Giant suppression of the activation rate in the presence of correlated white noise sources

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Abstract

We have studied the effect of two simultaneous correlated white noises, one additive and the other multiplicative, on the activation rate of a bistable system. It is proved that as a function of the (positive-valued) correlation strength between the two noise sources the activation rate can be suppressed by orders of magnitude (likewise, with a negative-valued correlation strength it can be enhanced). Hence, the correlation between the two noises provides a tool for a controlled slow-down (speed-up) of fast (slow) reactions by adding an appropriately chosen additional correlated noise force.

1. Introduction

The problem of evaluating activation rates in bistable systems has received considerable attention since the pioneering work of Kramers (see the recent review [1]). Among others, the situations so far studied include cases of additive or multiplicative white or colored noises [1]. Moreover, the case of simultaneously acting uncorrelated additive and multiplicative white (and colored) noises has been considered [2]. However, the impact of simultaneously correlated noise forces for the reaction rate has not been addressed previously.

The goal of this work is to show that the simultaneous consideration of additive and multiplicative – correlated – white noises induces a dramatic suppression of the escape rate in a double well system.

In order to fix ideas about the physical relevance of the present results, let us focus on a particular (realistic) model showing bistable behaviour. We will analyze the so called *ballast resistor* [3]. It consists of a metallic wire or ribbon immersed in a cooled gas. Both T_b, the temperature of the gas (or bath), and I, the electric current that flows along the wire, are externally controlled. Near a ferromagnetic or superconducting phase transition the metal shows an abrupt change (increase) in its resistivity as a function of the temperature, exhibiting an inflection point. Hence, the *characteristic* (current–voltage) curve looks very similar to a gas–liquid isotherm, i.e. contains a section of *negative resistance* where (through a Maxwell-like construction) the current remains constant as the voltage is varied. These special features have been exploited in the so called *hot-spot model* in connection with ex-

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periments in superconducting microbridges [4].

In order to describe the ballast resistor, it is necessary to write an equation for the conservation of the internal energy per unit length of the wire. After some approximations, using Onsager relations connecting, for instance, the differential thermoelectric power with the Peltier coefficient, and the fact that the internal energy of the wire can be taken as proportional to the local temperature of the wire, an equation for the temperature field $T(x,t)$ is obtained. The resulting equation is

$$c\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}K(x)\frac{\partial}{\partial x}T(x,t) - q(T(x,t) - T_{\mathrm{B}}) + I^2 R(T), \tag{1}$$

where c is the heat capacity of the wire (per unit length), K is the heat conductivity, q relates the energy flow into the bath to the difference in temperature between the wire and the bath, I is the electric current, and $R(T)$ the temperature dependent resistivity. The hot-spot model of the ballast resistor, assumes that c, K and q are independent of x, and also introduces a piecewise linearization of the cubic-like nonlinearities that arise in the original energy balance equation for the wire (i.e. $R(T) \sim \theta(T - T_{\mathrm{c}})$, where T_{c} is "the" critical temperature). As a matter of fact, it is possible to see this equation as *mimicking* the well known *Schlögl model* or a very general class of bistable reaction–diffusion systems [5].

Within this context of bistable systems, and particularly for the above indicated models, it is possible to see that fluctuations in some of the model parameters (for instance: in the bath temperature T_{B} and the resistivity for the hot-spot model; or in the direct and inverse reaction rates of the noncatalytic contribution for the Schlögl's model), should not be independent (i.e. must be correlated).

In addition, looking into the original equations of these models we find that fluctuations in the above indicated parameters lead us to noise contributions of both additive and multiplicative character. For instance, fluctuations in T_{B} will induce both, an additive noise contribution from the second term on the r.h.s. of the model equation, as well as a multiplicative noise contribution through its effect on the critical current and the resistivity [6] from the third term. Even though the first contribution can be (naturally) assumed as white, the second one, involving characteristic time scales of the system, would not necessarily be white. However, as in many other situations, it is reasonable to adopt as a first step in the modelization of this phenomenon the assumption that the second noise contribution is also white and, afterwards, relax this condition and include the finiteness of its correlation time.

Hence, the study of problems showing a simultaneous presence of additive and multiplicative – correlated – white noises could be of practical interest in several experimental situations where we face a transition between the two states of an intrinsically bistable system. For instance, the relevance of such an analysis for spatially extended systems is apparent from the results of Ref. [7].

In the next section we present our model system and show that the simultaneous consideration of additive and multiplicative correlated white noises induces a suppression of the transition rate. In the last section we present our conclusions.

