White-Noise–Induced Transport in Periodic Structures.

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Abstract. – Spatially periodic structures are exposed to additive Poissonian white shot noise of zero average. Because the underlying master equation no longer obeys the principle of detailed balance, these non-equilibrium fluctuations induce a macroscopic current—even *in the absence* of spatial asymmetry. The resulting current can be expressed in analytical closed form and we discuss its behaviour in the limits of very weak and very strong noise intensities. We find that the current *increases monotonically* and—in contrast to common intuition—saturates at infinitely strong noise intensity. The role of internal symmetries of the periodic structure is investigated from the viewpoint of optimizing the current amplitude at fixed noise intensity.

The fact that it is possible to obtain a macroscopic particle current in *periodic* structures without the application of any external bias force, field gradients or a spatially varying temperature has stimulated interest among many scientists working in such diverse areas as biology, chemistry or physics [1, 2]. The interest in considering such novel macroscopic transport mechanisms originated from biology [3]: Assemblies of tubulin molecules, which possess a periodic structure with intrinsically broken symmetry, allow motor proteins to «walk» along in a directed way. Similarly, lithographic techniques can be utilized to engineer periodic structures such as blazed gratings which, when exposed to oscillating or stochastic fields of zero average, separate or «pump» particles of micrometer size. Such schemes have recently been implemented experimentally by both electronic and optical means [4].

Usually it is assumed that additive white fluctuations are not capable of inducing a finite current due to the laws of equilibrium dynamics. It must be noted, however, that a Maxwell-Demon mechanism that extracts work from a single heat bath is possible with white noise possessing a Poissonian waiting-time statistics. Such noise, which commonly occurs in electronics and in microstructures, is characterized by a *temporal* asymmetry, *i.e.* sharp pulses of zero duration are followed by a constant negative value which lasts over an exponentially distributed waiting time.

The starting point of our analysis is an overdamped stochastic dynamics in a periodic potential $V(x) = V(x + L)$, with L denoting the spatial period, which is driven by white shot noise $\xi(t)$, *i.e.*

$$\dot{x} = -\frac{\partial V(x)}{\partial x} + \xi(t), \tag{1}$$

with

$$\xi(t) = \sum_{i=1}^{n(t)} y_i \delta(t - t_i) - \lambda \langle y_i \rangle. \tag{2}$$

The quantity $n(t)$ is a Poisson counting process with a parameter λ which determines the average sojourn time between two δ-kicks. The positive-valued weights of the δ-pulses $\{y_i\}$, being independent of $n(t)$, are exponentially distributed, *i.e.*

$$\varrho(y) = A^{-1} \exp[-y/A], \qquad y \geq 0, \tag{3}$$

with $\langle y_i \rangle \equiv A$. From (2) and (3) it follows that the noise $\xi(t)$ is of *zero average*, and possesses the correlation

$$\langle \xi(t)\, \xi(s) \rangle = 2\lambda A^2 \delta(t - s). \tag{4}$$

Equation (4) defines a total noise intensity $D \equiv \lambda A^2$ (1). The stochastic dynamics in (1)-(3) yields for the probability density $p(x, t)$ the Markovian integro-differential master equation [5, 6]

$$\frac{\partial p(x, t)}{\partial t} = -\frac{\partial}{\partial x}[f(x) - \lambda A]\, p(x, t) - \lambda p(x, t) + \lambda \int_0^\infty p(x - y, t)\, \varrho(y)\, \mathrm{d}y, \tag{5}$$

where we set $f(x) \equiv -\partial V(x)/\partial x$. Upon introducing the shift operator, $\exp[-y\, \partial/\partial x] \cdot p(x, t) = p(x - y, t)$, and performing a few manipulations involving the moments $\langle y_i^k \rangle = k!\, A^k$, (5) takes on the appealing form of a continuity equation for the probability

$$\frac{\partial p(x, t)}{\partial t} = -\frac{\partial}{\partial x} J(x, t), \tag{6}$$

with the probability current $J(x, t)$ given by

$$J(x, t) = f(x)\, p(x, t) - \lambda A^2 \frac{\partial}{\partial x} \int_{-\infty}^{x} \varrho(x - y)\, p(y, t)\, \mathrm{d}y. \tag{7}$$

