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# Transformations of nonlinear dynamical systems to jerky motion and its application to minimal chaotic flows

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Third-order explicit autonomous differential equations in one scalar variable or, mechanically interpreted, jerky dynamics constitute an interesting subclass of dynamical systems that can exhibit many major features of regular and irregular or chaotic dynamical behavior. In this paper, we investigate the circumstances under which three dimensional autonomous dynamical systems possess at least one equivalent jerky dynamics. In particular, we determine a wide class of three-dimensional vector fields with polynomial and non-polynomial nonlinearities that possess this property. Taking advantage of this general result, we focus on the jerky dynamics of Sprott's minimal chaotic dynamical systems and Rössler's toroidal chaos model. Based on the interrelation between the jerky dynamics of these models, we classify them according to their increasing polynomial complexity. Finally, we also provide a simple criterion that excludes chaotic dynamics for some classes of jerky dynamics and, therefore, also for some classes of three-dimensional dynamical systems. [S1063-651X(98)09710-4]

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## I. INTRODUCTION

Since Lorenz's discovery of the appearance of deterministic nonperiodic flow in 1963 and the emergence of chaos in the middle of the 1970s, autonomous three-dimensional dynamical systems play an outstanding role in modern nonlinear dynamics [1–6]. These systems are still low dimensional enough that their long-time behavior, the attractor, can be visualized in the three-dimensional phase space. They are, however, already complicated enough to exhibit a plethora of complex dynamical behavior such as quasiperiodic and irregular or chaotic oscillations. By virtue of the Poincaré-Bendixson theorem [2], two-dimensional nonlinear dynamical systems can only possess fixed points or periodic solutions as long-time solutions. Therefore, the transition from the phase space dimension two to the phase space dimension three opens a whole new world of dynamical behavior.

During the last two decades, there has been an immense effort and success towards the identification and understanding of irregular or chaotic dynamics, including the routes to irregularity. In this context, the *geometric theory of dynamics* [2–5] that analyzes dynamical vector fields in terms of their flow in phase space, and its numerical counterpart have been proven to be particularly powerful.

There are, however, still many open basic problems even for the case of three-dimensional dynamical systems. For example, how can we decide only on the basis of the functional form of a given three-dimensional dynamical system whether it might possess irregular dynamics for some ranges of its parameters? Another example deals with *minimal chaotic flows*: What are the minimal functional forms of nonlinearities in a three-dimensional dynamical system that are needed for a chaotic flow?

In 1994, a seminal investigation towards an identification of minimal chaotic systems was reported by Sprott [7]. Using a numerical search for three-dimensional vector fields with only quadratic nonlinearities, Sprott was able to identify nineteen distinct functional forms of dynamical systems (labeled A to S) that show irregular dynamics and are all functionally simpler than the paradigms of nonlinear dynamics, the Lorenz model [8], and the Rössler model [9].

In several recent papers [10–14], the problem of minimal chaotic dynamics has been attacked from a quite different point of view. Here, the starting point has not been dynamical systems or flows, but third-order explicit scalar ordinary differential equations or, suggestively speaking, *jerky dynamics*. It is well known that any explicit ordinary differential equation can be recast in the form of a dynamical system although the contrary does not hold in general. Therefore, jerky dynamics should also have the potential to show irregular evolution in time.

By performing a similar procedure as in Ref. [7], Sprott [10,11] was able to identify minimal polynomial dissipative and conservative jerky dynamics that show chaotic behavior. Surprisingly, one quadratic nonlinearity suffices to generate irregular evolution in time for some parameter values. Similar results have also been stated by Linz [12] on the basis of the jerky dynamics for Sprott's model R [7]. In this paper, it has also been shown that the jerky dynamics for the Lorenz [8] and the Rössler [9] model possess a functionally complicated form. For an interesting introduction into jerky dynamics with reference to the above-mentioned studies [10–12], we refer to the popular article by von Baeyer [15].

Not taken literally, jerky dynamics can also be found in nonmechanical disciplines of physics. Probably the first work concerning third-order differential equations that can show irregular dynamics traces back to Moore and Spiegel [16] and appeared in the context of a simple oscillator model of thermal convection. Jerky dynamics also appear, for example, in the context of the single-mode equations for a semiconductor laser subject to large optical injection, as reported by Erneux *et al.* [17], in geometric models for dendrite growth subject to special boundary conditions as discussed by Kruskal and Segur [18], and for the nonrelativistic motion of a radiative point charge subject to an external force (the Abraham-Lorentz equation) [19].

Since jerky dynamics (i) show all major features that three-dimensional vector fields possess, (ii) are conceptionally simpler than dynamical systems (as we will see in this study), and (iii) are the natural generalization of oscillator dynamics, they might serve as a useful tool to obtain further insight into nonchaotic and chaotic behavior including the routes to chaos.

In this context, the following questions that are the subject of our study arise. (i) Which three-dimensional dynamical systems can be recast into a jerky dynamics? (ii) Do apparently functionally different dynamical systems obey similar or even identical jerky dynamics? (iii) If so, can jerky dynamics be used as a tool to classify dynamical systems? (iv) Can we learn anything about the possible time evolution just by looking at the functional form of a jerky dynamics?

Our paper is organized as follows. In Sec. II we fix the notation and discuss some general circumstances under which a dynamical system cannot be transformed to uniquely determined jerky dynamics. Section III contains, as a major result, a wide class of nonlinear and not necessarily polynomial three-dimensional dynamical systems that can be recast into a jerky dynamics. Moreover, a *systematic* method of finding the jerky dynamics is also given. Taking advantage of the results of Sec. III, we derive in Sec. IV all existing jerky dynamics for Sprott's minimal chaotic models [7] and the toroidal Rössler model [20]. Based on similarities of their functional forms, we classify these jerky dynamics according to their polynomial complexity. We also present simple conditions under which chaotic behavior is excluded. These are based on an elementary no-chaos theorem given in Appendix C. In Sec. V we summarize our findings.

## II. BASICS

Generally speaking, an autonomous dynamical system is specified by a set of $n$ coupled first-order, ordinary differential equations (ODEs) that are not explicitly dependent on time $t$. In particular, three-dimensional dynamical systems are specified by

$$\dot{\mathbf{x}}=\mathbf{V}(\mathbf{x}), \tag{1}$$

where $\mathbf{x}=(x,y,z)^{\mathsf{T}}$ denotes a point in a three-dimensional phase space $\Gamma\subseteq\mathbb{R}^3$, $\mathbf{V}(\mathbf{x})$ the, in general, nonlinear vector field of the dynamical system and the overdot the derivative with respect to time. Specifying the initial conditions $\mathbf{x}(t=0)=\mathbf{x}_0$, $\mathbf{x}(t)$ represents the orbit or trajectory of the dynamical system (1) in the phase space. It is also a well known fact [2] that any autonomous $n$th order ODE that is given in an explicit form can be recast into an $n$-dimensional dynamical system. In particular, third-order explicit ODEs

$$\dddot{x}=J(x,\dot{x},\ddot{x}) \tag{2}$$

can be immediately transformed into a dynamical system (1) by introducing, for example, $\dot{x}=v$, $\dot{v}=a$, and $\dot{a}=J(x,v=\dot{x},a=\ddot{x})$ [2]. The contrary, however, is generally not true and constitutes the starting point of our investigation.

Motivated by the mechanical interpretation of Eq. (2) as evolution equation for the rate of change of the acceleration or the *jerk*, a third-order ODE of the form (2) that is (i) autonomous and (ii) explicit is called a *jerky dynamics*. Obviously, jerky dynamics (2) are a restricted class of all third-order ODEs and also of all third-order dynamical systems (1). As a necessary, but not sufficient requirement for a *nontrivial* jerky dynamics that is not just the derivative of a second-order explicit autonomous ODE, the jerk function must depend explicitly on $x$. Under certain constraints, in particular, when the acceleration $\ddot{x}$ enters only linearly into the jerky dynamics, it can also be interpreted as the derivative of a one-dimensional Newtonian equation with a memory term that depends on the dynamical history of the motion [13,14]. Throughout this paper, the jerk function is supposed to be an arbitrary and, in general, nonlinear function of its variables $x$, $\dot{x}$, and $\ddot{x}$ that is well defined for all $x$, $\dot{x}$, $\ddot{x}$.

One main subject of this paper is (i) to find classes of three-dimensional dynamical systems

$$\dot{x}=V_1(x,y,z), \tag{3a}$$

$$\dot{y}=V_2(x,y,z), \tag{3b}$$

$$\dot{z}=V_3(x,y,z) \tag{3c}$$

that can be recast into an *equivalent* jerky dynamics (2) and (ii) to determine a systematic transformation method to obtain Eq. (2) from Eq. (3) if it exists at all. We call a jerky dynamics in the variable $x$, Eq. (2), *equivalent* to the dynamical system (1) or (3) if, for the same initial conditions, the signals $x(t)$ generated by Eqs. (2) and (3) are identical.

Trying to calculate the equivalent jerky dynamics (2) for a dynamical system (3), four distinct situations can appear:

(i) There is no jerk function $J(x,\dot{x},\ddot{x})$ that is well defined in the sense that it is free of singularities for all $x$, $\dot{x}$, and $\ddot{x}$. Therefore, there is no equivalent jerky dynamics although a transformation to an implicit third-order ODE might be possible.

(ii) The equivalent jerky dynamics must be defined differently for distinct regions of the phase space $\Gamma$. As an example for this case consider the dynamical system $\dot{x}=x+y$, $\dot{y}=z^2$, $\dot{z}=x+xz$. Rewriting this system into a jerky dynamics (2) in $x$ leads to $\dddot{x}=\ddot{x}+2x(\ddot{x}-\dot{x})+2x\sqrt{\ddot{x}-\dot{x}}$ for the region $z\geqslant 0$ and $\dddot{x}=\ddot{x}+2x(\ddot{x}-\dot{x})-2x\sqrt{\ddot{x}-\dot{x}}$ for $z<0$. Such jerky dynamics are hard to handle and, therefore, will not be taken into consideration throughout this paper.

(iii) There is a well-defined and unique jerky dynamics in $x$ that is obtained from a dynamical system (3) by a *noninvertible* transformation of variables. An example for this case is $\dot{x}=x+y$, $\dot{y}=z^2$, and $\dot{z}=xz$. Deriving the $\dot{x}$ equation with respect to time and using $\dot{y}=z^2$, we obtain $\ddot{x}=\dot{x}+z^2$. Further derivation and insertion of the $\dot{z}$ equation yields the unique and well-defined jerky dynamics $\dddot{x}=\ddot{x}+2x(\ddot{x}-\dot{x})$. From the equations for $\dot{x}$ and $\ddot{x}$, however, we observe that two different points $(x,y,\pm z)$ of the phase space $\Gamma$ of the original dynamical system are mapped onto one single value of the jerk function $J(x,\dot{x},\ddot{x})=\ddot{x}+2x(\ddot{x}-\dot{x})$. Therefore, *two different* trajectories of the dynamical system correspond to *one* trajectory of its jerky dynamics if interpreted as a dynamical system.

(iv) There is a well-defined and unique jerky dynamics in $x$ that is obtained from a dynamical system (3) by a *globally invertible* transformation. In this case, it is clear that the topological structure, e.g., periodicity or irregularity, of the trajectories of the system (3) is transfered to the solutions of the equivalent jerky dynamics.

