Spatially periodic stochastic system with infinite globally coupled oscillators

Jing-hui Li, Peter Hänggi

Angaben zur Veröffentlichung / Publication details:

Li, Jing-hui, and Peter Hänggi. 2001. "Spatially periodic stochastic system with infinite globally coupled oscillators." *Physical Review E* 64 (1): 011106.
https://doi.org/10.1103/physreve.64.011106.

Spatially periodic stochastic system with infinite globally coupled oscillators

Jing-hui Li and Peter Hänggi
Lehrstuhl für Theoretische Physik I, Institut für Physik, Universität Augsburg, Universitässtrasse 1, D-86135 Augsburg, Germany
(Received 9 January 2001; published 18 June 2001)

In this paper we study a spatially periodic stochastic system with infinite globally coupled oscillators driven by a constant force F. With two typical models we show that when $F=0$ there is a nonequilibrium transition between the state with zero mean field ($s=0$) and the state with nonzero mean field ($s\neq 0$). For model I, the transition is not a phase transition, while for the model II it is (second order). In addition, we find that for coupled oscillators driven only by additive noises, when $F=0$ a transport may emerge if the nonzero mean field breaks the symmetry of the systems. With varying F a continuous or discontinuous transition between state $s>0$ and state $s<0$ will appear. The mean field or current sometimes exhibits hysteresis as a function of F.

DOI: 10.1103/PhysRevE.64.0611XX PACS number(s): 05.40.−a, 05.60.−k

I. INTRODUCTION

Noise-induced nonequilibrium phenomena in nonlinear systems have recently attracted a great deal of attention in a variety of contexts [1]. In general, these phenomena involve a response of the system that is not only produced or enhanced by the presence of the noise, but is optimized for certain values of the parameters of the noise. One example is the phenomenon of stochastic resonance [2], wherein the response of a nonlinear system to a signal is enhanced by the presence of noise, and maximized for certain values of the noise parameters. Another is the ''Brownian motor,'' wherein for Brownian motion in stochastic spatial periodic potentials the spatial asymmetry or noise asymmetry leads to a systematic transport whose magnitude and even direction can be tuned by the parameters of the noise [3]. A third is the nonequilibrium transition for systems with finite or infinite coupled oscillators, which is probably a phase transition (of first or second order) [4–6] or not [6,7]. For these systems, the most exciting factor is that a reentrant second order phase transition was found for a general spatially extended model by Van den Broeck *et al.* [4]. Afterward, this phenomenon was found in many systems with coupled oscillators. A fourth such phenomenon is resonant activation [8]. Here the mean first passage time (MFPT) of a particle driven by (usually white) noise over a fluctuating potential barrier exhibits a minimum as a function of the parameter of the fluctuating potential barrier (usually the flipping rate of the fluctuating potential barrier).

In this paper, we will study a spatially periodic system with infinite noise-driven overdamped oscillators which are globally coupled by the mean field and driven by a constant force. The nonequilibrium transition [4–7] and the transport [3] that probably occurs will be studied in detail. The setup of the problem is arranged as follows: We first consider a general model consisting of infinite globally coupled oscillators. Then, using formulas obtained by us, with two typical models we study the nonequilibrium transition and transport of particles.

II. A GENERAL MODEL

We consider a model whose Langevin equations of oscillators are (in dimensionless form)

$$\dot{x}_i=f(x_i)+g(x_i)es+\eta_i(t)+F,$$
$$f(x_i)=-\frac{dU_0(x_i)}{dx_i},\quad i=1,2,3,\ldots \tag{1}$$

where $U_0(x_i)$ is spatially periodic function of x with a period L, $g(x_i)$ is a linear or nonlinear function of x, $\eta_i(t)$ are Gaussian white noises with zero mean and correlation functions $\langle\eta_i(t)\eta_j(t')\rangle=2D\delta_{ij}\delta(t-t')$, e is a positive coupling constant, the mean field $s=\lim_{N\to\infty}(1/N)\Sigma_{i=1}^{N}f(x_i)$, and F is the constant force.

A coupling term such as the one in Eq. (1) appeared in some models for the coupled oscillators [6,7]. Now the coupling between the oscillators is not a constant, but a function of x (linear or nonlinear). In Sec. V, we will give a reason why $g(x)$ is taken as a function of x and not as a constant in the paper.

In the case of $N\to\infty$, all the oscillators have an identical evolution given by the nonlinear stochastic equation

$$\dot{x}=f(x)+g(x)es+\eta(t)+F, \tag{2}$$

where $s(t)=\langle f(x(t))\rangle$, which represents the time-dependent order parameter.

The Stratonovich interpretation of Eq. (2) yields the Fokker-Planck equation [9]

$$\partial_t P(x,s,t)=-\partial_x J(x,s,t), \tag{3}$$

with the probability current $J(x,s,t)$ given by

$$J(x,s,t)=A(x,s)P(x,s,t)-\partial_x B(x,s)P(x,s,t), \tag{4}$$

where $A(x,s)=f(x)+g(x)es+F$ and $B(x,s)=D$.