A stationary current J follows from (6) in the long-time limit $t \to \infty$ with periodic boundary conditions imposed on the corresponding stationary probability $p(x) = p(x + L)$. With $\varrho(y)$ given in (3), the current J is determined from (7) by (the prime denotes a derivative with respect to x)

$$J = A[f(x) - \lambda A]\, p'(x) + [f(x) + Af'(x)]\, p(x), \tag{8}$$

where the drift in (5) takes on the role of a diffusive part in (8). For $J = 0$, a homogeneous

(1) This white shot noise results also as a limit of *asymmetric* two-state noise of zero mean, *i.e.* $(a'/\lambda') - (a/\lambda) = 0$, where $(a', a > 0)$ denote the two-state values and (λ', λ) the corresponding transition rates. White shot noise is approached in the limit $a' \to \infty$, $\lambda' \to \infty$, such that $a'/\lambda' = a/\lambda = \mathrm{const}$ [5, 6]. Asymmetric two-state noise is exponentially correlated with correlation time $\tau_c = (\lambda + \lambda')^{-1}$, and its total integrated noise intensity equals $D = aa' \tau_c$ [6].

solution $p_0(x)$ reads

$$p_0(x) = \frac{1}{D_{\text{eff}}(x)} \exp[-\Psi(x)], \tag{9}$$

where the effective diffusion $D_{\text{eff}}(x)$ and the generalized potential $\Psi(x)$ are defined by the relations

$$D_{\text{eff}}(x) = D\left[1 - \frac{f(x)}{a}\right], \qquad \Psi(x) = -\int_0^x \frac{f(y)}{D_{\text{eff}}(y)}\,\mathrm{d}y, \tag{10}$$

with $a \equiv \lambda A = D/A$. In terms of $p_0(x)$ one readily finds for the normalized, spatially periodic and stationary probability, which carries a current J, the explicit result

$$p(x) = \frac{p_0(x) \int\limits_x^{x+L} \mathrm{d}y\, D_{\text{eff}}^{-1}(y)\, p_0^{-1}(y)}{\int\limits_0^L \mathrm{d}x\, p_0(x) \int\limits_x^{x+L} \mathrm{d}y\, D_{\text{eff}}^{-1}(y)\, p_0^{-1}(y)}. \tag{11}$$

For $p(x)$ to exist for all x-values we implicitly assume a positive-valued effective diffusion $D_{\text{eff}}(x)$, *i.e.* $a > \max|f(x)|$. With (11), the stationary current J has an appealing form (²): it is given by the double quadrature

$$J = \frac{1 - \exp[\Psi(L)]}{\int\limits_0^L \mathrm{d}x\, D_{\text{eff}}^{-1}(x) \exp[-\Psi(x)] \int\limits_x^{x+L} \mathrm{d}y \exp[\Psi(y)]}. \tag{12}$$

The closed form in this result allows to study various asymptotic regimes: For the important case of weak noise intensity $D \ll 1$, the steepest-descent approximation yields, up to terms of order $\mathcal{O}(D)$, the result

$$J(D \ll 1) \sim J_1 = \frac{\omega_0 \omega_u}{2\pi}\,[1 - \exp[\Psi(L)]]\, \exp[\Psi(x_0) - \Psi(x_u)], \tag{13}$$

where $\omega_0^2 = V''(x_0)$, $\omega_u^2 = -V''(x_u)$, and x_0 denotes the location of a minimum of $V(x)$, and x_u its neighbouring maximum $x_u > x_0$. Another interesting limit is given by allowing the noise

(²) The corresponding result for correlated, *temporal asymmetric* two-state noise, cf. footnote (¹), can again be expressed in closed form:

$$J = (1 - \exp[\Psi(L)])\left\{\int\limits_0^L D_{\text{eff}}^{-1}(x) \exp[-\Psi(x)] \int\limits_x^{x+L} [1 + \tau_c f'(y)] \exp[\Psi(y)]\,\mathrm{d}y\,\mathrm{d}x\right\}^{-1},$$

with $\Psi(x)$ defined as in (10), and $D_{\text{eff}}(x) = D[1 - f(x)/a][1 + a\tau_c f(x)/D]$, wherein $D \equiv aa'\tau_c \to a^2/\lambda$ as $\tau_c \to 0$. For the special case of symmetric two-state noise this result reduces to the one recently elucidated by Doering *et al.* [1e].

