Interplay of frequency-synchronization with noise: Current resonances, giant diffusion and diffusion-crests

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Abstract. – We elucidate how the presence of noise may significantly interact with the synchronization mechanism of systems exhibiting frequency-locking. The response of these systems exhibits a rich variety of behaviors, such as resonances and anti-resonances which can be controlled by the intensity of noise. The transition between different locked regimes provokes the development of a multiple enhancement of the effective diffusion. This diffusion behavior is accompanied by a crest-like peak-splitting cascade when the distribution of the lockings is self-similar, as occurs in periodic systems that are able to exhibit a Devil's staircase sequence of frequency-lockings.

The phenomenon of frequency synchronization or frequency-locking is generic for nonlinear dynamical systems where two or more frequencies compete. It has been observed in a great variety of situations including the cases of the driven pendulum, charge-density waves [1], chemical reactions [2] or Josephson junctions [3], to mention just a few. Its main characteristics is the appearance of a complex distribution of plateaus or staircase structure in the response of these systems. The presence of the plateaus is the signature of the locking at different frequencies which are rational multiples of the driving frequency.

A simple model exhibiting frequency-locking is the periodically driven washboard potential. This type of potential is particularly interesting since its ubiquity and simplicity makes it the archetype for modeling transport in periodic systems. A great variety of condensed-matter systems can be modeled using this potential. We may refer the cases of Josephson junctions [4], charge-density waves [5], superionic conductors [6], rotation of dipoles in external fields [7], phase-locked loops (PLL) [8], diffusion on surfaces [9] or separation of particles by electrophoresis [10], to mention just a few [11]. When the washboard potential is periodically driven it exhibits a great variety of nonlinear phenomena including phase-locking, hysteresis [12], stochastic resonance [13] and chaos [14].

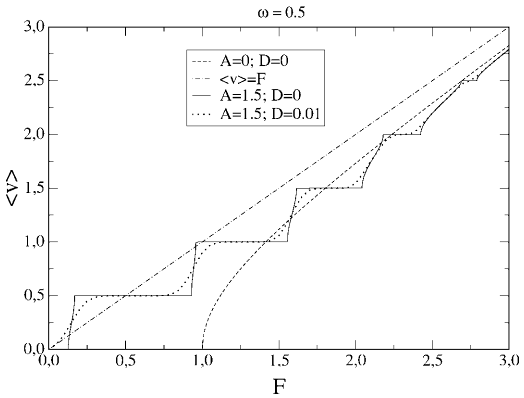

Fig. 1

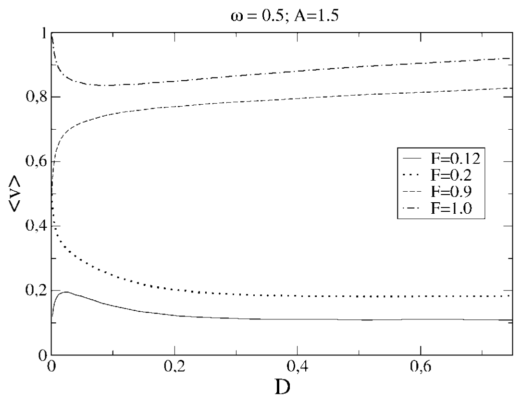

Fig. 2

Fig. 1 – Average velocity as a function of the tilt F, for angular driving frequency $\omega = 0.5$ for different driving strengths and noise intensities. The deterministic behavior in the absence of driving is depicted with the dashed line; the Shapiro step behavior is shown by the solid line. Small noise smooths the Shapiro steps (dotted line).

Fig. 2 – Velocity as a function of the level of noise for a driving strength $A = 1.5$ and angular frequency $\omega = 0.5$ for different values of the tilt F.

Our purpose in this letter is to report interesting new phenomena which take place in periodically driven multistable systems arising as a consequence of the cooperative effect between synchronization and noise. We shall demonstrate that the interplay between the frequency-locking and the noise gives rise to a multi-enhancement of the effective diffusion, and a rich behavior of the current as well, including partial suppression and characteristic resonances.