In the stationary state, the distribution $P(x,s,t)\to P(x,s)$, and the current $J(x,s,t)\to J(x,s)=\text{const}$. Then we have

$$J=A(x,s)P(x,s)-\partial_x DP(x,s). \tag{5}$$

Below we derive the constant probability current and the stationary probability density. The effective potential for Eq. (2) is $U(x,s)=U_0(x)+sU_1+U_2$, with U_0

$=-\int_0^x f(x)dx$, $U_1=-\int_0^x eg(x)dx$, and $U_2=-Fx$. From Eq. (5), and using the periodic boundary condition $P(0,s)=P(L,s)$ and $J(0)=J(L)=J=\text{const}$, we can easily obtain

$$J=M\{1-\exp[sU_1(L)/D-FL/D]\}, \tag{6}$$

where $M=P(L,s)B(L)/\int_0^L dx\exp[U(x,s)/D]$, which is the normalization constant for the stationary probability distribution. Here it needs to be explained that the symmetry or asymmetry of the original spatial potential $U_0(L)$ does not affect the transport, since $U_0(L)=0$. Substituting Eq. (6) into Eq. (5), and noting that $P(0,s)=P(L,s)$, we can obtain the stationary probability density

$$P(x,s)=M\frac{\exp[-U(x)/D]}{D}\oint\exp[U(x')/D-U(L)\eta(x-x')/D]dx'. \tag{7}$$

In the limit of $N\to\infty$, the self-consistent Weiss mean-field approach of Desai and Zwanzig is valid [4–7,10–12], and the Weiss mean field has to comply with the condition

$$s=\int_0^L f(x)P(x,s)dx=\overline{F}(s); \tag{8}$$

this is a self-consistency equation, whose solution yields the dependence of s with the system parameters.

First we consider the case of $F=0$. In the presence of spatial symmetry, Eq. (8) always has a solution $s=0$. With the appearance of multiple solutions, we can find $s\neq 0$. If $U_1(L)=0$, there is a nonequilibrium transition between the state $s=0$ and the state $s\neq 0$, which is not a phase transition since the symmetry is not broken; if $U_1(L)\neq 0$, a nonequilibrium phase transition with symmetry breaking will appear. For the former case, the current is zero; for the latter case, the current is probably not [the current $J=N[1-\exp(U_1(L)s/D)]$; please see Secs. III and IV]. In the presence of spatial asymmetry, Eq. (8) does not have the solution $s=0$, but only the solution $s\neq 0$. So there is not a nonequilibrium transition between the state $s=0$ and the state $s\neq 0$ (the system only has a state $s\neq 0$). Now the current of the transport is also determined by $J=N[1-\exp(U_1(L)s/D)]$. [If $U_1(L)=0$, we have $J=0$; if $U_1(L)\neq 0$, we probably have $J\neq 0$.]

If the inputting constant force is not zero, the system has only one state $s\neq 0$ with asymmetry. Now from corresponding formulas we can investigate the dynamic characteristic features of the system, including the nonequilibrium transition and the transport of particles (see the studies below). It needs to be explained that if F is large, the effect of the coupling between oscillators on the system will become small, in contrast with the case for a small value of F.

Below we consider two typical models. One is the case when $U_1(L)=0$; the other is the one when $U_1(L)\neq 0$. In order that we can clearly illustrate the effect of the mean field on the system, we only consider the spatially symmetric case.

III. MODEL I: CASE OF $U_1(L)=0$

In this section, we focus on the simplest possible examples: $f(x_i)=\cos x_i$ and $g(x_i)=-\sin x_i$ (in dimensionless form). From corresponding formulas in Sec. II, we can obtain $J=0$, $P(x,s)=N\{\exp[(-U_0(x)/D-sU_1(x))/D-U_2(x)/D]\}\oint\exp[U_0(x')/D+sU_1(x')/D+U_2(x')/D]dx'$, and $s=\oint f(x)P(x,s)dx=\overline{F}(x)$, with $U_0(x)=-\sin(x)$, $U_1=-(\cos x-1)$, and $U_2=-Fx$.

First we study the case of $F=0$. It can be verified that the function $\overline{F}(s)=\oint f(x)P(x,s)dx$ is a smooth, monotonic, and odd function. When $\partial_s\overline{F}(s)\leqslant 1$, the function $\overline{F}=\overline{F}(s)$ crosses the function $\overline{F}=s$ at $s=0$ (stable); when $\partial_s\overline{F}>1$, the function $\overline{F}=\overline{F}(s)$ crosses the function $\overline{F}=s$ at $s=0$ (unstable) and $s=\pm s^{(0)}$ (stable, $s^0>0$). In Fig. 1(a) we plot the function $\overline{F}=\overline{F}(s)$ versus s with $D=0.5$, and $e=1,\ 5$, and 10, respectively. In this figure the diagonal line is determined by $\overline{F}=s$. It is clear that the condition that the system transits from state $s=0$ to state $s\neq 0$, or vice versa, is $\partial_s\overline{F}(s)|_{s=0}=1$. The transition line is plotted in Fig. 1(b). The region below the curve corresponds to the zero mean field state, and that above the curve to the symmetric nonzero mean field state. At the transition line there is a bifurcation of the probability density. The nonzero value of the mean field is represented in Fig. 1(c) by the equation $s=\oint f(x)P(x,s)dx$ (the order parameter of this transition is $m=|s|$). The transition has the following characteristic features: (1) The transition is not a phase transition, since there is no symmetry breaking even if the order parameter changes continuously. (2) The state $s\neq 0$ is a bistable one with $s=\pm s^{(0)}$ ($s^{(0)}>0$). (3) With the increase of the noise strength (or the coupling constant) the transition occurs at a larger value of the coupling constant (or the noise strength).