As an aside we note that these considerations suggest a distinction between two different, more detailed definitions of equivalence of a dynamical system (3) and a jerky dynamics (2): (i) A dynamical system (3) and a jerky dynamics (2) are *dynamically equivalent*, if both describe the same dynamical behavior of the variable $x$. (ii) A dynamical system (3) and a jerky dynamics (2) are *topologically equivalent*, if any trajectory of the dynamical system belongs exactly to one trajectory of the jerky dynamics if interpreted as dynamical system and vice versa. Dynamical equivalence requires the existence of a unique and well-defined jerk function $J(x,\dot{x},\ddot{x})$. For topological equivalence there must also exist a globally invertible, (at least) continuous transformation between the dynamical system (3) and the jerky dynamics (2). If, moreover, this transformation is a diffeomorphism, i.e., invertible and differentiable, the dynamical system (3) and the jerky dynamics (2) should be called *diffeomorphically equivalent*. Obviously, topological or diffeomorphic equivalence implies dynamical equivalence. The converse, however, is not true.

Throughout this paper, the concept of *equivalent jerky dynamics* always refers to jerky dynamics of the latter type (iv), i.e., to topologically equivalent jerky dynamics (apart from Appendix D). Moreover, the guiding reasoning in constructing classes of dynamical systems (3) that possess at least one equivalent jerky dynamics (2) is based on the existence of an invertible transformation between both.

## III. TRANSFORMABLE DYNAMICAL SYSTEMS

In this section we present a wide class of *nonlinear* dynamical systems that can be recast into a jerky dynamics by invertible and, in general, *nonlinear* transformations. As already mentioned, any jerky dynamics (2) can be rewritten in the form of a dynamical system

$$\dot{\mathbf{u}}=\mathbf{W}(\mathbf{u}) \tag{4}$$

by introducing $\mathbf{u}=(x,v,a)^{\mathsf{T}}$ and $\mathbf{W}(\mathbf{u})=[v,a,J(x,v,a)]^{\mathsf{T}}$. If there is a jerky dynamics (2) or, equivalently, a dynamical system (4) for the system (3), then there must also be a transformation $\mathbf{T}=(T_1,T_2,T_3)^{\mathsf{T}}:\mathbf{x}\mapsto\mathbf{u}$ of variables

$$\mathbf{u}=\mathbf{T}(\mathbf{x}) \tag{5}$$

that converts the original dynamical system (3) to the dynamical system (4). From Eqs. (3), (4), and (5) we can read off the components of $\mathbf{T}$,

$$T_1(\mathbf{x})=x, \tag{6a}$$

$$T_2(\mathbf{x})=V_1(\mathbf{x}), \tag{6b}$$

$$T_3(\mathbf{x})=\dot{\mathbf{x}}\cdot\nabla V_1(\mathbf{x})=\mathbf{V}(\mathbf{x})\cdot\nabla V_1(\mathbf{x}), \tag{6c}$$

with $\nabla=(\partial_x,\partial_y,\partial_z)^{\mathsf{T}}$. Obviously, the transformation $\mathbf{T}$ depends on the structure of the vector field $\mathbf{V}(\mathbf{x})$ of the original dynamical system (3). It can be calculated for any arbitrary vector field. In order to find an equivalent jerky dynamics from $\dot{\mathbf{x}}=\mathbf{V}(\mathbf{x})$, however, there must also be a uniquely determined inverse transformation $\mathbf{T}^{-1}=(T_1^{-1},T_2^{-1},T_3^{-1})^{\mathsf{T}}$: $\mathbf{u}\mapsto\mathbf{x}$ given by

$$\mathbf{x}=\mathbf{T}^{-1}(\mathbf{u}) \tag{7}$$

such that $\mathbf{T}^{-1}$ maps the system (4) onto (3). By virtue of Eqs. (6), the condition of invertibility of $\mathbf{T}$ is effectively a constraint on the general form of the vector field $\mathbf{V}(\mathbf{x})$, and, therefore, defines the dynamical systems that possess an equivalent jerky dynamics.

In the following, it proves convenient to distinguish explicitly between the linear and nonlinear parts of the vector field. Therefore, we write it in the general form

$$\mathbf{V}(\mathbf{x})=\mathbf{c}+\mathsf{B}\mathbf{x}+\mathbf{n}(\mathbf{x}), \tag{8}$$

where $\mathbf{c}\in\mathbb{R}^3$ is a vector of constants, $\mathsf{B}\in\mathbb{R}^{3\times3}$ a matrix with constant coefficients $b_{ij}$ $(i,j=1,2,3)$ and $\mathbf{n}(\mathbf{x})=[n_1(\mathbf{x}),n_2(\mathbf{x}),n_3(\mathbf{x})]^{\mathsf{T}}$ a three-dimensional vector of solely nonlinear functions in $x,y,z$ that are at least twice differentiable and do not contain additive constants. Then, the following holds.

*Theorem.* Any dynamical system of the functional form

$$\dot{x}=c_1+b_{11}x+b_{12}y+b_{13}z+n_1(x), \tag{9a}$$

$$\dot{y}=c_2+b_{21}x+b_{22}y+b_{23}z+n_2(\mathbf{x}), \tag{9b}$$

$$\dot{z}=c_3+b_{31}x+b_{32}y+b_{33}z+n_3(\mathbf{x}) \tag{9c}$$

with $n_i$ $(i=1,2,3)$ being nonlinear functions of the indicated arguments can be reduced to a jerky dynamics, $\dddot{x}=J(x,\dot{x},\ddot{x})$, if the conditions

$$b_{12}n_2(\mathbf{x})+b_{13}n_3(\mathbf{x})=f(x,b_{12}y+b_{13}z) \tag{10a}$$

with $f$ being an arbitrary function of the indicated arguments and

$$b_{12}^2b_{23}-b_{13}^2b_{32}+b_{12}b_{13}(b_{33}-b_{22})\neq0 \tag{10b}$$

hold.

Before we prove the statement we remark the following. (i) From Eq. (9a) we see that the variables $y$ and $z$ are only allowed to enter *linearly* into the $\dot{x}$ equation. (ii) An important special case of the dynamical system (9) is obtained by setting $b_{13}=0$. Then the condition (10b) reduces to

$$b_{12}\neq0,\quad b_{23}\neq0. \tag{11}$$

Moreover, it follows from Eq. (10a) that in this case the nonlinearity $n_2(\mathbf{x})$ is solely a function of $x$ and $y$, while $n_3(\mathbf{x})$ can be an arbitrary function of $\mathbf{x}$. Altogether, the dynamical system (9) therefore reads

$$\dot{x}=c_1+b_{11}x+b_{12}y+n_1(x), \tag{12a}$$

$$\dot{y}=c_2+b_{21}x+b_{22}y+b_{23}z+n_2(x,y), \tag{12b}$$

$$\dot{z}=c_3+b_{31}x+b_{32}y+b_{33}z+n_3(\mathbf{x}). \tag{12c}$$

The functional form of Eqs. (12) can also be obtained from Eq. (9) by using the obviously invertible transformation $\eta=y+(b_{13}/b_{12})z$, since Eq. (10a) is valid and, therefore, the function $f(x,b_{12}y+b_{13}z)$ corresponds to the nonlinearity $n_2(x,\eta)$ in the new $\dot{\eta}$ equation.

*Proof.* The demonstration of the theorem requires three steps: the calculation of the transformation $\mathbf{T}$ and its inverse $\mathbf{T}^{-1}$ and, finally, the derivation of the jerky dynamics. The transformation $\mathbf{T}$ can immediately be obtained from the dynamical system (9) by virtue of Eqs. (6),

$$T_1(\mathbf{x})=x, \tag{13a}$$

$$T_2(\mathbf{x})=c_1+\mathbf{b}^1\cdot\mathbf{x}+n_1(x), \tag{13b}$$

$$T_3(\mathbf{x})=\mathbf{c}\cdot\mathbf{b}^1+\mathbf{b}^1\cdot\mathbf{b}_1x+\mathbf{b}^1\cdot\mathbf{b}_2y+\mathbf{b}^1\cdot\mathbf{b}_3z+\mathbf{b}^1\cdot\mathbf{n}(\mathbf{x})$$
$$+[c_1+\mathbf{b}^1\cdot\mathbf{x}+n_1(x)]\partial_x n_1(x). \tag{13c}$$

For convenience, we have introduced in Eq. (13) the notation

$$\mathbf{b}^i=(b_{i1},b_{i2},b_{i3})^{\mathsf{T}}\quad(i=1,2,3) \tag{14a}$$

for the row vectors and

$$\mathbf{b}_j=(b_{1j},b_{2j},b_{3j})^{\mathsf{T}}\quad(j=1,2,3) \tag{14b}$$

for the column vectors of the matrix $\mathsf{B}=(b_{ij})$ introduced in Eq. (8). The dot denotes the scalar product.

To calculate the inverse transformation $\mathbf{T}^{-1}$, we have to solve Eqs. (13) with respect to $x$, $y$, and $z$. Since $x=T_1(\mathbf{x})$ and Eq. (13a) hold and, therefore, $T_1$ maps $x$ only onto itself, solely the second and third components of $\mathbf{T}$ and the variables $y$ and $z$ need to be considered while $x$ can be handled like a simple parameter. To solve Eqs. (13b) and (13c) with respect to $y$ and $z$, both variables should enter only linearly into $T_2$ and $T_3$. In $T_3$, however, nonlinear terms are present that contain $y$ and $z$. Since we want to determine $y$ and $z$ as functions of $\mathbf{u}$, this does not cause a problem if these nonlinearities can be replaced by terms of $\mathbf{u}$. This explains why one has to demand the condition (10a) since in this case the part $\mathbf{b}^1\cdot\mathbf{n}(\mathbf{x})$ of Eq. (13c) can be written as $b_{11}n_1(x)+f(x,b_{12}y+b_{13}z)$. Using Eq. (13b) and $v=T_2(\mathbf{x})$, the second argument of $f$ can be substituted by an expression that solely depends on $x$ and $v$. Next, again using $v=T_2(\mathbf{x})$ and Eq. (13b) we can rewrite the second line of Eq. (13c) as $v\,\partial_x n_1(x)$. As a consequence, $T_3$ depends linearly on $y$ and $z$. Taking into account Eq. (5), the second and third component of the transformation $\mathbf{T}$, (13b) and (13c), can therefore be written as

$$\begin{pmatrix} v \\ a \end{pmatrix}=\begin{pmatrix} r(x) \\ s(x,v) \end{pmatrix}+\mathsf{M}\begin{pmatrix} y \\ z \end{pmatrix} \tag{15}$$

with the abbreviations

$$r(x)=c_1+b_{11}x+n_1(x), \tag{16a}$$

$$s(x,v)=\mathbf{c}\cdot\mathbf{b}^1+\mathbf{b}^1\cdot\mathbf{b}_1x+v\,\partial_x n_1(x)+b_{11}n_1(x)$$
$$+f(x,v-r(x)) \tag{16b}$$

and the matrix

$$\mathsf{M}=\begin{pmatrix} b_{12} & b_{13} \\ \mathbf{b}^1\cdot\mathbf{b}_2 & \mathbf{b}^1\cdot\mathbf{b}_3 \end{pmatrix}. \tag{17}$$