For the sake of simplicity, we will start our analysis considering the model of overdamped Brownian motion of a particle in a washboard potential with a modulated tilt. In scaled units, the dynamic equation is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\sin(x) + F + A\cos(\omega t) + \xi, \tag{1}$$

where F is the tilt, A is the amplitude of the periodic input, and ω its angular frequency. The system is under the influence of a Gaussian white noise ξ of zero mean and delta-correlated $\langle \xi(t)\xi(t')\rangle = 2D\delta(t-t')$, with D defining the noise level. Transport characteristics can be described in terms of two main quantities: the average velocity, or current, defined in the long-time limit as

$$\left\langle \frac{\mathrm{d}x}{\mathrm{d}t} \right\rangle \equiv \langle v\rangle \equiv \ \lim_{t\to\infty} \frac{\langle x(t)\rangle}{t}, \tag{2}$$

which is independent of the initial condition $x(0)$. The effective diffusion coefficient is defined as

$$D_{\mathrm{eff}} \equiv \ \lim_{t\to\infty} \frac{1}{2t}\left\langle [x(t) - \langle x(t)\rangle]^2 \right\rangle. \tag{3}$$

A main feature of the deterministic dynamics of the model is the appearance of a multiplicity of plateaus in the current when represented against the tilt, which have been referred to as Shapiro steps [3] (see fig. 1). This trait is a consequence of the locking of the particle velocity at the harmonics of the driving frequency. Naively speaking, the frequency-locking occurs because the particle tends to synchronize its motion with the periodic force to overcome an integer number of wells during one cycle of the force. The size of the jumps in the current is $\Delta \langle v \rangle = \omega$. The deterministic dynamics of this model has been widely characterized in the literature [15], and the existence of mode-locking has been corroborated experimentally [1,3].

A remarkable consequence of the locking is that the response of the system can be controlled by means of the modulated component of the tilt (for instance, mode-locked Josephson junctions are used as a voltage standard [16]). Particularly interesting is the fact that, for some values of the parameters, one can achieve values of the velocity larger than the tilt F, as occurs for $F = 0.25$, and $A = 1.5$ in the example depicted in fig. 1.

These peculiar nonlinear characteristics of the deterministic dynamics gives rise to interesting transport phenomena when noise is present. There has been a pronounced interest in the effect of noise in this model [17], especially in the context of Josephson junctions, laser-gyroscopes and charge-density waves. The linear and nonlinear response has been characterized as a function of the tilt [17], the driving frequency or the amplitude [18–20], and such intriguing phenomena as the reduction of the low-frequency broadband noise level in the locked regime of charge-density waves have been measured [1] and described [21]. Here we will focus on two different, previously unexplored aspects: the response as a function of the noise level and the behavior of the effective diffusion in terms of the control parameter (the tilt or the amplitude). Our results have been obtained from numerical simulations of eq. (1).

When the velocity is analyzed in terms of the noise level, it exhibits different regimes, which can be tuned through the value of the tilt F, as depicted in fig. 2. One of these regimes, corresponding to the curve $F = 0.9$ in fig. 2, reflects a monotonic increase of the current with the intensity of noise (see also below). This behavior just mimics the situation which occurs in the absence of driving (not shown). Then, the addition of noise facilitates the escape of the particle trapped in the potential, thus increasing its velocity downhill. However, when the periodic driving is present and synchronization occurs, a richer phenomenology emerge. For instance, the opposite behavior can also be found, *i.e.*, a monotonic decrease of the current *vs.* increasing noise intensity (as for $F = 0.2$ in fig. 2). Moreover, for another range of values of the tilt, the behavior is no longer monotonic: the current exhibits either an anti-resonance or a resonance, evidenced through the presence of a minimum (as for $F = 1.0$ in fig. 2) or a maximum ($F = 0.12$ in fig. 2), respectively, at characteristic values of the noise level.

