Spectral statistics of global avalanches along granular piles

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Abstract. – In a seminal experiment, Jaeger *et al.* (*Phys. Rev. Lett.*, **62** (1989) 40) measured the power spectrum of the velocity of avalanches along a granular pile in a rotating drum. Although their findings started an actively discussed debate on whether self-organized criticality rules granular avalanche dynamics, their experimental results have not been explained in theoretical detail yet. By the use of a simple dynamical model that incorporates the basic deterministic mechanisms of global avalanche flow as well as their stochastic aspects, we find almost perfect agreement between the model results and Jaeger *et al.*'s experimental data. We also discuss in detail the significance of the underlying physical mechanisms.

Granular matter such as beads, sand and powder can flow in a *non*-Newtonian way if driven out of its (metastable) static equilibrium by large enough external forces [1]. A striking example of the rheological complexity is that granular systems can exhibit stable static piling up to a *maximum angle of repose* φ_s of the inclined surface. For pilings with an inclination angle $\varphi > \varphi_\mathrm{s}$ the surface layer starts to slip in order to decrease the inclination angle until the *minimum angle of repose* $\varphi_\mathrm{r} < \varphi_\mathrm{s}$ has been reached. Then, the *avalanche* stops again. So far, avalanching seems to be a fact of common sense. Highly non-trivial, however, is the problem of the spectral properties of sequences of avalanches that are generated by continuous addition of grains to the top of the pile or by slow rotation of the pile, *e.g.*, in a drum [1]-[6].

A decade ago, Bak *et al.* [5] proposed the interesting scenario of self-organized criticality (SOC) that some spatially extended dissipative systems (such as sandpiles) might evolve into a critical state that does not possess characteristic time and length scales, and therefore, show a $1/f$ power spectrum [5]. In a subsequent seminal experiment, Jaeger *et al.* [2] probed the applicability of SOC to granular dynamics. They rotated a grain-filled semi-cylindrical drum (with a lower half being closed off) about its horizontally aligned axis with a constant rotation rate. By that, they created a well-defined sequence of discrete avalanches and measured the avalanche flow by detecting the flow over the rim at the lower end of the pile. The power spectrum of this velocity signal differs significantly from the $1/f$ power law decay suggested by SOC [5] (or the $1/f^2$ decay suggested in ref. [6]) and therefore this result started a still ongoing debate on the applicability of SOC to avalanches along granular piles [2]-[4], [6]. Despite its fundamental importance, there does not seem to exist a clear detailed simple theoretical explanation of Jaeger *et al.*'s result in the literature yet.

In this letter, we reconsider Jaeger *et al.*'s result [2] from a theoretical point of view, although we start from a different concept than SOC. Our focus is to show that the power spectrum found by Jaeger *et al.* [2] can be explained theoretically in all its major details by using a dynamical model that incorporates the basic macromechanical mechanisms of the avalanche dynamics such as viscoplastic yield, macromechanical friction, and small macromechanical stochastics. Moreover, our investigation leads to a global picture of the influence of stochastics on the power spectra of the global avalanche velocity for granular flow-over-the-rim experiments.

The model. – To model the statistics of sequences of discrete avalanches we start from the previously proposed deterministic minimal model (DMM) for the ensemble-averaged dynamics of avalanches [7] and combine it with the stochasticity of the *individual* avalanches. In the DMM (for details cf. ref. [7]), surface flow along granular piles is described by two global dynamical variables, the inclination angle $\varphi(t)$ of the pile and the characteristic velocity $v(t)$ of the avalanche flow. The latter is basically determined by the square root of the kinetic energy of the grain in motion. The DMM [7] generalizes Coulomb's frictional motion of bodies on inclined planes as follows: i) a velocity-dependent friction coefficient $k_{\rm d}(v)$ which interpolates between solid and Bagnold friction, $k_{\rm d}(v) = b_0 + b_2 v^2$ ($b_0 > 0$ and $b_2 > 0$) [1], ii) a viscoplastic yield condition such that an avalanche can only start if $\varphi > \varphi_{\rm s}$ and stops again if $v(t)$ reaches zero, and iii) a coupling of $\varphi(t)$ to the velocity dynamics which counteracts the acceleration of the avalanche.