Therefore, the problem of calculating $\mathbf{T}^{-1}$ is reduced to a simple matrix inversion. The condition that is necessary for invertibility of $\mathsf{M}$ is given by $\det\mathsf{M}=b_{12}^2b_{23}-b_{13}^2b_{32}+b_{12}b_{13}(b_{33}-b_{22})\neq 0$. This is exactly the condition (10b). Since we require that Eq. (10b) holds, it follows that the inverse of $\mathsf{M}$ and, therefore, also the inverse transformation $\mathbf{T}^{-1}$ exist. Consequently, from Eq. (15) one can finally calculate $\mathbf{T}^{-1}$ by additionally taking into account Eq. (7). The result reads

$$T_1^{-1}(\mathbf{u})=x, \tag{18a}$$

$$T_2^{-1}(\mathbf{u})=\{\mathbf{c}\cdot\mathbf{b}^1b_{13}-c_1\mathbf{b}^1\cdot\mathbf{b}_3+(b_{12}A_{32}-b_{13}A_{22})x$$
$$+[\mathbf{b}^1\cdot\mathbf{b}_3+b_{13}\partial_x n_1(x)]v-b_{13}a$$
$$-(b_{12}b_{23}+b_{13}b_{33})n_1(x)$$
$$+b_{13}f(x,v-r(x))\}/\det\mathsf{M}, \tag{18b}$$

$$T_3^{-1}(\mathbf{u})=\{c_1\mathbf{b}^1\cdot\mathbf{b}_2-\mathbf{c}\cdot\mathbf{b}^1b_{12}+(b_{12}A_{33}-b_{13}A_{23})x$$
$$-[\mathbf{b}^1\cdot\mathbf{b}_2+b_{12}\partial_x n_1(x)]v+b_{12}a$$
$$+(b_{12}b_{22}+b_{13}b_{32})n_1(x)$$
$$-b_{12}f(x,v-r(x))\}/\det\mathsf{M}, \tag{18c}$$

where $A_{ij}$ denotes the *adjunct* or the cofactor to the element $b_{ij}$ of the matrix $\mathsf{B}$ [21,22].

The general form of the jerky dynamics corresponding to the dynamical system (9) can be obtained by the following procedure. The derivative of the third component of Eq. (5) with respect to time reads

$$\dot{a}=\dddot{x}=\mathbf{V}(\mathbf{x})\cdot\nabla T_3(\mathbf{x}). \tag{19}$$

Since $\dot{a}=\dddot{x}=J(\mathbf{u})$ holds, Eq. (19) yields the jerk function $J(\mathbf{u})$. The expression (19), however, still depends on $y$ and $z$, so that we must insert the inverses $y=T_2^{-1}(\mathbf{u})$ and $z=T_3^{-1}(\mathbf{u})$ to obtain $J$ as a function of $\mathbf{u}$. A straightforward, but somewhat tedious calculation then leads to the final result for the jerky dynamics. Using the matrix $\mathsf{A}=(A_{ij})$ (with the adjuncts $A_{ij}$ of $\mathsf{B}$) and accordingly to Eq. (14) defined row and column vectors $\mathbf{A}^i$ and $\mathbf{A}_j$, it reads

$$J(\mathbf{u})=g(x,v)a+h(x,v)v+k(\mathbf{u}) \tag{20}$$

with

$$g(x,v)=\mathrm{tr}\mathsf{B}+\partial_x n_1(x)+f'[x,v-r(x)], \tag{21a}$$

$$h(x,v) = -\mathrm{tr}\mathsf{A} - (b_{22}+b_{33})\partial_x n_1(x) + \partial_x f(x,v-r(x)) + v\,\partial_x^2 n_1(x) + [b_{11}+\partial_x n_1(x)]f'(x,v-r(x)), \tag{21b}$$

$$k(\mathbf{u}) = \mathbf{c}\cdot\mathbf{A}_1 + \mathbf{b}^1\cdot\mathbf{A}^1 x + \mathbf{A}_1\cdot\mathbf{n}(x,T_2^{-1}(\mathbf{u}),T_3^{-1}(\mathbf{u})), \tag{21c}$$

and $\partial_x f$ denoting the derivative with respect to $x$ only of the first argument of $f$, $f'$ the derivative with respect to the second argument of $f$ and tr the trace of a matrix. If the nonlinear functions $n_2$ or $n_3$ in $k(\mathbf{u})$ depend on $y$ or $z$, one has to insert the inverse transformations $y=T_2^{-1}(\mathbf{u})$ and $z=T_3^{-1}(\mathbf{u})$, Eqs. (18b) and (18c). This completes the proof.

Several remarks are in order.

(i) If the condition (10b) does not hold, i.e., $b_{12}^2 b_{23} - b_{13}^2 b_{32} + b_{12}b_{13}(b_{33}-b_{22}) \equiv 0$, the corresponding dynamical system (9) describes an effectively two-dimensional dynamics. This can be seen as follows. If both $b_{12}$ and $b_{13}$ are equal to zero, the first component of Eq. (15) reduces to the first-order ODE $v=\dot{x}=r(x)$. If, however, e.g., $b_{12}\neq 0$ holds, one can solve the first of Eqs. (15) with respect to $y$ and insert the resulting equation into the second component of Eq. (15). Then, due to $b_{12}^2 b_{23} - b_{13}^2 b_{32} + b_{12}b_{13}(b_{33}-b_{22}) = 0$, the $z$ term vanishes and it remains the second-order ODE $a=\ddot{x}=s(x,\dot{x}) + (1/b_{12})\mathbf{b}^1\cdot\mathbf{b}_2[\dot{x}-r(x)]$.

(ii) For the special case $b_{13}=0$, i.e., no dependence on $z$ in Eq. (9a), Eq. (18b) does not depend on $a$ and $f(x,v-r(x))$. Moreover, in this case the nonlinear function $n_3(\mathbf{x})$ does not appear in the transformation $\mathbf{T}$, Eqs. (13). Therefore, also the inverse $\mathbf{T}^{-1}$ cannot contain $n_3(\mathbf{x})$. This can be seen from Eq. (10a), which reduces to $f(x,b_{12}y) = b_{12}n_2(x,y)$ for $b_{13}=0$. Consequently, Eqs. (21a) and (21b) do also not depend on $n_3(\mathbf{x})$ if $b_{13}=0$ holds. Additionally, one obtains $f' = \partial_y n_2$ for the derivative $f'$.

(iii) The part $g(x,v)a$ of the jerk function (20) does not contain all terms that are linear in $a$. Linear and nonlinear terms in $a$ can also appear in $k(\mathbf{u})$ after insertion of the inverses $T_2^{-1}$ and $T_3^{-1}$ into the nonlinear functions $\mathbf{n}(\mathbf{x})$. In the same way also linear and nonlinear terms in $v$ can be contained in $k(\mathbf{u})$.

(iv) From Eq. (21c) follows that an additive constant term in the jerk function (20) can only appear if $\mathbf{c}\neq\mathbf{0}$ and, therefore, if the original dynamical system also contains an additive constant term.

(v) Consider the functions $\mathbf{n}(\mathbf{x})$ to be polynomials of a certain degree $d>1$. Then, the transformation to the jerky dynamics does not necessarily conserve the degree of the polynomials entering into the jerk function.

(vi) Two special cases are included in the functional forms of the dynamical systems, Eqs. (9) and (12), the transformations, Eqs. (13) and (18), and the jerk function, Eq. (20). First, by setting $\mathbf{n}(\mathbf{x})\equiv\mathbf{0}$ in Eq. (9), one directly infers that any *linear* dynamical system can always be converted into an equivalent jerky dynamics in $x$ if it fulfills the condition (10b). Then, we can read off the jerk function from Eqs. (20) and (21),

$$J(\mathbf{u}) = \mathbf{c}\cdot\mathbf{A}_1 + \mathbf{b}^1\cdot\mathbf{A}^1 x - \mathrm{tr}\mathsf{A}v + \mathrm{tr}\mathsf{B}a. \tag{22}$$

Needless to mention, the jerk function of linear dynamical systems is also linear. The second special case refers to *nonlinear* dynamical systems that can be converted into a jerky dynamics by *linear* transformations. Only if the conditions $n_1(x)\equiv 0$ and $n_2(x,y)\equiv 0$ hold for the system (12), the transformation $\mathbf{T}$, Eq. (13), and its inverse $\mathbf{T}^{-1}$, Eq. (18), are linear. This agrees with results found in Ref. [13] where, however, only Newtonian jerky dynamics were considered and, therefore, also the general functional form of $n_3(\mathbf{x})$ has been restricted.

(vii) It is possible that a dynamical system that is contained in the class specified by (9) can be converted simultaneously into two or even three jerky dynamics in different variables (the jerky dynamics in one certain variable is unique, if it exists). Then, however, additional restrictions apply. From the structure of the dynamical systems that possess a jerky dynamics in $x$, Eqs. (9), we see that for a simultaneous existence of a jerky dynamics in $y$ and/or $z$, it is necessary that $n_2(\mathbf{x})=n_2(y)$ and/or $n_3(\mathbf{x})=n_3(z)$ hold. Moreover, there are additional conditions for each variable $y$ and/or $z$ that are similar to Eq. (10). Details are given in Appendix B. For the case of three simultaneous jerky dynamics, the set of all conditions leads to two formally different dynamical systems (apart from certain permutations of variables). Also here, we refer to Appendix B, in particular, Eqs. (B4) and (B6).

The above theorem constitutes an important tool in two respects. (i) Given a specific dynamical system, one can decide only on the basis of the functional form of its vector field $\mathbf{V}(\mathbf{x})$ if it belongs to the class specified by Eq. (9) and, therefore, possesses an equivalent jerky dynamics in $x$. Here, also an exchange of the variables $y$ and $z$ and the indices 2 and 3, respectively, has to be taken into account. The jerky dynamics can immediately be calculated by using Eqs. (20) and (21). (ii) It can be possible that the given dynamical system possesses a jerky dynamics in $y$ or $z$, but not in $x$. This can also be verified with the help of the Eqs. (9) by considering all permutations of variables $(x,y,z)$ and indices $(1,2,3)$, respectively. If, e.g., after exchanging $x$ and $y$ (and the indices 1 and 2), the given dynamical system is of the form (9), we can conclude that it possesses a jerky dynamics in $y$. Equation (20) can be used to determine it.

## IV. MINIMAL CHAOTIC FLOWS

In this section, we apply the results of Sec. III on the transformability of the dynamical systems, Eq. (9), to the nonlinear dynamical systems A to S found by Sprott [7] and a system of Rössler [20] that exhibits toroidal chaos (denoted by TR). These models are minimal dynamical systems that can show chaotic behavior for some parameter range where minimal is understood in an algebraic sense. They have only five terms with two quadratic nonlinearities (models A to E) or six terms with one quadratic nonlinearity (models F to S and TR). Moreover, Sprott also has found dynamical systems with five terms and only one quadratic nonlinearity that are chaotic in a certain parameter range [10,11]. These models are already given in form of a jerky dynamics. Zhang and Heidel [23] have shown that three-dimensional dissipative quadratic systems with less than five terms cannot exhibit chaotic behavior.