If $F\neq 0$, superficially the particles will move along the direction of the force. However, owing to the coupling among different particles, some anomalous properties, such as negative mobility, hysteresis, and so on, probably appear [11]. Now the mean field is also determined by the equation $s=\oint f(x)P(x,s)dx=\overline{F}(x)$ with the parameters F, e, and D. We have studied the mean field as a function of F when the noise strength is definite but the coupling is varied. Studies showed that there are two kinds of coupling: in one, the mean field is a continuous function of F [see Fig. 2(a)]; in the other, the mean field is a discontinuous function of F [to see Figs. 2(b) and 2(c)]. For the former, there is a continuous transition from state $s>0$ to state $s<0$, or vice versa [see Fig. 2(a)]. For the latter, there is a discontinuous transition from state $s>0$ to state $s<0$, or vice versa [see Fig. 2(b)]. The transition diagram (or the transition line) is given in Fig. 3(a). In the upper region the system is in state $s>0$; in the lower region it is in state $s<0$; in the shadowed region it is a state composed of $s>0$ and $s<0$, where hysteresis for the mean field versus the constant force appears. Below we give the characteristic features of the transitions. (1) There is a critical value e_0 of the coupling e. When $e<e_0$ a continuous transition occurs; while when $e>e_0$ a discontinuous transition occurs. The critical value e_0 for the appearance of the

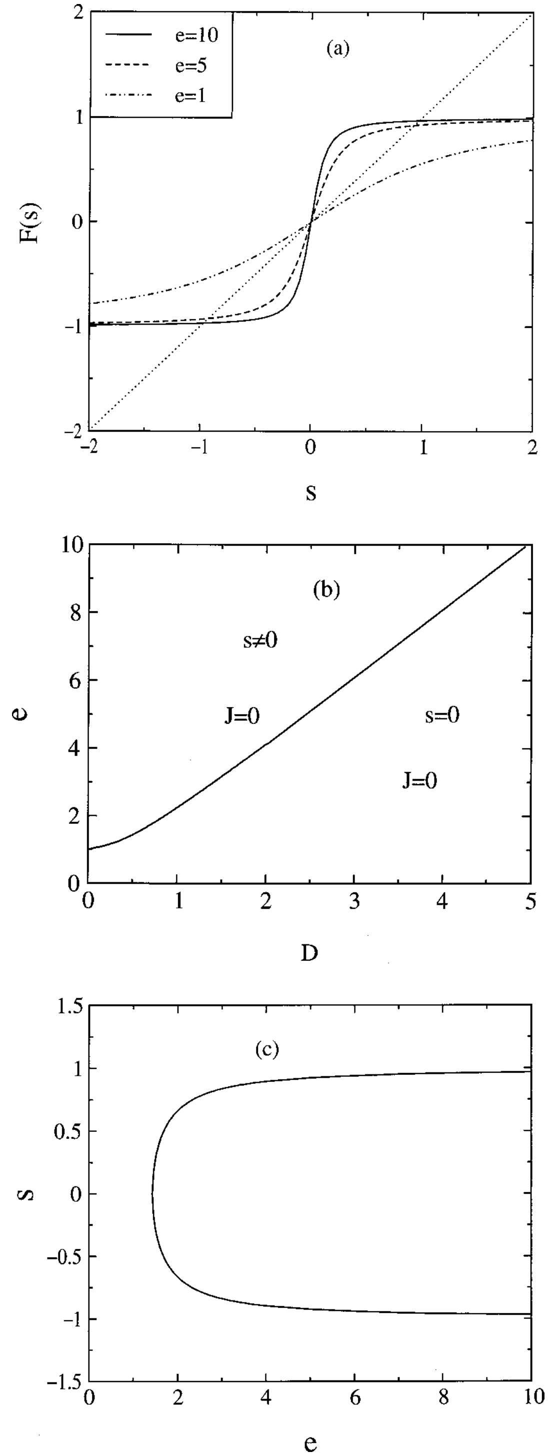

FIG. 1. When $F=0$ for model I, (a) the function $\overline{F}=\overline{F}(s)$ vs s with $D=0.5$ and $e=1$, 5, and 10, respectively (the diagonal line is determined by $\overline{F}=s$); (b) the transition line for e vs D; and (c) the nonzero mean field vs e with $D=0.5$.