Macromechanical stochasticity is rooted in the fact that moving granular matter consists of closely packed grains which interact by inelastic collisions and friction. This leads to local and (due to the finite extension of a pile) also to global fluctuations during avalanching. This effect, nicely demonstrated in the experiments [8], leads effectively to a macromechanical stochastics which can be modeled by a Langevin term, $\tilde{\zeta}(t)$, in the velocity equation. For the sake of simplicity, we suppose that $\tilde{\zeta}(t)$ represents Gaussian white noise fluctuations with zero mean and a correlation function $\langle\tilde{\zeta}(t)\tilde{\zeta}(t')\rangle = \tilde{\varDelta}^2\delta(t-t')$. Here, $\tilde{\varDelta}$ denotes the fluctuation strength.

The resulting model reads explicitly

$$\dot{v} = g\big[\sin\varphi - (b_0 + b_2 v^2)\cos\varphi + \tilde{\zeta}(t)\big]\chi(\varphi, v)\,, \tag{1}$$

$$\dot{\varphi} = -av + \varpi\,, \tag{2}$$

with the cut-off function (the viscoplastic yield conditions) $\chi(\varphi, v) = \Theta(v) + \Theta(\varphi - \varphi_{\rm s}) - \Theta(v)\Theta(\varphi - \varphi_{\rm s})$. Here, $\Theta(y)$ denotes Heaviside's step function, a, b_0 and b_2 all denote positive parameters, g is the gravitational acceleration, and ϖ the external constant rotation rate of the semi-cylindrical drum. The dynamics of the avalanches of eqs. (1) and (2) is centered around the angle $\varphi_{\rm d} = \tan b_0$ [7]. Introducing the deviation from this angle, $\varPhi(t) = \varphi(t) - \varphi_{\rm d}$, non-dimensionalizing time by $t \to t/\sqrt{ga}$ and velocity by $v \to v\sqrt{g/a}$, setting $\omega = \varpi/\sqrt{ga}$, and performing a small-angle approximation in $\varPhi$ (since $\varphi_{\rm s} - \varphi_{\rm r} \simeq 2°$), we obtain the following stochastic extension of the DMM:

$$\dot{v} = [-\delta v^2 + \varOmega_0^2\varPhi + \zeta(t)]\chi(\varPhi, v)\,, \tag{3}$$

$$\dot{\varPhi} = -v + \omega\,, \tag{4}$$

where $\chi(\varPhi, \dot{\varPhi}) = \Theta(-\dot{\varPhi} + \omega) + \Theta(\varPhi - \varPhi_{\rm s}) - \Theta(-\dot{\varPhi} + \omega)\Theta(\varPhi - \varPhi_{\rm s})$, $\varPhi_{\rm s} = \varphi_{\rm s} - \varphi_{\rm d}$, $\delta = (gb_2/a)\cos\varphi_{\rm d} > 0$, and $\varOmega_0^2 = 1/\cos\varphi_{\rm d} > 0$. After non-dimensionalization, the fluctuation strength is given by $\varDelta = \tilde{\varDelta}/g$. In the deterministic limit ($\varDelta = 0$) and for small rotation rates

ω, the model shows periodic global avalanches which start from $\Phi_\mathrm{s} = \varphi_\mathrm{s} - \varphi_\mathrm{d}$ with $v = 0$ and decay to $\Phi_\mathrm{r} = \varphi_\mathrm{r} - \varphi_\mathrm{d} \simeq -\Phi_\mathrm{s}$ when $v = 0$ has been reached again. They are separated by rigid pile rotations until Φ_s is reached again. The duration of the rigid pile rotation is determined by $(\Phi_\mathrm{s} - \Phi_\mathrm{r})/\omega$. For small fluctuation strength Δ, this basic mechanism is still present in the extended model, eqs. (3) and (4), however, with superimposed small stochastic variations of the velocity of the avalanches and the inclination angle of the surface. Note that the fluctuations $\zeta(t)$ in eq. (3) only act when there is flow, $v(t) \neq 0$.

