Comment on "Breathers and Kink-Antikink Nucleation"

In a recent Letter [1] Büttiker and Christen (BC) computed the kink-antikink pair nucleation rate for the overdamped sine-Gordon (SG) theory

$$\gamma \phi_t - c_0^2 \phi_{xx} + \omega_0^2 \sin\phi = -F + \zeta(x,t), \quad (1)$$

where $\zeta(x,t)$ is a zero mean-valued Gaussian noise with $\langle \zeta(x,t)\zeta(x',t')\rangle = 2\gamma kT\delta(x-x')\delta(t-t')$ and F denotes a constant external drive with $F \ll \omega_0^2$. In the BC approach a thermal pair is deemed fully nucleated after the kink-antikink partners drift or diffuse a *phenomenological distance* s_0 apart, with s_0 much larger than the kink (antikink) size $d = c_0/\omega_0$, but much smaller than the equilibrium kink (antikink) free path $1/n_0$, with $n_0 = (2E_0/\pi kTd^2)^{1/2}\exp(-E_0/kT)$ and $E_0 = 8\omega_0 c_0$. As a consequence, BC nucleation rate Γ_{BC} and lifetime τ_{BC} are given by [1]

$$\Gamma_{BC} = 2n_0/\tau_{BC} = (4\omega_0^2/\pi\gamma s_0)\exp(-2E_0/kT), \quad (2)$$

and depend explicitly on the otherwise undetermined cutoff length s_0.

In order to help readers to compare the BC result with prior calculations by the present authors [2,3], we feel compelled to comment as follows

(i) From a statistical mechanical viewpoint, the kink (antikink) density $n_0(T)$ is a well-defined equilibrium observable [4], no matter what the damping regime (or the thermalization mechanism) of Eq. (1). Let us set initial conditions $\phi_t(x,0) = \phi(x,0) = 0$ in Eq. (1) and define the kink (antikink) lifetime τ as the time constant for the kink (antikink) density to grow from 0 at $t = 0$ up to its equilibrium value $n_0(T)$. Such a nonequilibrium definition of τ is sound (at least for small F values [2]) and does not depend and any *ad hoc* cutoff length. This property is all the more valid for equilibrium $n_0(T)$ fluctuations.

(ii) The *ad hoc* cutoff length s_0 of Ref. [1] might be motivated, indeed, by the operational definition of kink (antikink), e.g., in a digital simulation [5]. The question then arises whether a finite nucleation rate Γ can be obtained for $s_0 \to \infty$, this limit being required by the mathematical definition of nucleated pair. In contrast with BC result (2), we proved [2] that a *natural* cutoff length $n_0^{-1}[1 + \sqrt{1 + (F/F_c)^2}\,]^{-1}$, with $F_c = kTn_0/2\pi$, comes about in a dilute gas of kinks and antikinks at thermal equilibrium [4]. Accordingly, for the weakly driven regime [with the rate Γ_0 of Eq. (20) in Ref. [2]], $s_0 = kT/2\pi F$; for the purely diffusive regime [with the rate Γ_D of Eq. (19) in Ref. [2]], $s_0 = 1/2n_0$–for numerical evidence see [5]. The physical interpretation of s_0 in these two regimes is revealing. In the absence of a drive, $1/2n_0$ is the average kink-antikink separation (mean free path), whereas $kT/2\pi F$ is the kink-antikink distance when the drift dominates over diffusion.

(iii) The definition of s_0 in Ref. [1] leads to mistaking a "standard breather" solution [6], $\phi_B(x,t) = 4tg^{-1}[(q/\omega)\sin\omega t/\mathrm{ch}qx]$ for a nucleated pair [7]. The unperturbed solution ϕ_B represents a kink-antikink bound state with total energy $2E_0\sqrt{1 - \omega^2/\omega_0^2}$, where kink and antikink oscillate, crossing each other with angular frequency ω, such that $\omega^2 + q^2c_0^2 = \omega_0^2$. Because the amplitude of such oscillations diverges for $\omega \to 0$, we can define a value ω_s of ω so that for $\omega \le \omega_s$ the breather width exceeds the kink width d. At low temperature, the nucleation rate of standard breathers with $\omega \le \omega_s$ is [7]

$$\Gamma_B = (4\omega_s^2/\pi\gamma d_s)\exp(-2E_0/kT), \quad (3)$$

with $d_s = c_0/\omega_s$. Not surprisingly, Γ_B *coincides with* Γ_{BC} *after substituting* s_0 *in Eq. (2) with the effective breather length* $d_s(\omega_0^2/\omega_s^2)$. A certain ambiguity in the definition of the SG degrees of freedom is thus introduced by BC, which was consistently ruled out in the earlier literature [5–7]. In passing, we notice that the space-time plot for the time evolution of an overdamped standard breather may well resemble the closed loops (bubbles), hand-drawn in Fig. 1 of Ref. [1].

In conclusion, the consistent use of the dilute gas approximation yields a stationary kink nucleation rate [2], where the prefactor is uniquely defined as a function of both the temperature T and the external drive F.

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