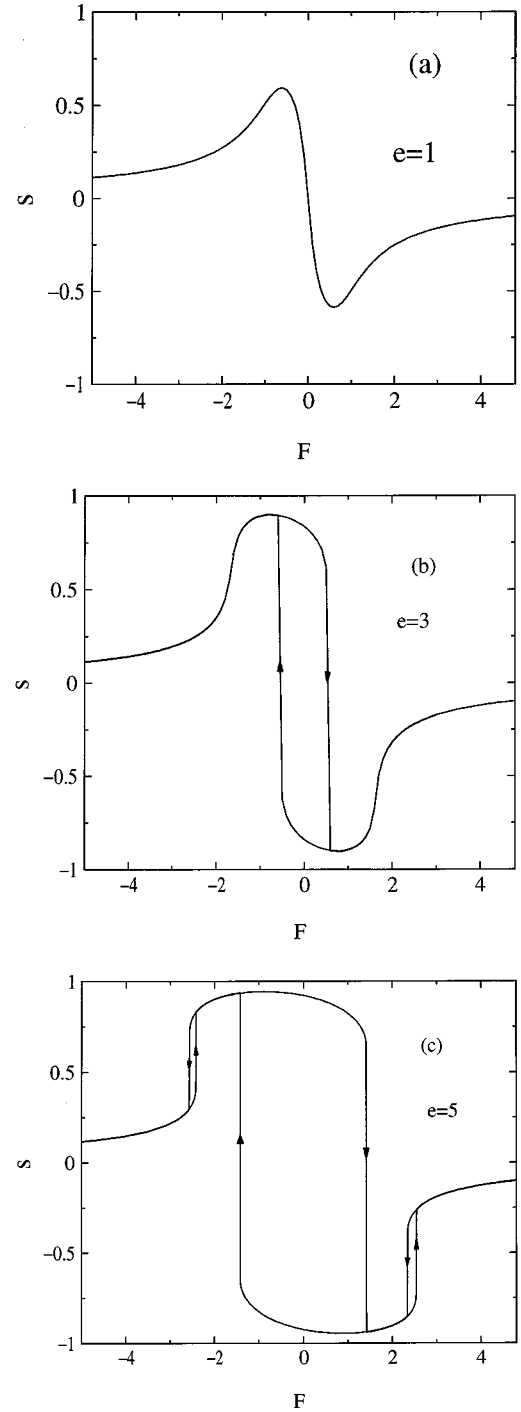

FIG. 2. The mean field vs F for model I for different values of e, (a) $e=1$, (b) $e=3$, and (c) $e=5$, with $D=0.5$.

discontinuous transition as a function of the noise strength D is depicted in Fig. 3(b). From this figure we can find that increasing the noise strength leads to a raise of the critical value e_0. (2) The transition is not a phase transition, since with the appearance of the transition there is no symmetry breaking. (3) The discontinuous transition is doubly unidirectional, which can be observed from Fig. 2(b). The line for the mean field versus the constant force presents an anomalous hysteresis loop. (4) The transitions from state $s>0$ to state $s<0$, and vice versa, are symmetric with respect to $F=0$. (5) For the discontinuous transition, with increasing coupling strength the transition requires a greater value of the constant force F (correspondingly, the anomalous hysteresis will grow with the increase of the coupling). In addition, if the coupling strength is large enough, in addition to the

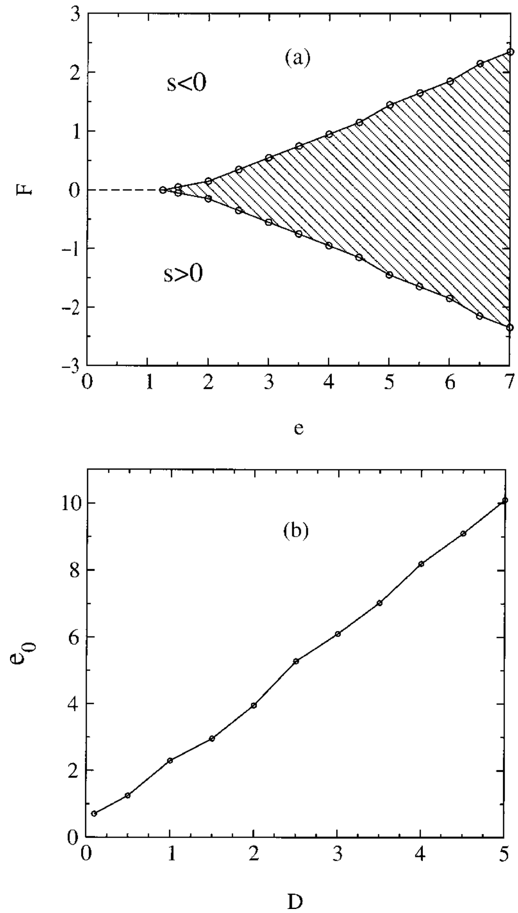

FIG. 3. $F\neq 0$ for model I. (a) The transition diagram for the transition between the state $s<0$ and the state $s>0$, where the dashed line presents the continuous transition and the solid lines the discontinuous one transition. (b) The critical value e_0 vs D for the appearance of the discontinuous transition.

original discontinuous transition from state $s>0$ to state $s<0$, or vice versa, two other double discontinuous transitions will appear which are in state $s>0$ and state $s<0$, respectively [see Fig. 2(c)]. In the case of large F, the absolute value of the mean field will become small, and there will be no doubly discontinuous transition for the mean field.

The transport of particles is also studied for the current versus the constant force. It is shown that when the coupling is not large enough, there are no hysteresis and negative mobility for the current versus the constant force, as observed in Ref. [7] [see Figs. 4(a) and 4(b)]. However, when the coupling is large enough, normal and anomalous hysteresis will appear [see Figs. 4(c), 5(a) and 5(b); Figs. 5(a) and 5(b) are enlargements or the corresponding parts in Fig. 4(c)], but no negative mobilities appear. From Figs. 2(c), 4(c), 5(a), and 5(b), we can note that the appearance of the hysteresis for the current is due to the hysteresis for the mean field, and the direction of the hysteresis for the current is opposite to that for the mean field. In Fig. 2(b), a hysteresis for the mean field appears, but owing to the fact that the spatial coupling among different oscillators is not large enough, there is no hysteresis for the current. If F is large,