Power spectrum for zero fluctuations. – Here, we discuss the deterministic limit of the power spectrum of $v(t)$ of a sequence of discrete avalanches. Since the non-linearity δ in eq. (3) is typically of the order 10^{-1} for the experiments [7], we approximate the model eqs. (3) and (4) with $\delta = 0$ by its deterministic limit $\Delta = 0$. For small rotation rates $\omega \ll \Omega_0$, the resulting velocity signal $v(t)$ is a sequence of equally shaped successive avalanches (or half-oscillations with a period $T_\mathrm{av} = \pi/\Omega_0$) separated by rigid pile rotations with $v = 0$ that last $T_\mathrm{rpr} = 2\Phi_\mathrm{s}/\omega$. One obtains for a sequence of N avalanches

$$v(t) = \begin{cases} \Omega_0 \Phi_\mathrm{s} \sin[\Omega_0 (t - nT)], & nT \le t \le nT + \dfrac{\pi}{\Omega_0}\,, \\ 0, & \text{elsewhere}\,, \end{cases} \tag{5}$$

where $n = 0, \ldots, N-1$, with $T = T_\mathrm{av} + T_\mathrm{rpr}$ being the period of the avalanching process. With the Fourier transform of the velocity, $\tilde{v}(f) = \int_{-\infty}^{\infty} \mathrm{d}t\; v(t) e^{2\pi i f t}$, given by

$$\tilde{v}(f) = \frac{\Omega_0^2 \Phi_\mathrm{s}}{\Omega_0^2 - 4\pi^2 f^2} \left[1 + e^{2\pi^2 i f/\Omega_0} \right] \sum_{n=0}^{N-1} e^{2\pi i f n T}, \tag{6}$$

the power spectrum $S_N(f) \propto |\tilde{v}(f)|^2$ of a sequence of N avalanches reads (up to an arbitrary normalization factor)

$$S_N(f) \propto \frac{2\Omega_0^4 \Phi_\mathrm{s}^2}{(\Omega_0^2 - 4\pi^2 f^2)^2} \left[1 + \cos\left(\frac{2\pi^2 f}{\Omega_0} \right) \right] \left[\frac{\sin(N\pi T f)}{\sin(\pi T f)} \right]^2 . \tag{7}$$

In deriving eq. (7), we have assumed that the first avalanche starts at $t = 0$. Started at $t = \tau$, an additional factor $\exp[2\pi i f \tau]$ enters into $\tilde{v}(f)$; the power spectrum $S_N(f)$, however, remains unchanged. The first two terms in the product on the rhs of eq. (7) determine the power spectrum $S_1(f)$ of one single avalanche. For this contribution, $\log S_1(f)$ is almost flat for intermediate frequencies, $f < \Omega_0/2\pi$; for larger frequencies, $f > \Omega_0/2\pi$, the global structure (not taking into account the spiky higher resonances caused by the zeroes of $S_1(f)$) is a linear decay in f with a slope of four. This implies a $1/f^4$ decay of $S_N(f)$ for large f. The contribution of the subsequent $N-1$ avalanches is determined by the third and last term of the product on the rhs of eq. (7). This term has a pronounced peak at $f = 1/T$ and further resonances at all multiples of $f = 1/T$. Due to the various zeroes and divergences appearing in $S_N(f)$ as a function of f, the structure of $\log S_N(f)$ looks rather rugged. To extract the characteristic features of the power spectrum, a coarse-grained averaging over the fine structure of $S_N(f)$ must be performed. In fig. 1, we show the logarithm of the *smoothed* power spectrum, $S(f)$, resulting from eq. (7) (dashed line) for parameter values extrapolated from the experiment [2] and, for comparison, the experimental result (full line). There is some similarity between these two curves; the deterministic limit of the power spectrum, however, shows a more sharply pronounced transition from the flat shoulder to the roll-off and differs in its decay behavior.