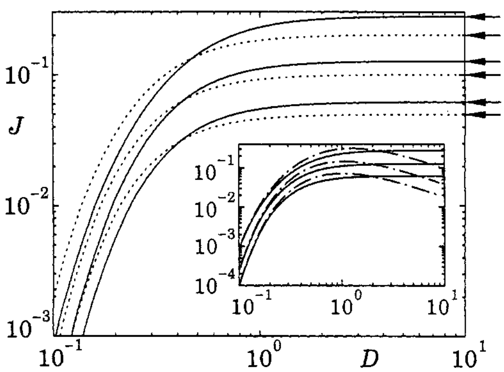

Fig. 1.

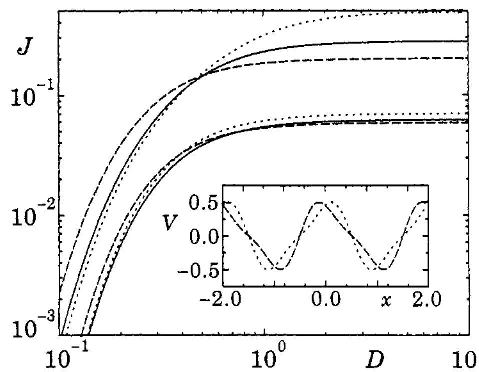

Fig. 2.

Fig. 1. – The current J is shown *vs.* shot noise intensity D for three values of the asymmetry parameter $\lambda A = a = 2.5, 5$ and 10 (top to bottom). The solid lines represent results for a cosine potential (see below eq. (18)), while dotted lines are for the piecewise linear potential defined in eq. (17). The arrows indicate the asymptotic limits for $D \to \infty$, given in eq. (14). The inset compares the steepest-descent approximation (dashed-dotted) to the exact result (solid) for the cosine potential at the same a values as in the main figure.

Fig. 2. – The current is shown for different shapes of the potential: Solid lines are for $V(x) = (1/2)\cos(\pi x)$, dashed and dotted lines for $V = 0.454[\cos(\pi x) \pm (1/4)\sin(2\pi x)]$, the latter potentials are depicted in the inset. The parameter a is 2.5 in the upper curves and $a = 10$ for the lower ones. The limit $D \to \infty$ again agrees within linewidth with (14) for $D = 10$.

intensity D to become very large. In this case, the current *saturates* at the value

$$J(D \gg 1) \sim J_2 = L^{-1} \frac{\int_0^L f(x)\, D_{\text{eff}}^{-1}(x)\, \mathrm{d}x}{\int_0^L D_{\text{eff}}^{-1}(x)\, \mathrm{d}x}\,. \tag{14}$$

Hence, in clear contrast to the case with Brownian ratchets driven by additive, *symmetric* forces [1, 2], strong white shot noise does not blur the directed motion, but rather enhances the current towards a finite maximal value, cf. fig. 1 and 2. This feature can be traced back to the role of the effective diffusion $D_{\text{eff}}(x)$ in (10), being non-homogeneous over the spatial period L. In this sense, white shot noise resembles from a mathematical viewpoint the current mechanism that characterizes transport in *Gaussian* white-noise–driven systems with a state-dependent diffusion that is not in phase with the periodic potential [7]. With $a \to \infty$, the white shot noise approaches the additive Gaussian white noise [6]. In this latter limit the master equation in (5) reduces to a Smoluchowski equation that obeys detailed balance. Hence, the current J must vanish as $a \to \infty$, *i.e.*

$$J(a \to \infty) \sim B/a + \mathcal{O}(1/a^2), \tag{15}$$

with a slope

$$B = \int_0^L f^2(x)\, \mathrm{d}x \left[\int_0^L \exp[-V(x)/D]\, \mathrm{d}x \int_0^L \exp[V(x)/D]\, \mathrm{d}x \right]^{-1}. \tag{16}$$

