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Surmounting a fluctuating double well: A numerical study

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For a system which is simultaneously driven by both *white* and *colored* noise, we present results for the stationary probability and the smallest eigenvalue, which are calculated numerically for the exact non-Markovian dynamics. By use of the method of matrix continued fractions we study the stochastic motion in a symmetric double well where the barrier height fluctuates due to colored noise. Our findings can be invoked as a benchmark in testing present and future approximate theories for such *two-noise driven* systems.

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The study of dynamical systems perturbed by noise sources is of wide-ranging significance to the detailed understanding of transport coefficients and the characterization of nonlinear phenomena. In this context, the effects of nonwhite noise sources (i.e., colored noise) has attracted a great deal of interest in recent years [1]. The objective here is the study of a class of stochastic systems that are characterized by the fact that two noise sources act simultaneously on the system dynamics. We shall assume that—apart from thermal noise—additional colored noise (being either of intrinsic or external origin) perturbs the system dynamics. This renders the theoretical analysis rather complex due to the inherent non-Markovian nature of the problem. Due to this non-Markovian character the majority of authors restricted the investigation of the dynamics to the case of a single colored noise source, hence neglecting the influence of an additional (possibly strong) white noise force. The situation of two noise sources acting simultaneously is, however, generic for many applications: A first situation is given by nonlinear optical systems where both colored pump fluctuations and rapidly fluctuating spontaneous emission noise are present [1]. Another situation arises in the actively studied field of "resonance activation" [2] where thermal Nyquist noise together with slow stochastic barrier modulations govern the escape from metastable states. This latter situation is characteristic for Brownian ratchets [3] which exhibit noise induced transport in periodic potentials without reflection symmetry.

In this study we consider the class of stochastic flows which are driven by additive white noise and multiplicative, exponentially correlated Gaussian noise (i.e., Ornstein-Uhlenbeck noise). The relevance of such two-noise driven non-Markovian systems has triggered a large activity among theorists in constructing approximation schemes [4–6] covering limiting and intermediate parameter regimes. In order to *test* such approximation schemes one is in need of precise numerical results which we provide herein by use of matrix continued fraction (MCF) expansions [1,7]. As a concrete model we use the archetypal system of a symmetric bistable Ginzburg-Landau flow, being driven by white thermal noise and colored, symmetric barrier fluctuations of arbitrary correlation strength τ. This non-Markovian bistable dynamics tests, within exponential sensitivity, both the bimodal stationary probability density $W_{st}(x;\tau)$ and the lowest nonvanishing eigenvalue $\lambda_1(\tau)$. This eigenvalue equals twice the rate of reaction.

The stochastic process under investigation is described by a stochastic differential equation (SDE) with two noise forces, i.e.,

$$\dot{x}=f(x)+g(x)\zeta(t)+\sqrt{D}\xi_1(t) \qquad (1)$$

with $f(x)$ in dimensionless variables given by $f(x)=x-x^3$. The symmetric barrier fluctuations are modeled by $g(x)=x$. The term $\zeta(t)$ represents a colored noise with its characteristics given by the Ornstein-Uhlenbeck process

$$\dot{\zeta}=-\frac{\zeta}{\tau}+\frac{\sqrt{Q}}{\tau}\xi_2(t). \qquad (2)$$

In both equations (1) and (2) the stochastic forces $\xi_i(t)$, $i=1,2$, denote Gaussian white noise (with multiplicative noise being interpreted in the Stratonovich sense) of zero mean and correlation $\langle\xi_i(t)\xi_j(s)\rangle=2\delta_{ij}\delta(t-s)$; $i,j=1,2$. Hence, the Gaussian colored noise is exponentially correlated,

$$\langle\zeta(t)\zeta(s)\rangle=\frac{Q}{\tau}e^{-|t-s|/\tau}, \qquad (3)$$

yielding a *constant* integrated intensity $2Q$.

Systematic analytical approaches in the theory of this class of stochastic processes are possible [4,5], but they possess the caveat of being restricted to limited parameter regimes only, e.g., limits of very small or very large correlation times τ. In contrast, approximation schemes that cover a broad parameter regime, such as small-to-intermediate-to-large noise color, are generally not of systematic nature, but nevertheless can yield reliable results. With one colored noise source only, this fact has repeatedly been demonstrated with the use of the so called unified-colored-noise approximation [8]. Following a similar reasoning, a generalization to the situation in (1) has been put forward recently in [6]. The approximation scheme in that latter work has been tailored in such a way that it yields the exact stationary probability for the x dynamics in piecewise parabolic wells. In this sense, a Markovian approximation covering the behavior at small correlation times $\tau\lesssim 1$ can be obtained. The corresponding (Stratonovich) SDE reads [6]

