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Angaben zur Veröffentlichung / Publication details:

Klepel, Konrad, Dirk Blömker, and Wael W. Mohammed. 2014. "Amplitude equation for the generalized Swift–Hohenberg equation with noise." *Zeitschrift für angewandte Mathematik und Physik* 65 (6): 1107–26. https://doi.org/10.1007/s00033-013-0371-8.

Amplitude Equation for the generalized Swift-Hohenberg Equation with Noise

Konrad Klepel, Wael W. Mohammed and Dirk Blömker

Abstract. We derive an amplitude equation for a stochastic partial differential equation (SPDE) of Swift-Hohenberg type under the simplifying assumption that the noise acts uniformly on the whole system. Due to the natural separation of timescales, solutions are well approximated by a stochastic differential equation (SDE), the so called amplitude equation, describing the evolution of the dominant pattern.

Although the slow dominant modes are not forced directly, via the nonlinearity the noise gets transmitted through the system to those modes, too, and multiplicative noise appears in the amplitude equation. Moreover, additional linear and cubic terms appear due to averaging. This leads to either noise induced stabilization or destabilization effects in the dominating pattern.

Mathematics Subject Classification (2010). 60H10, 60H15, 35B36, 35R60, 35Q35, 35K91.

Keywords. amplitude equation, pattern formation, generalized Swift-Hohenberg model, additive noise, multi-scale analysis, stabilization due to noise.

1. Introduction

The Swift-Hohenberg equation is a model equation used to study pattern formation in driven systems. It was originally derived in [SH77] as a qualitative description of the convective instability in the Rayleigh Benard model. Today it is one of the celebrated models in pattern formation. In the simplest case it takes the form

$$\partial_t u = ru - (1+\nabla^2)^2 u - u^3 \ , \tag{1}$$

where $r \in \mathbb{R}$ is the bifurcation parameter, corresponding to the Rayleigh number in convection problems. At $r = 0$ is the change of stability that corresponds to the convective instability. A variant, arising from a different regime in the Rayleigh Benard model, is the so called generalized Swift-Hohenberg model with quadratic and cubic nonlinearity:

$$\partial_t u = ru - (1 + \nabla^2)^2 u + \alpha u^2 - u^3, \tag{2}$$

where $\alpha > 0$ is an additional parameter, measuring the strength of the quadratic instability. Equation (2) is also derived, when a general nonlinearity is expanded via Taylor's formula. The dynamics of (2) was studied in [CH93], [HMBD95], [BK06] and recently [BD12] among others. In these articles the usual approach of amplitude equations is the derivation of a simplified model in the vicinity of the change of stability at $r = 0$.

This technique is well-known in the physics literature, and it is a kind of normal form, related to the center manifold, describing the essential dynamics of the pattern forming system by a simple ordinary differential equation. To be more precise, both (1) and (2) are very well approximated by

$$u(t, x) \approx \sqrt{|r|} \cdot A(|r|t) \cdot e^{ix} + \sqrt{|r|} \cdot \overline{A(|r|t)} \cdot e^{-ix}. \tag{3}$$

where the complex amplitude $A(T)$ of the dominant mode e^{ix} is the solution of

$$\partial_T A = \text{sgn}(r)A + 3(\tfrac{38}{27}\alpha^2 - 1)|A|^2 A, \tag{4}$$

which is accordingly named amplitude equation (AE, for short) of (2). Note that $T = |r|t$ denotes the *slow time*.

For the deterministic Swift-Hohenberg equation on an unbounded domain solutions are approximated via the Ginzburg-Landau PDE, as a whole band of uncountably many eigenvalues changes stability. This is also well-known in the physics literature, but for rigorous results on the deterministic Swift-Hohenberg equation, see for instance the results in [KMS92], [CE90], [MSZ00] and [Sch96].

It is the aim of this article to provide rigorous error estimates and to verify the existence of an amplitude equation for (2). We only add noise constant in space, which does not cover the physically important case of thermal noise, but only x-independent perturbations acting uniformly on the whole system. This has the interpretation of shaking the whole Rayleigh-Bénard experiment uniformly. The assumption on the noise is only for simplicity of presentation, as noise with more spatial structure complicates the result significantly. Completely analogous, we could treat several different kinds of noise with spatial dependence, which is not acting on the dominant modes directly. But here the question arises, how this noise would be realized in experiments.

In contrast to that, if the additive noise acts on the dominant modes, then we need to change scaling and consider smaller noise. See for example [BH04] or [Blö07]. We comment later in section 3.1 on this in more detail.

Thus as an example, here we only consider the following stochastic generalized Swift-Hohenberg equation:

$$\partial_t u = \nu\varepsilon^2 u - (1 + \Delta)^2 u + \alpha u^2 - u^3 + \varepsilon\sigma\partial_t\beta, \tag{SH}$$

where $\beta(t)$ is a real valued standard Brownian motion. For simplicity of presentation we focus only on one specific example of boundary conditions and consider (SH) with periodic boundary conditions on $[0, 2\pi]$ only. Dirichlet and Neumann conditions would yield similar results. We comment later in Section 3.1 on other types of boundary conditions in more detail.

Here α, σ and ν are real-valued constants. The small parameter $\varepsilon > 0$ relates the distance from bifurcation to the noise strength. In experiments the noise strength is often considered to be given and the distance from bifurcation can by adjusted, for example by changing the temperature and thus the Reynolds number in Rayleigh-Bénard convection. Of course different scalings are possible, but then in the final result for the amplitude equation, either the noise or the linear term disappears.

We show that in our scaling, although the constant mode is non-dominant, and does not play a role in the deterministic result, the noise appears also in the amplitude equation through coupling between Fourier modes induced by the nonlinear terms. Additional terms on the right-hand side are created and the noise appears multiplicative. To be more precise, (SH) is well approximated by

$$u(t,x) \approx \varepsilon A(\varepsilon^2 t) \cdot e^{ix} + \varepsilon \overline{A(\varepsilon^2 t)} \cdot e^{-ix} , \tag{5}$$

where the complex-valued amplitude $A(T)$ solves the Itô differential equation

$$dA = (\nu A + 3(\alpha^2 - \tfrac{1}{2})\sigma^2 A + 3(\tfrac{38}{27}\alpha^2 - 1)A|A|^2)dT + 2\alpha\sigma A d\tilde{\beta} . \tag{AE}$$

Here $\tilde{\beta}(T) := \varepsilon\beta(\varepsilon^{-2}T)$ is a rescaled version of $\beta(t)$. It is an interesting observation, that the equation contains only multiplicative noise instead of additive noise. This is due to the fact that the noise acting not directly on the dominant modes is mapped by the (in this case quadratic) nonlinearity back to the dominant modes. In order to obtain additive noise only in the amplitude equation, one needs to force directly the dominant modes. But in that case it is essential to have smaller noise, in oder to have a meaningful result. See [BH04] for Swift-Hohenberg without quadratic terms or [BM09] for an equation of Burgers type.

A surprising observation is that due to our choice of the quadratic nonlinearity unstable terms both cubic and linear arise in the amplitude equation. The additional terms arise from nonlinear interaction, where squares of the noise actually average to a constant.

This is has also interesting interpretation for the pattern formation in a physical system. Depending on the size of α, sufficiently strong noise has the potential to either stabilize or destabilize the dominant pattern close to the convective instability. Or in other words, adding a noise could lead either to an earlier change of stability, or to a delay of the first instability.

This is significantly different to other models with quadratic nonlinearities like Burgers equation, for example, where the additional linear terms are always stabilizing. See [BHP07] for a rigorous treatment of a large class of equations that contain the Burgers equation and [BMNW11] for numerical experiments, showing that in some cases the approximation remains true for surprisingly long times.

Our research was initiated originally by the observations of Axel Hutt and collaborators [HLSG07, Hut08, HLSG08], who treated the case with $\alpha = 0$. By numerical simulations of the equation on very large domains and formal arguments based on the non-rigorous application of center manifold theory they derived the amplitude equation for the standard Swift-Hohenberg equation with noise constant in space. Moreover, they already pointed out that additive noise has the potential to stabilize the dynamics. For a rigorous result in this direction see [BM12] on bounded domains and [MBK12] on unbounded domains.
A similar stabilization effect of the Burgers equation was also observed by A. Roberts [Ro03] with a single noise forcing the sine-Fourier mode. This was later established rigorously in [BHP07], even in the case of higher dimensional noise. The main difference for Burgers is that the amplitude equation still contains multiplicative noise, while in the situation of standard Swift-Hohenberg, no noise remains in the amplitude equation.
The case of quadratic nonlinearities (as in the Burgers equation) is much more involved as the case of cubic nonlinearities (as in the standard Swift-Hohenberg equation). For quadratic nonlinearities the interaction of the noise and the nonlinearity complicates the problem significantly, as non-dominant Fourier-modes have a significant impact on the dominant modes. See [BHP07]. While in [BM12] only the dominant modes survive and all other Fourier modes are treated as error terms. Additional terms only arise due to averaging of noise and nonlinear interaction of noise.
The paper is organized as follows. Section 2 provides the setting of the problem, while Section 3 states the main result. In Section 4 we collect all proofs.