Before we study the behaviour of the current J for smooth periodic structures, we remark that for piecewise linear potentials the current can be obtained explicitly. For example, for the symmetric, and periodically continued potential

$$V(x) = \begin{cases} 1 + x, & x \in [-1, 0] \bmod L, \\ 1 - x, & x \in [0, 1] \bmod L, \end{cases} \tag{17}$$

with $L = 2$, the resulting current reads

$$J = \frac{1}{4D} \frac{b_+ - b_-}{(b_+ - 1)(b_- - 1)}, \tag{18}$$

with $b_\pm = \exp[a/[D(a \mp 1)]]$.

The dependence of J on the *shape* of the periodic potential is studied numerically in fig. 1. We evaluated the current for the piecewise linear potential in (17) and compared the results with a smooth symmetric potential $V(x) = V(-x) = (1/2)[\cos(\pi x) + 1]$ having the same period $L = 2$, and the same bare barrier height $\Delta V = V(x_u) - V(x_0) = 1$. We find that for both periodic structures the current *increases monotonically* to the maximal value J_2, cf. (14). The asymptotic values are approached already at $D \geqslant \mathcal{O}(1)$. We note that for a smooth barrier the current is enhanced over the piecewise linear case. The inset compares the steepest-descent approximation in (13) (dashed-dotted line) with the numerical precise result (solid line) for the $\cos(\pi x)$-structure. It should be stressed that the weak-noise approximation fails above $D \sim \mathcal{O}(1)$: It even predicts a bell-shaped behaviour at moderate noise intensities, being in clear contradiction to the exact behaviour. Therefore, the *a priori* use of a steepest-descent approximation comprises pitfalls!

The combination of both a spatial asymmetry for $V(x)$, *i.e.* a ratchet-type potential [1-4], and temporal asymmetric shot noise is depicted in fig. 2. The potentials are again chosen to possess the identical period $L = 2$ and the same bare potential barrier height of $\Delta V = 1$. We compare the same cos-potential as in fig. 1 with two ratchet potentials of opposite «polarity», *i.e.* $V(x) = 0.454[\cos(\pi x) \pm (1/4)\sin(2\pi x)]$, see inset in fig. 2. We find that for ratchet potentials driven by white shot noise a differing polarity does not affect the sign of the current. This fact is in contrast to the case of a rocking ratchet [2] and ratchets driven by symmetric coloured noise [1]. With positive-valued weights for the δ-kicks, we obtain a strictly positive current J, independent of internal symmetries of the periodic structure. Put differently, if we separate $\xi(t)$ into a constant negative tilt $-a = -\lambda\langle y_i \rangle$, and positive, infinite high δ-spikes, one finds that the latter ones dominate the current. This result is *not* evident *a priori*, because the δ-spikes act over zero duration. Comparing the asymmetric with the symmetric potential, we observe that the ratchet with the positive (forward) polarity (dotted lines) yields an enhanced current at moderate-to-large noise intensity D; this behaviour is reversed at small noise intensities $D \ll 1$. (In fig. 2 the solid and the dotted lines cross for $a = 2.5$ at $D \approx 0.425$ and for $a = 10$ at $D \approx 0.085$.) At present, we are not able to provide a vivid physical explanation of this result; the features of $\Psi(x)$ cannot explain these observations.