The jumps in the current, which are the result of the locking of the velocity at integer multiples of the driving frequency, yield the underlying mechanism for the appearance of the different regimes. Their existence can be qualitatively understood from the analysis of the deterministic dynamics (see fig. 1). A small amount of noise tends to smooth the steps because the noise helps the particle to escape from its trapped, phase-locked state. Consequently, at small values of the noise, the current increases for values of the tilt F on the right half of the plateaus (*i.e.*, before the jump for the velocity has occurred) and decreases on the other half of the plateaus (*i.e.*, after the jump has occurred). For high noise levels, the particle does not notice the presence of the potential and its velocity thus tends to a limiting value which corresponds to the value of the tilt F, *i.e.* $\langle v \rangle \to F$ as $D \to \infty$. These two limiting tendencies determine the appearance of the different regimes. For instance, a monotonic increase of the current may occur for values of the tilt F on the right half of synchronization plateaus (as for $F = 0.9$ in fig. 2) because the addition of noise consistently increases the velocity up to the limiting value of $\langle v \rangle = F$. Similarly, the anti-resonance occurs for values of the tilt on the left half of the plateaus (as for $F = 1.0$ in fig. 2), because the smoothing of the steps leads to an initial decrease of the current with the noise, followed by an increase in order to reach the asymptotic $\langle v \rangle \to F$. The occurrence of the remaining regimes requires special conditions. In

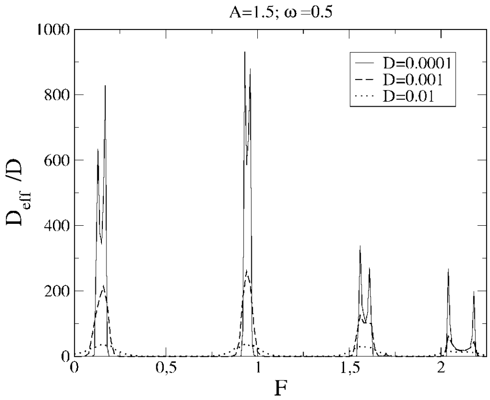

Fig. 3

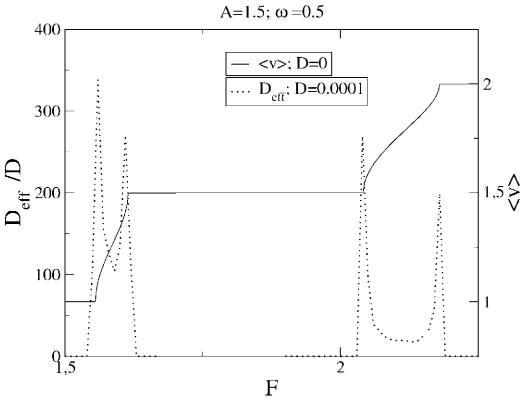

Fig. 4

Fig. 3 – Effective diffusion $D_{\rm eff}/D$ as a function of the tilt F, for a driving strength $A = 1.5$, angular frequency $\omega = 0.5$ for different values of the level of noise D.

Fig. 4 – Close-up of the behavior of $D_{\rm eff}/D$ in fig. 3 superimposed to the plot $\langle v \rangle$ *vs.* F. The peaks in $D_{\rm eff}$ occur at the onset of transition between the locked and nonlocked regimes.

particular, a maximum of the velocity or a monotonic decreasing regime may only appear for values of the tilt before and after a step in which $\langle v \rangle$ exceeds the value $\langle v \rangle = F$ for $D = 0$ (as $F = 0.12$ and $F = 0.2$ in fig. 2, respectively). The necessity of this requirement can be inferred from the above reasoning. It is also important to remark that the behavior of the current *vs.* the tilt F is similar for values of the driving strength $A < 1$, thus manifesting that these anomalous behaviors are not a peculiarity of strong driving.

The transition between the locked and nonlocked regimes in the deterministic dynamics gives rise to a peculiar behavior of the effective diffusion. In the numerical simulations of eq. (1) one observes the presence of multiple peaks in the effective diffusion when represented as a function of the tilt, as depicted in fig. 3. As the intensity of noise diminishes, the ratio $D_{\rm eff}/D$ significantly increases, which is the signature of an enhancement of the effective diffusion. The location of the peaks corresponds to the values of the tilt F at which the jumps in the current *vs.* F occur. Consequently, by means of the periodic forcing one can control the conditions under which the enhancement of diffusion occurs. That is, by selecting the proper amplitude or frequency of the periodic driving, one can obtain a significant enhancement of the diffusion at an arbitrary value of the tilt. Moreover, the effect is "robust" in the sense that there exist a multiplicity of values of the tilt for which an enhancement of the diffusion can be obtained.

