Stochastic resonance driven by time-modulated correlated white noise sources

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We analyze the effects caused by the simultaneous presence of correlated additive and multiplicative noises for stochastic resonance. Besides the standard potential modulation we also consider a time-periodic variation of the correlation between the two noise sources. As a foremost result we find that stochastic resonance, as characterized by the signal-to-noise ratio and the spectral amplification, becomes characteristically broadened. The broadening can be controlled by varying the relative phase shift between the two types of modulation force.

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I. INTRODUCTION

There is evidence from many recent theoretical and experimental studies that fluctuations are essential and play a constructive role in a variety of intriguing noise-induced phenomena. Some key examples are problems related to self-organization and dissipative structures [1,2], noise-induced transitions [3], noise-induced *phase* transitions [4], *thermal ratchets* or Brownian motors [5], combinations of the latter two phenomena [6], noise sustained patterns [7], and stochastic resonance in zero-dimensional and spatially extended systems [8,9].

The last phenomenon, that is, *stochastic resonance* (SR), has attracted considerable interest in the last decade due, among other aspects, to its potential technological applications for optimizing the transmission of information such as the output signal-to-noise ratio (SNR) and amplification factor (η) in nonlinear dynamical systems. The phenomenon shows the counterintuitive role played by noise in nonlinear systems as it harnesses the fluctuations to enhance the output response of a system subjected to a weak external signal. There is a wealth of papers, conference proceedings, and reviews on this subject; for a comprehensive recent review see Ref. [9], showing the large number of applications in science and technology, ranging from paleoclimatology to electronic circuits, lasers, and noise-induced information flow in sensory neurons in living systems, to name a few.

Several recent papers have aimed at achieving an enhancement of the system response (that is, obtaining a larger output SNR) by means of the coupling of several SR units [10–14] in what forms an ‘‘extended medium’’ [15]. Yet another aspect that has attracted interest is the construction of systems or arrangements where the SNR becomes mostly independent of external parameters such as the noise intensity (SR without tuning) [12,16].

In this work we focus on the latter aspect. For this we study a bistable system which is subject to both an additive and a multiplicative noise source but—at variance with the work in Ref. [17]—we consider the case when both noise sources are *correlated*. In addition to the modulation of the bistable potential by a weak external signal, we consider that this very correlation between both noises is modulated as well. We have found that this extra noise correlation contributes to a remarkable widening of the SNR's maximum as a function of the additive noise intensity, making the detection of the signal less sensitive to the actual value of that noise. In previous preliminary work [19] we analyzed the case when both modulation frequencies are equal; here we extend the study to the most general case; i.e., when (i) both frequencies are equal and possess either zero or a finite relative phase shift $\phi \neq 0$, or (ii) there are different driving frequencies. We have also done numerical simulations and have considered the evaluation of not only the SNR, but also another characterization of the SR phenomenon that relates SR to stochastic synchronization, namely, the spectral amplification factor [18]. It is worth remarking here that the additive (external) noise can be assumed to be white, while the (internal) multiplicative noise source generally involves time scales characteristic of the system; therefore it is generally far from being white. However, as discussed previously in [20], as a first step we can approximate the multiplicative colored noise by a white one.

The organization of the paper is as follows. In the next section we set up the model. In Sec. III we present the results for the case of unequal modulation frequencies, while Sec. IV contains the case of equal frequencies with both a zero and a finite relative phase shift. In Sec. V we present the results of our numerical simulations while in Sec. VI we discuss the spectral amplification factor. The last section contains the final discussion and some conclusions.

II. THEORETICAL APPROACH

The model system we consider here corresponds to an overdamped bistable system described by the Langevin

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equation (written in properly scaled, dimensionless variables; see [9])

$$\dot{x}=\varepsilon(t)+x-x^3+\xi_\varepsilon(t)+x\xi_\rho(t), \tag{1}$$

where $\varepsilon(t)=\varepsilon_0\cos(\Omega_\varepsilon t)$, and ε_0 and Ω_ε are, respectively, the intensity and frequency of the dipole-type potential modulation $-x\varepsilon(t)$. The additive and multiplicative Gaussian white noise sources, indicated by $\xi_\varepsilon(t)$ and $\xi_\rho(t)$, respectively, obey

$$\begin{aligned}\langle \xi_i(t)\rangle&=0,\quad i=1,2,\\ \langle \xi_\varepsilon(t)\xi_\varepsilon(t')\rangle&=2D_\varepsilon\delta(t-t'),\\ \langle \xi_\rho(t)\xi_\rho(t')\rangle&=2D_\rho\delta(t-t'),\\ \langle \xi_\varepsilon(t)\xi_\rho(t')\rangle&=2\sqrt{D_\varepsilon D_\rho}\rho(t)\,\delta(t-t').\end{aligned} \tag{2}$$

The strength of the correlation between the two noises is measured by the parameter $\rho(t)$, fulfilling the condition $|\rho(t)|\leqslant 1$. The associated Fokker-Planck equation (in the Stratonovich prescription) reads [21–23]

$$\begin{aligned}\partial_t P(x,t)=&-\partial_x\{[x-x^3+D_\rho x+\rho(t)\sqrt{D_\rho D_\varepsilon}-\varepsilon(t)]P(x,t)\}\\ &+D_\varepsilon\partial_x^2\{[1+Rx^2+2\rho(t)\sqrt{R}]P(x,t)\}.\end{aligned} \tag{3}$$

Here we have defined $R=D_\rho/D_\varepsilon$. In what follows we assume that ρ is a time dependent periodic function of the form $\rho(t)=\rho_0\cos(\Omega_\rho t+\phi)$, where ϕ is an arbitrary but fixed relative phase. In previous work [19] we studied the case where $\Omega_\rho=\Omega_\varepsilon$. We shall shortly review this situation but will mainly elaborate on the most general case with $\Omega_\rho\neq\Omega_\varepsilon$.

In order to evaluate the correlation function and the *power spectral density* (PSD) to obtain the SNR, we exploit the results of the two-state approach [9,24,25]. The problem of obtaining the SNR of a nonlinear and essentially bistable symmetric system subject to a weak periodic signal is then reduced to a description where the transitions occur between the two minima of the deterministic potential. Besides using linear response theory the main approximation involves an adiabatic approximation in the sense that the relaxation time around each minimum is much shorter than the characteristic time for transitions between the two stable states and the corresponding slow driving periods.

In the absence of any signal, the deterministic potential of the system has two minima located at the points $x_\pm=\pm 1$. Let $n_\pm(t)$ be the populations in each state, defined as $n_+(t)=\int_0^{+\infty}P(x,t)dx$, and $n_-(t)=1-n_+(t)$, respectively. It has been shown that these minima do not coincide with the maxima of the steady state probability distribution [26]. However, if $\rho(t)\sqrt{D_\rho D_\varepsilon}$ is sufficiently small it is justified to neglect this fact.