The nineteen models of Sprott and the toroidal Rössler model are given in the second column of Table I. In Sprott's models [7] we have substituted all coefficients that are not equal to $\pm 1$ as well as numerical constants by parameters that are denoted by greek letters. Using the results of Sec. III, we infer that the models A to C and E do not belong to the class of dynamical systems (9). On the other hand, we can analytically calculate all existing equivalent jerky dynamics for each of the systems D and F to S and TR. The resulting jerky dynamics as well as the corresponding transformations $\mathbf{T}$ and their inverses $\mathbf{T}^{-1}$ are also given in the third, fourth, and fifth columns of Table I. The method of comprehensive Gröbner bases described in appendix D has been used to verify whether there are additionally equivalent jerky dynamics that are not contained in the class (9) and, simultaneously, to check the analytical results. It turns out that *all* existing jerky dynamics are of the type being described by Eqs. (9).

The entry ''none'' for the models A, B, C, and E means that there are *no* equivalent jerky dynamics for these systems in the sense that the jerk function is a nonsingular and polynomial expression. Therefore, only one of the models with two quadratic nonlinearities, namely, system D, can be converted into an equivalent jerky dynamics. This is mainly due to the fact that two of the total of five terms contained in these models are nonlinear. This leads, in general, to a lack of sufficiently many linear terms that are necessary for the existence of the inverse transformation. On the other hand, any system with six terms and only one nonlinearity (F to S and TR) can be recast into at least one jerky dynamics. The models F, I, and L possess even two.

If, for a certain model, Eq. (10b) yields a condition on the parameters, it is possible that the inverse transformation, and, therefore, also the jerky dynamics, does not exist for some parameter values. In fact, this occurs for the system F if $\alpha=-1$ and for I, L, and S if $\alpha=0$. In the cases I, L, and S, this can also be read off from the form of the dynamical system since, for $\alpha=0$, they do not describe an effectively three-dimensional dynamical behavior. Therefore, a jerky dynamics is not well defined in these cases. This fact transfers to the specific structure of the corresponding jerky dynamics, which reduce to one- or two-dimensional dynamical equations or are not defined if $\alpha=0$. As well, the dynamics of model F is effectively two dimensional if $\alpha=-1$. This can be seen from the corresponding transformation that leads to the jerky dynamics in $x$.

From Table I one can also see that most of the transformations are linear (models D, F (the jerky dynamics in $x$), G, I, L, M, O, Q, R, S, and TR), solely one contains a cubic term (model P) and the remaining are quadratic. In contrast to that, all derived jerky dynamics have only quadratic nonlinearities. Moreover, it is interesting that not all of the six possible quadratic combinations of $x$, $\dot{x}$, $\ddot{x}$ appear in the models. Terms like $\dot{x}\ddot{x}$ and $\ddot{x}^2$ are missing. The absence of the $\ddot{x}^2$ term means that all models are even Newtonian jerky [14].

### A. Relations between the jerky dynamics

Comparing the jerky dynamics of Table I, one observes that some of them look rather similar as far as the functional form of the jerk functions is concerned. Consider, e.g., the jerky dynamics of system J and the jerky dynamics in $y$ of model I. Both consist of the same terms; they only differ with respect to their coefficients. Also the jerky dynamics of model L (in $x$) and of N are of the same functional form apart from an additional nonzero constant. The systems F and H possess even *identical* jerky dynamics. Furthermore, there are three models (F, I, and L) that possess two equivalent jerky dynamics that contain similar terms. These observations raise the question of whether there are relations between different (but similar) jerky dynamics, or if there are even invertible transformations that map them onto each other.

To discuss this point in more detail, we consider a transformation $\mathbf{T}_J:(x,\dot{x},\ddot{x})\mapsto(\xi,\dot{\xi},\ddot{\xi})$ with

$$\xi=T_{J,1}(x,\dot{x},\ddot{x}), \tag{23a}$$

$$\dot{\xi}=T_{J,2}(x,\dot{x},\ddot{x}), \tag{23b}$$

$$\ddot{\xi}=T_{J,3}(x,\dot{x},\ddot{x}) \tag{23c}$$

that maps a jerky dynamics $\dddot{x}=J_x(x,\dot{x},\ddot{x})$ onto another jerky dynamics $\dddot{\xi}=J_\xi(\xi,\dot{\xi},\ddot{\xi})$. Such a transformation is completely determined by its first component $T_{J,1}$ and the jerk function $J_x$. The second and third components $T_{J,2}$ and $T_{J,3}$ are only derivatives of $T_{J,1}$ with respect to time and the jerk function $J_x$ must be used to substitute $\dddot{x}$ terms that appear after each derivation. Therefore, the transformation (23) contains only one independent component that can be chosen as $\xi=T_{J,1}(x,\dot{x},\ddot{x})$. This property and the condition of invertibility strongly restrict the class of possible transformations (23) between different jerky dynamics. Therefore, it is only possible for very special cases to convert similar jerky dynamics to each other by invertible transformations.

However, *different* jerky dynamics that belong to the *same* dynamical system can always be transformed to each other by an invertible transformation. This follows from the fact that the jerky dynamics themselves are obtained from the dynamical system via invertible transformations. For illustration, assume that we have a dynamical system that possesses two equivalent jerky dynamics (like the models F, I, and L) for its variables, say, $x$ and $y$. Suppose that the jerky dynamics are calculated from the dynamical system by the invertible transformations $\mathbf{T}_x:(x,y,z)\mapsto(x,\dot{x},\ddot{x})$ and $\mathbf{T}_y:(x,y,z)\mapsto(y,\dot{y},\ddot{y})$. Then, we can immediately write down the transformation $\mathbf{T}_J$ that maps the jerky dynamics in $x$ to the one in $y$ as a combination of $\mathbf{T}_x^{-1}$ and $\mathbf{T}_y$, $\mathbf{T}_J=\mathbf{T}_y\circ\mathbf{T}_x^{-1}$. Moreover, since we have $y=T_{J,1}(x,\dot{x},\ddot{x})$ and also $y=T_{x,2}^{-1}(x,\dot{x},\ddot{x})$, one can immediately read off the characteristic first component of $\mathbf{T}_J$ from $\mathbf{T}_x^{-1}$. Analoguous arguments are valid for the inverse $\mathbf{T}_J^{-1}$.

For model F, the first component of the transformation between its jerky dynamics reads (cf. Table I) $x=-\dot{y}+\alpha y$ (or $y=[1/(1+\alpha)](\ddot{x}+\dot{x}+x-x^2)$ for the inverse), where $\alpha\neq-1$ must hold. Invertibility of this transformation is based on the property of the jerky dynamics for $y$ that it can be written as $\dddot{y}=(\alpha-1)\ddot{y}+(\alpha-1)\dot{y}-y-(-\dot{y}+\alpha y)^2$.

TABLE I. Minimal models and their jerky dynamics.

| Model | Equations | Jerky dynamics | Transformations | Inverse transformations |
|---|---|---|---|---|
| A | $\dot{x}=y$<br>$\dot{y}=-x+yz$<br>$\dot{z}=\alpha-y^2$ | none | none | none |
| B | $\dot{x}=yz$<br>$\dot{y}=x-y$<br>$\dot{z}=\alpha-xy$ | none | none | none |
| C | $\dot{x}=yz$<br>$\dot{y}=x-y$<br>$\dot{z}=\alpha-x^2$ | none | none | none |
| D | $\dot{x}=-y$<br>$\dot{y}=x+z$<br>$\dot{z}=xz+\alpha y^2$ | $\dddot{x}=x\ddot{x}-\dot{x}-\alpha\dot{x}^2+x^2$ | $\dot{x}=-y$<br>$\ddot{x}=-x-z$ | $y=-\dot{x}$<br>$z=-x-\ddot{x}$ |
| E | $\dot{x}=yz$<br>$\dot{y}=x^2-y$<br>$\dot{z}=\alpha-\beta x$ | none | none | none |
| F | $\dot{x}=y+z$<br>$\dot{y}=-x+\alpha y$<br>$\dot{z}=x^2-z$ | $\dddot{x}=(\alpha-1)\ddot{x}+(\alpha-1)\dot{x}-x$<br>$-\alpha x^2+2x\dot{x}$<br>$\dddot{y}=(\alpha-1)\ddot{y}+(\alpha-1)\dot{y}-y$<br>$-\alpha^2y^2+2\alpha y\dot{y}-\dot{y}^2$ | $\dot{x}=y+z$<br>$\ddot{x}=-x+x^2+\alpha y-z$<br>$\dot{y}=\alpha y-x$<br>$\ddot{y}=(\alpha^2-1)y-\alpha x-z$ | $y=\frac{1}{1+\alpha}(\ddot{x}+\dot{x}+x-x^2)$<br>$z=\frac{1}{1+\alpha}(-\ddot{x}+\alpha\dot{x}-x+x^2)$<br>$x=-\dot{y}+\alpha y$<br>$z=-\ddot{y}+\alpha\dot{y}-y$ |
| G | $\dot{x}=\alpha x+z$<br>$\dot{y}=xz-y$<br>$\dot{z}=-x+y$ | $\dddot{x}=(\alpha-1)\ddot{x}+(\alpha-1)\dot{x}-x$<br>$-\alpha x^2+x\dot{x}$ | $\dot{x}=\alpha x+z$<br>$\ddot{x}=(\alpha^2-1)x+y+\alpha z$ | $y=\ddot{x}-\alpha\dot{x}+x$<br>$z=\dot{x}-\alpha x$ |
| H | $\dot{x}=-y+z^2$<br>$\dot{y}=x+\alpha y$<br>$\dot{z}=x-z$ | $\dddot{z}=(\alpha-1)\ddot{z}+(\alpha-1)\dot{z}-z$<br>$-\alpha z^2+2z\dot{z}$ | $\dot{z}=-z+x$<br>$\ddot{z}=z+z^2-x-y$ | $x=\dot{z}+z$<br>$y=-\ddot{z}-\dot{z}+z^2$ |
| I | $\dot{x}=-\alpha y$<br>$\dot{y}=x+z$<br>$\dot{z}=x+y^2-z$ | $\dddot{x}=-\ddot{x}-\alpha\dot{x}-2\alpha x-\frac{1}{\alpha}\dot{x}^2$<br>$\dddot{y}=-\ddot{y}-\alpha\dot{y}-2\alpha y+2y\dot{y}$ | $\dot{x}=-\alpha y$<br>$\ddot{x}=-\alpha x-\alpha z$<br>$\dot{y}=x+z$<br>$\ddot{y}=-\alpha y+y^2+x-z$ | $y=-\frac{1}{\alpha}\dot{x}$<br>$z=-\frac{1}{\alpha}\ddot{x}-x$<br>$x=\frac{1}{2}(\ddot{y}+\dot{y}+\alpha y-y^2)$<br>$z=\frac{1}{2}(-\ddot{y}+\dot{y}-\alpha y+y^2)$ |
| J | $\dot{x}=\alpha z$<br>$\dot{y}=-\beta y+z$<br>$\dot{z}=-x+y+y^2$ | $\dddot{y}=-\beta\ddot{y}+(1-\alpha)\dot{y}-\alpha\beta y+2y\dot{y}$ | $\dot{y}=-\beta y+z$<br>$\ddot{y}=(1+\beta^2)y+y^2-x-\beta z$ | $x=-\ddot{y}-\beta\dot{y}+y+y^2$<br>$z=-\ddot{y}-\dot{y}+y\dot{y}+y^2$ |
| K | $\dot{x}=xy-z$<br>$\dot{y}=x-y$<br>$\dot{z}=x+\alpha z$ | $\dddot{y}=(\alpha-1)\ddot{y}+(\alpha-1)\dot{y}-y$<br>$+y\ddot{y}+(2-\alpha)y\dot{y}-\alpha y^2+\dot{y}^2$ | $\dot{y}=-y+x$<br>$\ddot{y}=y-x-z+xy$ | $x=\dot{y}+y$<br>$z=-\ddot{y}-\dot{y}+y\dot{y}+y^2$ |
| L | $\dot{x}=y+\alpha z$<br>$\dot{y}=\beta x^2-y$<br>$\dot{z}=\gamma-x$ | $\dddot{x}=-\ddot{x}-\alpha\dot{x}-\alpha x+2\beta x\dot{x}+\alpha\gamma$<br>$\dddot{z}=-\ddot{z}+(2\beta\gamma-\alpha)\dot{z}-\alpha z$<br>$-\beta\dot{z}^2-\beta\gamma^2$ | $\dot{x}=y+\alpha z$<br>$\ddot{x}=-\alpha x+\beta x^2-y+\alpha\gamma$<br>$\dot{z}=\gamma-x$<br>$\ddot{z}=-\alpha z-y$ | $y=-\ddot{x}-\alpha x+\beta x^2+\alpha\gamma$<br>$z=\frac{1}{\alpha}(\ddot{x}+\dot{x}+\alpha x-\beta x^2-\alpha\gamma)$<br>$x=\gamma-\dot{z}$<br>$y=-\ddot{z}-\alpha z$ |
| M | $\dot{x}=-z$<br>$\dot{y}=-x^2-y$<br>$\dot{z}=\alpha+\beta x+y$ | $\dddot{x}=-\ddot{x}-\beta\dot{x}-\beta x+x^2-\alpha$ | $\dot{x}=-z$<br>$\ddot{x}=-\beta x-y-\alpha$ | $y=-\ddot{x}-\beta x-\alpha$<br>$z=-\dot{x}$ |