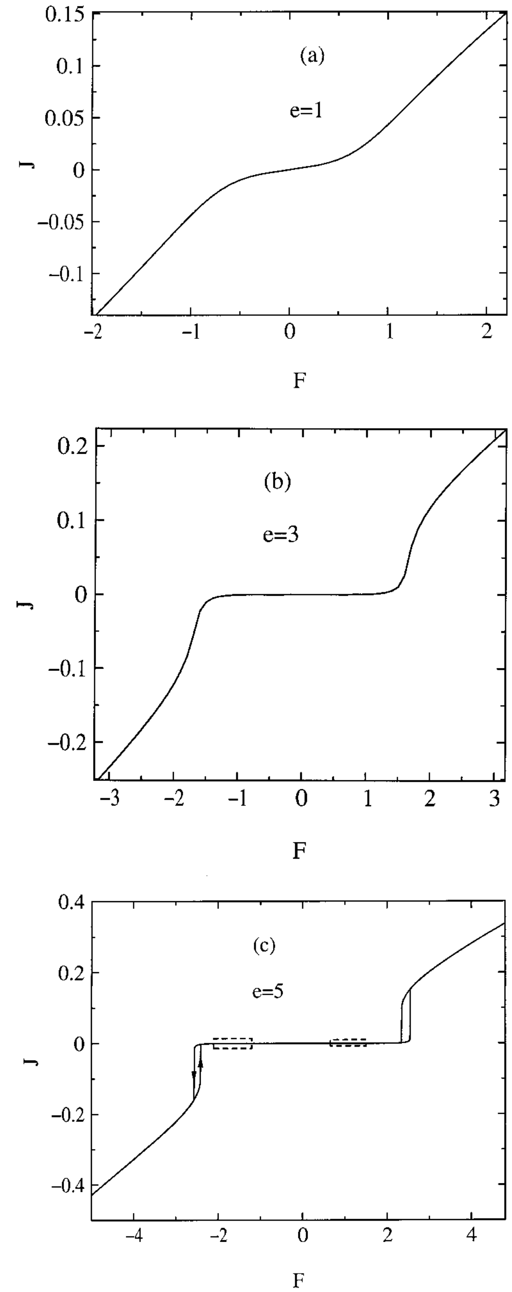

FIG. 4. The probability current J vs F for different values of e. (a) $e=1$, (b) $e=3$, and (c) $e=5$, with $D=0.5$ for model I.

the particles will move along the direction of the force, and there will be no hysteresis for the current.

IV. MODEL II: CASE OF $U_1(L)\neq 0$

Now we consider the case of $U_1(L)\neq 0$. The special example studied by us is (in dimensionless form)

$$\dot{x}_i=\cos x_i+(-\sin x_i+1)es+\eta_i(t)+F, \tag{9}$$

where $i=1,2,3,\ldots$, and the mean field s and noises $\{\eta_i(t)\}$ are the same as those in Eq. (1). Then we can obtain $U_0=-\sin x$, $U_1=-e(\cos x-1+x/(2\pi))$, and $U_2=-Fx$; here

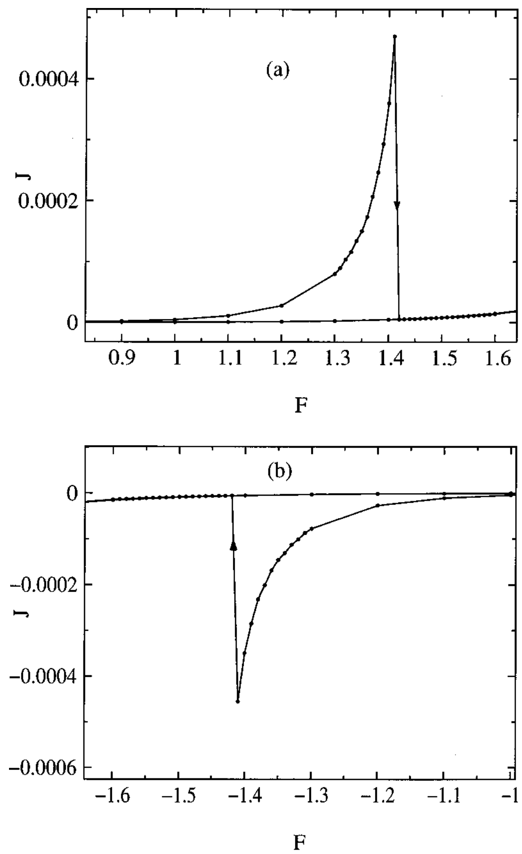

FIG. 5. Enlargements for the corresponding parts (surrounded by dashed lines) in Fig. 4(c). (a) corresponds to the left part of Fig. 4(c) and 4(b) to the right part.

we have dropped the subscript i for simplicity, since, when $N\rightarrow\infty$, all the oscillators have an identical evolution.

We now turn to a more detailed analysis of Eq. (8) in the case of $F=0$. Obviously, the trivial solution $s=0$ always exists (the system is in a symmetric state). With the appearance of multiple solutions, we can find ''ordered'' phases with an order parameter $m=|s|\neq 0$ (the symmetry of the system is broken). The critical condition should be

$$\overline{F'}(s=0)=1. \tag{10}$$

In Fig. 6(a), we plot the phase transition line in light of Eq. (10). The nonzero mean field is depicted in Fig. 6(b) with $D=0.5$ (the dashed line is that of model I). From Figs. 6(a) and 6(b) we can observe that the transition line and its order parameter line are basically similar to those of model I. But there are differences between them. (1) The former is a phase transition (of second order); while the latter is not. (2) For the former, in the state $s\neq 0$, the current is probably not zero, while for the latter it is.