2. Setting

We consider mild solutions of (SH) with values in the space $C^0 = C^0_{per}([0, 2\pi])$, i.e. the space of 2π periodic continuous functions, defined by

Definition 1. *A stochastic process $u(t)$, $t \in [0, T_0]$ with continuous paths in C^0 is a mild solution of* (SH) *if the following variation of constants formula holds in C^0 for all $t \in [0, T_0]$:*

$$u(t) = e^{-t(1+\partial_x^2)^2}u(0) + \int_0^t e^{-(t-s)(1+\partial_x^2)^2}[\nu\varepsilon^2 u(s) + \alpha u^2(s) - u^3(s)]ds \\ + \varepsilon \int_0^t e^{-(t-s)(1+\partial_x^2)^2}\sigma d\beta(s) \ , \tag{6}$$

where $e^{-t(1+\partial_x^2)^2}$ is the semigroup created by the operator $-(1+\partial_x^2)^2$ (cf. [Paz83]*).*

Using standard theory given in [DPZ92], it is straightforward to verify that such a mild solution exists. This is, for example, achieved via Banach's fixed-point theorem for unique local solutions and energy estimates for global solutions.

Remark 2. *The stochastic integral on the right-hand side of* (6) *can be simplified to*

$$Z(t) := \varepsilon\sigma \int_0^t e^{-(t-s)(1+\partial_x^2)^2} 1 d\beta(s) = \varepsilon\sigma \int_0^t e^{-(t-s)} d\beta(s), \tag{7}$$

which is a simple real-valued Ornstein-Uhlenbeck process.

Our approximation result states the error in terms of the distance to the bifurcation point ($r = \nu = 0$) using big $\mathcal{O}$ notation modified for random variables. This is defined by the following:

Definition 3. *Let X_ε with $\varepsilon > 0$ be a family of stochastic processes and $f(\varepsilon)$ be a function of ε. Then X_ε is of order $f(\varepsilon)$, which we abbreviate by*

$$X_\varepsilon = \mathcal{O}(f(\varepsilon)),$$

if and only if for every p-th moment of X_ε there is a constant C_p such that the following is valid for all $\varepsilon > 0$:

$$\mathbb{E}\left(|X_\varepsilon|^p\right) \leq C_p |f(\varepsilon)|^p.$$

3. Main result

The main result is the following approximation theorem for the stochastic generalized Swift-Hohenberg equation (SH).

Theorem 4. *Let $T_0 > 0$ be a time of order 1, $\alpha \in \mathbb{R}$ with $\alpha^2 < \frac{27}{38}$ and $0 < \kappa < \frac{1}{17}$. Let u be a stochastic process with continuous paths in C^0 that is a mild solution of* (SH) *with $\|u(0)\|_\infty = \mathcal{O}(\varepsilon^{1-\kappa})$.*
Furthermore, let $A(T)$, $T \in [0, T_0]$ be a stochastic process with continuous paths in $\mathbb{C}$ that solves (AE) *with*

$$A(0) = \frac{1}{2\pi} \int_0^{2\pi} \varepsilon^{-1} u(0,x) e^{ix} dx = \mathcal{O}(\varepsilon^{-\kappa}),$$

Then for all $p \in \mathbb{N}$ there is a constant C_p such that the following holds:

$$\mathbb{P}\Big(\sup_{t \in [0, T_0/\varepsilon^2]} \|u(t) - u_A(t) - \varepsilon Z_\varepsilon(\varepsilon^2 t) - e^{-t(1+\partial_x^2)^2} u_s(0)\|_\infty > \varepsilon^{2-19\kappa} \Big) \leq C_p \varepsilon^p, \tag{8}$$

with the approximation

$$u_A(t,x) = \varepsilon A(\varepsilon^2 t) e^{ix} + \varepsilon \bar{A}(\varepsilon^2 t) e^{-ix}$$

where Z_ε is the Ornstein-Uhlenbeck process defined by

$$Z_\varepsilon(T) := \varepsilon^{-1}\sigma \int_0^T e^{-\varepsilon^{-2}(T-s)} d\tilde{\beta}(s). \tag{9}$$

Here we easily see that $Z_\varepsilon(\varepsilon^2 t) = Z(t)$ with Z defined in (7).
Let us comment that $\kappa > 0$ introduced in the theorem above mainly takes care of the fact, that we cannot bound the fast stochastic convolution Z_ε uniformly in time by a constant with high probability, but by a bound that is slightly

worse depending on ε. As the final error bound will thus be slightly worse than order $\mathcal{O}(\varepsilon^2)$, we can also allow for initial conditions $u(0)$ that are not of order $\mathcal{O}(\varepsilon)$, but slightly worse. This in turn means that the amplitude equation can be studied with slightly larger initial conditions than order $\mathcal{O}(1)$. It turns out in the proofs that in combination with the stability of the nonlinearity, the very weakly unstable linear part is not strong enough to prevent the solution of the amplitude equation from being bounded at a slightly larger order than $\mathcal{O}(\varepsilon^{-\kappa})$, at least on the timescales under consideration.

Let us finally remark, that although we think of κ being very small, the previous theorem is still true for $\kappa \in [1/19, 1/17]$, but useless, as we lose a full order of ε in the final result.

Remark 5. *We see in* (AE) *surprising deterministic terms induced by the noise that can lead to stabilization or destabilization effects for the dominant pattern.*

There is a stabilizing linear term from the cubic term, that was already observed in [Hut08]. *The quadratic term leads to destabilizing terms both cubic and linear. But if α is not too large, increasing the noise strength σ may lead to a stabilization effect. Obviously, stabilizing the amplitude equation means destroying the dominant pattern in the original equation, while destabilization implies that the pattern arises due to the presence of noise. The origin of these lie in the nonlinear interaction of the noise together with averaging results (see Lemma* 11*).*

Remark 6. *In the case of $\alpha^2 > \frac{27}{38}$, which is not treated in Theorem* 4*, the amplitude equation* (AE) *has an unstable cubic nonlinearity, and thus exhibits blow up in finite time, while in the case $\alpha^2 = \frac{27}{38}$* (AE) *loses the cubic completely. In both cases the scaling of the parameters is such that the model turns unphysical, and different scalings should be considered.*

Nevertheless as long as the solution A to (AE) *is not too large (for example $|A(T)| \leq \varepsilon^{-\kappa}$) our approximation result still holds, up to a stopping time, where A fails to be bounded.*

The proof is basically the same except the fixed time $T_0 > 0$ is replaced everywhere by the stopping time $\tau_A = \inf\{t : |A(t)| \geq \varepsilon^{-\kappa}\} \wedge T_0$. For simplicity of presentation, we refrain from giving more details here.

Remark 7. *The interesting case $\alpha^2 = \frac{27}{38}$ was studied in the deterministic case. See for example* [BD12]*, where an even more general case was treated. In this case* (AE) *loses its cubic nonlinearity, and turns out to be a linear equation only. Thus we can consider larger solutions and hence larger noise. By changing the scaling still a meaningful amplitude equation is obtained but now with a quintic nonlinearity.*

Using the methods presented in this paper it is straightforward but lengthy to derive the quintic amplitude equation also in the stochastic case. We refrain from giving details here.

3.1. Possible extensions of the result

Let us remark on further extensions of the results presented here. First of all, it is straightforward to consider different kinds of cubic or quadratic nonlinearities, as was already done in [BM12] or [BHP07]. The main focus of our result presented here was to discuss a specific example that exhibits the potential destabilization of the amplitude equation via unstable cubic terms that arise from the presence of the quadratic nonlinearity.

Closeness to bifurcation: An interesting new approach was presented in [KPPS11, KPPS13]. While the linear perturbation shifting the bifurcation is mostly of lower order, as the $\nu\varepsilon^2 u$ term in (SH), they consider a perturbation in the differential operator of highest order. This seems to lead to similar results, as the lower order perturbations, but the methods of proof have to be different. A further interesting question, which is not yet fully settled, is whether it causes major problems when the lower order forcing term does not commute with the linear operator. In order to avoid this, discussion, we consider only the somewhat simplest case of $\nu\varepsilon^2 u$.

Boundary conditions and domains: Different boundary conditions in many cases yield similar results. For instance, in the case of Dirichlet or Neumann conditions for equation (SH) we can consider the Fourier basis given in terms of $\sin(kx)$ or $\cos(kx)$, where only a single mode is changing stability. The amplitude A of the dominant Mode $k = 1$ is in that case only real valued, but apart from that the main result would be the same. The amplitude equation) is only a one-dimensional ODE containing similar terms as (AE). Only the constants do change.

We could also treat with similar methods other higher dimensional domains for the underlying SPDE. The main feature for domain and boundary conditions is that the linear operator (in our case $-(1 + \partial_x^2)^2$) has a non-negative spectrum and exhibits a basis of eigenfunctions, where the dominating space is given by its finite dimensional kernel. Nevertheless, we then need additional technical conditions, how the non-linearities interact with the eigenfunctions. See [BHP07] for an example of Burgers type in full abstract generality. In order to avoid these technicalities, we consider only our specific example using complex Fourier series. The convolution structure of the nonlinearity in Fourier-space simplifies the results slightly, but as seen in [BHP07], the convolution structure is not essential for the results.