TABLE I. (*Continued*).

| Model | Equations | Jerky dynamics | Transformations | Inverse transformations |
| --- | --- | --- | --- | --- |
| N | $\dot{x}=-\alpha y$<br>$\dot{y}=x+z^2$<br>$\dot{z}=\beta+y-\gamma z$ | $\dddot{z}=-\gamma\ddot{z}-\alpha\dot{z}-\alpha\gamma z+2z\dot{z}+\alpha\beta$ | $\dot{z}=-\gamma z+y+\beta$<br>$\ddot{z}=\gamma^2 z+z^2+x-\gamma y-\beta\gamma$ | $x=\ddot{z}+\gamma\dot{z}-z^2$<br>$y=\dot{z}+\gamma z-\beta$ |
| O | $\dot{x}=y$<br>$\dot{y}=x-z$<br>$\dot{z}=x+xz+\alpha y$ | $\dddot{x}=x\ddot{x}+(1-\alpha)\dot{x}-x-x^2$ | $\dot{x}=y$<br>$\ddot{x}=x-z$ | $y=\dot{x}$<br>$z=-\ddot{x}+x$ |
| P | $\dot{x}=\alpha y+z$<br>$\dot{y}=-x+y^2$<br>$\dot{z}=x+y$ | $\dddot{y}=2y\ddot{y}+(1-\alpha)\dot{y}-y+2\dot{y}^2-y^2$ | $\dot{y}=y^2-x$<br>$\ddot{y}=2y^3-\alpha y-2yx-z$ | $x=-\dot{y}+y^2$<br>$z=-\ddot{y}+2y\dot{y}-\alpha y$ |
| Q | $\dot{x}=-z$<br>$\dot{y}=x-y$<br>$\dot{z}=\alpha x+y^2+\beta z$ | $\dddot{y}=(\beta-1)\ddot{y}+(\beta-\alpha)\dot{y}-\alpha y-y^2$ | $\dot{y}=-y+x$<br>$\ddot{y}=y-x-z$ | $x=\dot{y}+y$<br>$z=-\ddot{y}-\dot{y}$ |
| R | $\dot{x}=\alpha-y$<br>$\dot{y}=\beta+z$<br>$\dot{z}=xy-z$ | $\dddot{x}=-\ddot{x}-\alpha x+x\dot{x}-\beta$ | $\dot{x}=-y+\alpha$<br>$\ddot{x}=-z-\beta$ | $y=-\dot{x}+\alpha$<br>$z=-\ddot{x}-\beta$ |
| S | $\dot{x}=-x-\alpha y$<br>$\dot{y}=x+z^2$<br>$\dot{z}=\beta+x$ | $\dddot{z}=-\ddot{z}-\alpha\dot{z}-\alpha z^2+\alpha\beta$ | $\dot{z}=x+\beta$<br>$\ddot{z}=-x-\alpha y$ | $x=\dot{z}-\beta$<br>$y=\frac{1}{\alpha}(-\ddot{z}-\dot{z}+\beta)$ |
| TR | $\dot{x}=-y-z$<br>$\dot{y}=x$<br>$\dot{z}=\alpha(y-y^2)-\beta z$ | $\dddot{y}=-\beta\ddot{y}-\dot{y}-(\alpha+\beta)y+\alpha y^2$ | $\dot{y}=x$<br>$\ddot{y}=-y-z$ | $x=\dot{y}$<br>$z=-\ddot{y}-y$ |

The jerky dynamics of model I can be converted to each other by a transformation with $y=-(1/\alpha)\dot{x}$ as its first component (cf. Table I) [or $x=\frac{1}{2}(\ddot{y}+\dot{y}+\alpha y-y^2)$ for the inverse], where $\alpha\neq 0$ must hold. Here, the corresponding transformation is invertible, because nonlinear terms containing $x$ do not appear in the jerky dynamics for $x$. The transformation for the jerky dynamics of model L is of the same type as for model I.

As already mentioned, the dynamical systems F and H possess identical jerky dynamics apart from the labels of the variables. Therefore, these systems must be equivalent and there must be an invertible transformation between both. Relabeling the variables $(x,y,z)$ of model H by $(\xi,\eta,\zeta)$, the corresponding transformation $\mathbf{T}_{\rm FH}:(x,y,z)\mapsto(\xi,\eta,\zeta)$ reads

$$\xi=x+y+z,\quad \eta=x-(1+\alpha)y,\quad \zeta=x. \tag{24}$$

Since all transformations we use here are invertible and model F can be recast into two equivalent jerky dynamics, the second jerky dynamics of F must also be equivalent to system H. To obtain it from system H, however, one has to transform *all* variables. Therefore, it is not a jerky dynamics for H. Using the above notation, the transformation $\mathbf{T}:(\xi,\eta,\zeta)\mapsto(y,\dot{y},\ddot{y})$ reads explicitly

$$y=\frac{1}{1+\alpha}(\zeta-\eta), \tag{25a}$$

$$\dot{y}=-\frac{\alpha}{1+\alpha}\eta-\frac{1}{1+\alpha}\zeta, \tag{25b}$$

$$\ddot{y}=-\xi-\frac{\alpha^2}{1+\alpha}\eta+\frac{1}{1+\alpha}\zeta, \tag{25c}$$

where $\alpha\neq -1$ must hold.

### B. Classification of simple chaotic jerky dynamics

To discuss the relations of jerky dynamics that belong to *different* dynamical systems, we consider the linear transformation

$$\xi=k(x+c),\quad \dot{\xi}=k\dot{x},\quad \ddot{\xi}=k\ddot{x} \tag{26}$$

(with $k,c\in\mathbb{R}$, and $k\neq 0$) that moves the origin and simultaneously rescales variables. This transformation is the only one that (i) converts a jerky dynamics (for a variable $x$) into another equivalent jerky dynamics (for a new dynamical variable $\xi$), (ii) is invertible and, (iii) independent of the specific form of the jerk function $J_x(x,\dot{x},\ddot{x})$. In general, Eq. (26) does not convert the different jerky dynamics of Table I to each other, but transforms them to seven basic classes of jerky dynamics that differ by the type and the number of terms appearing in the corresponding seven jerk functions. Transformations between these different classes have not been found. In Table II the basic classes (denoted by $JD_1$ to $JD_7$) are listed as well as the models that belong to each

TABLE II. Basic classes of dissipative jerky dynamics.