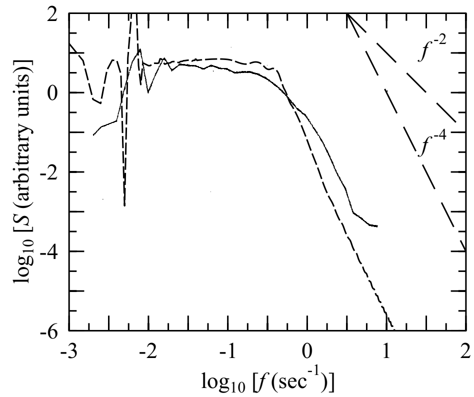

Fig. 1. – Dashed line: logarithm of the smoothed deterministic power spectrum as a function of the (re-dimensionalized) frequency. Full line: experimental data by Jaeger *et al.* [2] (adjusted in such a way that the flat shoulders roughly agree). From the experiment [2], the following data are known: maximum angle of repose $\varphi_s = 27.8°$, mean minimal angle of repose $\varphi_r = 25.6°$, rotation rate $\overline{\omega} = 1.3°/\text{min}$, and avaraged duration of an avalanche (with dimensions) $\overline{T}_{av} = 1.3\,\text{s}$. This implies for the model parameters used here $\varphi_d = (1/2)(\varphi_s + \varphi_r) = 26.7°$, $b_0 = \tan\varphi_d = 0.503$, $\Omega_0 = 1/\cos\varphi_d = 1.12$, $a = \pi^2/g\Omega_0^2\overline{T}_{av}^2 = 0.5/\text{m}$, $\omega = \overline{\omega}g^{-1/2}a^{-1/2} = 1.65 \times 10^{-4}$.

Power spectrum for non-zero fluctuations. – The sequence of avalanches for small non-zero Δ is distinct from its deterministic limit by several facts: i) The minimal angle of repose Φ_r and the avalanche duration are not sharply determined, but basically Gaussian distributed about their mean values $\langle\Phi_r\rangle \simeq -\Phi_s$ and $\langle T_{av}\rangle \simeq \pi/\Omega_0$. ii) The duration of the rigid pile rotation is not constant, but also basically Gaussian distributed about its mean $\langle T_{rpr}\rangle \simeq (\Phi_s - \langle\Phi_r\rangle)/\omega$. The velocity signal of the individual avalanches and the intervals between two successive avalanches are fluctuating too. To demonstrate these features, we show in fig. 2 a) the numerically integrated velocity signal $v(t)$ resulting from the stochastic model, eqs. (3) and (4), for a sequence of 17 successive avalanches (as in ref. [2]), and a fluctuation strength $\Delta = 5 \times 10^{-4}$. All other model parameters have been extrapolated from the experiment [2], except for δ that has been estimated from the experiment in ref. [8].

To investigate the power spectrum of the avalanching process, we take the fast-Fourier transform of the velocity signal and compute the power spectrum by taking the square of modulus of the Fourier-transformed velocity. The power spectrum $S(f)$ of the successive avalanches is depicted in fig. 2 b). Although the non-smoothed logarithm of $S(f)$ looks rather rugged, some basic structural features are visible: i) a pronounced first peak at low frequencies, ii) a broad flat shoulder at intermediate frequencies and iii) a roll-off for large frequencies that is weaker than a $1/f^4$ decay.