Noise: The assumption that the noise is spatially constant, is easily changed to noise acting on any other Fourier-mode. Unless the dominant modes are forced, the main result would be the same. Only constants in (AE) might change. Nevertheless, the constant forcing has the nice physical interpretation of uniformly shaking the whole experimental set up uniformly. If any other Fourier-mode is forced, then it is not easy to see how a similar interpretation can be given

Infinite dimensional noise, i.e. noise acting on infinitely many Fourier modes like thermal fluctuations, can be treated by similar methods. However, one

needs many assumptions that various infinite series appearing in the calculations do converge. This can be regarded as a sort of regularity assumptions on the noise. Nevertheless there is a key problem with noise driven Fourier modes interacting via the nonlinearity with other noise driven modes. In some cases a fast OU-process inside stochastic integrals needs to be averaged. This can not be averaged directly with strong error estimates as done here in Lemma 11, for example. The main result in principle remains still true, but only weak convergence of approximation to the solution in the limit $\varepsilon \to 0$ is available.
A remarkable result was introduced in [BHP07]. If the dominant space is one-dimensional one can verify a martingale-approximation lemma, which is based on Levy-representation theorems of Brownian motions. Using this it is possible in some cases to still verify an approximation result with explicit error terms. See [BHP07] or [BM12] for a detailed discussion. However, as there are numerous nonlinear interactions of noise terms, the stochastic forcing in the amplitude equation is of the type $\sqrt{c_1|A|^2 + c_2}\, d\beta$, which is somewhat a combination of additive and multiplicative noise.
Another crucial point in our approach is also that quadratic nonlinearities do not map back noisy modes to the dominant ones. For example, if we change the linear operator slightly to $-(4+\partial_x^2)^2$ such that the second Fourier-mode is dominant and force only the first Fourier mode with a forcing term $\varepsilon\sigma\sin(x)\partial_t\beta$, then the approach presented here would fail, as new terms appear in the amplitude equation, that are much larger than order one.
In order to obtain a meaningful result we need to consider smaller noise or larger distance from bifurcation. In that case we conjecture that this leads to a constant deterministic forcing terms in the amplitude equation. This is due to the fact that the quadratic nonlinearity maps the square of the noisy Fourier mode to the dominant mode, which is then averaged to a constant.

4. Proof of the main result

We start by rescaling $u(t,x)$ to the slow time-scale by

$$v(T,x) := \varepsilon^{-1}u(\varepsilon^{-2}T,x)\ .$$

Its stochastic differential is given by

$$dv = (-\varepsilon^{-2}(1+\partial_x^2)^2 v + \nu v + \varepsilon^{-1}\alpha v^2 - v^3)dT + \varepsilon^{-1}\sigma d\tilde{\beta}\ .$$

The mild formulation is:

$$\begin{aligned} v(T) &= e^{-T\varepsilon^{-2}(1+\partial_x^2)^2}v(0) + Z_\varepsilon(T) \\ &\quad + \int_0^T e^{-(T-s)\varepsilon^{-2}(1+\partial_x^2)^2}[\nu v(s) + \varepsilon^{-1}\alpha v^2(s) - v^3(s)]ds\ . \end{aligned} \tag{10}$$

Here Z_ε is the fast Ornstein-Uhlenbeck process defined in (9). It is the solution of

$$dZ_\varepsilon = -\varepsilon^{-2}Z_\varepsilon dT + \sigma\varepsilon^{-1}d\tilde{\beta}, \quad Z_\varepsilon(0) = 0\ . \tag{11}$$

Also we define the stopping time

$$\tau^* = \inf\{T > 0 : \|v(T)\|_\infty > \varepsilon^{-\kappa_0}\} \wedge T_0, \tag{12}$$

where κ is defined in Theorem 4 and κ_0 is any small real value with $\kappa_0 > \kappa$, which asserts that $\tau^* > 0$ almost surely. Later we fix $\kappa_0 = \frac{9}{8}\kappa$ in the proof of Theorem 4. Expanding $v(T,x)$ as a complex Fourier series yields

$$v(T,x) = \sum_{k=-\infty}^{\infty} v_k(T)e^{ikx} . \tag{13}$$

Define a splitting of the Fourier modes into the non-dominant modes

$$v_s(T,x) = \sum_{|k|\neq 1} v_k(T)e^{ikx} \tag{14}$$

and the dominant modes

$$v_c(T,x) = v(T,x) - v_s(T,x) = v_1(T)e^{ix} + c.c. . \tag{15}$$

Finally for technical reasons, we define

$$v_\infty(T,x) = \sum_{|k|\geq 3} [v_k(T) - e^{-T\varepsilon^{-2}(1-k^2)^2} v_k(0)] \cdot e^{ikx} \tag{16}$$

For $|k| \geq 1$ from the mild solution (10), each v_k is given by

$$\begin{aligned} v_k(T) &= e^{-\varepsilon^{-2}(1-k^2)^2 T} v_k(0) \\ &\quad + \int_0^T e^{-\varepsilon^{-2}(1-k^2)^2(T-s)} \Big[\nu v_k(s) + \varepsilon^{-1}\alpha(\widehat{v^2})_k(s) - (\widehat{v^3})_k(s) \Big]\, ds, \end{aligned} \tag{17}$$

where the hat indicates the discrete Fourier transform and the lower index k denotes its k-th mode.

4.1. Removing non-dominant modes

We show first that the non-dominant modes ($|k| \neq 1$) can be approximated by the fast OU-process Z_ε. With a slight abuse of the $\mathcal{O}$-notation, our result states:

$$v_s(T) = e^{-T\varepsilon^{-2}(1+\partial_x^2)^2} v_s(0) + Z_\varepsilon(T) + \mathcal{O}(\varepsilon^{1-2\kappa_0}) .$$

Or, to be more precise:

Lemma 8. *Under the assumptions of Theorem* 4*, with stopping time* τ^* *defined by* (12) *and* v_k *as in* (13)*, the following statements are true:*

$$\sup_{T\in[0,\tau^*]} \| \sum_{|k|\geq 2} [v_k(T) - e^{-T\varepsilon^{-2}(1-k^2)^2} v_k(0)] \cdot e^{ikx} \|_\infty = \mathcal{O}(\varepsilon^{1-2\kappa_0}) , \tag{18}$$

$$\sup_{T\in[0,\tau^*]} \|v_0(T) - Z_\varepsilon(T) - e^{-T\varepsilon^{-2}} v_0(0)\| = \mathcal{O}(\varepsilon^{1-2\kappa_0}) . \tag{19}$$

Proof. Since $\|v\|_\infty \leq \varepsilon^{-\kappa_0}$, it follows that for any $k \in \mathbb{Z}$ and $n \in \mathbb{N}$

$$|(\widehat{v^n})_k| \leq \Big(\sum_{k\in\mathbb{Z}} |(\widehat{v^n})_k|^2 \Big)^{1/2} = \|\widehat{v^n}\|_{L_2} = \|v^n\|_{L_2} \leq \sqrt{2\pi}\|v^n\|_\infty \leq \sqrt{2\pi}\varepsilon^{-n\kappa_0}. \tag{20}$$

In combination with the simple inequality (for $|k| \neq 1$)

$$\int_0^T e^{-\varepsilon^{-2}(1-k^2)^2(T-s)} ds \leq (1-k^2)^{-2}\varepsilon^2,$$

we derive the following by bounding the integral term in (17)

$$\left| v_k(T) - e^{-T\varepsilon^{-2}(1-k^2)^2} v_k(0) \right| \leq \varepsilon^{1-2\kappa_0} \cdot (1-k^2)^{-2} \cdot (2 + |\nu| + |\alpha|). \tag{21}$$

Therefore with $\sum_{|k|\geq 2}(1-k^2)^{-2} \leq \sum_{k=1}^{\infty} k^{-2} = \frac{\pi^2}{6}$ we obtain (using $\kappa_0 < 1$ for the cubic term)

$$\sum_{|k|\geq 2} \left| v_k(T) - e^{-T\varepsilon^{-2}(1-k^2)^2} v_k(0) \right| \leq \varepsilon^{1-2\kappa_0} \cdot \frac{\pi^2}{3}(2 + |\nu| + |\alpha|),$$

which proves (18). Projecting the mild solution (6), the constant mode v_0 has the form

$$\begin{aligned} v_0(T) =& e^{-\varepsilon^{-2}T} v_0(0) + Z_\varepsilon(T) \\ &+ \int_0^T e^{-\varepsilon^{-2}(T-s)} (\nu v_0(s) + \varepsilon^{-1}\alpha(\widehat{v^2})_0(s) - (\widehat{v^3})_0(s)) ds \; . \end{aligned} \tag{22}$$