| Model | | Basic classes<br>coefficients = values for which there is irregular behavior | | | | Transformation |
|---|---|---|---|---|---|---|
| $JD_1$ | | $\dddot{\xi}=k_1\ddot{\xi}+k_2\xi+\xi\dot{\xi}+k_3$ | | | | |
| | I | $k_1=-1$ | $k_2=-2\alpha=-0.4$ | $k_3=-2\alpha^2=-0.08$ | | $\xi=2y-\alpha$ |
| | J | $k_1=-\beta=-2$ | $k_2=-\alpha\beta=-4$ | $k_3=\alpha\beta(1-\alpha)=-4$ | | $\xi=2y+(1-\alpha)$ |
| | L | $k_1=-1$ | $k_2=-\alpha=-3.9$ | $k_3=\alpha(2\beta\gamma-\alpha)=-8.19$ | | $\xi=2\beta x-\alpha$ |
| | N | $k_1=-\gamma=-2$ | $k_2=-\alpha\gamma=-4$ | $k_3=\alpha(2\beta-\alpha\gamma)=-4$ | | $\xi=2z-\alpha$ |
| | R | $k_1=-1$ | $k_2=-\alpha=-0.9$ | $k_3=-\beta=-0.4$ | | $\xi=x$ |
| | SJ | $k_1=-A=-2.017$ | $k_2=-1$ | $k_3=0$ | | $\xi=v$ |
| $JD_2$ | | $\dddot{\xi}=k_1\ddot{\xi}+k_2\dot{\xi}+\xi^2+k_3$ | | | | |
| | M | $k_1=-1$ | $k_2=-\beta=-1.7$ | $k_3=-\alpha-\frac{\beta^2}{4}=-2.4225$ | | $\xi=x-\frac{\beta}{2}$ |
| | Q | $k_1=\beta-1=-0.5$ | $k_2=\beta-\alpha=-2.6$ | $k_3=-\frac{\alpha^2}{4}=-2.4025$ | | $\xi=-y-\frac{\alpha}{2}$ |
| | S | $k_1=-1$ | $k_2=-\alpha=-4$ | $k_3=-\alpha^2\beta=-16$ | | $\xi=-\alpha z$ |
| | TR | $k_1=-\beta=-0.2$ | $k_2=-1$ | $k_3=-\frac{1}{4}(\alpha+\beta)^2\approx-0.0858$ | | $\xi=\alpha y-\frac{1}{2}(\alpha+\beta)$ |
| $JD_3$ | | $\dddot{\xi}=k_1\ddot{\xi}+k_2\dot{\xi}+k_3\xi^2+\xi\dot{\xi}+k_4$ | | | | |
| | F | $k_1=\alpha-1=-0.5$ | $k_2=\alpha-\frac{1}{\alpha}-1=-2.5$ | $k_3=-\frac{\alpha}{2}=-0.25$ | $k_4=\frac{1}{2\alpha}=1$ | $\xi=2x+\frac{1}{\alpha}$ |
| | G | $k_1=\alpha-1=-0.6$ | $k_2=\alpha-\frac{1}{2\alpha}-1=-1.85$ | $k_3=-\alpha=-0.4$ | $k_4=\frac{1}{4\alpha}=0.625$ | $\xi=x+\frac{1}{2\alpha}$ |
| | H | $k_1=\alpha-1=-0.5$ | $k_2=\alpha-\frac{1}{\alpha}-1=-2.5$ | $k_3=-\frac{\alpha}{2}=-0.25$ | $k_4=\frac{1}{2\alpha}=1$ | $\xi=2z+\frac{1}{\alpha}$ |
| $JD_4$ | | $\dddot{\xi}=k_1\ddot{\xi}+k_2\dot{\xi}+k_3\xi^2+\xi\ddot{\xi}+k_4$ | | | | |
| | O | $k_1=-\frac{1}{2}$ | $k_2=1-\alpha=-1.7$ | $k_3=-1$ | $k_4=\frac{1}{4}$ | $\xi=x+\frac{1}{2}$ |
| $JD_5$ | | $\dddot{\xi}=k_1\dot{\xi}+k_2\xi^2+k_3\dot{\xi}^2+\xi\ddot{\xi}$ | | | | |
| | D | $k_1=-1$ | $k_2=1$ | $k_3=-\alpha=-3$ | | $\xi=x$ |
| $JD_6$ | | $\dddot{\xi}=k_1\ddot{\xi}+k_2\dot{\xi}+k_3\xi^2+k_4\dot{\xi}^2+\xi\ddot{\xi}+k_5$ | | | | |
| | P | $k_1=-1$<br>$k_5=\frac{1}{2}$ | $k_2=1-\alpha=-1.7$ | $k_3=-\frac{1}{2}$ | $k_4=1$ | $\xi=2y+1$ |
| $JD_7$ | | $\dddot{\xi}=k_1\ddot{\xi}+k_2\dot{\xi}+k_3\xi^2+k_4\dot{\xi}^2+k_5\xi\dot{\xi}+\xi\ddot{\xi}+k_6$ | | | | |
| | K | $k_1=\alpha-\frac{1}{2\alpha}-1\approx-2.37$<br>$k_5=2-\alpha=1.7$ | $k_2=\alpha-\frac{1}{\alpha}-\frac{1}{2}\approx-3.53$<br>$k_6=\frac{1}{4\alpha}\approx0.83$ | $k_3=-\alpha=-0.3$ | $k_4=1$ | $\xi=y+\frac{1}{2\alpha}$ |

class. Moreover, the concrete realization of the first component of the transformation (26) for each model is given. Also the simplest dissipative chaotic jerky dynamics

$$\dddot{x}=-A\ddot{x}-x+\dot{x}^2, \tag{27}$$

which Sprott has reported in Ref. [11] fit into these classes. Its jerk function consists only of three terms with one quadratic nonlinearity. Rewriting Eq. (27) as [11]

$$\dddot{v}=-A\ddot{v}-v+v\dot{v} \tag{28}$$

by differentiating it with respect to time and defining the new variable $v=2\dot{x}$, it is of the basic form $JD_1$. Equation (28) is chaotic over the same range of $A$ as is Eq. (27). Consequently, the simple jerky dynamics (28) is also listed in Table II where it is denoted by SJ. Since the jerky dynamics SJ is the simplest model of the class $JD_1$ as far as the number of terms appearing in the jerk function is concerned ($k_3=0$), it is interesting to check if there are other $JD_1$ models where the parameters $\alpha$, $\beta$, and $\gamma$ can be chosen such that they are identical to SJ, i.e., $k_1=-A$, $k_2=-1$ and $k_3=0$ are valid. It turns out that this is the case only for model N with $\alpha=A^{-1}$, $\beta=\frac{1}{2}$, and $\gamma=A$. However, by rescaling time of the jerky dynamics SJ, Eq. (28), by $t\to A^{-1}t$ (which does not change the direction of time in the relevant parameter range, since chaotic dynamics appears for positive $A$) and substituting $v$ by $A^2v$ (for $A\neq 0$), Eq. (28) turns into

$$\dddot{v}=-\ddot{v}-A^{-3}v+v\dot{v} \tag{29}$$

and still belongs to the basic class $JD_1$. Equation (29) is again equivalent to model N [by choosing $\alpha=A^{-3}$, $\beta=1/(2A^3)$, $\gamma=1$] but also to two other jerky dynamics of the class $JD_1$. These are model L [for $\alpha=A^{-3}$, $\beta\gamma=1/(2A^3)$] and model R (for $\alpha=A^{-3}$ and $\beta=0$). In particular for model R, this means that the underlying dynamical system can exhibit irregular behavior if $\beta=0$. In this case, model R has only five terms with one quadratic nonlinearity. However, the range of the parameter $\alpha=A^{-3}$ for which a route to chaotic behavior appears via a Feigenbaum scenario is rather narrow. It can be determined from the range of $A$ given in Refs. [10,11]. We infer that the jerky dynamics (29) has a period-doubling Feigenbaum route to chaos for $0.111\lesssim\alpha\lesssim 0.121$.

The jerky dynamics that belong to the same basic class are not necessarily identical. Their jerk functions do consist of the same functional form, but, in general, with different parameters or combinations of parameters as coefficients. Using the values of the parameters $\alpha$, $\beta$, and $\gamma$ for which Sprott found chaotic behavior [7], one can easily determine values of the coefficients $k_i$ of the basic jerky dynamics $JD_1$ to $JD_7$ that lead to a chaotic dynamics. These values of the coefficients are also shown in Table II. For the classes with several models, i.e., $JD_1$ to $JD_3$, we accordingly find several distinct points in the parameter space of the $k_i$'s where the corresponding jerky dynamics exhibit chaotic behavior. To find numerically such irregular behavior, the initial conditions $\xi(0)$, $\dot{\xi}(0)$, and $\ddot{\xi}(0)$ have to be chosen appropriately for each jerky dynamics. These initial conditions can be found by applying successively the transformations of Table I and Table II to the original initial conditions $x(0)=y(0)=z(0)=0.05$ (for the models D and F to S) [7] and $x(0)=0.4$, $y(0)=-0.4$, $z(0)=-0.7$ (for the model TR) [20] that has been used to generate a chaotic solution of the original models.

It should be noted that the algebraic structure of the seven classes is not uniquely determined. For reasons that will become clear below, we have chosen them such that no linear $\dot{\xi}$ term appears in $JD_1$ and no linear $\xi$ term in $JD_2$ to $JD_7$. Moreover, the coefficient of one quadratically nonlinear term of each model is chosen to be equal to $+1$. This corresponds to a rescaling of the variable $\xi$ and is achieved by an appropriate choice of $k$ in the transformation (26).

It is interesting that the transformations that convert the models of Table I to the, in general, algebraically simpler basic classes of jerky dynamics generate more complicated dynamical systems with a larger total number of terms if substituted into the original systems. Moreover, the total number of terms on the right-hand side of the basic jerky dynamics of Table II varies from four to seven and the number of nonlinear terms from one to four although all the corresponding minimal dynamical systems (except from model D) have the same number of terms and nonlinearities. Therefore, simplicity of the dynamical systems does, at least in general, not correspond to simplicity of the equivalent jerky dynamics.

### C. Conditions that exclude chaos

For the seven basic classes of jerky dynamics $JD_1$ to $JD_7$ listed in Table II, we show in this section that chaotic dynamics is excluded for some ranges of the parameters $k_i$.

According to the Appendix C, a jerky dynamics can, in general, be written as an integro-differential equation

$$\ddot{\xi}+\Omega(\xi,\dot{\xi})=\int^{t}f(\xi(\tau),\xi'(\tau),\xi''(\tau))d\tau, \tag{30}$$

where the prime denotes the derivative with respect to $\tau$. It can be proven (cf. Appendix C) that the jerky dynamics that underlies Eq. (30) cannot exhibit irregular dynamics if the integrand $f(\xi,\xi',\xi'')$ of the memory term is either positive semidefinite or negative semidefinite for all $\xi$, $\xi'$ and $\xi''$. Next, we apply this statement to the jerky dynamics of the basic models $JD_1$ to $JD_7$.

Since all these models do not contain a $\ddot{x}^2$ term, the integrand $f$ of Eq. (30) does not depend on $\xi''$, i.e., $f=f(\xi,\xi')$. For the model $JD_1$ the function $f$ reads

$$f(\xi,\xi')=k_2\xi+k_3 \tag{31}$$

and is therefore also independent of $\xi'$. Taking into account the above statement, we can immediately infer that chaotic dynamics cannot appear for $k_2=0$. For $k_3$ no condition can be given; in particular, $k_3=0$ is not excluded. In fact, Sprott's simplest dissipative chaotic jerky dynamics SJ (cf. Table II) serves as example for a $JD_1$ model with $k_3=0$ that can exhibit chaotic dynamics.

In this context, it is interesting that the jerk model

$$\dddot{\xi}=k_1\ddot{\xi}+k_2\dot{\xi}+\xi\dot{\xi}+k_3 \tag{32}$$

cannot show irregular behavior at all. This also follows from the theorem in Appendix C. Eq. (32) is very similar to the basic model $JD_1$; the term linear in $\xi$ has only been substituted by a term linear in $\dot{\xi}$.

For the remaining basic models $JD_2$ to $JD_7$ the structure of the jerky dynamics has been chosen such that the integrand of the memory term is of the functional form

$$f(\xi,\xi')=A\xi^2+B\xi'^2+C. \tag{33}$$

The parameters $A$, $B$, and $C$ for each model are determined by the coefficients of the nonlinear terms $\xi^2$, $\dot{\xi}^2$, and $\xi\ddot{\xi}$ and by the additive constant of the underlying jerky dynamics. If the corresponding terms are not present $A$, $B$, and/or $C$ are zero. From the specific form (33) of $f(\xi,\xi')$ we obtain conditions on the (relative) signs of the coefficients $A$, $B$, and $C$ that exclude chaotic behavior of the corresponding jerky dynamics. In particular, irregular dynamics cannot appear if either $A\geqslant 0$, $B\geqslant 0$, $C\geqslant 0$ or $A\leqslant 0$, $B\leqslant 0$, $C\leqslant 0$ hold simultaneously. For the jerky dynamics $JD_2$, e.g., we have $A=1$, $B=0$, $C=k_3$ and we can infer that for $k_3>0$ chaotic dynamics cannot appear. This condition translates into conditions for the parameters $\alpha$ and $\beta$ of each model that is contained in the class $JD_2$. For the model M, e.g., we obtain $-\alpha-\frac{1}{4}\beta^2>0$. Similar considerations hold for the other basic classes.