In conclusion, in this study we have investigated the role of white shot noise of zero average for *directed transport* in periodic structures, in the presence and in the absence of an internal asymmetry. We demonstrate that the white-noise character of *additive, temporally asymmetric* fluctuations is sufficient to generate a net macroscopic current (3). For vanishing

(3) In contrast, temporally *symmetric* white shot noise (*i.e.* with both positive and negative y_i in (2), (3)) can induce a current in periodic structures only if the reflection symmetry is broken.

temporal asymmetry the white shot noise with Poissonian jump statistics smoothly approaches a Gaussian limit [6], see (15), which in turn restores the detailed balance symmetry characteristic for equilibrium systems (4). As a major result we find that white shot noise induces a current that monotonically increases with the noise intensity, and undergoes a saturation at extremely strong noise intensity. This counterintuitive result is due to an effective, inhomogeneous diffusion, which becomes more homogeneous with increasing level a, at which the shot noise impulse $\xi(t)$ dwells during an exponentially distributed sojourn time.

With shot noise processes being abundant in electrical engineering and in life sciences [9] these exist a variety of potential applications in making use of shot-noise-induced directed transport, and to devise several novel separation mechanisms. Hence, we encounter another example where white noise with temporal asymmetry can be a useful tool—rather than a nuisance.

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(4) For equilibrium systems driven by white shot noise it is necessary to consider Langevin equations driven by *state-dependent* white noise [8], which are tailored to restore the detailed balance symmetry.

REFERENCES

[1] *a*) AJDARI A. and PROST J., *C. R. Acad. Sci. Paris*, **315** (1992) 1635; *b*) MAGNASCO M. O., *Phys. Rev. Lett.*, **71** (1993) 1477; *c*) ASTUMIAN R. D. and BIER M., *Phys. Rev. Lett.*, **72** (1994) 1766; *d*) CHAUWIN J. F., AJDARI A. and PROST J., *Europhys. Lett.*, **27** (1994) 421; *e*) DOERING C. R., HORSTHEMKE W. and RIORDAN J., *Phys. Rev. Lett.*, **72** (1994) 2984.

[2] BARTUSSEK R., HÄNGGI P. and KISSNER J. G., *Europhys. Lett.*, **28** (1994) 459; BARTUSSEK R. and HÄNGGI P., *Phys. Blätter*, **51** (1995) 506.

[3] AJDARI A. and PROST J., *Proc. Natl. Sci. USA*, **88** (1991) 4468; SIMON S. M., PESKIN C. S. and OSTER G. F., *Proc. Natl. Sci. USA*, **89** (1992) 3770; LEIBLER S., *Nature*, **370** (1994) 412.

[4] ROUSSELET J., SALOME L., AJDARI A. and PROST J., *Nature*, **370** (1994) 446; FAUCHEUX L. P., BOURELIEU L. S., KAPLAN P. D. and LIBCHABER A. J., *Phys. Rev. Lett.*, **74** (1995) 1504.

[5] HÄNGGI P., *Z. Phys. B*, **31** (1978) 407; **36** (1980) 271; VAN DEN BROECK C., *J. Stat. Phys.*, **31** (1983) 467; RODRIGUEZ M. A., PESQUERA L., SAN MIGUEL M. and SANCHO J. M., *J. Stat. Phys.*, **40** (1985) 6691.

[6] VAN DEN BROECK C. and HÄNGGI P., *Phys. Rev. A*, **30** (1984) 2730.

[7] BÜTTIKER M., *Z. Phys. B*, **68** (1987) 161; VAN KAMPEN N. G., *IBM J. Res. Devel.*, **32** (1988) 107; LANDAUER R., *J. Stat. Phys.*, **53** (1988) 233.

[8] HÄNGGI P., TALKNER P. and BORKOVEC M., *Rev. Mod. Phys.*, **62** (1990) 251, Sect. V; HÄNGGI P., *Z. Phys. B*, **36** (1980) 271, relations (5.4) and (6.15)-(6.21).

[9] PAPOULIS A., *Probability, Random Variables, and Stochastic Processes* (McGraw-Hill, New York, N.Y.) 1984, Chapt. 10-14; LEVI B. G., *Phys. Today*, 48 (1995) 17; SCHOLEY J. M., *Motility Assays for Motor Proteins* (Academic Press, New York, N.Y.) 1993.