Thus with similar arguments as before, for all $T < \tau^*$ the left side of (19) is bounded by

$$\left| v_0(T) - Z_\varepsilon(T) - e^{-\varepsilon^{-2}T} v_0(0) \right| \leq \varepsilon^{1-2\kappa_0}(2 + |\nu| + |\alpha|).$$

□

4.2. Rewriting the first Fourier-Mode

The next step is to show that the dominant mode $v_1(T)$ is well approximated by $A(T)$. For simplicity of presentation let us define the following functions:

$$a(T) := v_1(T), \qquad \Phi(T) := \varepsilon^{-1}\left(v_2(T) - e^{-9T\varepsilon^{-2}} v_2(0) \right),$$
$$\Psi(T) := \varepsilon^{-1}\left(v_0(T) - Z_\varepsilon(T) - e^{-T\varepsilon^{-2}} v_0(0) \right).$$

Lemma 9. *Under the assumptions of Lemma 8, the stochastic differential of $a(T)$ is given by*

$$da = (\nu a + 3(\tfrac{38}{27}\alpha^2 - 1)a|a|^2 + 6(\alpha^2 - \tfrac{1}{2})aZ_\varepsilon^2)dT + 2\alpha\sigma a d\tilde{\beta} + dR, \tag{23}$$

where $R(t)$ is a stochastic processes with $\sup_{t\in[0,\tau^*]} |R(t)| = \mathcal{O}(\varepsilon^{1-8\kappa_0})$.

Proof. In Lemma 8 in (20) and (21) we established:

$$\sup_{T\in[0,\tau^*]} |v_1(T)| \leq \varepsilon^{-\kappa_0} \tag{24}$$

$$\sup_{T\in[0,\tau^*]} \Big(\sup_{|k|\geq 2} |v_k(T) - e^{-\varepsilon^{-2}(1-k^2)^2} v_k(0)| \Big) = \mathcal{O}(\varepsilon^{1-2\kappa_0}). \tag{25}$$

This readily implies

$$\sup_{T\in[0,\tau^*]} |a(T)| = \mathcal{O}(\varepsilon^{-\kappa_0}), \quad \sup_{T\in[0,\tau^*]} |\Phi(T)| = \mathcal{O}(\varepsilon^{-2\kappa_0}), \quad \sup_{T\in[0,\tau^*]} |\Psi(T)| = \mathcal{O}(\varepsilon^{-2\kappa_0}).$$

The slightly better bound on a unfortunately does not improve the final result. We could just bound all three terms by $\mathcal{O}(\varepsilon^{-2\kappa_0})$.
The infinite-dimensional part is bounded by

$$\sup_{T\in[0,\tau^*]} \|v_\infty(T)\|_\infty = \mathcal{O}(\varepsilon^{1-2\kappa_0}). \tag{26}$$

The OU-process can be bounded by

$$\sup_{T\in[0,\tau^*]} |Z_\varepsilon(T)| = \mathcal{O}(\varepsilon^{-\zeta}) \tag{27}$$

for all positive $\zeta > 0$. For a proof of this well-known result see for example [BM12] p. 9 (Lemma 14).
Now we can directly calculate the stochastic differentials $da, d\Phi$ and $d\Psi$ by writing v as

$$v = ae^{ix} + \varepsilon\Phi e^{i2x} + \bar{a}e^{-ix} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + Z_\varepsilon + v_\infty + e^{-T\varepsilon^{-2}(1+\varepsilon^2\partial_x^2)^2} v_s(0)$$

and multiplying it with itself to bound $(\widehat{v^2})_k$ and $(\widehat{v^3})_k$ for $k \in \{0,1,2\}$. Note that we can bound the Fourier transform by the L^∞ norm. We have

$$\begin{aligned} v^2 &= 2(ae^{ix} + \bar{a}e^{-ix} + Z_\varepsilon)(\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + v_\infty) \\ &\quad + (ae^{ix} + \bar{a}e^{-ix} + Z_\varepsilon)^2 + r_1 \\ v^3 &= (ae^{ix} + \bar{a}e^{-ix} + Z_\varepsilon)^3 + r_2 \end{aligned} \tag{28}$$

with

$$\begin{aligned} r_1 &= (\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + v_\infty)^2 + (e^{-\varepsilon^{-2}T\mathcal{L}} v_s(0))^2 \\ &\quad + 2(ae^{ix} + \varepsilon\Phi e^{i2x} + \bar{a}e^{-ix} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + Z_\varepsilon + v_\infty)e^{-\varepsilon^{-2}T\mathcal{L}} v_s(0) \\ r_2 &= (\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + v_\infty)^3 + (e^{-\varepsilon^{-2}T\mathcal{L}} v_s(0))^3 \\ &\quad + 3(ae^{ix} + \bar{a}e^{-ix} + Z_\varepsilon)(\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + v_\infty)^2 \\ &\quad + 3(ae^{ix} + \bar{a}e^{-ix} + Z_\varepsilon)^2(\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + v_\infty) \\ &\quad + 3(ae^{ix} + \bar{a}e^{-ix} + Z_\varepsilon)(e^{-\varepsilon^{-2}T\mathcal{L}} v_s(0))^2 + 3(ae^{ix} + \bar{a}e^{-ix} + Z_\varepsilon)^2(e^{-\varepsilon^{-2}T\mathcal{L}} v_s(0)) \\ &\quad + 3(\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + v_\infty)(e^{-\varepsilon^{-2}T\mathcal{L}} v_s(0))^2 \\ &\quad + 3(\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + v_\infty)^2(e^{-\varepsilon^{-2}T\mathcal{L}} v_s(0)) \\ &\quad + 6(ae^{ix} + \bar{a}e^{-ix} + Z_\varepsilon)(\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + v_\infty)(e^{-\varepsilon^{-2}T\mathcal{L}} v_s(0)). \end{aligned}$$

Here we used for shorthand notation

$$\mathcal{L} = -\varepsilon^{-2}(1 + \varepsilon^2\partial_x^2)^2 \ .$$

Because of

$$\begin{aligned} \sup_{T\in[0,\tau^*]} \|\varepsilon\Phi(T)e^{i2x} + \varepsilon\bar{\Phi}(T)e^{-i2x} + \varepsilon\Psi(T) + v_\infty(T)\|_\infty &= \mathcal{O}(\varepsilon^{1-2\kappa_0}), \\ \sup_{T\in[0,\tau^*]} \|a(T)e^{ix} + \bar{a}(T)e^{-ix} + Z_\varepsilon(T)\|_\infty &= \mathcal{O}(\varepsilon^{-\kappa_0}), \end{aligned}$$

which follows from (24), (25), (26) and (27), together with

$$\begin{aligned}\|\int_0^T e^{-\varepsilon^{-2}s\mathcal{L}} v_s(0)ds\|_\infty &\leq \varepsilon^2 \sum_{|k|\neq 1} (1-k^2)^{-2} |(\widehat{v_s(0)})_k| \\ &\leq \varepsilon^2 \sqrt{2\pi} \sum_{|k|\neq 1} (1-k^2)^{-2} \|v_s(0)\|_\infty = \mathcal{O}(\varepsilon^{2-\kappa_0})\end{aligned}$$

we can bound the integral in time of r_1 and r_2 by

$$\sup_{T\in[0,\tau^*]} \|\int_0^T r_1 dt\|_\infty = \mathcal{O}(\varepsilon^{2-6\kappa_0})$$

$$\sup_{T\in[0,\tau^*]} \|\int_0^T r_2 dt\|_\infty = \mathcal{O}(\varepsilon^{1-6\kappa_0}).$$

Analogously we can bound integrals of any power of $\|r_i\|_\infty$. Inserting (28) into the mild solution formulas (17) respectively (22) gives

$$da = (\nu a + 2\alpha\bar{a}\Phi + 2\alpha a\Psi - 3a|a|^2 - 3aZ_\varepsilon^2 + \varepsilon^{-1}2\alpha a Z_\varepsilon + R_1)dT \qquad (29)$$

$$d\Phi = (-9\varepsilon^{-2}\Phi + \varepsilon^{-2}\alpha a^2 + R_2)dT \qquad (30)$$

$$d\Psi = (-\varepsilon^{-2}\Psi + \varepsilon^{-2}\alpha|a|^2 + \varepsilon^{-2}\alpha Z_\varepsilon^2 + R_3)dT \qquad (31)$$

where

$$R_1(t) = \varepsilon^{-1}\alpha(\widehat{r_1})_1 - (\widehat{r_2})_1,$$

$$R_2(t) = \nu\Phi + 2\varepsilon^{-1}\alpha Z_\varepsilon\Phi - 3\varepsilon^{-1}a^2 Z_\varepsilon + 2\varepsilon^{-2}\alpha v_3\bar{a} + \varepsilon^{-2}\alpha(\widehat{r_1})_2 - \varepsilon^{-1}(\widehat{r_2})_2$$

and

$$R_3(t) = \nu\Psi + \varepsilon^{-1}\alpha\Psi Z_\varepsilon - \varepsilon^{-1}Z_\varepsilon^3 + 6\varepsilon^{-1}|a|^2 Z_\varepsilon + \varepsilon^{-2}\alpha(\widehat{r_1})_0 - \varepsilon^{-1}(\widehat{r_2})_0$$

are stochastic processes with

$$\sup_{T\in[0,\tau^*]} \int_0^T |R_1|ds = \mathcal{O}(\varepsilon^{1-6\kappa_0}), \quad \sup_{T\in[0,\tau^*]} \int_0^T |R_2| + |R_3|ds = \mathcal{O}(\varepsilon^{-1-6\kappa_0}).$$