2. Model

We consider a one dimensional fluctuating double well system, driven by correlated additive and multiplicative noises, i.e. the Stratonovich–Langevin equation

$$\dot{x} = x - x^3 + \sqrt{Q}\,x\xi_1(t) + \sqrt{D}\,\xi_2(t), \tag{2}$$

$$\langle \xi(t) \rangle = 0, \tag{3}$$

$$\langle \xi_i(t)\xi_j(s) \rangle = 2C_{ij}\delta(t - s), \tag{4}$$

where

$$\{C_{ij}\} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}. \tag{5}$$

The noises $\xi_1(t)$ and $\xi_2(t)$ denote white Gaussian noises with cross-correlation intensity ρ. Here $|\rho| \leqslant 1$. It is easy to see [8–10] that the associated Fokker–Planck equation (FPE) is given by

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}[(x - x^3 + Qx + \rho\sqrt{QD})P(x,t)] + D\frac{\partial^2}{\partial x^2}[(1 + Rx^2 + 2\rho\sqrt{R}x)P(x,t)], \tag{6}$$

where $R \equiv Q/D$.

The stationary probability density of the FPE in Eq. (5) is then readily obtained as

$$P_{st}(x) = \frac{Z^{-1}}{(1 + Rx^2 + 2\rho\sqrt{R}\,x)^{1/2}} \exp[-\Phi(x)/D], \tag{7}$$

where the generalized potential is given in terms of a quadrature by

$$\Phi(x) = -\int^{x} \frac{(y - y^3)\,\mathrm{d}y}{1 + Ry^2 + 2\rho\sqrt{R}\,y}. \tag{8}$$

The form of the stationary probability has been investigated previously in Ref. [9]. In particular, these authors considered the change from a unimodal shape to a bimodal shape for the stationary probability in Eq. (6) and showed that the transition depends on a characteristic threshold value, a_c, for the linear relaxation rate in Eq. (1). In our situation a_c is given by

$$a_c = RD + 3(\tfrac{1}{4}\rho^2 RD^2)^{1/3},$$

and the (positive) unit linear relaxation rate in (1) always lies above this threshold, i.e. $1 > a_c$, for $0 \leqslant R \leqslant R_{bm}$, where R_{bm} is the threshold value for bimodality of the stationary probability. For our chosen parameter sets, the minimum in the transition rate occurs always *within* the bimodal range for the stationary probability (see the figures). Above this threshold value, the mean first passage time still exists, but is no longer dominated by an Arrhenius-like bottleneck.

Our prime concern here is the study of the transition rate κ, defined here as $1/T$, the inverse of the mean-first-passage-time (MFPT) from the left stable state $x_- = -1$, to the right stable state $x_+ = +1$ ($x_- \to x_+$). This quantity crucially determines the transport in the bistable system. It is given by

$$T^+(R,\rho) = D^{-1} \int_{-1}^{1} \mathrm{d}x\, H(x) \exp[\Phi(x)/D] \times \int_{-\infty}^{x} \mathrm{d}y\, H(y) \exp[-\Phi(y)/D], \tag{9}$$

where

$$H(x) = (1 + Rx^2 + 2\rho\sqrt{R}\,x)^{-1/2}. \tag{10}$$

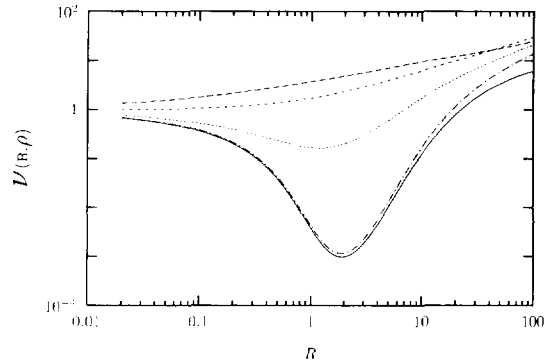

Fig. 1. The relative rate $\nu(R,\rho) = \kappa(R,\rho)/\kappa_S$ is shown at $D = 0.1$, for different values of the cross-correlation intensity ρ. The long-dashed line represents a case with anti-correlation ($\rho = -0.9$). The short-dashed line is the case without correlation ($\rho = 0$). The curves for positive cross-correlation intensity are depicted by the dotted line for $\rho = 0.7$ and by the dash-dotted line for ($\rho = 0.9$). The full line gives the steepest descent prediction for this latter case.

One can deduce from the symmetry of the system that $\kappa^+(R,\rho) = \kappa^-(R,-\rho)$.

We plot the curves of the relative rate $\nu = \kappa/\kappa_S$ as a function of R, see Figs. 1 and 2. The symbol κ_S indicates the rate in the Smoluchowski limit ($R = 0$). One can see that for fixed R, the generalized force with $\rho > 0$, as given by the integrand in Eq. (7), increases within the interval $x_- < x < 0$, when increasing the value of ρ. Also the barrier for the transition $x_- \to x_+$, increases. Thus, this causes an increase for the MFPT and consequently the transition rate *decreases* exponentially.