## V. CONCLUSION AND DISCUSSION

In summary, our investigations consist of two major parts. (i) In Sec. III, we have provided a broad class of three-dimensional dynamical systems with polynomial and even nonpolynomial nonlinearities that possess at least one uniquely determined jerky dynamics [24]. (ii) In Sec. IV, we have shown that fifteen of Sprott's minimal chaotic dynamical systems [7] are in fact jerky. Moreover, we have shown that these models, the toroidal Rössler model [20], and Sprott's minimal dissipative jerk model [10,11] can be classified in seven classes of jerky dynamics with increasing polynomial complexity. Based on the description of nonlinear three-dimensional dynamical systems as jerky dynamics, we also have been able to derive criteria for the functional form of the jerk function that exclude chaotic behavior.

So far, our investigation has only been applied to known vector fields that are algebraically very simple and is, therefore, far from being complete. We expect that the classification scheme for three-dimensional dynamical systems proposed above can also be used for many other three-dimensional vector fields that appear in physics, chemistry, and ecology. Also transformations of fourth and higher-dimensional vector fields to fourth and higher-order scalar differential equations do not pose a conceptual problem.

In perspective, we think that a sound understanding of jerky dynamics might also be important in the following respects. (i) Quite natural experimental realizations of jerky dynamics are obviously electric circuits with internal feedback. Here, a basic understanding of jerky dynamics can help to systematically ''manufacture'' simple nonchaotic and chaotic electric circuits. (ii) An interesting, albeit technical point is that systematic analytical perturbation methods are easier to handle if a three-dimensional dynamical system is available in a jerky form. As an example, we refer to the work by Erneux *et al.* [17]. (iii) Since jerky dynamics exhibit many major features of chaotic dynamical behavior, a comprehensive investigation of jerky dynamics could lead to a deeper understanding of chaos and the routes to chaos.

Particularly challenging is a thorough understanding of the simplest classes of jerky dynamics that can exhibit nonchaotic and chaotic dynamics depending on the values of the entering parameters. Systematic investigations of the jerky dynamics $JD_1$,

$$\dddot{\xi}=k_1\ddot{\xi}+k_2\xi+\xi\dot{\xi}+k_3, \tag{34}$$

and $JD_2$,

$$\dddot{\xi}=k_1\ddot{\xi}+k_2\dot{\xi}+\xi^2+k_3, \tag{35}$$

will be reported in a subsequent study [25]. Although these two models are comparably simple they differ in an important point. The model (34) possesses *one* fixed point independent on the specific values of the parameters $k_1$, $k_2$, and $k_3$ (except for $k_2=0$), while the model (35) possesses *two* fixed points for $k_3<0$ and *no* fixed point for $k_3>0$ independent of $k_1$ and $k_2$.

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## APPENDIX A: CONSTRUCTION OF THE SYSTEM (9)

The problem of finding dynamical systems that possess an equivalent jerky dynamics consists of finding criteria for the transformation (6) such that it is *globally* invertible. In the following, we present considerations that guided us to find the classes of transformable dynamical systems stated in Eqs. (9). Restricting to multivariate polynomial vector fields $\mathbf{V}(\mathbf{x})$ in Eq. (3) for the moment, the transformation $\mathbf{T}$, Eq. (6), is also polynomial. Then we can take advantage of the Jacobian conjecture [26] to find criteria for the vector field $\mathbf{V}(\mathbf{x})$, such that an inverse of $\mathbf{T}$ exists.

The Jacobian conjecture [26] states that a polynomial transformation $\mathbf{T}$ is globally invertible if and only if its functional determinant fulfills

$$\frac{\partial(T_1,T_2,T_3)}{\partial(x,y,z)}=\begin{vmatrix}\partial_x T_1 & \partial_x T_2 & \partial_x T_3\\ \partial_y T_1 & \partial_y T_2 & \partial_y T_3\\ \partial_z T_1 & \partial_z T_2 & \partial_z T_3\end{vmatrix}=\text{const}\neq 0. \tag{A1}$$

The Jacobian conjecture, which has not been soundly proven yet, is related to the following well-known theorem about inverse functions [21]. An arbitrary multivariate transformation is uniquely invertible in a vicinity of the point $\mathbf{x}_0$ if the functional determinant is nonzero at $\mathbf{x}_0$. This theorem, however, provides only *local* invertibility in the neighborhood of a point. The Jacobian conjecture constitutes a *global* extension of this theorem but only for polynomial transformations.

Using Eqs. (6), the functional determinant reads

$$\frac{\partial(T_1,T_2,T_3)}{\partial(x,y,z)}=(\partial_y V_1)\partial_z(\mathbf{V}\cdot\nabla V_1)-(\partial_z V_1)\partial_y(\mathbf{V}\cdot\nabla V_1). \tag{A2}$$

Inserting Eq. (8) yields

$$\begin{aligned}\frac{\partial(T_1,T_2,T_3)}{\partial(x,y,z)}=&(b_{12}+\partial_y n_1)[(\mathbf{b}_3+\partial_z\mathbf{n})\cdot(\mathbf{b}^1+\nabla n_1)\\&+(\mathbf{c}+\mathbf{b}_1x+\mathbf{b}_2y+\mathbf{b}_3z+\mathbf{n})\cdot(\partial_z\nabla n_1)]\\&-(b_{13}+\partial_z n_1)[(\mathbf{b}_2+\partial_y\mathbf{n})\cdot(\mathbf{b}^1+\nabla n_1)\\&+(\mathbf{c}+\mathbf{b}_1x+\mathbf{b}_2y+\mathbf{b}_3z+\mathbf{n})\cdot(\partial_y\nabla n_1)].\end{aligned} \tag{A3}$$

Next, assuming that the polynomial $n_1(\mathbf{x})$ is only a function of $x$, $n_1(\mathbf{x})=n_1(x)$, one obtains from Eq. (A3)

$$\frac{\partial(T_1,T_2,T_3)}{\partial(x,y,z)}=b_{12}[b_{12}(b_{23}+\partial_z n_2)+b_{13}(b_{33}+\partial_z n_3)]$$
$$-b_{13}[b_{12}(b_{22}+\partial_y n_2)+b_{13}(b_{32}+\partial_y n_3)]$$
$$=\text{const}\neq 0. \tag{A4}$$

As a consequence, one has to demand further conditions on the nonlinear polynomials $n_2(\mathbf{x})$ and $n_3(\mathbf{x})$ such that the functional determinant is a nonzero constant. This yields the following condition for the terms in Eq. (A4) that contain $n_2(\mathbf{x})$ and $n_3(\mathbf{x})$,

$$b_{12}(b_{12}\partial_z n_2+b_{13}\partial_z n_3)-b_{13}(b_{12}\partial_y n_2+b_{13}\partial_y n_3)=C, \tag{A5}$$

with $C\in\mathbb{R}$ being a constant. Equation (A5) might have many solutions, but, taking into account the requirement of invertibility of $\mathbf{T}(\mathbf{x})$, it is necessary that the sum $b_{12}n_2(\mathbf{x})+b_{13}n_3(\mathbf{x})$ is a function only of $x$ and $b_{12}y+b_{13}z$, as is stated in Eq. (10a). With this constraint, Eq. (A5) is fulfilled and the constant $C$ is zero. This can easily be shown by taking the derivative of Eq. (10a) once with respect to $y$ and once with respect to $z$, multiplicating the first resulting equation by $-b_{13}$, the second by $b_{12}$ and finally summing up both equations. Since the constant in Eq. (A5) is zero, Eq. (A4) reduces to the condition (10b).

In the above considerations, we have not made explicit use of the assumption that the nonlinearities $\mathbf{n}(\mathbf{x})$ only consist of *polynomials* apart from the fact that the Jacobian conjecture has been used as starting point. Therefore, we conjecture that they might also be valid for arbitrary nonlinear functions $n_1(x)$, $n_2(\mathbf{x})$, and $n_3(\mathbf{x})$ that are at least twice differentiable and fulfill Eq. (10a). The theorem of Sec. III and its rigorous proof show the validity of this conjecture and also provide some evidence for the validity of the Jacobian conjecture.

## APPENDIX B: DYNAMICAL SYSTEMS WITH TWO OR THREE JERKY DYNAMICS

To obtain dynamical systems of the class (9) with *two* simultaneously existing jerky dynamics, e.g., in $x$ and $y$, one has to restrict the nonlinear function $n_2(\mathbf{x})$ such that it is only a function of $y$, $n_2(\mathbf{x})=n_2(y)$. This follows directly from Eqs. (9). In addition to the conditions (10),

$$b_{12}n_2(y)+b_{13}n_3(\mathbf{x})=f_1(x,b_{12}y+b_{13}z), \tag{B1a}$$

$$b_{12}^2b_{23}-b_{13}^2b_{32}+b_{12}b_{13}(b_{33}-b_{22})\neq 0, \tag{B1b}$$

that ensure the existence of the jerky dynamics in $x$, there are also corresponding constraints for the jerky dynamics in $y$ that read explicitly

$$b_{21}n_1(x)+b_{23}n_3(\mathbf{x})=f_2(y,b_{21}x+b_{23}z), \tag{B2a}$$

$$b_{23}^2b_{31}-b_{21}^2b_{13}+b_{21}b_{23}(b_{11}-b_{33})\neq 0, \tag{B2b}$$

where $f_1$ and $f_2$ are functions of the indicated arguments. Any dynamical system of the functional form (9) with $n_2(\mathbf{x})=n_2(y)$ that fulfills the conditions (B1) and (B2) can be recast into an equivalent jerky dynamics in its variables $x$ *and* $y$. For simultaneously existing jerky dynamics in two other variables one has to take into account permutations of variables and indices, respectively.

For dynamical systems that possess simultaneously *three* jerky dynamics, further constraints apply. Clearly, $n_3(\mathbf{x})=n_3(z)$ must hold. Furthermore, in addition to Eqs. (B1) and (B2) there is a third condition reading explicitly

$$b_{31}n_1(x)+b_{32}n_2(y)=f_3(z,b_{31}x+b_{32}y), \tag{B3a}$$

$$b_{31}^2b_{12}-b_{32}^2b_{21}+b_{31}b_{32}(b_{22}-b_{11})\neq 0. \tag{B3b}$$

Combining the conditions (B1), (B2), and (B3), one obtains two distinct dynamical systems that simultaneously possess *three* equivalent jerky dynamics, namely, first

$$\dot{x}=c_1+b_{11}x+b_{12}y+b_{13}z+n_1(x), \tag{B4a}$$

$$\dot{y}=c_2+b_{21}x+b_{22}y, \tag{B4b}$$

$$\dot{z}=c_3+b_{31}x+b_{33}z, \tag{B4c}$$

with the constraints

$$b_{12}\neq 0,\quad b_{13}\neq 0,\quad b_{21}\neq 0,\quad b_{31}\neq 0, \tag{B5}$$

and second

$$\dot{x}=c_1+b_{11}x+b_{12}y+n_1(x), \tag{B6a}$$

$$\dot{y}=c_2+b_{22}y+b_{23}z+n_2(y), \tag{B6b}$$

$$\dot{z}=c_3+b_{31}x+b_{33}z+n_3(z) \tag{B6c}$$

with the constraints

$$b_{12}\neq 0,\quad b_{23}\neq 0,\quad b_{31}\neq 0. \tag{B7}$$

In general, for both systems (B4) and (B6) one should also consider permutations of variables and indices, respectively. Therefore, there are dynamical systems with nonlinearities in each component of the vector field that possess three equivalent jerky dynamics. Even if a three-dimensional dynamical system can be transformed to a jerky dynamics in each of its variables $x$, $y$, and $z$, the resulting three scalar differential equations are, at least in general, not of the same functional form.