In order to eliminate Φ and Ψ on the right side of (29) we apply the Itô formula to $\bar{a}\Phi$, $a\Psi$ and aZ_ε. Note that there is no Itô correction at this point, as a, Φ, and Ψ are random but differentiable, which follows from the representation in (29) – (31).

$$d(\bar{a}\Phi) = (d\bar{a})\Phi + \bar{a}(d\Phi) = (\bar{a}(-9\varepsilon^{-2}\Phi + \varepsilon^{-2}\alpha a^2) + R_4)dT$$

$$d(a\Psi) = (da)\Psi + a(d\Psi) = (a(-\varepsilon^{-2}\Psi + 2\varepsilon^{-2}\alpha|a|^2 + \varepsilon^{-2}\alpha Z_\varepsilon^2) + R_5)dT$$

$$d(aZ_\varepsilon) = (da)Z_\varepsilon + a(dZ_\varepsilon) = (\varepsilon^{-1}2\alpha a Z_\varepsilon^2 - \varepsilon^{-2}aZ_\varepsilon + R_6)dT + a\varepsilon^{-1}\sigma d\tilde{\beta}$$

where

$$R_4(t) = \bar{a}R_2 + \Phi(\nu\bar{a} + 2\alpha a\bar{\Phi} + 2\alpha\bar{a}\bar{\Psi} - 3\bar{a}|a|^2 - 3\bar{a}Z_\varepsilon^2 + \varepsilon^{-1}2\alpha\bar{a}Z_\varepsilon + \bar{R}_1),$$

$$R_5(t) = aR_3 + \Psi(\nu a + 2\alpha\bar{a}\Phi + 2\alpha a\Psi - 3a|a|^2 - 3aZ_\varepsilon^2 + \varepsilon^{-1}2\alpha a Z_\varepsilon + R_1)$$

and

$$R_6(t) = Z_\varepsilon(\nu a + 2\alpha\bar{a}\Phi + 2\alpha a\Psi - 3a|a|^2 - 3aZ_\varepsilon^2 + R_1)$$

are stochastic processes with

$$\sup_{t\in[0,\tau^*]}\int_0^T |R_4| + |R_5|ds = \mathcal{O}(\varepsilon^{-1-8\kappa_0}), \quad \sup_{t\in[0,\tau^*]}\int_0^T |R_6|ds = \mathcal{O}(\varepsilon^{-8\kappa_0}).$$

Therefore we have:

$$\bar{a}\Phi dT = (\frac{1}{9}\alpha a|a|^2 + \frac{1}{9}\varepsilon^2 R_4)dT - \frac{1}{9}d(\varepsilon^2\bar{a}\Phi) \tag{32}$$

$$a\Psi dT = (2\alpha a|a|^2 + \alpha a Z_\varepsilon^2 + \varepsilon^2 R_5)dT - d(\varepsilon^2 a\Psi) \tag{33}$$

$$\varepsilon^{-1}aZ_\varepsilon dT = (2\alpha a Z_\varepsilon^2 + \varepsilon R_6)dT + \sigma a d\tilde{\beta}(T) - d(\varepsilon a Z_\varepsilon). \tag{34}$$

By substituting (32) – (34) into (29) we get the desired result for da with

$$dR = 2\alpha\varepsilon^2((\frac{1}{9}R_4 dT + R_5 dT - \frac{1}{9}d(\bar{a}\Phi) - d(a\Psi)) + 2\alpha\varepsilon(R_6 dT - d(aZ)).$$

□

4.3. Averaging with error bounds

Next we have to get the equation for da to match the amplitude equation (AE). For this we need to remove $aZ_\varepsilon^2 dT$. This is done in this section. First we need the following technical Lemma:

Lemma 10. *Let $X(t,\omega) \in \mathbb{C}$ be a stochastic process with*

$$X(t) = \int_0^t f(s)ds + \int_0^t g(s)d\tilde{\beta},$$

where $\sup_{t\in[0,T_0]}|f(t)| = \mathcal{O}(\varepsilon^\gamma)$ *and* $\sup_{t\in[0,T_0]}|g(t)| = \mathcal{O}(\varepsilon^\gamma)$ *with* $\gamma \in \mathbb{R}$. *Then $X(t)$ has the same bound as $f(t)$ and $g(t)$:*

$$\sup_{t\in[0,T_0]}|X(t)| = \mathcal{O}(\varepsilon^\gamma) \tag{35}$$

Let us remark that the same result is true, if we replace T_0 by the stopping time τ^*.

Proof. The proof is straightforward using Burkholder-Davis-Gundy, Hölder, and Young's inequality. □

Now we can substitute the aZ^2 term in (23). This is done by using the averaging property of Z_ε described in the next Lemma.

Lemma 11. *Let $X(t) \in \mathbb{C}$ be a stochastic process with $dX = f(T)dT + g(T)d\tilde{\beta}$, where* $\sup_{T\in[0,T_0]}|f(T)| = \mathcal{O}(\varepsilon^{-\gamma})$ *and* $\sup_{T\in[0,T_0]}|g(T)| = \mathcal{O}(\varepsilon^{-\gamma})$ *with $\gamma > 0$. Then with Z_ε as defined by* (11) *the following holds:*

$$\sup_{T\in[0,T_0]}|\int_0^T X(s)Z_\varepsilon(s)^2 ds - \int_0^T \frac{1}{2}\sigma^2 X(s)ds| = \mathcal{O}(\varepsilon^{1-\kappa_0-\gamma}). \tag{36}$$

Again the same result is true, if we replace T_0 by the stopping time τ^*.

Proof. By using Itô's formula we get

$$d(XZ_\varepsilon^2) = (dX)Z_\varepsilon^2 + X(dZ_\varepsilon^2) + (dX)(dZ_\varepsilon^2)$$

and

$$d(Z_\varepsilon^2) = 2(dZ_\varepsilon)Z_\varepsilon + (dZ_\varepsilon)^2 = 2Z_\varepsilon(-\varepsilon^{-2}Z_\varepsilon dT + \varepsilon^{-1}\sigma d\tilde{\beta}) + \varepsilon^{-2}\sigma^2 dT.$$

This gives

$$d(XZ_\varepsilon^2) = fZ_\varepsilon dT + gZ_\varepsilon d\tilde{\beta} - \varepsilon^{-2}2XZ_\varepsilon^2 dT + \varepsilon^{-1}2\sigma XZ_\varepsilon d\tilde{\beta} + \varepsilon^{-2}\sigma^2 XdT + \varepsilon^{-1}\sigma g dT.$$

We already know from the proof of Lemma 9 that $\sup_{T\in[0,T_0]}|Z_\varepsilon(T)| = \mathcal{O}(\varepsilon^{-\kappa_0})$ and it follows from Lemma 10 that $\sup_{T\in[0,T_0]}|X(T)| = \mathcal{O}(\varepsilon^{-\gamma})$. Therefore $d(XZ_\varepsilon^2)$ can be written as

$$d(XZ_\varepsilon^2) = -\varepsilon^{-2}2XZ_\varepsilon^2 dT + \varepsilon^{-2}\sigma^2 XdT + R_7 dT + R_8 d\tilde{\beta},$$

where $R_7(T)$ and $R_8(T)$ are stochastic processes with

$$\sup_{[0,\tau^*]}|R_7| = \mathcal{O}(\varepsilon^{-1-\kappa_0-\gamma}), \qquad \sup_{[0,\tau^*]}|R_8| = \mathcal{O}(\varepsilon^{-1-\kappa_0-\gamma}).$$

By multiplying with ε^2 and integrating from 0 to T we get

$$\int_0^T \tfrac{1}{2}\sigma^2 X ds - \int_0^T XZ_\varepsilon^2 ds = \tfrac{1}{2}\varepsilon^2 XZ_\varepsilon^2\Big|_0^T - \varepsilon^2\int_0^T R_7 ds - \varepsilon^2\int_0^T R_8 d\tilde{\beta}$$

and the application of Hölder and Burkholder-Davis-Gundy yields the desired result. □

4.4. SDE Lemma

With Lemma 11 we have closed the gap between the SDEs (AE) and (23) down to some error on the right side which is of order $\varepsilon^{1-8\kappa_0}$. But to be able to compare the first Fourier mode a and the solution of the amplitude equation A we need the following Lemma.