Smoothing of the small-scale variations in the power spectrum leads to the dashed curve in fig. 3 a). Apparently, there is an *almost perfect agreement* of the smoothed *computed* power spectrum (*dashed* curve) resulting from the model, eqs. (3) and (4), for $\Delta = 5 \times 10^{-4}$ and the *experimental* data of Jaeger *et al.*'s power spectrum [2] (*solid* curve) in the experimentally measured frequency range. Due to the high time resolution in our stochastic simulations, we can calculate $S(f)$ up to frequencies that are ten times larger than in ref. [2]. By that, we find a saturation into a $1/f^2$ decay for $S(f)$ for large frequencies.

The changes of $\log S(f)$ in comparison to the $\Delta = 0$ limit in fig. 1 are as follows. i) The peak at low frequencies is broadened because the interval between two successive avalanches is distributed about its mean value and not sharply determined. ii) The transition from

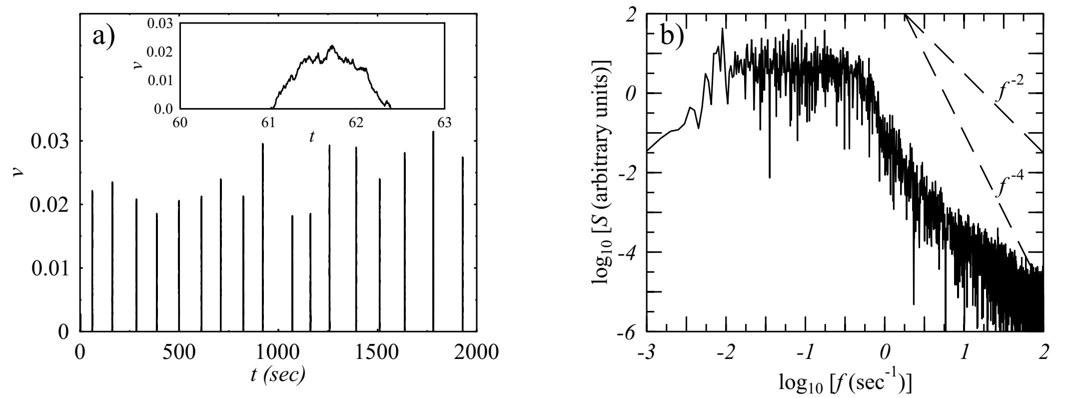

Fig. 2. – a) Velocity $v(t)$ for a sequence of 17 avalanches on a re-dimensionalized time scale, obtained by numerical integration of eqs. (3) and (4). Model parameters as in fig. 1 except that $\Delta = 5 \times 10^{-4}$ and $\delta = 0.1$. Inset: a blow up of the first avalanche to demonstrate that $v(t)$ consists basically of non-linear half-oscillations with small superimposed stochastics. b) Logarithm of the power spectrum $S(f)$ of the velocity signal $v(t)$ from a). The dashed lines represent decays $\propto f^{-2}$ and $\propto f^{-4}$.

the flat shoulder to the roll-off close to the inverse of the avalanche duration is less sharply pronounced. This rounding effect originates from the fact that the avalanche duration varies stochastically about its mean. iii) For slightly larger frequencies, the roll-off of $S(f)$ decays like $1/f^4$. This is the remnant of the decay behavior of the deterministic limit. iv) For even larger frequencies, there is a crossover to a $1/f^2$ decay of $S(f)$. This algebraic decay results from the small-scale fluctuations of the velocity during flow and is analogous to the $1/f^2$ decay of the power spectrum of a harmonic oscillator with a Langevin term.