To apply the two-state approach, let us introduce $W_+(t)$ and $W_-(t)$, these being the (adiabatic) nonstationary transition rates from the state x_+ to x_- and from the state x_- to x_+, respectively. Then we can write the following *master equation* for the probability distribution:

$$\frac{dn_-}{dt}=-\frac{dn_+}{dt}=W_+(t)n_+-W_-(t)n_-. \tag{4}$$

The reduction from a bistable continuous system, whose probability density evolves through a Fokker-Planck equation, to a discrete system driven by a master equation such as Eq. (4) is well known [27]. To evaluate the statistical moments within such an approximation, the probability density has the form $p(x,t)=n_+(t)\delta(x-x_+)+n_-(t)\delta(x-x_-)$.

Usually, the time dependence of $W_\pm$ is such that the exact solution for Eq. (4) cannot be found. If, however, both modulations are small compared with the barrier height, i.e., $\rho_0\sqrt{D_\rho D_\varepsilon}\ll V(0)-V(\pm 1)$ and $\varepsilon_0\ll V(0)-V(\pm 1)$, then it is possible to make a Taylor expansion of the functions $W_\pm(t)$ around $\rho_0=\varepsilon_0=0$. We thus obtain within linear response theory the result

$$\begin{aligned}W_\pm(t)=W_0&+\left.\frac{dW_\pm}{d\varepsilon}\right|_{\rho_0=\varepsilon_0=0}\varepsilon(t)+\left.\frac{dW_\pm}{d\rho}\right|_{\rho_0=\varepsilon_0=0}\rho(t)\\ &+O(\varepsilon_0^2)+O(\rho_0^2)+\cdots,\end{aligned} \tag{5}$$

where W_0 is the transition rate evaluated in absence of modulation ($\rho_0=\varepsilon_0=0$). The latter may be calculated by means of the mean first-passage time $\mathcal{T}(R,\rho_0,\varepsilon_0)$ [20], yielding

$$\begin{aligned}\frac{1}{W_0}&=\mathcal{T}(R)|_{\rho_0=\varepsilon_0=0}\\ &=\frac{1}{D}\int_{-1}^{1}dxH(x)\exp[\Phi(x)/D]\\ &\quad\times\int_{-\infty}^{x}dyH(y)\exp[-\Phi(y)/D]\end{aligned} \tag{6}$$

with the function $H(x)\equiv[1+Rx^2+2\rho(t)\sqrt{R}x]^{-1/2}$. The determination of the effective potential $\Phi(x)$ is from the adiabatic asymptotic (nonstationary) probability density; yielding

$$\Phi(x)=\int_{-\infty}^{x}H(x')^2[x'-x'^3+\varepsilon(t)+2\rho(t)\sqrt{D_\rho D_\varepsilon}]dx'.$$

For the chosen sinusoidal form of the modulations the expansion has the explicit form

$$\begin{aligned}W_\pm(t)=\tfrac{1}{2}[&W_0\mp\alpha_\varepsilon\cos(\Omega_\varepsilon t)\mp\alpha_\rho\cos(\Omega_\rho t+\phi)\\ &+O(\varepsilon_0^2)+O(\rho_0^2)+\cdots],\end{aligned} \tag{7}$$

The factors α_ρ and α_ε are given by

$$\frac{\alpha_\varepsilon}{2}=-\left.\frac{dW_\pm}{d\varepsilon}\right|_{\rho_0=\varepsilon_0=0}\varepsilon_0\quad\text{and}\quad\frac{\alpha_\varepsilon}{2}=-\left.\frac{dW_\pm}{d\rho}\right|_{\rho_0=\varepsilon_0=0}\rho_0; \tag{8}$$

with

$$\frac{dW_\pm(R)}{d\varepsilon}=-\frac{1}{\mathcal{T}(R)^2}\frac{d\mathcal{T}(R)}{d\varepsilon}, \tag{9}$$

$$\frac{dW_\pm(R)}{d\rho}=-\frac{1}{\mathcal{T}(R)^2}\frac{d\mathcal{T}(R)}{d\rho}. \tag{10}$$

After a somewhat cumbersome calculation for the derivatives of $\mathcal{T}(R)$ we end up with

$$D_\varepsilon^2 \frac{d\mathcal{T}}{d\varepsilon}\bigg|_{\rho,\varepsilon=0} = -\int_{-1}^{1} dx H(x) \exp[\Phi(x)/D_\varepsilon] \times \left(H(x)^2 x \int_{-1}^{x} dy H(y) \exp[-\Phi(y)/D_\varepsilon] + \int_{-1}^{x} dy H(y) \exp[-\Phi(y)/D_\varepsilon] \times \int_{-1}^{y} dz H(z)^2 z \right) \tag{11}$$

for the contribution of the potential modulation. The corresponding contribution of the correlation modulation is

$$\frac{D_\varepsilon^2}{\sqrt{D_\rho D_\varepsilon}} \frac{d\mathcal{T}}{d\rho}\bigg|_{\rho,\varepsilon=0} = -\int_{-1}^{1} dx H(x) \exp[\Phi(x)/D_\varepsilon] \times \Bigg\{ H(x)^2 x \left[1 - H(x) + H(x)^2 x^2 \left(\frac{x^2}{2} - 1 \right) \right] \int_{-1}^{x} dy H(y) \exp[-\Phi(y)/D_\varepsilon] - \frac{1}{D_\varepsilon} \int_{-1}^{x} dy H(y)^3 y \exp[-\Phi(y)/D_\varepsilon] \times \left[1 + H(y) + H(y)^2 y^2 \left(\frac{y^2}{2} - 1 \right) \right] \Bigg\}. \tag{12}$$

It is also important to note here that, if the modulation is made around a value $\rho_0 \neq 0$, it becomes necessary to extend the two-state approach in order to take into account the lack of symmetry of the potential [28].

In order to simplify the notation, let us define $\Upsilon_\varepsilon(t)$ and $\Upsilon_\rho(t)$ as

$$\Upsilon_\varepsilon(t) = \alpha_\varepsilon \frac{\varepsilon(t)}{\sqrt{W_o^2 + \Omega_\varepsilon^2}}, \quad \Upsilon_\rho(t) = \alpha_\rho \frac{\rho(t)}{\sqrt{W_o^2 + \Omega_\rho^2}}. \tag{13}$$

Integrating Eq. (4) up to first order in the variables ρ_0 and ε_0, we obtain

$$n_+(t|x_0,t_0) = 1 - \Upsilon_\varepsilon(t) - \Upsilon_\rho(t) + e^{-W_o|t-t_0|} \times [\Upsilon_\varepsilon(t_0) + \Upsilon_\rho(t_0) + 2\delta_{x_0 1} - 1], \tag{14}$$

where $n_+(t|x_0,t_0)$ is the conditional probability that $x(t) = x_+$, given that $x(t_0) = x_0$. The funtion $\delta_{x_0 1}$ is equal to 1 if the particle is initially located at x_+ and 0 otherwise, and similarly for $n_-(t|x_0,t_0)$.