Lemma 12. *Let $X_1(t), X_2(t) \in \mathbb{C}$ be stochastic processes given by*

$$\begin{aligned} X_1(t) &= X_1(0) + \int_0^t f(X_1)ds + \int_0^t g(X_1)d\beta \\ X_2(t) &= X_1(0) + \int_0^t f(X_2)ds + \int_0^t g(X_2)d\beta + R(t) \end{aligned} \tag{37}$$

with $\sup_{t\in[0,\tau_0]}|R(t)| = \mathcal{O}(\varepsilon^\gamma)$, where $\gamma \in \mathbb{R}$ and $\tau_0 \le T_0$ is a stopping time. Let there be a constant $C > 0$ and a process $\hat{R}(t)$ with $\sup_{t\in[0,\tau_0]}|\hat{R}(t)| = \mathcal{O}(\varepsilon^\gamma)$ such that the functions f and g satisfy the following conditions:

$$\mathrm{Re}\{(f(X_1) - f(X_2))\overline{\varphi}\} \le C(|\varphi|^2 + |\hat{R}(t)|^2) \tag{38}$$

$$\forall x, y \in \mathbb{C}: \quad |g(x) - g(y)|^2 \le C|x-y|^2, \tag{39}$$

where $\varphi := X_1 - (X_2 - R)$. Then the difference between X_1 and X_2 can be bounded by

$$\sup_{t\in[0,\tau_0]}|X_1(t) - X_2(t)| = \mathcal{O}(\varepsilon^\gamma). \tag{40}$$

Note that condition (38) can be established by a bound of the type

$$\mathrm{Re}\{(f(x)-f(y))(x-y-z)\} \le C|x-y-z|^2 + p(y,z)$$

with polynomial p provided we have additional bounds on the process X_2.

Proof. Because of the unknown derivative of R it is much easier to split $X_1 - X_2$ into

$$X_1 - X_2 = \varphi - R \tag{41}$$

and bound $|\varphi|$ rather than the actual term.
Due to the stopping time the process φ is not easily bounded directly. Thus we extend all processes to $[0, T_0]$ and define

$$\tilde{R}(t) := \begin{cases} R(t) & \text{for } t \le \tau_0 \\ R(\tau_0) & \text{for } t > \tau_0 \end{cases}$$

and modify X_1 and X_2:

$$\begin{aligned} \tilde{X}_1(t) &:= X_1(0) + \int_0^{\tau_0 \wedge t} f(\tilde{X}_1)ds + \int_0^t g(\tilde{X}_1)d\beta \\ \tilde{X}_2(t) &:= X_1(0) + \int_0^{\tau_0 \wedge t} f(\tilde{X}_2)ds + \int_0^t g(\tilde{X}_2)d\beta + \tilde{R}(t). \end{aligned}$$

With this we can define a suitable replacement for φ:

$$\begin{aligned} \varphi_{\tau_0}(t) &:= \tilde{X}_1(t) - (\tilde{X}_2(t) - \tilde{R}(t)) \\ &= \int_0^{\tau_0 \wedge t} (f(X_1) - f(X_2))ds + \int_0^{\tau_0 \wedge t} (g(\tilde{X}_1) - g(\tilde{X}_2))d\beta. \end{aligned}$$

Note that $\sup_{t\in[0,T_0]} |\tilde{R}(t)| = \mathcal{O}(\varepsilon^\gamma)$ and for any stopping time $\tau \le \tau_0$ we have $\varphi_{\tau_0}(\tau) = \varphi(\tau)$, $\tilde{X}_1(\tau) = X_1(\tau)$ and $\tilde{X}_2(\tau) = X_2(\tau)$. This means

$$\sup_{t\in[0,\tau_0]} |\varphi(t)| = \sup_{t\in[0,\tau_0]} |\varphi_{\tau_0}(t)|.$$

Now in order to bound the moments of $\sup_{t\in[0,\tau_0]} |\varphi_{\tau_0}|$ we first need a bound on the moments of $|\varphi_{\tau_0}|$. We start by taking the differential of $|\varphi_{\tau_0}|^{2p}$ for $p \in \mathbb{N}$:

$$\begin{aligned} d|\varphi_{\tau_0}|^{2p} &= d(\overline{\varphi_{\tau_0}}\varphi_{\tau_0})^p = p(\overline{\varphi_{\tau_0}}\varphi_{\tau_0})^{p-1} d(\overline{\varphi_{\tau_0}}\varphi_{\tau_0}) \\ &= p|\varphi_{\tau_0}|^{2p-2}\left((d\overline{\varphi_{\tau_0}})\varphi_{\tau_0} + \overline{\varphi_{\tau_0}}(d\varphi_{\tau_0}) + (d\overline{\varphi_{\tau_0}})(d\varphi_{\tau_0})\right). \end{aligned}$$

The derivative of φ_{τ_0} is given by

$$d\varphi_{\tau_0} = \chi_{[0,\tau_0\wedge t]}(f(X_1) - f(X_2))dt + (g(\tilde{X}_1) - g(\tilde{X}_2))d\beta\ .$$

Therefore

$$\begin{aligned} d|\varphi_{\tau_0}|^{2p} = p|\varphi_{\tau_0}|^{2p-2}[&\chi_{[0,\tau_0\wedge t]} 2\,\mathrm{Re}\left\{\overline{\varphi_{\tau_0}}(f(X_1) - f(X_2))\right\} dt \\ &+ 2\,\mathrm{Re}\left\{\overline{\varphi_{\tau_0}}(g(\tilde{X}_1) - g(\tilde{X}_2))\right\} d\beta + |g(\tilde{X}_1) - g(\tilde{X}_2)|^2 dt]. \end{aligned}$$

Next we integrate and split the right side into three parts:

$$
\begin{aligned}
|\varphi_{\tau_0}(t)|^{2p} = &\int_0^{\tau_0\wedge t} p|\varphi_{\tau_0}|^{2p-2} 2\,\mathrm{Re}\,\{\overline{\varphi_{\tau_0}}(f(X_1)-f(X_2))\}\,ds \\
&+ \int_0^t p|\varphi_{\tau_0}|^{2p-2} 2\,\mathrm{Re}\left\{\overline{\varphi_{\tau_0}}(g(\tilde{X}_1)-g(\tilde{X}_2))\right\} d\beta \\
&+ \int_0^t p|\varphi_{\tau_0}|^{2p-2} |g(\tilde{X}_1)-g(\tilde{X}_2)|^2 ds \\
:= I_1 + I_2 + I_3
\end{aligned}
$$

For the first part we can exchange φ and φ_{τ_0} freely because the integral goes only up to the stopping time τ_0. Doing this and using (38) we get

$$
\begin{aligned}
I_1 &= \int_0^{\tau_0\wedge t} p|\varphi|^{2p-2} 2\,\mathrm{Re}\,\{\overline{\varphi}(f(X_1)-f(X_2))\}\,ds \\
&\leq \int_0^{\tau_0\wedge t} p|\varphi|^{2p-2} 2C(|\varphi|^2 + |\hat{R}|^2) ds \\
&\leq \int_0^{\tau_0\wedge t} C_p(|\varphi_{\tau_0}|^{2p} + |\hat{R}|^{2p}) ds \leq C_p\Big(\int_0^t (|\varphi_{\tau_0}|^{2p} ds + \int_0^{\tau_0} |\hat{R}|^{2p}) ds,
\end{aligned}
$$

where C_p is a constant depending on p and we used Young's inequality in the last step. The third part can be bounded from above by using (39) and a simple application of the triangle inequality:

$$
\begin{aligned}
I_3 &\leq \int_0^t p|\varphi_{\tau_0}|^{2p-2} |\tilde{X}_1 - \tilde{X}_2|^2 ds \\
&\leq \int_0^t p|\varphi_{\tau_0}|^{2p-2} (|\varphi_{\tau_0}|^2 + |\tilde{R}|^2) ds \leq \int_0^t C_p(|\varphi_{\tau_0}|^{2p} + |\tilde{R}|^{2p}) ds
\end{aligned}
$$

Again we used Young's inequality in the last step. Now since stochastic integration preserves the local martingale property, taking the expectation value of $|\varphi_{\tau_0}|^{2p}$ yields, for all $t \leq T_0$,

$$
\begin{aligned}
\mathbb{E}(|\varphi_{\tau_0}(t)|^{2p}) &= \mathbb{E}(I_1) + \mathbb{E}(I_2) \\
&\leq C_p \mathbb{E}\Big(\int_0^t |\varphi_{\tau_0}|^{2p} + |\tilde{R}|^{2p} ds + \int_0^{\tau_0} |\hat{R}|^{2p} ds\Big) \\
&\leq \int_0^t C_p \mathbb{E}(|\varphi_{\tau_0}|^{2p}) ds + C_p T_0 R_{\mathrm{sup}}^{2p},
\end{aligned}
$$

where $R_{sup}^{2p} := \mathbb{E}(\sup_{t\in[0,\tau_0]} |\hat{R}(t)|^{2p} + \sup_{t\in[0,T_0]} |\tilde{R}(t)|^{2p})$. We apply Gronwall's Lemma to get

$$
\begin{aligned}
\mathbb{E}(|\varphi_{\tau_0}(t)|^{2p}) &\leq C_p T_0 R_{sup}^{2p} + \int_0^t C_p^2 T_0 R_{sup}^{2p} e^{(T_0 - s)C_p} ds \\
&\leq C_p T_0 R_{sup}^{2p} + C_p^2 T_0^2 R_{sup}^{2p} e^{T_0 C_p}. \qquad (42)
\end{aligned}
$$