## APPENDIX C: NO-CHAOS THEOREM

Looking at the functional form of a jerky dynamics $\dddot{x}=J(x,\dot{x},\ddot{x})$, it is highly nontrivial to decide whether it can have chaotic solutions for some parameter ranges or not. On a pragmatic level, chaotic dynamics means that the long-time evolution of the underlying system is (i) bounded, i.e., $|x(t)|<\infty$ for all $t$, and (ii) neither a fixed point nor a periodic or quasiperiodic solution.

For some subclasses of jerky dynamics, however, one can derive a simple criterion under what circumstances aperiodic or chaotic solutions cannot appear. Consider the following integro-differential equation

$$\ddot{x}+\Omega(x,\dot{x})=\int^{t} f(x(\tau),x'(\tau),x''(\tau))d\tau \qquad \text{(C1)}$$

with $\Omega$ and $f$ being differentiable functions with respect to their arguments and the prime denoting the derivative with respect to $\tau$. Taking the time derivative of Eq. (C1), one obtains a jerky dynamics

$$\dddot{x}+p(x,\dot{x})\ddot{x}+q(x,\dot{x},\ddot{x})=0 \qquad \text{(C2)}$$

with

$$p(x,\dot{x})=\partial_{\dot{x}}\Omega(x,\dot{x}), \qquad \text{(C3a)}$$

$$q(x,\dot{x},\ddot{x})=\dot{x}\partial_{x}\Omega(x,\dot{x})-f(x,\dot{x},\ddot{x}). \qquad \text{(C3b)}$$

In turn, any jerky dynamics $\dddot{x}=J(x,\dot{x},\ddot{x})$ can be recast in the functional form (C2). Moreover, if $p(x,\dot{x})$ and $q(x,\dot{x},\ddot{x})$ are integrable functions with respect to their arguments, it can also be rewritten in form of Eq. (C1). Then the following holds.

*Theorem.* Any jerky dynamics (C2) with integrable $p(x,\dot{x})$ and $q(x,\dot{x},\ddot{x})$ cannot show chaotic behavior if $f(x,\dot{x},\ddot{x})$ is either a positive or a negative semidefinite function for all $x$, $\dot{x}$ and $\ddot{x}$.

*Proof.* To demonstrate the statement, we first write Eq. (C1) as

$$\ddot{x}+\Omega(x,\dot{x})=h, \qquad \text{(C4a)}$$

$$\dot{h}=f(x,\dot{x},\ddot{x}). \qquad \text{(C4b)}$$

The condition that $f(x,\dot{x},\ddot{x})$ is positive (or negative) semidefinite for all $x$, $\dot{x}$, and $\ddot{x}$ and, therefore, also for all $t$, implies that $\dot{h}(t)\geqslant 0$ [or $\dot{h}(t)\leqslant 0$] holds for all $t$. Consequently, $h(t)$ is a monotonically increasing (or decreasing) function of $t$. In the long-time limit, the modulus of $h(t)$ can only attain zero, a finite nonzero constant or infinity.

If $\lim_{t\to\infty}|h(t)|=C<\infty$ holds, the time evolution of Eq. (C4a) reduces to an effectively second-order dynamics,

$$\ddot{x}+\Omega(x,\dot{x})=\pm C, \qquad \text{(C5)}$$

in the long-time limit $t\to\infty$. By virtue of the Poincaré-Bendixson theorem [2], the time evolution of Eq. (C5) can only approach a fixed point (including infinity) or be periodic.

If $\lim_{t\to\infty}|h(t)|=\infty$ holds, the time evolution of Eq. (C4a) eventually escapes to infinity. Fixed points and bounded solutions cannot be attained, since the left-hand side of Eq. (C4a) also has to diverge. Therefore, the proof is complete.

Two remarks are in order. (i) The theorem generalizes a previously presented theorem in Ref. [14] in two respects. (a) It does not require the boundedness of $\ddot{x}+\Omega(x,\dot{x})=0$. (b) It is not restricted to Newtonian jerky dynamics, i.e., $f$ is also allowed to depend on $x''$. (ii) Under the conditions stated in the theorem, not only chaotic solutions are excluded, but also quasiperiodic and even period-doubling solutions cannot exist in the long-time limit.

## APPENDIX D: GRÖBNER BASES TECHNIQUE

In this section, we present a computational method that can be used to check the existence and to compute symbolically the jerky dynamics of a given dynamical system (3). This method is based on an algebraic elimination procedure for nonlinear polynomial equations known as *(comprehensive) Gröbner bases* technique. For details and a mathematically rigorous treatment of this technique we refer to the literature, especially the two monographs [27,28]. Here we only summarize some facts and results about Gröbner bases that are needed to solve our problem.

To apply the *algebraic* theory of (comprehensive) Gröbner bases to the problem whether a dynamical system (3) possesses an equivalent jerky dynamics, we reformulate it in an algebraic way. From Eqs. (3) we obtain the seven equations

$$f_1=\dot{x}-V_1(x,y,z)=0, \qquad \text{(D1a)}$$

$$f_2=\dot{y}-V_2(x,y,z)=0, \qquad \text{(D1b)}$$

$$f_3=\dot{z}-V_3(x,y,z)=0, \qquad \text{(D1c)}$$

$$f_4=\ddot{x}-(\dot{x}\partial_x V_1+\dot{y}\partial_y V_1+\dot{z}\partial_z V_1)=0, \qquad \text{(D1d)}$$

$$f_5=\ddot{y}-(\dot{x}\partial_x V_2+\dot{y}\partial_y V_2+\dot{z}\partial_z V_2)=0, \qquad \text{(D1e)}$$

$$f_6=\ddot{z}-(\dot{x}\partial_x V_3+\dot{y}\partial_y V_3+\dot{z}\partial_z V_3)=0, \qquad \text{(D1f)}$$

$$f_7=\dddot{x}-(\dot{x}^2\partial_x^2 V_1+\dot{y}^2\partial_y^2 V_1+\dot{z}^2\partial_z^2 V_1+\ddot{x}\partial_x V_1+\ddot{y}\partial_y V_1 +\ddot{z}\partial_z V_1)=0. \qquad \text{(D1g)}$$

Considering $x,\dot{x},\ddot{x},\dddot{x},y,\dot{y},\ddot{y},z,\dot{z},\ddot{z}$ as ten independent variables, the problem of finding a third-order differential equation for the variable $x$, $P(x,\dot{x},\ddot{x},\dddot{x})=0$, requires the elimination of the six variables $y,\dot{y},\ddot{y},z,\dot{z},\ddot{z}$ from the seven Eqs. (D1). In general, $P(x,\dot{x},\ddot{x},\dddot{x})=0$ is not necessarily an explicit equation. Moreover, there is no general *systematic* strategy to find $P(x,\dot{x},\ddot{x},\dddot{x})=0$ for arbitrary nonlinearities in the vector field $\mathbf{V}(\mathbf{x})$. For *polynomial* nonlinearities, however, the Gröbner bases technique applies.

The basic idea behind this technique is as follows. For a given finite set of polynomials $F=\{f_1,f_2,\ldots,f_m\}$ in several variables $x_1,x_2,\ldots,x_q$ find a set of polynomials $G=\{g_1,g_2,\ldots,g_n\}$ (with $n\neq m$ in general), the *Gröbner basis*, that possess the same common zeros as $F$ and are the multivariate generalization of the greatest common divisor of a finite set of polynomials in one variable. The explicit form of the Gröbner basis polynomials depends on the choice of a *term order* of the variables $x_1,x_2,\ldots,x_q$ that one has to fix, e.g., one can choose the lexicographical order $x_1\prec x_2\prec\cdots\prec x_q$. Gröbner bases, however, are only well defined for polynomials with coefficients that are real numbers. For polynomials containing real parameters, as is mostly the case for dynamical systems, the normal Gröbner basis can possibly lose its property of being Gröbner basis for certain values of the parameters. This problem can be overcome by the construction of a *comprehensive* Gröbner basis for $F$ that remains stable under any specialization of the parameters [29]. In Ref. [29] an algorithm for the symbolic computation of comprehensive Gröbner bases is given. Moreover, this algorithm is implemented in the experimental computer algebra system MAS (Modula-2 Algebra System) by Weispfenning and co-workers that is freely available by ftp [30].

The jerky dynamics for a given dynamical system can be found with the help of the *elimination theorem* [27,28] for (comprehensive) Gröbner bases. From this theorem one can extract the following statement: If there is *exactly one* polynomial $P(x,\dot{x},\ddot{x},\dddot{x})$ in the (comprehensive) Gröbner basis $G$ for the set of polynomials $\{f_1,f_2,\ldots,f_7\}$ given in Eqs. (D1) that does not depend on the variables $y,\dot{y},\ddot{y},z,\dot{z},\ddot{z}$, then there is a (possibly implicit) third-order ODE given by $P(x,\dot{x},\ddot{x},\dddot{x})=0$ that is equivalent to the dynamical system (3) that determines (D1). If, moreover, $P(x,\dot{x},\ddot{x},\dddot{x})$ is of the form $c\dddot{x}-Q(x,\dot{x},\ddot{x})$ (where $c\in\mathbb{R}$ is a nonzero constant), then there exists a *unique* and polynomial jerky dynamics for the dynamical system that leads to Eq. (D1). The jerky dynamics is given by $\dddot{x}=J(x,\dot{x},\ddot{x})$ with $J(x,\dot{x},\ddot{x})=Q(x,\dot{x},\ddot{x})/c$. The constant $c$ can contain parameters of the original dynamical system. Since $c\neq 0$ must hold, we obtain a condition on these parameters that corresponds to Eq. (10b) or Eq. (11). The above statement only holds for the term order $x,\dot{x},\ddot{x},\dddot{x}\prec y,\dot{y},\ddot{y},z,\dot{z},\ddot{z}$ of the independent variables $x$, $\dot{x}$, $\ddot{x}$, $\dddot{x}$, $y$, $\dot{y}$, $\ddot{y}$, $z$, $\dot{z}$, $\ddot{z}$, where the order of the right-hand variables and the left-hand variables among themselves is irrelevant.

Using MAS, one can symbolically compute the Gröbner basis of the polynomials (D1) and, therefore, determine the jerky dynamics in $x$ of the dynamical system that underlies Eqs. (D1) if it exists. The existence of a jerky dynamics in the other variables $y$ or $z$ can be checked by taking into account the $\dddot{y}$ or $\dddot{z}$ equation instead of the polynomial (D1g) and choosing appropriate term orders. This computational approach is especially of advantage for dynamical systems that do not belong to the class (9). According to our experience, the comprehensive Gröbner bases method is hard to use to derive criteria for the existence of an equivalent jerky dynamics for a general dynamical system with a polynomial vector field that contains all linear and nonlinear combinations up to some degree with arbitrary real parameters as coefficients. Here, the computational effort is still too high, because of the large number of parameters (even for polynomials of degree two).


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