$$\dot{x}=\frac{1}{\tau\gamma(x,\tau)}[x-x^3+x\sqrt{Q}\xi_2(t)+\sqrt{D}\xi_1(t)], \qquad (4)$$

with $\gamma(x,\tau)$ given by

$$\gamma(x,\tau)=\left\{\frac{(1+2\tau x^2)(1+x^2Q/D)}{1+(Q/D+2\tau)x^2}-\frac{2\tau x^2Q/D(1-x^2)}{[1+(Q/D+2\tau)x^2]^2}\right\}\Big/\tau. \quad (5)$$

Given this Markovian approximation we readily can evaluate the stationary probability density or the mean first passage time (MFPT).

In the limit of large noise color $\tau \gg 1$, the corresponding approximation, which again reproduces the exact behavior for parabolic potentials, reads [6]

$$\dot{x}=\frac{(x-x^3)(1+2\tau x^2)}{1+(Q/D+2\tau)x^2}+\sqrt{D}\xi_1(t). \quad (6)$$

For the case in which the white noise approaches zero, i.e., $D \to 0$, both approximations agree with the unified-colored-noise approximation used in previous literature [1,8].

In our numerical calculations we start from the exact equations, (1) and (2), and derive the corresponding (exact) Fokker-Planck equation for the joint probability density $W(x,\zeta;t)$, i.e.,

$$\frac{\partial}{\partial t}W(x,\zeta;t)=\left[-\frac{\partial}{\partial x}(x-x^3+x\zeta)+D\frac{\partial^2}{\partial x^2}+\frac{1}{\tau}\frac{\partial}{\partial \zeta}\zeta+\frac{Q}{\tau^2}\frac{\partial^2}{\partial \zeta^2}\right]W(x,\zeta;t). \quad (7)$$

We are primarily interested in the colored noise dependent stationary density in x, i.e., $W_{st}(x;\tau)$, which follows from the stationary joint density $W_{st}(x,\zeta)=\lim_{t\to\infty}W(x,\zeta;t)$ by

$$W_{st}(x;\tau)=\int_{-\infty}^{\infty}\zeta W_{st}(x,\zeta). \quad (8)$$

This function can be evaluated numerically by use of the method of MCF expansions [1,7]. Another quantity of the system which we can access by means of the latter method is the lowest nonvanishing eigenvalue $\lambda_1(\tau)$ of the time-dependent Fokker-Planck solution in (7), being of the structure

$$W(x,\zeta;t)=\sum_{n=0}^{\infty}W_n(x,\zeta)e^{-\lambda_n t}, \quad (9)$$

where $W_0=W_{st}$, and $\lambda_0=0$. For the case of escape in a symmetric double well this eigenvalue is related to the MFPT T of the particle from one minimum to the other by

$$\lambda_1(\tau)\simeq\frac{2}{T}. \quad (10)$$

The MFPT T can be obtained from the approximative SDEs in terms of two quadratures [9], which are evaluated numerically. The eigenvalue $\lambda_1(\tau)$ equals $2/T$ in the limit of small noise intensities D and Q, where λ_1 is clear-cut separated from the remaining set of eigenvalues.

Next we test the quality of the approximation schemes in (4) and (6) against precise numerical results. In doing so we choose equal noise strengths D and Q for the white and colored noise since otherwise the results are dominated by the well known cases of a SDE with only one white or only one colored noise, respectively [1]. In all figures the error of the MCF results lies within the line thickness.