With this we can now bound the moments of $\sup_{t\in[0,\tau_0]} |\varphi_{\tau_0}(t)|$. We start with $\mathbb{E}(\sup_{t\in[0,\tau_0]} I_3(t))$:

$$\begin{aligned}
\mathbb{E}(\sup_{t\in[0,\tau_0]} I_3(t)) &= \mathbb{E} \sup_{t\in[0,\tau_0]} \Big(\int_0^t 2\,\mathrm{Re}\Big\{\overline{\varphi_{\tau_0}}(g(\tilde{X}_1)-g(\tilde{X}_2))\Big\}\, d\beta\Big) \\
&\le \mathbb{E}\Big(\int_0^{\tau_0} C_p^2 |\varphi_{\tau_0}|^{4p-2} |g(\tilde{X}_1)-g(\tilde{X}_2)|^2 ds\Big)^{1/2} \\
&\le \Big(\mathbb{E}\int_0^{T_0} C_p^2 |\varphi_{\tau_0}|^{4p-2}(|\varphi_{\tau_0}|^2+|\tilde{R}|^2) ds\Big)^{1/2} \\
&\le C_p\Big(\mathbb{E}\int_0^{T_0} |\varphi_{\tau_0}|^{4p}+|\tilde{R}|^{4p} ds\Big)^{1/2},
\end{aligned}$$

where we used the Burkholder Davis Gundy theorem in the second step, the Hölder inequality in the third and Young's inequality in the last step.
The whole term is now easily bounded by

$$\begin{aligned}
\mathbb{E}(\sup_{t\in[0,\tau_0]} |\varphi(t)|)^{2p} &= \mathbb{E}(\sup_{t\in[0,\tau_0]} (I_1+I_2+I_3)) \\
&\le C_p\mathbb{E}\Big(\int_0^{T_0} (|\varphi_{\tau_0}|^{2p}+|\tilde{R}|^{2p}) ds + \int_0^{\tau_0} |\hat{R}|^{2p} ds\Big) \\
&\qquad + C_p\Big(\mathbb{E}\int_0^{T_0} |\varphi_{\tau_0}|^{4p}+|\tilde{R}|^{4p} ds\Big)^{1/2} \\
&\le C_p\Big(\int_0^{T_0} \mathbb{E}|\varphi_{\tau_0}|^{2p} ds\Big) + C_p\Big(\int_0^{T_0} \mathbb{E}|\varphi_{\tau_0}|^{4p} ds\Big)^{1/2} \\
&\qquad + C_p(T_0+T_0^{1/2}) R_{\sup}^{2p}.
\end{aligned}$$

Using (42) we get

$$\begin{aligned}
\mathbb{E}(\sup_{t\in[0,\tau_0]} |\varphi_{\tau_0}(t)|)^{2p} &\le C_p T_0 (C_p T_0 R_{\sup}^{2p} + C_p^2 T_0^2 R_{\sup}^{2p} e^{T_0 C_p}) \\
&\quad + C_p T_0^{1/2}(C_2 p T_0 R_{\sup}^{4p} + C_2 p^2 T_0^2 R_{\sup}^{4p} e^{T_0 C_2 p}) + C_p T_0^{3/2} R_{\sup}^{2p}.
\end{aligned}$$

Finally any moment can be bounded by even moments through Hölder interpolation, which proves that $\sup_{t\in[0,\tau_0]} |\varphi(t)| = \sup_{t\in[0,\tau_0]} |\varphi_{\tau_0}(t)| = \mathcal{O}(\varepsilon^\gamma)$. By assumption we also have that $\sup_{t\in[0,\tau_0]} |R(t)| = \mathcal{O}(\varepsilon^\gamma)$, so the result follows from (41). □

From what we have proven it is easily shown that the theorem holds at least until the time τ^*, but we still need to show that τ^* is large enough. For this we prove bounds on moments of A which are a direct application of Lemma 12.

Corollary 13. *Let $A(t)$ be the solution to the amplitude equation* (AE) *from Theorem* 4*, then the following holds:*

$$\sup_{t\in[0,T_0]} |A(t)| = \mathcal{O}(\varepsilon^{-\kappa}). \tag{43}$$

Note that κ was defined in Theorem 4 such that $A(0) = \mathcal{O}(\varepsilon^{-\kappa})$.

Proof. We define f, g and R by

$$\begin{aligned} R(t) &:= -A(0) \\ f(A) &:= \nu A + 3(\tfrac{38}{27}\alpha^2 - 1)A|A|^2 + 3(\alpha^2 - \tfrac{1}{2})\sigma^2 A \\ g(A) &:= 2\sigma\alpha A. \end{aligned} \tag{44}$$

With this we can write A and zero as in (37):

$$\begin{aligned} A(t) &= A(0) + \int_0^t f(A)dt + \int_0^t g(A)d\beta \\ 0 &= A(0) + \int_0^t f(0)dt + \int_0^t g(0)d\beta + R. \end{aligned}$$

Since $f(0) = g(0)$ we obtain $\sup_{t\in[0,T_0]} |R(t)| = \sup_{t\in[0,T_0]} |A(0)| = \mathcal{O}(\varepsilon^{-\kappa})$, and we derive the desired result directly from Lemma 12, provided we can prove the conditions (38) and (39). Because g is linear (39) is readily verified:

$$|g(x) - g(y)|^2 = |2\sigma(x - y)|^2 \le 4\sigma^2|x - y|^2. \tag{45}$$

This leaves (38). For better readability we write f as

$$f(X) = C_1X - C_2|X|^2X$$

with positive constants C_1 and C_2. For the linear part of f we are in the same position as for g, there is no dependency on X_1 or X_2:

$$\mathrm{Re}\{(\overline{X_1} - (\overline{X_2} - \overline{R}))(C_1X_1 - X_2)\} \le 3C_1(|X_1 - (X_2 - R)|^2 + |R|^2). \tag{46}$$

For the cubic term, to keep this proof simple, we note that it is sufficient to bound it here just for the special case $X_1 = A$ and $X_2 = 0$.

$$\begin{aligned} \mathrm{Re}\{(\overline{A} - (0 - \overline{R}))(-C_2|A|^2A - 0)\} &= -C_2|A|^4 + \mathrm{Re}\{\overline{R}A\} \\ &\le 2(|A - (0 - R)|^2 + |R|^2) \end{aligned}$$

□

4.5. Removing the error

Combining the lemmas of the previous sections, we are now able to prove Theorem 4.

Proof of theorem 4. By Lemma 8 $u(t)$ can be approximated by $a = v_1$ and Z_ε until the time τ^*:

$$\sup_{t\in[0,\tau^*]} \|u(t) - \varepsilon a(\varepsilon^2 t)e^{ix} - \varepsilon\overline{a}(\varepsilon^2 t)e^{-ix} - \varepsilon Z_\varepsilon - e^{T\varepsilon^{-2}(1+\partial_x^2)^2} v_s(0)\|_\infty = \mathcal{O}(\varepsilon^{2-8\kappa_0}).$$

Now we bound the difference between a and A until time τ^*. The initial condition $A(0)$ is exactly the coefficient of the first Fourier mode of $v(0, x)$).

This means $A(0) = a(0)$, thus by Lemma 9 and Lemma 10 we know that a is given by

$$a(t) = A(0) + \int_0^t (\nu a + 3(\tfrac{38}{27}\alpha^2 - 1)a|a|^2 + 6(\alpha^2 - \tfrac{1}{2})aZ_\varepsilon^2)ds + \int_0^t 2\sigma a d\tilde{\beta} + R_9,$$

where $\sup_{[0,\tau^*]} |R_9| = \mathcal{O}(\varepsilon^{1-8\kappa_0})$. Next we split the aZ_ε^2 term into

$$aZ_\varepsilon^2 = (a - R_9)Z_\varepsilon^2 + R_9 Z_\varepsilon^2.$$

The second part is bounded by $\sup_{[0,\tau^*]} |R_9 Z_\varepsilon^2| = \mathcal{O}(\varepsilon^{1-10\kappa_0})$ and the first part can be exchanged by using Lemma 11. Set as indicated after (12) $\kappa_0 = \frac{9}{8}\kappa$. Because

$$\sup_{[0,\tau^*]} |\nu a + 3(\frac{38}{27}\alpha^2 - 1)a|a|^2 + 6(\alpha^2 - \tfrac{1}{2})aZ_\varepsilon^2| = \mathcal{O}(\varepsilon^{-6\kappa_0}) \tag{47}$$

$$\sup_{[0,\tau^*]} |2\sigma a| = \mathcal{O}(\varepsilon^{-6\kappa_0}) \tag{48}$$

and $10\kappa_0 = \frac{45}{4}\kappa \leq 12\kappa$ we get

$$a(t) = A(0) + \int_0^t (\nu a + 3(\tfrac{38}{27}\alpha^2 - 1)a|a|^2 + 3(\alpha^2 - \tfrac{1}{2})\sigma^2 a)ds + \int_0^t 2\sigma a d\tilde{\beta} + R_{10},$$

where $\sup_{t\in[0,\tau^*]} |R_{10}(t)| = \mathcal{O}(\varepsilon^{1-12\kappa})$.