From Eq. (14), all the moments of the distribution $p(x,t)$ may be determined and the conditioned autocorrelation function, averaged over noise $\langle\rangle$, and uniformly over time $\langle\rangle_t$, reads

$$\begin{aligned} \mathcal{K}(\tau,t_0) &= \langle x(t)x(t+\tau)|x_0,t_0\rangle_t \\ &:= \frac{1}{T}\int_0^T \langle x(t)x(t+\tau)|x_0,t_0|x_0,t_0\rangle dt \\ &= e^{-W_o|\tau|} - e^{-W_o|\tau|}\Upsilon_\varepsilon(t_0)^2 - e^{-W_o|\tau|}\Upsilon_\rho(t_0)^2 \\ &\quad - 2e^{-W_o|\tau|}\Upsilon_\varepsilon(t_0)\Upsilon_\rho(t_0) + \Upsilon_\varepsilon(t_0)\Upsilon_\varepsilon(t_0+\tau) \\ &\quad + \Upsilon_\rho(t_0)\Upsilon_\rho(t_0+\tau) + \Upsilon_\rho(t_0)\Upsilon_\varepsilon(t_0+\tau) \\ &\quad + \Upsilon_\varepsilon(t_0)\Upsilon_\rho(t_0+\tau). \end{aligned} \tag{15}$$

In Eq. (15) it is possible to see that the autocorrelation function depends explicitly on the modulation frequencies Ω_ε and Ω_ρ, as well as on t_0. Here, t_0 represents the time when the output PSD of a system is measured and the data acquisition begins. The PSD $\mathcal{S}(\Omega)$, is the time-averaged Fourier transform (over the time span T) of the autocorrelation function $\mathcal{K}(\tau,t_0)$,

$$\begin{aligned} \langle \mathcal{S}(\Omega)\rangle_t &= \frac{1}{2\pi}\left\langle \int_{-\infty}^{\infty} \mathcal{K}(\tau,t_0)\exp i\Omega\tau d\tau \right\rangle_{t_0} \\ &:= \frac{1}{T}\int_0^T \mathcal{S}(\Omega,t)dt. \end{aligned} \tag{16}$$

By means of the two-state approach, using Eq. (15) and Eq. (16), the general expression for the PSD is

$$\begin{aligned} \mathcal{S}(\Omega) &= \left(\frac{2W_o}{W_o^2+\Omega^2}\right)\left(1 - \frac{\alpha_\varepsilon^2\varepsilon_0^2}{2(W_o^2+\Omega_\varepsilon^2)} - \frac{\alpha_\rho^2\rho_0^2}{2(W_o^2+\Omega_\rho^2)}\right) \\ &\quad + \frac{\pi\alpha_\varepsilon^2\varepsilon_0^2}{2(W_o^2+\Omega_\varepsilon^2)}[\delta(\Omega-\Omega_\varepsilon)+\delta(\Omega+\Omega_\varepsilon)] \\ &\quad + \frac{\pi\alpha_\rho^2\rho_0^2}{2(W_o^2+\Omega_\rho^2)}[\delta(\Omega-\Omega_\rho)+\delta(\Omega+\Omega_\rho)] \\ &\quad + \delta_{\Omega_\varepsilon-\Omega_\rho}\frac{\pi}{2}\left(\frac{2\alpha_\rho\rho_0\alpha_\varepsilon\varepsilon_0\cos(\phi)}{(W_o^2+\Omega_\varepsilon^2)}\right) \\ &\quad \times[\delta(\Omega-\Omega_\varepsilon)+\delta(\Omega+\Omega_\varepsilon)], \end{aligned} \tag{17}$$

where $\delta_x = 1$ if $x=0$ and 0 otherwise.

Hence, we can distinguish between two distinct cases according to the relation between the frequencies, i.e., (i) different frequencies and (ii) equal frequencies.

To determine the output SNR, denoted by $\mathcal{R}$, we use the standard definition

$$\mathcal{R} = 10\log_{10}\left(\frac{\int_{\Omega-\Delta}^{\Omega+\Delta}\mathcal{S}(\omega)d\omega}{\mathcal{S}_n(\omega=\Omega)}\right). \tag{18}$$

Here $\mathcal{S}_n$ is the PSD in absence of the signal. The parameter Δ is introduced in order to tune the theoretical result when it is compared with a numerical simulation or an experiment. Such a parameter is related to the bandwidth of sampling frequencies.

III. DIFFERENT MODULATION FREQUENCIES

In spite of the result being the same, when the frequencies are unequal the calculations differ slightly depending on whether the ratio between driving frequencies is rational or irrational. This difference in the calculation procedure arises due to the evaluation of T. If the ratio between the modulation frequencies is a rational number, that is, $\Omega_\varepsilon/\Omega_\rho = q/p$, with $q,p \in \mathcal{N}$, then $T = 2\pi q/\Omega_\varepsilon = 2\pi p/\Omega_\rho$, while when the ratio is an irrational number T increases without limit ($T \to \infty$). However, as we have already remarked, the final result remains the same, and using Eq. (16) the PSD emerges as

$$\mathcal{S}(\Omega) = \left(\frac{2W_o}{W_o^2+\Omega^2}\right)\left(1 - \frac{\alpha_\varepsilon^2\varepsilon_0^2}{2(W_o^2+\Omega_\varepsilon^2)} - \frac{\alpha_\rho^2\rho_0^2}{2(W_o^2+\Omega_\rho^2)}\right) + \frac{\pi\alpha_\varepsilon^2\varepsilon_0^2}{2(W_o^2+\Omega_\varepsilon^2)}[\delta(\Omega-\Omega_\varepsilon)+\delta(\Omega+\Omega_\varepsilon)] + \frac{\pi\alpha_\rho^2\rho_0^2}{2(W_o^2+\Omega_\rho^2)}[\delta(\Omega-\Omega_\rho)+\delta(\Omega+\Omega_\rho)]. \quad (19)$$

Introducing an arbitrary phase shift between the signals leads to the same result, namely, the appearance of two *distinct* SR effects, each one due to a different modulation. As in the case when only one parameter is modulated, a Lorentzian-like dependence arises in the output PSD due to the noisy dynamics of the system. The signals are amplified, yielding δ functions at the modulation frequencies. This is a counterintuitive result since (as is already known) SR is a consequence of the nonlinearity of the system. In spite of the latter fact, it is apparent that the indicated effect on SR appears via two separate—*linearly* superimposed—events, suggesting that there is no cooperative effect between the two modulations, no matter how similar they are. One might think that this fact is due to the linear response approximation we have used within the two-state approach and to the fact that the modulation amplitude we used is very small. Indeed, the numerical simulations support this result, as will be shown later.