The probability current versus the coupling constant is plotted in Fig. 6(c) with different values of D ($D=0.5$, 2, and 5, respectively) and $F=0$. The figure shows the following. (1) For given values of the noise and coupling strength, the nonzero current J does not have a definite sign ($J=\pm J_0$, $J_0>0$). One unavoidably wants to ask a question, i.e., if the noise and the coupling constant are definite, will the current be positive or negative? To answer this question, we must first consider the hysteresis or the current versus the constant force F. In Fig. 6(d), we plot the line for the current as a function of the constant force with $D=0.5$ and $e=1.5$. From the figure we can see that there is a normal hysteresis for the current versus the constant force. Thus, if the constant force changes from positive to zero, when $F=0$ we have $J>0$; otherwise, i.e., if the constant force changes from negative to zero, when $F=0$ we have $J<0$. If the constant force F is not added to every oscillator, the net current will be zero. The reason for this is that there are two currents ($J_e=\pm J_0$, $J_0>0$) produced for the oscillators, and consequently the net current must be that $J_{net}=(J_0-J_0)=0$. (2) Only in the symmetry-breaking phase state ($s\neq 0$) is there a nonzero current. (3) The current versus the coupling constant attenuates to zero very quickly. Thus, even if in the state $s\neq 0$, when e is large enough, the current is almost zero (for example, when $D=0.5$, if $e>4$, the current $J\approx 0$). (4) In the phase state $s\neq 0$, the small value e of the coupling constant plays a role of destructive influence on the asymmetry of the system, so the $|J|$-e response curve will have a positive slope. However, for a larger value of e, the central role will be to produce coherent motion with increases as e increases; then the $|J|$-e curve goes down. Thus, finally, we can obtain a peaked $|J|$-e response curve, at the peak of which a phenomenon of resonance will occur. (5) With increasing noise strength greater values of the coupling constant are required to induce the current.

Below we analyze the current that emerges when $F=0$. For uncoupled oscillators, we know that, if a spatially periodic system is driven by only thermal additive noises (the temperature is a constant), no transport can occur [transport occurring with thermal additive noises means that thermal fluctuation (only one heat source) is converted into work, and implies a violation of the second law of thermodynamics]. This is only for uncoupled oscillators. If the oscillators are coupled together globally or locally with the mean field, and the nonzero mean field can break the symmetry of the system, transport will probably be produced even if the system is only driven by the additive noises. Now the energy for the transport stems from the noises and the nonzero constant force. Superficially, it seems that when $F=0$, an isolated system can transfer energy to the surroundings. Obviously, this is very incorrect, since it violates the second law of thermodynamics. For the current to be nonzero when $F=0$ requires a precondition. This is that the constant force changes from nonzero to zero. If there is not this condition, when $F=0$ no transport occurs. With the change of the constant force, when $F=0$ there are still some energies contained in the system, which are produced when $F\neq 0$. Thus a nonzero net jet still exists when $F=0$. The reason for this phenomenon occurring here is that when $F=0$ the nonzero mean field can break the symmetry of the system.

With varying F, there are still the same continuous and discontinuous transitions for the mean field as studied in Sec. III. Here, in order to avoid unnecessary repetition, we do not present corresponding figures that are basically similar to the ones in Sec. III; nor do we give discussions of the continuous and discontinuous transitions occurring here.