The crossover to the $1/f^2$ decay and the frequency range with a decay being stronger than $1/f^2$ depends on the magnitude of Δ. For zero Δ, the $1/f^4$ decay is present. For larger Δ

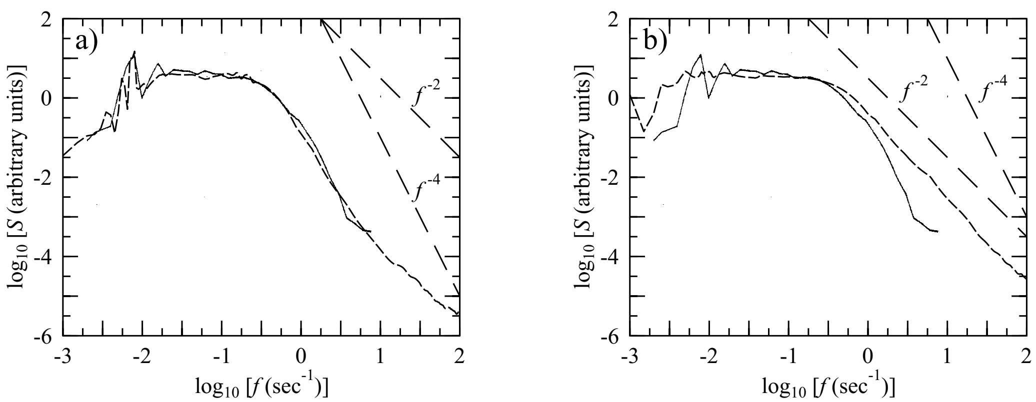

Fig. 3. – Dashed lines: logarithm of the smoothed power spectrum $S(f)$ of a sequence of 17 successive avalanches obtained by numerical integration of the model eqs. (3) and (4) for a fluctuation strength a) $\Delta = 5 \times 10^{-4}$, b) $\Delta = 5 \times 10^{-3}$ and all other model parameters as in fig. 2. Full line: for comparison the experimental result obtained by Jaeger *et al.* [2]. For the fluctuation strength Δ in case a), there is an almost perfect agreement with the experimental result [2]. The *smoothed* power spectrum $S(f)$ does not possess any significant dependence on the number of avalanches N if $N > 5$.

(as in fig. 3 a)), the roll-off possesses both decay behaviors. Increasing Δ further, the range with a $1/f^4$ decay shrinks and finally disappears. For a fluctuation strength $\Delta = 5 \times 10^3$, the logarithm of the power spectrum shown in fig. 3 b) only consists of a flat shoulder that crosses over to a roll-off with a $1/f^2$ decay of $S(f)$. The fluctuation strength Δ in our model is thus far only a parameter, but it reflects aspects of the spatially averaged complex microdynamics of the grains. Thus, the fluctuation strength Δ depends on material properties such as inelasticity and on the ratio of system size and typical grain size.

Setting the non-linearity δ equal to zero has *no significant* effect on the velocity signal and the power spectrum. The power spectra for $\delta = 0$ agree within line width with the power spectra for $\delta = 0.1$ depicted in figs. 2 and 3. Therefore, unlike in fast rotated drums [7], the solid-like component in the friction coefficient is a dominant macromechanical friction mechanism. Moreover, the fluctuation strength Δ is *effectively* the *only adjustable parameter* in our model; all other model parameters values are taken from the experiment [2].

Conclusions. – We have investigated the power spectrum of the velocity of global avalanches along granular piles in slowly rotated drums within a comparably simple (physically motivated) stochastic dynamical system. The power spectrum reflects a combination of deterministic mechanisms (frictional surface flow, viscoplastic yield) and small macromechanical (but micromechanically generated) stochasticity. The inclusion of stochastics leads to a *non-universal* roll-off of the power spectrum for large frequencies. The roll-off varies from a $1/f^4$ decay for $\Delta = 0$ to a $1/f^2$ decay for comparably large $\Delta = 5 \times 10^{-3}$. For intermediate Δ there is a crossover of these to different decay behaviors. For $\Delta = 5 \times 10^{-4}$, there is an almost perfect agreement of our model results and experimental results [2]. We hope that our results stimulate further work on power spectra in the flow-over-the-rim experiments as well as investigations on the micromechanical roots of the macromechanical stochastics.

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The authors acknowledge support by the "Graduiertenkolleg: Nichtlineare Probleme in Analysis, Geometrie und Physik" (GRK 283) financed by DFG and the state of Bavaria.

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