The expressions for each independent contribution to SR are

$$\mathcal{R}_\varepsilon = 10\log_{10}\left[\frac{\pi^2\alpha_\varepsilon^2\varepsilon_0^2}{W_o\Delta}\left(1 - \frac{\alpha_\varepsilon^2\varepsilon_0^2}{2(W_o^2+\Omega_\varepsilon^2)} - \frac{\alpha_\rho^2\rho_0^2}{2(W_o^2+\Omega_\rho^2)}\right)^{-1}\right], \quad (20)$$

$$\mathcal{R}_\rho = 10\log_{10}\left[\frac{\pi^2\alpha_\rho^2\rho_0^2}{W_o\Delta}\left(1 - \frac{\alpha_\rho^2\rho_0^2}{2(W_o^2+\Omega_\rho^2)} - \frac{\alpha_\varepsilon^2\varepsilon_0^2}{2(W_o^2+\Omega_\varepsilon^2)}\right)^{-1}\right]. \quad (21)$$

Of particular interest is the case in which there is a single modulation of either $\varepsilon(t)$ or $\rho(t)$. For instance, in Fig. 1 we depict the SNR when only the potential is modulated, as a function of D_ε and D_ρ. It is apparent that the SR phenomenon disappears for large noise intensities. Indeed, if the multiplicative noise D_ρ is kept constant, and D_ε is increased, the SNR starts from a small value, increases reaching a maximum (for an additive noise intensity $D_\varepsilon \sim 10^{-1}$), and afterwards decreases. It is also possible to see that if D_ρ becomes large enough ($D_\rho \sim 10$) the phenomenon of SR vanishes. In the opposite limit, $D_\rho \to 0$, we recover the well known SR with an additive noise only, and in this case $\mathcal{R}$ reaches its maximum.

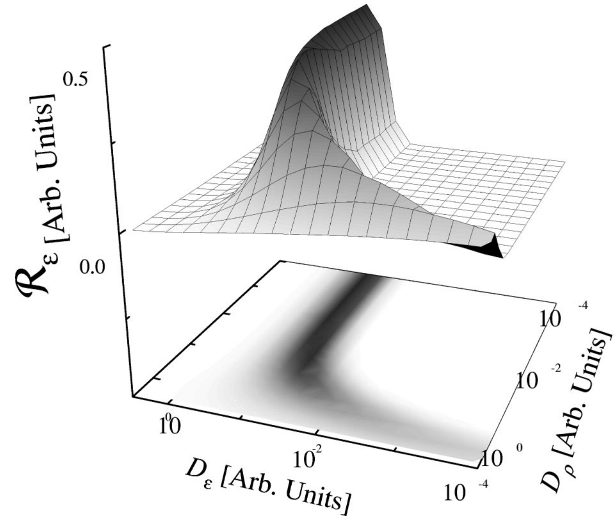

FIG. 1. We show the SNR ($\mathcal{R}_\varepsilon$) as a function of the additive and multiplicative noise intensities D_ε and D_ρ, when only the potential is modulated. We fixed $\varepsilon_0 = 0.05$ and $\Omega_\varepsilon = 0.008$. In the limiting case $D_\rho \to 0$, the usual SR phenomenon is recovered. It is easy to see that, when $D_\rho \sim 10^{-1}$, the typical width of SR increases greatly.

From Fig. 1, and for fixed D_ρ, we can see that it is possible to associate a characteristic width to the SNR as a function of D_ε. This width indicates the range of values of additive noise intensities D_ε where the SR phenomenon is more apparent. As indicated in the Introduction, it is of interest to widen this range as much as possible because then it will be possible to find a great insensitivity to external parameters, such as the additive noise intensity D_ε. As shown in Fig. 1, the maximum of the SNR is lower for large D_ρ intensities, while its width increases.

In Fig. 2, $\mathcal{R}$ is depicted as a function of D_ε and D_ρ when the signal is introduced only through a modulation on the correlation. In this case, the SR phenomenon exhibits a more localized behavior in (D_ε, D_ρ) parameter space. In fact, the effect vanishes not only when the additive noise intensity approaches infinity (i.e., $D_\varepsilon \to +\infty$) or zero (i.e., $D_\varepsilon \to 0$), but also when $D_\rho \to +\infty$ or $D_\rho \to 0$.

This aspect of the phenomenon is explained through the Fokker-Planck equation [Eq. (3)]. Both signals enter that equation as contributions to the convective term. Therefore, the potential modulation contributes as $\varepsilon(t)$, while the correlation modulation contributes as $\rho(t)\sqrt{D_\rho D_\varepsilon}$. Then (if $\varepsilon_0 = 0$), the signal disappears only if at least one of the noise intensities is zero.

In Fig. 2 we also show the curves $\mathcal{R}(D_\varepsilon)$, for different values of D_ρ. Here it is apparent that the characteristic width of SNR is modified when the multiplicative noise intensity is

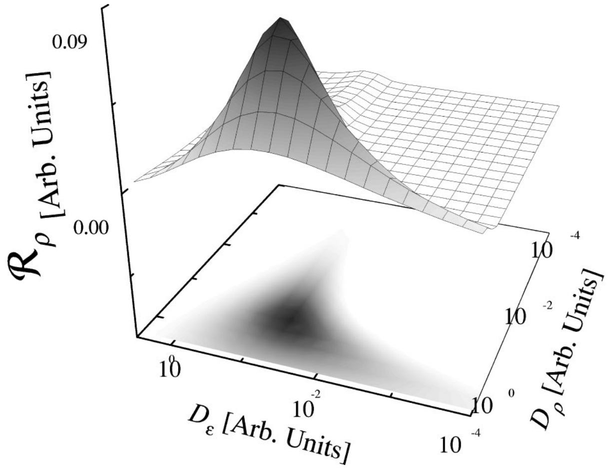

FIG. 2. In this three-dimensional plot we show the output SNR ($\mathcal{R}_\rho$) as a function of D_ε and D_ρ. Here, only the correlation parameter is modulated; thus, $\rho_0 = 0.08$ and $\Omega_\rho = 0.008$. The SR phenomenon diminishes when both noises disappear or increase enough, at variance with the previous case.

varied. We may note that, depending on this intensity, the width of $\mathcal{R}$ may change by nearly three orders of magnitude. In spite of this, as the width of the SNR grows, its maximum falls.

It is worthwhile remarking that the functional dependence on the modulation frequency is small. When the modulation is applied simultaneously for $\varepsilon(t)$ and $\rho(t)$, all the results shown thus far remain robust. This fact is due to the *unexpected* linear superposition of both modulation effects.