With f and g defined as in (44) we show that there exists a process $\hat{R}$ with

$$\sup_{t\in[0,\tau^*]} |\hat{R}(t)| = \mathcal{O}(\varepsilon^{1-18\kappa}) \tag{49}$$

such that the conditions (38) and (39) are fulfilled and we can apply Lemma 12. Since $\sup_{t\in[0,\tau^*]} |R_{10}| = \mathcal{O}(\varepsilon^{1-9\kappa})$ the condition on g and the linear term of f are already covered by (45) respectively (46). Because of this we only need show that there is a positive constant C and a process $\hat{R}$ conforming to (49) such that

$$\rho := \mathrm{Re}\left\{-C_2(\overline{A} - (\overline{a} - \overline{R_{10}}))(|A|^2 A - |a|^2 a)\right\} \leq C(|A - (a - R_{10})|^2 + |\hat{R}|^2),$$

where $C_2 = -3(\frac{38}{27}\alpha^2 - 1)$ is a positive constant. We do this by splitting ρ into two parts:

$$\begin{aligned} \rho &= \mathrm{Re}\left\{-C_2\left(\overline{A} - (\overline{a} - \overline{R_{10}})\right)\left(|A|^2 A - |a|^2 a\right)\right\} \\ &= \mathrm{Re}\left\{-C_2\left(\overline{A} - (\overline{a} - \overline{R_{10}})\right)\left(|A|^2 A - |a - R_{10}|^2(a - R_{10})\right)\right\} \\ &\quad + \mathrm{Re}\left\{-C_2\left(\overline{A} - (\overline{a} - \overline{R_{10}})\right)\left(|a - R_{10}|^2(a - R_{10}) - |a|^2 a\right)\right\} \\ &=: \rho_1 + \rho_2. \end{aligned}$$

The first term is negative because for any two complex numbers z, w we have

$$\begin{aligned}
&2\,\mathrm{Re}\{(\overline{z}-\overline{w})(|z|^2z-|w|^2w\}\\
&\quad= 2|z-w|^2(|z|^2+|w|^2)+2\,\mathrm{Re}\{(z-w)^2\overline{zw}\}\\
&\quad\geq 2|z-w|^2(|z|^2+|w|^2)-|z-w|^2(|z|^2+|w|^2)\\
&\quad\geq |z-w|^2(|z|^2+|w|^2)\geq 0.
\end{aligned}$$

This means ρ_1 can be bounded from above by 0. The second term can be bounded by

$$\begin{aligned}
|\rho_2| &\leq C_2|\overline{A}-(\overline{a}-\overline{R_{10}})|(3|a|^2|R_{10}|+3|a||R_{10}|^2+|R_{10}|^3)\\
&\leq C_2(|\overline{A}-(\overline{a}-\overline{R_{10}})|^2+(3|a|^2|R_{10}|+3|a||R_{10}|^2+|R_{10}|^3)^2)
\end{aligned}$$

and since $\sup_{t\in[0,\tau^*]}|a(t)|=\mathcal{O}(\varepsilon^{-3\kappa})$ we obtain (as $\kappa<\frac{1}{17}$)

$$\sup_{t\in[0,\tau^*]}\big(3|a|^2|R_{10}|+3|a||R_{10}|^2+|R_{10}|^3\big)=\mathcal{O}(\varepsilon^{1-18\kappa}).$$

Therefore Lemma 12 yields the following bound on $|A-a|$:

$$\sup_{t\in[0,\tau^*]}|A(t)-a(t)|=\mathcal{O}(\varepsilon^{1-18\kappa}).$$

Combining this with Corollary 13 we obtain

$$\sup_{t\in[0,\tau^*]}|a(t)|\leq\sup_{t\in[0,\tau^*]}|A(t)-a(t)|+\sup_{t\in[0,\tau^*]}|A(t)|=\mathcal{O}(\varepsilon^{-\kappa}). \tag{50}$$

Next we show that the probability $\mathbb{P}(\tau^*<T_0)$ is small. Define the following subset of the probability space Ω:

$$M:=\{\omega\in\Omega:\tau^*(\omega)<T_0\}.$$

If $\omega\in M$ then it follows from the definition of τ^* that $\|v(\tau^*(\omega))\|_\infty=\varepsilon^{-\kappa_0}$. Therefore the moments of $\|v(\tau^*)\|_\infty$ can be written as follows

$$\mathbb{E}\|v(\tau^*)\|_\infty^p=\int_{M^c}\|v(\tau^*)\|_\infty^p d\mathbb{P}+\int_M(\varepsilon^{-\kappa_0})^p d\mathbb{P}\geq\mathbb{P}(M)\varepsilon^{-p\kappa_0},$$

where $M^c:=\Omega\setminus M$ is the complement set of M. From (50), (27),(19) and (18) we have

$$\begin{aligned}
\mathbb{E}\|v(\tau^*)\|_\infty^p &\leq C_p\mathbb{E}\sup_{t\in[0,\tau^*]}(|a(t)|^p+|Z_\varepsilon(t)|^p+|v_0(t)-Z_\varepsilon(t)-e^{-\varepsilon^{-2}T}v_0(0)|^p)\\
&\quad+C_p\mathbb{E}\sup_{t\in[0,\tau^*]}\|\sum_{k\geq 2}v_k-e^{-\varepsilon^{-2}T(1-k^2)^2}v_k(0))e^{ikx}\|_\infty^p\\
&\quad+C_p\mathbb{E}\sup_{t\in[0,\tau^*]}\|e^{-\varepsilon^{-2}T\mathcal{L}}\sum_{k\neq 1}(v_k(0))e^{ikx}\|_\infty^p\\
&\leq C_p\varepsilon^{-p\kappa}
\end{aligned}$$

with a constant C_p depending on p, where we used that there is a constant C such that for all $u\in C^0$,

$$\|e^{-\varepsilon^{-2}T\mathcal{L}}u\|_\infty\leq C\|u\|_\infty.$$

This is a direct consequence of Lemma 4.5 in [MBK12] which follows the ideas of Collet and Eckmann in [CE90]. Therefore the probability of M is bounded by

$$\mathbb{P}(M) \leq C_p \varepsilon^{p(\kappa_0-\kappa)}.$$

Define

$$\xi := \sup_{t\in[0,T_0]} \|u(t) - \varepsilon A(\varepsilon^2 t)e^{ix} - \varepsilon \bar{A}(\varepsilon^2 t)e^{-ix} + \varepsilon Z_\varepsilon(\varepsilon^2 t) - e^{-t(1+\partial_x^2)^2} u_s(0)\|_\infty$$

The last step is now to bound the probability of $\sup_{t\in[0,T_0]}\|\xi\|_\infty$ being too large (i.e. $\mathbb{P}(\sup_{t\in[0,T_0]}\|\xi\|_\infty > \varepsilon^{2-19\kappa})$). We can split this into

$$\begin{aligned}\mathbb{P}(\sup_{t\in[0,T_0]} \|\xi\| > \varepsilon^{2-19\kappa})) &= \mathbb{P}(M \cap \{\sup_{t\in[0,T_0]} \|\xi\| > \varepsilon^{2-19\kappa})\}) \\ &\quad + \mathbb{P}(M^c \cap \{\sup_{t\in[0,T_0]} \|\xi\| > \varepsilon^{2-19\kappa})\}) \\ &=: P_1 + P_2.\end{aligned}$$

P_1 is easily bounded by

$$\mathbb{P}(M \cap \{\sup_{t\in[0,T_0]} \|\xi\| > \varepsilon^{2-19\kappa})\}) \leq \mathbb{P}(M) \leq C_p \varepsilon^{p(\kappa_0-\kappa)},$$

so the only thing left to do is to bound P_2. We get

$$P_2 = \mathbb{P}(M^c \cap \{\sup_{t\in[0,T_0]} \|\xi\| > \varepsilon^{2-19\kappa}\}) \leq \mathbb{P}(\sup_{t\in[0,T_0]} \|\xi\| > \varepsilon^{2-19\kappa}).$$

Using the Chebychev inequality gives

$$P_2 \leq C_q \frac{1}{\varepsilon^{q(2-19)\kappa}} \mathbb{E}(\sup_{t\in[0,T_0]} \|\xi\|^q) \leq C_q \varepsilon^{q\kappa},$$

where q is any positive number and C_q is a constant depending on q. By choosing $q = p/\kappa$ we get the desired result. □

Acknowledgement

This work is supported by the Deutsche Forschungsgemeinschaft (DFG BL535/9-1, "Mehrskalenanalyse stochastischer partieller Differentialgleichungen"). We thank both anonymous referees for very helpful remarks.

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