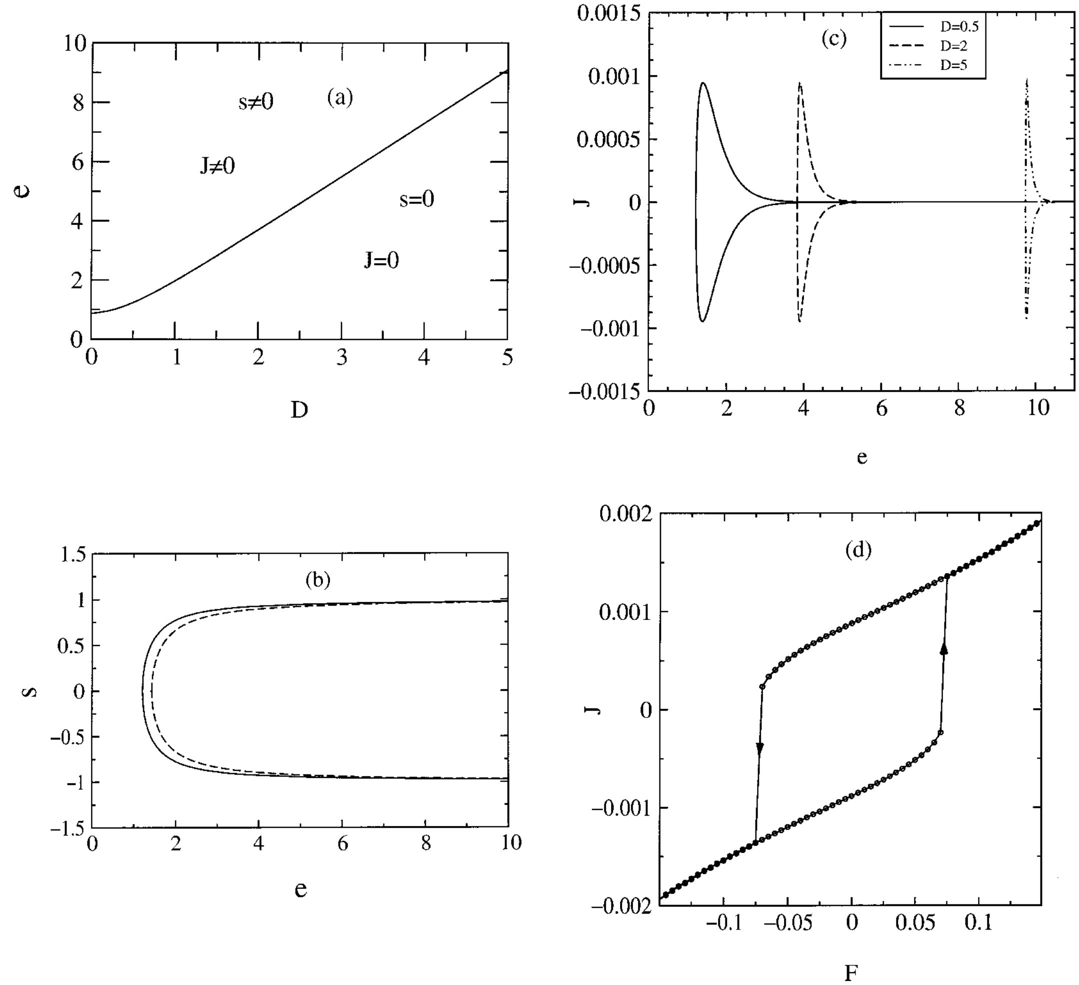

FIG. 6. Model II. (a) The phase transition line when $F=0$. (b) The nonzero mean field vs e (the solid line); the dashed line stands for the model I, with $D=0.5$ and $F=0$. (c) When $F=0$, the current vs e for different values of D ($D=0.5$, 2, and 5, respectively). (d) The normal hysteresis for the current vs F with $e=1.5$ and $D=0.5$.

The transport of particles for the current versus the constant force is studied. We find the following. (1) There is no negative mobility for the current versus the constant force. (2) If the coupling strength is small, there are also no hysteresis. (3) When the coupling strength is near a critical value $e=e_1$, where when $F=0$, the absolute value of the current has a peak as the function of e, and a normal hysteresis will arise [see Fig. 6(d)]. (4) When the coupling strength is large enough, normal and anomalous hysteresis will appear simultaneously. We note that for the above features of the current versus the constant force of model II, only feature (3) is different from that of model I. This is because when $F=0$ a nonzero current for model II will be produced, owing to the symmetry breaking induced by the nonzero mean field.

V. CONCLUSION AND DISCUSSION

In conclusion, we have studied a spatially periodic stochastic system with infinite globally coupled oscillators subject to a constant force F. With two typical models we have found that when $F=0$ there is a nonequilibrium transition between state $s=0$ and state $s\neq 0$. For model I, the transition is not a phase transition, since the symmetry of the system has not been broken, even though it is between the ''disorder'' state and the ''order'' state. For model II, the transition is second order, since the symmetry of the system has been broken and the order parameter changes continuously, and possesses features similar to those observed at the second order equilibrium phase transitions: a divergence of the correlation length and the susceptibility, a critical slowing down, and a scaling behavior. In addition, we have found that for coupled oscillators, even if they are only driven by additive symmetric noises (in this paper, we set them as Gaussian white noises), when $F=0$ a net current for the particles may emerge if the current versus F has a hysteresis near $F=0$. With varying F a continuous or discontinuous transition between state $s>0$ and state $s<0$ will appear. This transition is not a phase transition, since no symmetry breaking occurs even though the order parameter changes continuously or discontinuously. Moreover, hysteresis for the mean field or the current can sometimes be found as functions of F. This is because a nonzero mean field can break the symmetry of the system.

Model (1) given in the paper is theoretically mathematical and physical. It makes us reminiscent of the work of Shiino [12]. In Ref. [12], Shiino proved the H theorem in an asymptotic approach, and showed a critical slowing down of order-parameter fluctuations for a bistable ensemble and $g(x)=1$. We suppose that our models also satisfy the H theorem (we will give the proof elsewhere). For our model II, when $F=0$ there is a nonequilibrium second phase transition, which has an interesting feature: a critical slowing down (since this feature is general for the second order phase transition).

In our paper, the function $g(x)$ is set as a linear or nonlinear function of x. If $g(x)=\text{const}$, Eq. (8) in the case of $F=0$, has only the trivial solution $s=0$, and there is no nonequilibrium transition between state $s=0$ and state $s\neq 0$ for the system (see Secs. III and IV); when $F\neq 0$, although s is nonzero and is a function of F, e, and D, the appearance of characteristic features such as the discontinuous transition between $s>0$ and $s<0$ and the hysteresis for the mean field and the current (see Secs. III and IV) will not exist; this is because the coupling between the oscillators is too simple, and there is not enough effect on the system with changing x. We have made a numerical simulation in the case of $g(x)=1$, and found no phenomena of discontinuous transition and hysteresis.

For a single oscillator we can find spontaneous oscillations (a running solution). The effect of the coupling term [i.e., $esg(x)$] on this behavior depends on the structure of the effective potential when adding the coupling. For example, in the case of the same period for an effective potential with coupling and the one without coupling, when adding the coupling, if the potential barrier becomes lower and the potential well more shallow, the coupling term can enhance the spontaneous oscillations; if the barrier becomes higher and the well deeper, the coupling term can weaken this behavior. If the external constant force is large, the effective potential function will have a large average slope, and the spontaneous oscillations will become weaker.

The system considered here consists of an infinite number of globally coupled oscillators driven by noises. When the oscillators are finite, the features of the system will change. For example, in Ref. [7], when the oscillators are finite, the system has a transition between a state with zero mean field and a state with nonzero mean field. However, when the oscillators are infinite, no transition occurs in the system. Thus, in our paper, the case when the oscillators are finite remains to be studied. In addition, the systems we studied in this paper are globally coupled and driven only by additive noises. If the oscillators are locally coupled, the results are the same. If introducing multiplicative noises in our systems, we suppose that the reentrant transition found by Broeck *et al.* [4] will probably appear.

ACKNOWLEDGMENT

This work was supported by the Alexander von Humboldt Foundation.

[1] *Noise and Order: The New Synthesis*, edited by M. Millonas (Springer-Verlag, New York, 1996).