IV. EQUAL MODULATION FREQUENCIES

When both driving frequencies are equal ($\Omega_\varepsilon = \Omega_\rho$) we can distinguish two possibilities: the signals possess either a zero or a finite relative phase shift ϕ. The general form for the modulations is

$$\rho(t) = \rho_0 \cos(\Omega_\varepsilon t + \phi),$$

$$\varepsilon(t) = \varepsilon_0 \cos(\Omega_\varepsilon t),$$

with the period being $T = 2\pi/\Omega_\varepsilon$. The PSD is given by

$$\mathcal{S}(\Omega) = \left(\frac{2W_o}{W_o^2+\Omega^2}\right)\left(1 - \frac{\alpha_\varepsilon^2\varepsilon_0^2 + \alpha_\rho^2\rho_0^2 + 2\alpha_\varepsilon\varepsilon_0\alpha_\rho\rho_0 \cos(\phi)}{2(W_o^2+\Omega_\varepsilon^2)}\right) + \frac{\pi}{2}\left(\frac{\alpha_\varepsilon^2\varepsilon_0^2 + \alpha_\rho^2\rho_0^2 + 2\alpha_\varepsilon\varepsilon_0\alpha_\rho\rho_0 \cos(\phi)}{(W_o^2+\Omega_\varepsilon^2)}\right) \times [\delta(\Omega-\Omega_\varepsilon) + \delta(\Omega+\Omega_\varepsilon)]. \tag{22}$$

and the output SNR reads

$$\mathcal{R} = 10\log_{10}\left[\frac{\pi[\alpha_\varepsilon^2\varepsilon_0^2 + \alpha_\rho^2\rho_0^2 + 2\alpha_\varepsilon\varepsilon_0\alpha_\rho\rho_0 \cos(\phi)]}{\Delta W_o} \times\left(1 - \frac{\alpha_\varepsilon^2\varepsilon_0^2 + \alpha_\rho^2\rho_0^2 + 2\alpha_\varepsilon\varepsilon_0\alpha_\rho\rho_0 \cos(\phi)}{2(W_o^2+\Omega_\varepsilon^2)}\right)^{-1}\right]. \tag{23}$$

A. Signals with zero relative phase shift

From Eq. (22), the corresponding expression for $\phi = 0$ emerges as

$$\mathcal{S}(\Omega) = \left(\frac{2W_o}{W_o^2+\Omega^2}\right)\left(1 - \frac{(\alpha_\varepsilon\varepsilon_0 + \alpha_\rho\rho_0)^2}{2(W_o^2+\Omega_\varepsilon^2)}\right) + \frac{\pi}{2}\left(\frac{(\alpha_\varepsilon\varepsilon_0 + \alpha_\rho\rho_0)^2}{(W_o^2+\Omega_\varepsilon^2)}\right)[\delta(\Omega-\Omega_\varepsilon) + \delta(\Omega+\Omega_\varepsilon)]. \tag{24}$$

From this expression, we note that the two SR phenomena associated with each separate modulation become indistinguishable: the system behaves as if only one signal acted on it. The expression for the SNR thus reads

$$\mathcal{R} = 10\log_{10}\left[\frac{\pi^2(\alpha_\varepsilon\varepsilon_0 + \alpha_\rho\rho_0)^2}{W_o\Delta}\left(1 - \frac{(\alpha_\varepsilon\varepsilon_0 + \alpha_\rho\rho_0)^2}{2(W_o^2+\Omega_\varepsilon^2)}\right)^{-1}\right]. \tag{25}$$

To simplify notation, we shall call the output PSD due to the potential modulation $\mathcal{S}_\varepsilon$, while the output PSD due to modulation of the correlation will be indicated as $\mathcal{S}_\rho$.

When there is no relative shift between the signals, the two SR phenomena are linearly superimposed. This is a quite interesting fact because (as indicated above) the SR peak is wider when the correlation is modulated than when there is only a modulation of the potential. Thus, by modulating both parameters simultaneously, the total SR effect may achieve a greater independence relative to the (external) additive noise intensity D_ε.

In Fig. 3 we show SNR surfaces in (D_ε, D_ρ) parameter space when both modulations are simultaneously present, with the same driving frequency and with $\phi = 0$. It is apparent that (keeping the noise intensity D_ρ constant) we obtain a widening of the SNR function. This effect is larger if the phenomena due to both modulations are of similar amplitude. Indeed, if $\mathcal{S}_\varepsilon \gg \mathcal{S}_\rho$, the characteristic peak in the SNR function is approximately the same as that obtained if the signal is injected through the potential. In the opposite case, $\mathcal{S}_\rho \gg \mathcal{S}_\varepsilon$, the effect is more dependent on the noise intensities.

If $\mathcal{S}_\rho \cong \mathcal{S}_\varepsilon$, a (very) considerable widening of the SNR function is found. It is seen that for multiplicative noise intensities around 0.8 the characteristic peak of $\mathcal{R}$ increases its width up to three orders of magnitude in comparison with the case of modulation of the potential.

When the two signals have a phase shift of $\phi = \pi$ we find a different behavior. The linear superposition of the signals then weakens the total SR phenomenon. An interesting as-

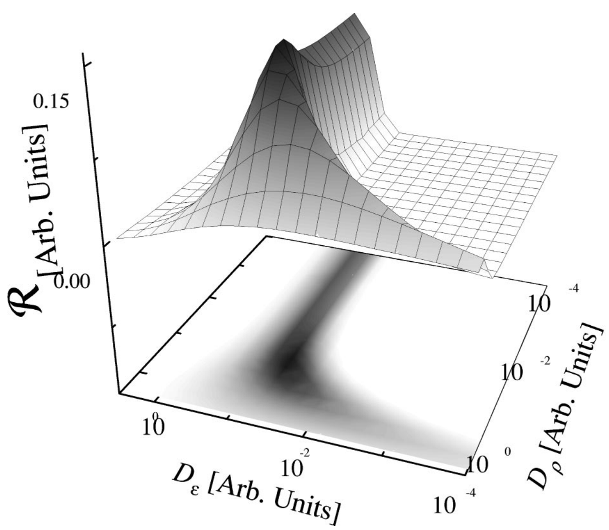

FIG. 3. Here we show the SNR ($\mathcal{R}$) in (D_ε,D_ρ) space. We have fixed $\varepsilon_0=0.05$, $\rho_0=0.08$, $\Omega_\varepsilon=\Omega_\rho=0.005$, and $\phi=0$. A contribution between the two phenomena depicted in the previous figures is apparent.

pect is that, for specially chosen modulation intensities, the SR exhibits *two* peaks vs D_ε, instead of the usual situation with only one peak.

B. Signals with a finite relative phase shift

The second case is when there is a finite relative phase shift ϕ between the signals. We will consider only the case $\phi=\pi/2$. The PSD becomes

$$\mathcal{S}(\Omega)=\left(\frac{2W_o}{W_o^2+\Omega^2}\right)\left(1-\frac{\alpha_\varepsilon^2\varepsilon_0^2+\alpha_\rho^2\rho_0^2}{2(W_o^2+\Omega_\varepsilon^2)}\right)+\frac{\pi}{2}\left(\frac{(\alpha_\varepsilon^2\varepsilon_0^2+\alpha_\rho^2\rho_0^2)}{(W_o^2+\Omega_\varepsilon^2)}\right)[\delta(\Omega-\Omega_\varepsilon)+\delta(\Omega+\Omega_\varepsilon)], \tag{26}$$

i.e., the PSD splits into two separate contributions. In this case, the *powers* of both signals are added, at variance with the previous case in which the *modulations* were added. This is due to the fact that the two signals are orthogonal (in Fourier space). The SNR is given by

$$\mathcal{R}=10\log_{10}\left(\pi^2\frac{\alpha_\varepsilon^2\varepsilon_0^2+\alpha_\rho^2\rho_0^2}{W_o\Delta}\left[1-\frac{\alpha_\varepsilon^2\varepsilon_0^2+\alpha_\rho^2\rho_0^2}{2(W_o^2+\Omega_\varepsilon^2)}\right]^{-1}\right). \tag{27}$$

It is also worth remarking that $\mathcal{R}$ is independent of the sign of the amplitudes ε_0, ρ_0, unlike in the case with $\phi=0$, [see Eq. (25)].

