y) dy = \alpha D - a\langle x \rangle_{\rm st} \equiv 0 \tag{38}$$

and $p(x) \to 0$ as $x \to \infty$.

In Fig. 1 the distribution $p(x)$—which by construction coincides precisely with a stationary probability of the Fokker-Planck approximation in (31) with $K(x,t) \to K(x)$—is shown for $a = 4\alpha$ and three values of

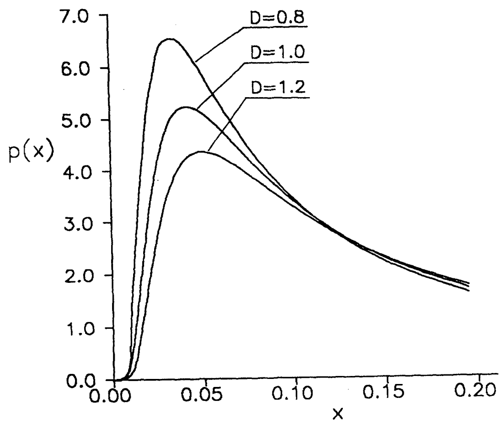

FIG. 1. Some selected examples of the stationary probability distribution (8) for $a=4\alpha$ are depicted at various values of the noise intensity D.

the noise intensity D. In Figs. 2 and 3 the corresponding diffusion function $K(x)$ is presented for small and "large" x, respectively. It is seen that for small x, $K(x)$ is a nonlinear function [a good approximation of it is $K(x)\sim x^2$]. On the other hand, for large x, $K(x)$ seems to be linear in its argument. It is a result of our numerical analysis. Unfortunately, we have not been able to get these findings from analytical evaluation of the Fourier integral in (8) (its integrand is two valued).

Equation (31) with the diffusion function $K(x)$ is related to the Ito stochastic differential equation

$$\frac{d\tilde{x}}{dt}=-a\tilde{x}+\alpha D+\sqrt{2K(\tilde{x})}\eta(t) , \tag{39}$$

where $\eta(t)$ is standard Gaussian white noise with unit intensity. Equation (39) is *an equation with multiplicative noise.* The property (36) guarantees that noise cannot drive the system below the boundary $\tilde{x}=0$ because noise tends to zero as $\tilde{x}\to 0$.

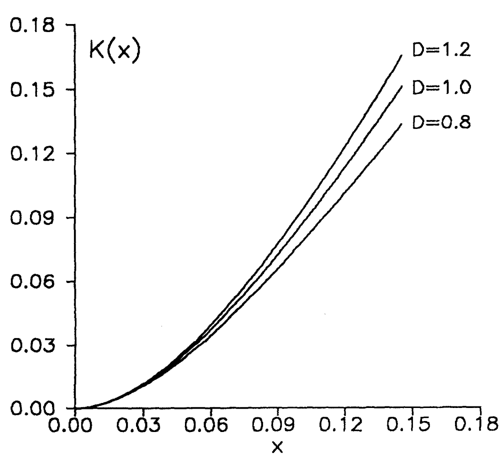

FIG. 2. The diffusion function $K(x)$ for $a=4\alpha$, $\alpha=1$, i.e., $\nu=-\frac{1}{2}$, and various values of the noise intensity D: the small-x case. For fixed $\nu=-2\alpha/a$, $K(x)$ depends linearly on α.

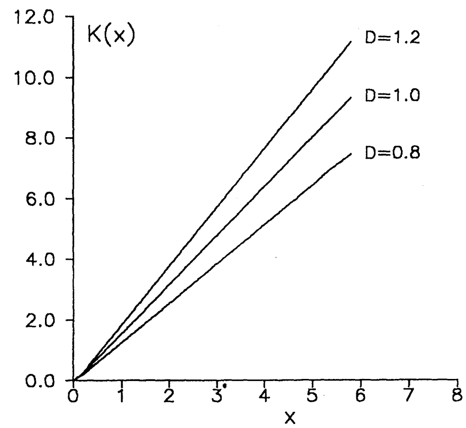

FIG. 3. Same as Fig. 2, but for "large" x.

VII. FINAL REMARKS

In this paper we have studied the non-Markovian process (2) driven by quadratic exponentially correlated Gaussian noise $\xi(t)$. The evolution equations (4) and (10) for its probability density have been constructed. In the Langevin-type equation (2), the phase space of the process $x(t)$ is a real line $\mathbb{R}^1$. The noise $\xi(t)$ causes that the state space of $x(t)$ is restricted to a half line (26) and is bounded from below. This feature is not visible in the master equation (4) or in its Kramers-Moyal expansion (10). It is reflected in the exact solution (8) because the relation (27) holds for (8). Nevertheless, this fact is not so obvious when looking at (8). It would be simple (by applying the Jordan lemma [26]) if one could prove that the integrand in (8) had singularities on the lower half complex plane [18].

We have shown that there does not exist a nontrivial limit that simplifies the Kramers-Moyal expansion (10). The weak noise intensity limit ($D \ll 1$) yields the power series in D and the (deterministic) limit $D\to 0$ is trivial. For short correlation times ($\tau_c \ll 1$), the expansion (10) is not a power series in $1/\tau_c$. The limit $\tau_c\to 0$ does not exist unless the constant a is proportional to $\alpha=1/\tau_c$. Truncation of the Kramers-Moyal expansion after the second term produces a diffusion-type modeling based on the Fokker-Planck equation (24) with a constant (x-independent) diffusion coefficient. This equation was derived and discussed in [10]. If one applies the Stratonovich theorem [6] to Eq. (2), then Eq. (24) is rederived. The same approximation was used in [9].

Truncation of the series (10) at any order higher than the second leads to incorrect description because of the Pawula theorem [22]. So, the problem seems to be unsolved. Fortunately, an alternative Fokker-Planck description is accessible and acceptable [23,24] assuming an x *dependence of a diffusion coefficient* $K(x)$. The function $K(x)$ in (34) guarantees that the stationary distribution and boundary conditions are correctly reconstructed. The diffusion function is fixed by the correct drift term and the correct stationary probability as in [23]. The corrected Fokker-Planck equation (31) corresponds to the Ito equation (39). One can say that the Markovian process $\bar{x}(t)$ defined by (39) is an approximation of the non-Markovian process $x(t)$ in (2).

Unfortunately, a general scheme of construction for the x-dependent diffusion coefficient for general flows driven by nonlinear noise functions has not been found. This task thus remains an open problem which we hope will attract future attention.

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