With an anti-correlation , i.e. $\rho < 0$ the barrier decreases with decreasing ρ. This in turn causes the opposite effect, namely an exponential *increase* for the forward rate. When $\rho = +1$, the indicated force becomes divergent at $x = 1/\sqrt{R}$ and, if $R > 1$, the transition is quenched down to zero, i.e. ($\kappa^+ = 0$). Figs. 1 and 2 depict those behaviours. For decreasing values of intensity of the additive noise D, the suppression of the reaction rate becomes more drastic. We also plot the steepest descent approximation of the relative ratio for $\rho = 0.9$. It is worthwhile using this approximation in order to obtain some analytical results. For small values of D we have to leading order

$$\nu(R,\rho) = \frac{\kappa(R,\rho)}{\kappa_S} = \exp[(\tfrac{1}{4} - \Delta\Phi)], \tag{11}$$

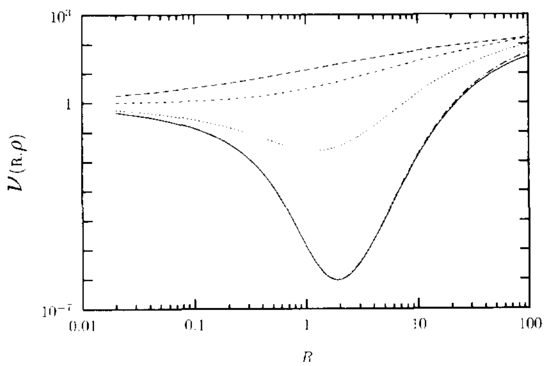

Fig. 2. The relative rate $\nu(R,\rho) = \kappa(R,\rho)/\kappa_S$ is shown at $D = 0.05$, for different values of the cross-correlation intensity ρ. The long-dashed line represents the case with anti-correlation ($\rho = -0.9$). The short-dashed line is the case with zero correlation ($\rho = 0$). The curves for positive cross-correlation intensity are shown by the dotted line for $\rho = 0.7$ and by the dash-dotted line for ($\rho = 0.9$). The full line denotes the corresponding leading steepest descent prediction ($\rho = 0.9$).

where

$$\Delta\Phi = \Phi(0) - \Phi(-1). \tag{12}$$

For $R \ll 1$ we have

$$\nu(R \ll 1, \rho) = \exp\left(-\frac{4}{5D}\rho\sqrt{R} + \frac{R(1-4\rho^2)}{12D}\right), \tag{13}$$

while for $R \gg 1$ we have

$$\nu(R \gg 1, \rho) = \nu(R, \rho = 0)\exp\left(\frac{2\rho}{RD\sqrt{R}}\right). \tag{14}$$

It is known that the transition rate increases with increasing R when the noises are uncorrelated [2]. Now, with correlated noises, the denominator of the generalized force

$$\frac{x - x^3}{1 + Rx^2 + 2\rho\sqrt{R}x},$$

has a ρ-dependent contribution that plays a very strong role for small and moderate values of R. Depending whether ρ is negative or positive, the generalized force can be smaller or larger than for the case when $\rho = 0$, respectively. For larger values of R, the correlation between the noises becomes negligible ($Rx^2 \gg 2\rho\sqrt{R}x$), and the transition rate approaches an asymptotic behaviour that corresponds to the case of uncorrelated noise ($\rho = 0$); a fact that follows readily from Eq. (13).

3. Conclusions

We have shown that the simultaneous consideration of additive and multiplicative correlated white noises can induce a very *giant* suppression (or with an anti-correlation $\rho < 0$ the opposite effect of a *giant* enhancement) of the forward transition rate (e.g. six orders of magnitude in Fig. 2) in bistable systems. It is clear that, from a mathematical point of view, we could transform the original problem (with two correlated noises) into another one with transformed variables and two uncorrelated noises, and then use the standard approach. However, such a procedure would clearly blur the physics of the escape dynamics that occurs in the original problem.

As indicated in the introduction, there exist realistic models of systems showing bistability, where the fluctuations in the model parameters are not independent and, in those cases at least, such fluctuations do lead to noise contributions of both additive and multiplicative character. Yet another potential physical application is given by the switching of magnetization in single-domain ferromagnetic particles which can be described by the noisy Gilbert equation [11]. Here, external and internal magnetic field fluctuations are generally correlated and mutually influence the bistable relaxation dynamics of the magnetic moment [11].

A question to be raised is if this behaviour can be also found when one of the noise sources is colored. For instance, some initial analysis can be done within the framework of approximations like the UCNA or related ones [2,12,13]. This will be the subject of further work. Using the UCNA approximation [12], for example, a preliminary analysis shows that the effect of *giant suppression* of the reaction rate is robust against small amounts of noise color. It will be also interesting to perform numerical or analogue simulations of this latter situation in order to have an independent confirmation supporting the novel behaviour found in this work. However, the most relevant question is whether this novel phenomenon can be implemented in real physical systems. We hope that the present study and our discussion of possible candidate systems will stimulate experimentalists to search (realize) for this technologically useful effect.

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