V. NUMERICAL SIMULATIONS

In order to verify our analytic predictions of the previous section, Eq. (1) was numerically integrated. We have used the Runge-Kutta-Helfand [29] method for solving stochastic differential equations. Such a method is fast enough, and of higher order in the integration step h [in fact $\mathcal{O}(h^3)$ [30]]. The integration gives a trajectory whose statistical moments are the same as those for the formal solution of the equation

When modulating the correlation parameter, the same dynamical behavior as that obtained with a potential modulation was observed. This phenomenon has been previously studied in Ref. [17]. By means of numerical simulations, we verified the existence of SR when a modulation is applied only over the correlation parameter between the two noises.

A prediction of our approach is that, when both parameters are modulated with different driving frequencies, the two contributions seem to appear independently from each other. However, as indicated previously, this counterintuitive result is due to the linear response simplifications made within the two-state approach. In Figs. 4(a) through 4(d), we depict the PSD $\mathcal{S}(\Omega)$ obtained from numerical experiments, where the independent contributions from the two separate effects are apparent. Clearly, this is due to the smallness of the modulation amplitudes, and for larger values of ε_0 and ρ_0 we do find not only peaks corresponding to the harmonics but also those associated with the sum or difference of the two frequencies.

When both frequencies are equal, with zero relative phase shift between the signals, a net contribution due to the two signals arises with an enhanced SR effect. In Fig. 4(e) we depict the case when both frequencies are equal and with $\phi=0$. Finally, Fig. 4(f) allows us to verify another feature predicted by the two-state approach: when the modulation frequencies are equal, but with $\phi=\pi$, a weakening in the output spike at $\Omega=\Omega_\varepsilon=\Omega_\rho$ indeed occurs.

The characterization of the SR phenomenon given by the output SNR $\mathcal{R}$, is depicted in Fig. 5. In part (a) of this figure we compare $\mathcal{R}$ versus the additive noise intensity D_ε when there is a modulation $\varepsilon(t)$. In contrast, in Fig. 5(b) we show $\mathcal{R}$ when the correlation $\rho(t)$ is varied periodically. Although the functional dependence on D_ε looks the same in both plots, it is important to remark that the characteristic width in each of the two cases is quite different. When a modulation is simultaneously applied in both parameters (with $\phi=0$), we can expect that the effects reinforce each other. Figure 5(c) shows the existence of such behavior corroborating the SNR dependence as a function of the additive noise intensity D_ε for fixed D_ρ.

Lastly, Fig. 6 shows the dependence of the output SNR on multiplicative noise intensity. Note that the function $\mathcal{R}(D_\rho)$ increases for small D_ρ intensities and assumes a maximum at $D_\rho\sim0.2$. Thus, we can conclude that in the presence of a correlation modulation SR can become enhanced through correlated ($\rho\neq0$) multiplicative noise.

In order to analyze the possibility of using the simultaneous modulation of potential and correlation as a way of controlling SR [32], it is important to study the behavior of the SNR as a function of the relative phase shift ϕ when both modulation frequencies are equal. The numerical results are given in Fig. 7, and as predicted by the two-state approach the maximum in SNR occurs when $\phi=0$.