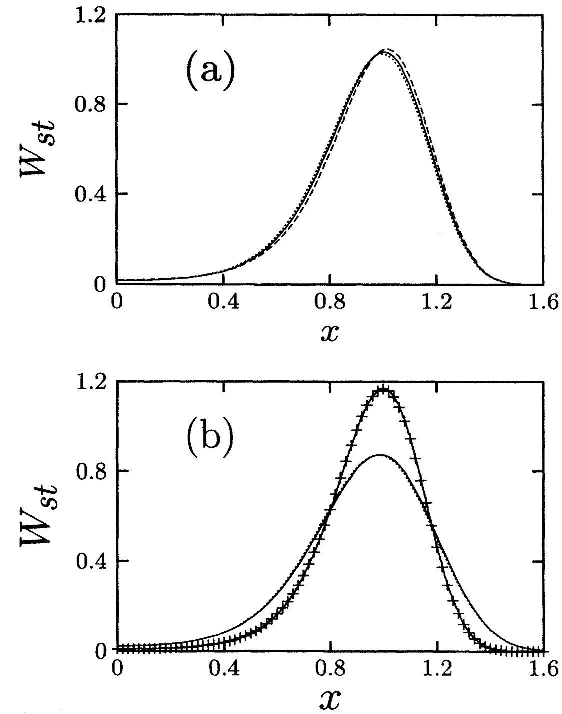

FIG. 1. (a) The stationary probability density for positive-valued x, $W_{st}(x;\tau)=W_{st}(-x;\tau)$, is shown for white and colored noise strengths $D=Q=0.05$. The correlation time of the colored noise is $\tau=1$. The solid line gives the numerical (MCF) result, the dashed line represents the small-τ theory, while the dotted line depicts the crossover approximation which essentially agrees with the large-τ approximation. (b) The same as in (a) for different correlation times: The results with $\tau=10$ are depicted by crosses for the crossover approximation and as a solid line for the numerical (MCF) results. The results of the large-τ approximation in (6) agree within line thickness with the numerical results for $\tau=10$. The other set of lines shows the results for $\tau=0.1$: The numerical results (solid) and the crossover approximation (dotted); here, the small-τ theory in (4) agrees almost within line thickness with the numerical results.

The stationary probability densities $W_{st}(x;\tau)$ are shown in Fig. 1. We depict only the range $x \geq 0$, since $W_{st}(x;\tau)=W_{st}(-x;\tau)$. In Fig. 1(a) we depict the results for $\tau=1$. This value of noise color is most "critical," because the regime of validity of the approximations (4) and (6) are least justifiable for an intermediate noise color of the order $\tau \sim O(1)$. Nevertheless the numerical result (solid line) agrees rather well with the small-τ theory (dashed line). In addition to the above given approximations we consider a SDE which covers approximatively the limiting probability densities $W_{st}(x;\tau)$ for $\tau \to 0$ and $\tau \to \infty$ simultaneously, termed *crossover approximation* [6]. Such a crossover SDE reads

$$\dot{x}=x-x^3+\sqrt{\tilde{D}(x,\tau)}\xi_1(t) \quad (11)$$

with

$$\tilde{D}(x,\tau)=[(1+2\tau x^2)D+Qx^2]/(1+2\tau x^2). \quad (12)$$

The resulting stationary probability density $W_{st}(x;\tau) = A(x,\tau)\exp(-\Phi(x,\tau)/D)$ possesses the identical generalized potential $\Phi(x,\tau)$ as the large τ approximation in (6). The only difference lies in the prefactor $A(x,\tau)$. The results of the crossover approximation are depicted in the figures by dotted lines. In Fig. 1(a) this crossover approximation essentially agrees with the large-τ approximation in (6). In Fig. 1(b) we show the stationary probability for small noise color $\tau=0.1$, and large noise color $\tau=10$. The exact result agrees very well (within line thickness) with the scheme in (4) for the case of small noise color, while the crossover approximation exhibits small deviations (dotted line). For large noise color, the numerics (solid line) coincide very well with the large-τ approximation in (6) and the crossover approximation (crosses) is also practically indistinguishable from the exact numerical results. The density at $\tau=10$ is peaked more narrowly due to the effective reduction of the influence of colored noise, i.e., $\langle\zeta^2\rangle = Q/\tau$ decreases with increasing τ (freezing out of colored noise).

The results for the exponentially small eigenvalue $\lambda_1(\tau)$ are depicted in Fig. 2. In Fig. 2(a) we use the same noise strengths as for Fig. 1. The eigenvalue increases monotonically from its minimum at $\tau=\infty$—where the colored noise is completely frozen out—to its maximal value at $\tau=0$—where the dynamics is driven by two white noise sources $\xi_1(t)$ and $\xi_2(t)$. In the latter limit the total effective noise strength S amounts at the stable states $x=\pm 1$ to $S=D+Q=0.1$. In this case of nonweak noise the relationship involving the inverse MFPT in (10) consistently underestimates the exact eigenvalue at small noise color; see Fig. 2(a). For $\tau\to\infty$ this difference becomes increasingly less detectable due to the effective lowering of the total noise intensity $S \overset{\tau\to\infty}{\longrightarrow} D=0.05$. Indeed, at even lower noise intensities $Q=D=0.02$, cf. panel 2(b), the difference at $\tau\to 0$ now also vanishes. The crossover approximation for twice the inverse MFPT is depicted in Fig. 2 by the dotted line: It smoothly approaches the corresponding approximative results for τ small [dashed-dotted line in panel 2(a)] and τ large [dashed line in panel 2(a)], and bridges accurately the regime of intermediate noise color $\tau\sim O(1)$. As a matter of fact, the maximal error between exact result and approximation occurs near $\tau=1$. In panel 2(b) the maximal error occurs near $\tau\sim O(1)$ and it does not exceed 18%. At small noise intensities the steepest descent approximation, see [10], to the quadrature formulas of the MFPT can be invoked. With $W_{st}(x;\tau)$ more strongly peaked at large noise color the agreement between steepest descent and exact result improves for decreasing noise color.