[2] B. McNamara, K. Wiesenfeld, and R. Roy, Phys. Rev. Lett. **60**, 2626 (1988); L. Gammaitoni, F. Marchesoni, E. Menichella-Saetta, and S. Santucci, *ibid.* **62**, 349 (1989); P. Jung and P. Hänggi, Phys. Rev. A **44**, 8032 (1991); J. K. Douglass, L. Wilkens, E. Pautazelou, and F. Moss, Nature (London) **365**, 337 (1997); K. Wiesenfeld, D. Pierson, E. Pantazelou, C. Dames, and F. Moss, Phys. Rev. Lett. **72**, 2125 (1994); M. Grifoni and P. Hanggi, *ibid.* **76**, 1611 (1995); J. M. G. Vilar and J. M. Rubí, *ibid.* **77**, 2863 (1996); **78**, 2882 (1997); **78,** 2886 (1997); A. Krawiecki, A. Sukiennicki, and R. A. Kosinski, Phys. Rev. E **62**, 7683 (2000).

[3] M. O. Magnasco, Phys. Rev. Lett. **71**, 1477 (1993); R. Bartussek, P. Hänggi, and J. G. Kissner, Europhys. Lett. **28**, 459 (1994); C. R. Doering, W. Horsthemke, and J. Riordan, Phys. Rev. Lett. **72**, 2984 (1994); R. D. Astumian and M. Bier, *ibid.* **72**, 1766 (1994); M. M. Millonas and C. Ray, *ibid.* **75**, 1110 (1995); J. Luczka, B. Bartussek, and P. Hängii, Europhys. Lett. **31**, 431 (1995); I. Zapata, J. Luezka, F. Sols, and P. Hänggi, Phys. Rev. Lett. **80**, 829 (1998); Phys. Rev. E **58**, 139 (1998); R. H. Luchsinger, *ibid.* **62**, 272 (2000); Jing-Hui Li and Z. Q. Huang, *ibid.* **57**, 3917 (1998).

[4] C. Van den Broeck, J. M. R. Parrondo, and R. Toral, Phys. Rev. Lett. **73**, 3395 (1994).

[5] F. Castro, A. D. Sanchez, and H. S. Wio, Phys. Rev. Lett. **75**, 1691 (1995); C. Van den Broeck, J. M. R. Parrondo, R. Toral, and R. Kawai, Phys. Rev. E **55**, 4084 (1997); R. Müller, K. Lippert, A. Kühnel, and U. Behn, *ibid.* **56**, 2658 (1997); **58**, 2838 (1998); S. E. Mangioni, R. R. Deza, R. Toral, and H. S. Wio, *ibid.* **61**, 223 (2000); Jung-Hui Li and Z. Q. Huang, *ibid.* **53**, 3315 (1996).

[6] Jing-Hui Li and Z. Q. Huang, Phys. Rev. E **58**, 2838 (1998).

[7] A. Pikovsky, K. Rateitschak, and J. Kurths, Z. Phys. B **95**, 541 (1994); Jing-Hui Li and Z. Q. Huang, Phys. Rev. E **58**, 2760 (1998).

[8] C. R. Doering and J. C. Gadoua, Phys. Rev. Lett. **69**, 2318 (1992); M. Bier and R. D. Astumian, Phys. Rev. Lett. **71**, 1649 (1993); U. Zürcher and C. R. Doering, Phys. Rev. E **47**, 3862 (1993); C. Van den Broeck, *ibid.* **47**, 4579 (1993); P. Pechukas and P. Hänggi, Phys. Rev. Lett. **73**, 2772 (1994); A. J. R. Madureira, P. Hänggi, V. Buonomano, and W. A. Rodrigues, Jr., Phys. Rev. E **51**, 3849 (1995); P. Hänggi, Chem. Phys. **180**, 157 (1994); P. Reimann, Phys. Rev. E **49**, 4938 (1994); **52**, 1579 (1995); Phys. Rev. Lett. **74**, 4576 (1995); M. Marchi, F. Marchesoni, L. Gammaitoni, E. Menichella-Saetta, and S. Santucci, Phys. Rev. E **54**, 3479 (1996); J. Iwaniszewski, *ibid.* **54**, 3173 (1996); Marián Boguñá, Josep M. Porrà, Jaume Masoliver, and Katja Lindenberg, *ibid.* **57**, 3990 (1998); Tomas Novotny and Petr Chvosta, *ibid.* **63**, 012102 (2001); Jing-Hui Li, D. Y. Xing, J. M. Dong, and B. Hu, *ibid.* **60**, 1324 (1999); Jing-Hui Li, B. Hu, D. Y. Xing, and J. M. Dong, *ibid.* **60**, 6443 (1999).

[9] C. W. Gardiner, *Handbook of Stochastic Methods for Physics,*

Chemistry and the Natural Sciences (Springer-Verlag, Berlin, 1983); H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1984); N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1992).

[10] R. C. Desai and R. Zwanzig, J. Stat. Phys. **19**, 1 (1978); J. J. Brey, J. M. Casado, and M. Morillo, J. Phys. A **128**, 497 (1984); L. L. Bonilla, J. Stat. Phys. **46**, 659 (1987); L. L. Bonilla, J. M. Casado, and M. Morillo, *ibid.* **48**, 571 (1987).

[11] P. Reimann, R. Kawai, C. Van den Broeck, and P. Hänggi, Europhys. Lett. **45**, 545 (1999); P. Reimann, C. Van den Broeck, and R. Kawai, Phys. Rev. E **60**, 6402 (1999); J. Buceta, J. M. Parrondo, C. Van den Broeck, and F. J. de la Rubia, *ibid.* **61**, 6287 (2000).

[12] M. Shiino, Phys. Rev. A **36**, 2393 (1987).