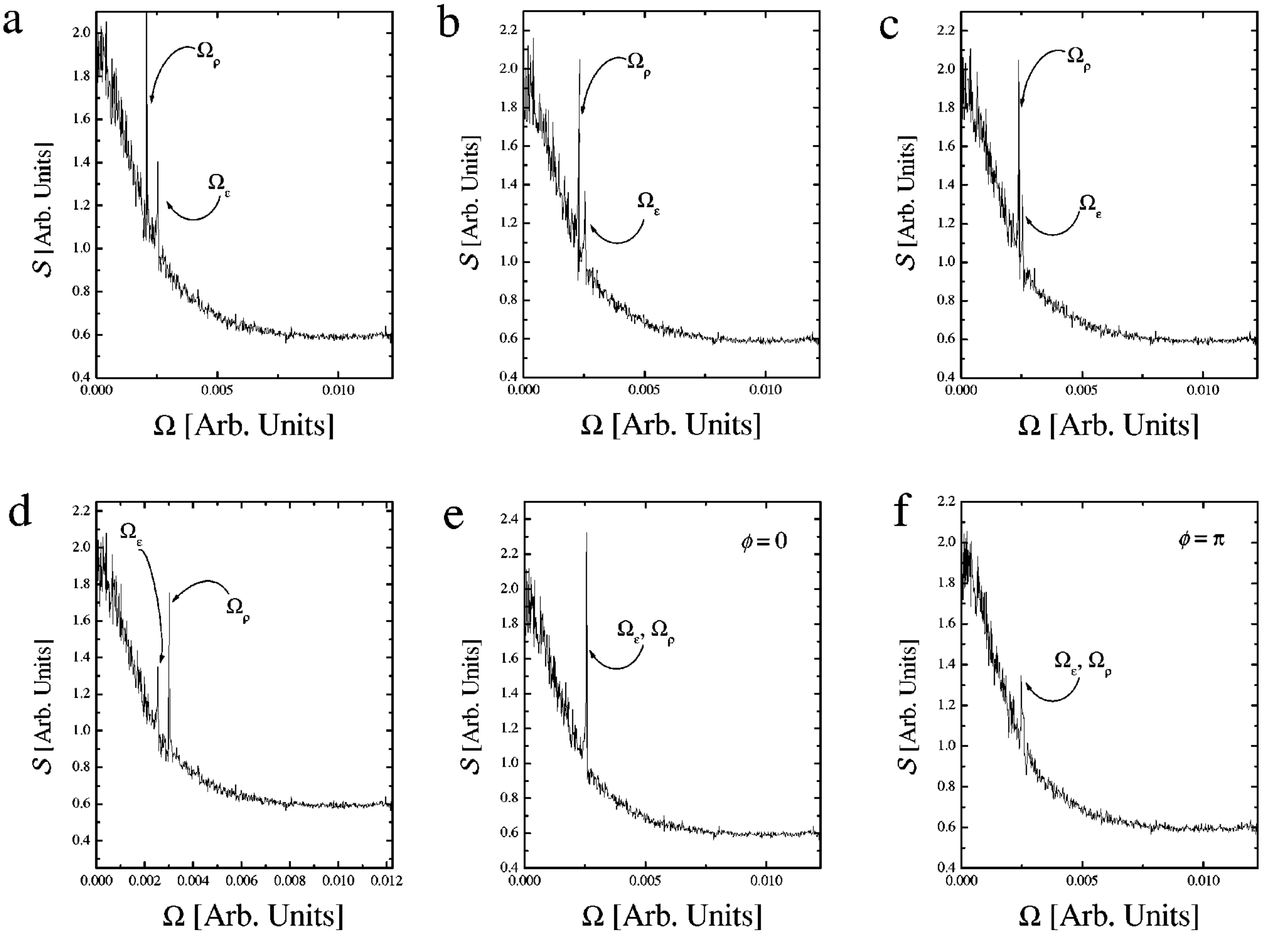

FIG. 4. We show the output PSD of the system when both amplitudes are different from zero. In all the plots, $D_\varepsilon = 0.1$, $D_\rho = 0.1$, $\varepsilon_0 = 0.04$, $\Omega_\varepsilon = 2.5\times 10^{-3}$, and $\rho_0 = 0.06$. In the plots (a) through (e) there is no relative phase shift between the signals ($\phi = 0$), while in (f) $\phi = \pi$. The values of the frequency modulation of the correlation parameter are (a) $\Omega_\rho = 2.0\times 10^{-3}$, (b) $\Omega_\rho = 2.3\times 10^{-3}$, (c) $\Omega_\rho = 2.4\times 10^{-3}$, (d) $\Omega_\rho = 2.9\times 10^{-3}$, (e) and (f) $\Omega_\rho = 2.5\times 10^{-3}$.

VI. SPECTRAL AMPLIFICATION FACTOR

The previous results, based on the two-state approach, do not contain the whole dynamics. Furthermore, they are restricted to an adiabatic regime and to small modulation amplitudes. This fact motivated researchers to propose other characterizations of the SR phenomenon. A particularly interesting point of view was proposed in Ref. [18], exploiting a Floquet expansion for the steady state solution of the associated (nonstationary) Fokker-Planck equation [see Eq. (3)]. For large times, and independently of the initial distribution $P(x,t_0)$, the Fokker-Planck equation tends asymptotically to a periodic function $P_{as}(x,t)$. This asymptotic solution may be expanded into Floquet states: a basis of time-dependent, periodic eigenfunctions of such an equation. The same approach can be used with the asymptotic expression of the autocorrelation function. For such an expansion to be possible, a common characteristic frequency (which we denote $\bar{\Omega}$) must exist in the system so that the function can be written as an expansion in its higher harmonics,

$$\bar{K}_{as}(\tau) = \sum_{n=-\infty}^{\infty} |M_n|^2 \exp(in\bar{\Omega}\tau) = 2\sum_{n=0}^{\infty} |M_n|^2 \cos(n\bar{\Omega}\tau). \tag{28}$$

Projecting this expression on the first harmonic of the series, we obtain

$$\int_0^{+\infty} \bar{K}_{as}(\tau)\cos(\bar{\Omega}_\varepsilon \tau)\,d\tau = 2\pi |M_1|^2. \tag{29}$$

This expression will be used to evaluate $|M_1|^2$. Hence we obtain

$$P_1 = 4\pi |M_1|^2, \tag{30}$$

while the total power at the input [by inspection of the Fokker-Planck equation (3)] is

$$P_{in} = \pi(\varepsilon_0 + \rho_0\sqrt{D_\varepsilon D_\rho})^2. \tag{31}$$

The *spectral amplification factor* (SAF) η is defined as

$$\eta(\bar{\Omega}, D_\varepsilon, D_\rho, \varepsilon_0, \rho_0) = \frac{P_1}{P_{in}} = 4\left(\frac{|M_1|}{\varepsilon_0 + \rho_0\sqrt{D_\varepsilon D_\rho}}\right)^2. \tag{32}$$

This SAF is an indicator for the SR phenomenon [18], indicating to what extent the signal is able to entrain the noise (stochastic synchronization).

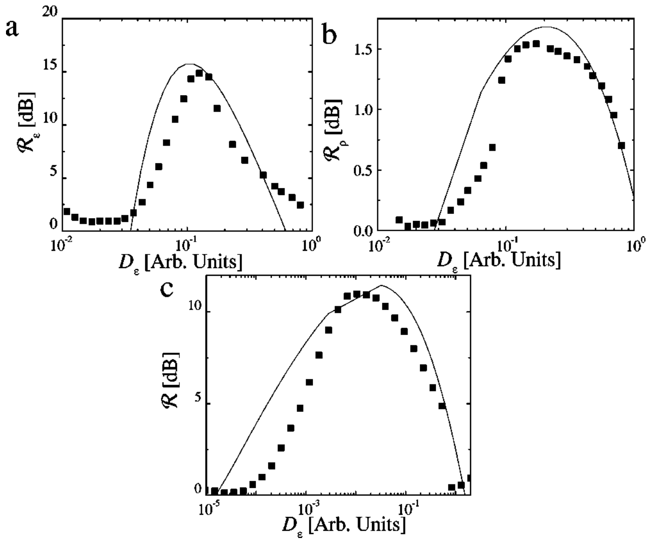

FIG. 5. SNR vs D_ε when modulating both parameters with the same driving frequency $\Omega_\varepsilon=\Omega_\rho=2.5\times10^{-3}$, and with relative phase shift $\phi=0$. In the three plots, we compare the theoretical predictions (lines) with the numerical simulations (filled squares). In plot (a), the multiplicative noise intensity is $D_\rho=0.1$, and only the potential is modulated: $\varepsilon_0=0.06$. In plot (b) there is modulation only over the correlation parameter, $D_\rho=0.1$, $\rho_0=0.03$. Finally, plot (c) shows the case in which there is a signal injected through both parameters. In this figure, $D_\rho=0.8$ $\varepsilon_0=0.05$, $\rho_0=0.07$, and $\phi=0$. It is important to remark here that we used a different scale in case (c) due to the large widening obtained in the SR peak, compared with the ones obtained in plots (a) and (b).

In order to verify the results obtained by means of the output SNR for the system under study, we have also evaluated this SAF η. The numerical results can be obtained by a discrete version of the asymptotic autocorrelation function from Eq. (29). We restrict our analysis to the case when the two modulation frequencies are equal. For this reason, we consider in the present section only the case with $\phi=0$.

In Fig. 8 we depict the results for η as a function of D_ε when the potential [case (a)] and the noise correlation [case (b)] are modulated. Once again, the characteristic width of the numerically evaluated SR phenomenon becomes broadened when we have $\rho_0\neq0$ as compared to the case $\rho_0=0$. However, we observed that the functional shape of $|M_1|^2$ remains numerically (mostly) unaffected by the value of the modulating strength. Thus, the observed widening is due to the fact that η depends inversely on the product $D_\rho D_\varepsilon$. Hence, when $D_\varepsilon\to0$, $|M_1|^2$ increases more when modulating the correlation $\rho(t)$ than when modulating $\varepsilon(t)$. Finally, Fig. 8(c) shows the η factor when both parameters are modulated simultaneously. It corroborates our previous result, that its width is notably larger than in the case of Fig. 8(a). As was observed in the SNR results, this yields a higher maximum in the SR indicator (in this case η).