In summary, we have presented precise MCF calculations for the archetypal bistable flow driven by Gaussian white noise and multiplicative, exponentially correlated Gaussian noise. The approximation schemes for this class of two noise driven stochastic flows in (4), (6), and (11) which are tailored to cover extended regimes of noise correlation times τ compare favorably with the exact results for the whole range of small-to-large noise color τ. These approximations provide good results even for the regime of intermediate noise color $\tau\sim O(1)$ where no systematic analytical estimates are known.

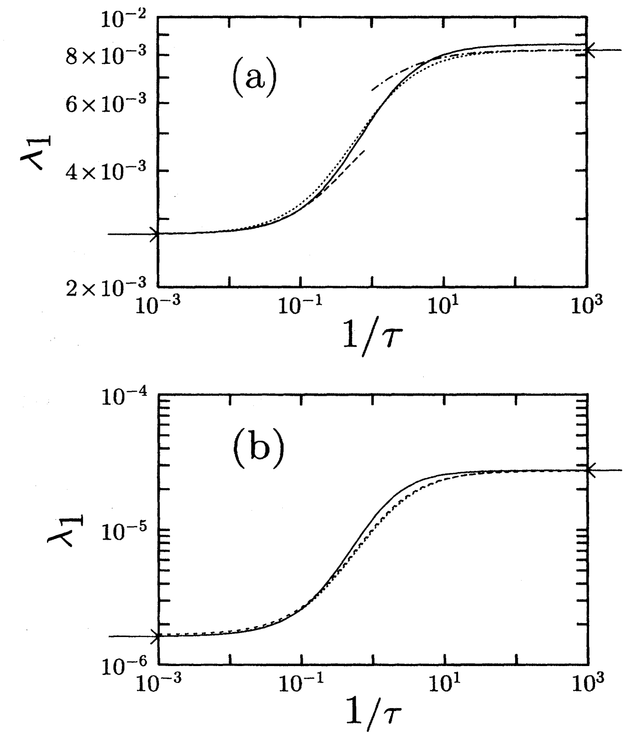

FIG. 2. (a) First nonvanishing eigenvalue $\lambda_1(\tau)$ of the exact non-Markovian dynamics vs inverse correlation time. Panel (a) is for $D=Q=0.05$. The numerical result is depicted by the solid line, while approximative theoretical results from the inverse MFPT are shown as a dashed line for the large-τ theory in (6), a dashed-dotted line for small-τ theory in (4), and a dotted line for the crossover approximation in (11) and (12). Arrows indicate the exact limits for twice the inverse MFPT at $\tau=0$ and $\tau\to\infty$, respectively. (b) $\lambda_1(\tau)$ for noise strengths $D=Q=0.02$. The numerical result is depicted by a solid line and the crossover approximation by a dotted line. In addition we show the steepest descent approximation for twice the inverse MFPT to the crossover approximation as a dashed line, see footnote [10].

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[10] With $R=Q/D$, twice the inverse MFPT reads, up to order $O(D^2)$, as

$$\lambda_{cross} \simeq \frac{2}{T} = \frac{\sqrt{2}}{\pi} \left\{ 1 + \frac{D[(18+36R+72\tau+17(R+2\tau)^2]}{12(1+2\tau)^3(1+R+2\tau)} \right\}^{-1} \times \exp\left\{ -\left[\frac{(R+2\tau)(2\tau-1)+\tau}{2(R+2\tau)^2} - \frac{\tau}{2(R+2\tau)} + \frac{[(R+2\tau)^2-(R+2\tau)(2\tau-1)-2\tau]\ln(1+R+2\tau)}{2(R+2\tau)^3} \right] \Big/ D \right\}.$$

For small D, and R finite, the prefactor $\{1+D\times\cdots\}^{-1}$ can be approximated by 1: In Fig. 2(b) the first order correction is not detectable; in contrast, for larger noise D, cf. Fig. 2(a), the correction in the prefactor gives rise to a change in λ_{cross} of approximately 10% as $\tau\rightarrow 0$, but effectively already a zero correction for $\tau\rightarrow\infty$.