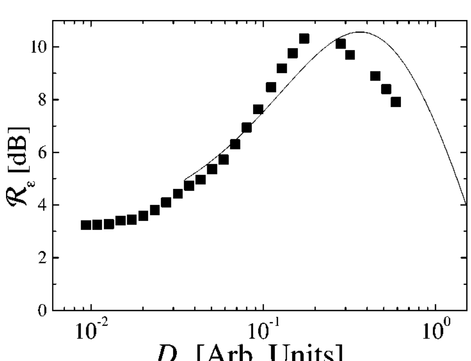

FIG. 6. SNR as a function of multiplicative noise intensity, when modulating the potential ($\varepsilon_0=0.04$) and the noise correlation parameter $\rho(t)$ ($\rho_0=0.02$). The remaining parameters are $D_\varepsilon=0.1$, $\Omega_\varepsilon=\Omega_\rho=2.5\times10^{-3}$, and $\phi=0$. In this plot we see that the output SNR may grow even with increasing multiplicative noise intensity. The numerical result (squares) is in good agreement with the two-state theory (lines).

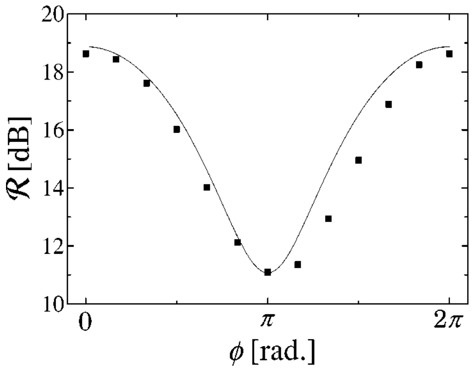

FIG. 7. Dependence of output SNR on phase shift ϕ between the two modulations $\varepsilon(t)$ and $\rho(t)$. We have fixed $D_\varepsilon=0.1$, $D_\rho=0.7$, $\varepsilon_0=0.05$, $\rho_0=0.07$, and $\Omega_\varepsilon=\Omega_\rho=2.5\times10^{-3}$. We observe that the maximum occurs for $\phi=0$, when no relative phase shift exists between the signals, while a minimum is assumed for $\phi=\pi$. As in the previous plots, we compare the numerical results (black squares) with our theoretical predictions (solid line).

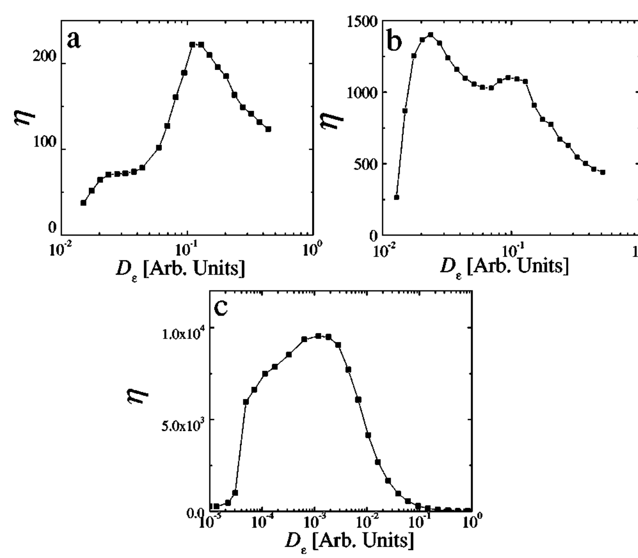

FIG. 8. Behavior of the spectral amplification factor. In (a), only the bistable potential is modulated, i.e., $\rho(t)=0$ and $\varepsilon_0=0.05$. In (b), the correlation $\rho(t)$ is the only modulated parameter ($\rho_0=0.04$). In this case as in the previous one, the driving frequency is $\Omega_0=\Omega_\varepsilon=2.5\times10^{-3}$, and the multiplicative noise intensity is $D_\rho=0.1$. In (c) we depict the case when both parameters are modulated with amplitudes $\varepsilon_0=0.05$, $\rho_0=0.7$, while $D_\rho=0.8$, and $\Omega_\varepsilon=\Omega_\rho=2.5\times10^{-3}$. The scale D_ε is enlarged to emphasize visibly the widening of the SR peak.

VII. CONCLUSIONS

In this work we have studied a bistable symmetric system, driven by two white Gaussian noise sources that are correlated: one of them is associated with an additive white noise and the other with a multiplicative white noise. As in other systems exhibiting SR, a periodic signal was introduced through a weak potential modulation. In systems that exhibit SR, the phenomenon depends strongly on the tuning of an external (noise) parameter, which usually cannot be controlled. In order to improve this effect we have added a second signal to the system, in the form of a modulation of the correlation between the two white noises. In this way we obtain a SR effect similar to that when the potential is modulated.

Through the two-state approach, the output SNR was calculated as a function of the various parameters of the system. Numerical simulations of the system under study were consistent with all theoretical results, including the evaluation of the SAF, a synchronization measure of the SR in our system. The results are briefly summarized as follows.

On modulating only the noise correlation $\rho(t)$ the SR phenomenon becomes less sensitive to the external additive noise. This is the reason why, when modulating *simultaneously* both parameters with signals that have the same driving frequency and with a relative phase shift of $\phi=0$, what we find is a large degree of independence of the external parameter. As a consequence, by appropriately tuning the (internal) multiplicative noise intensity, the range where the SR appears (as a function of the external additive noise) can be extended by almost two orders of magnitude. This aspect was found in both the SNR and the spectral amplification measure. Unfortunately, such behavior occurs only when the two driving frequencies are equal. It is interesting to note that, for some multiplicative noise intensities, the SR phenomenon can increase with increasing multiplicative noise intensity D_ρ.

A fact worth remarking is the prominent prediction obtained within the two-state approach: when both parameters are modulated with different driving frequencies, the two SR phenomena associated with each driving source can appear independently of each other. One might think that this counterintuitive result is due to the linear response approximation we have used within the two-state approach and to the fact that the modulation amplitude we used is very small. Indeed, the numerical simulations support this view.

Regarding the physical relevance of the present results, in Ref. [20] some examples of realistic models showing bistable behavior plus the possibility of correlated additive and multiplicative noise sources were discussed. However, in those cases the modulation of the correlation does not seem to be at all that simple. A simpler way to physically realize the situation we have described here is by means of an electronic circuit with two different white noise sources. Noise sources could be combined in such a way as to produce a third correlated one. Hence, one of the original sources together with the engineered one can be used to produce the two correlated noises. In addition, the modulation of this correlation can be appropiately introduced through the parameter defining the correlation [31]. Furthermore, using this method it should be possible to introduce the very idea described here into other experimental situations where multiplicative and additive noise have been introduced previously.

What we intend to point out in this work is the fact that with an appropriate design it is possible to achieve a remarkable widening of the range of values of the fluctuation parameter where the SR phenomenon can be detected. Also, a remarkable aspect of our present findings indicates an alternative route of controlling the SR phenomenon along the reasoning put forward in Ref. [32]. We hope that the present results can awaken the interest of theoreticians and experimentalists in the search for alternative forms of the SR phenomenon that can produce effects similar to those indicated above and contribute to different ways of controlling the phenomenon.

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