The mechanism for the enhancement of the effective diffusion can be traced back to the extreme sensitivity of the dynamics upon the addition of a small amount of noise. In the frequency-locked regime, the motion of the particle is enslaved by the periodic forcing, hence the effective dispersion is small. In fact, the random perturbations are effectively suppressed in the locked regime [1, 21], and the effective, barrier-activated diffusion $D_{\rm eff}$ is smaller than the free diffusion D. On the contrary, in the nonlocked regime, the behavior of the stochastic particle dynamics becomes strongly diffusive with large fluctuations. These extreme fluctuations emerge because of a drift-assisted splitting mechanism of the Brownian particle dynamics. The giant enhancement of the effective diffusion takes place close to the parameter region where the particle gets rid of the captivity of the periodic driving. It is precisely at those transition points where the dynamics of the particle becomes particularly sensitive to the addition

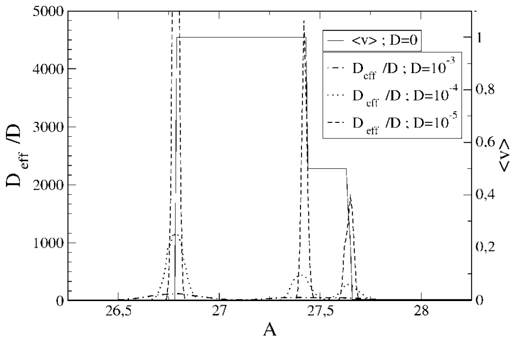

Fig. 5

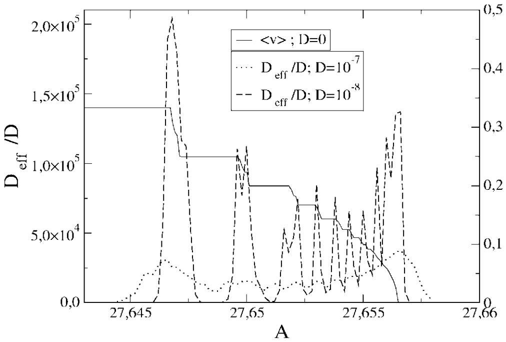

Fig. 6

Fig. 5 – Effective diffusion D_{eff}/D as a function of the amplitude strength A, for the model described by eq. (4) at different values of the level of noise D. The plot of the deterministic response $\langle v\rangle$ *vs.* A is overlaid.

Fig. 6 – Magnified view of parts of fig. 5. The plot $\langle v\rangle$ *vs.* A exhibits a Devil's staircase structure. The splitting of the peaks of the effective diffusion, occurring at the onset of transition between different locked regimes, becomes increasingly more manifested as we decrease the level of noise.

of a small dose of noise. This happens because the synchronization is efficiently nourishing this drift-assisted splitting mechanism. Similar features been reported in refs. [22, 23] for the tilted washboard potential in the absence of periodic modulation, and in refs. [24, 25] for other systems at slow driving.

Qualitatively, it is quite suggestive that a very sensitive dependence of the current $\langle v\rangle$ upon small changes of the static force F will be intimately connected with a very sensitive dependence of the particle trajectory $x(t)$ on the random force $\xi(t)$, resulting in a greatly enhanced diffusion D_{eff}/D. In other words, we may expect a qualitative (but not quantitative) proportionality between $\mathrm{d}\langle v\rangle/\mathrm{d}F$ and D_{eff}/D, which is indeed confirmed by our numerical findings. In particular, a careful analysis of the effective diffusion at very low levels of noise reveals the existence of a splitting of the peaks, showing that the enhancement occurs at *every* transition between locked and nonlocked regimes, and *vice versa*. This characteristic is illustrated in fig. 4. In the limit $D \to 0$ there is no further splitting, and the ratio D_{eff}/D assumes giant values while at the same time the corresponding peak width narrows, thus tending to a collection of very sharp singularities which are located at every onset and every end of the locked regimes.

This effect is not restricted to the behavior of the effective diffusion as a function of the tilt F. The enhancement of the effective diffusion also takes place at the onset of frequency-locking as a function of other control parameters, such as the amplitude A and the angular frequency ω.

The results reported above reveal the basic mechanism for the occurrence of multi-enhancement of the effective diffusion. In the model we have implemented, the locking takes place only at integer values of the driving frequency. We can now analyze the implications of our results in more complex models exhibiting different types of synchronization mechanisms. A particularly interesting situation is the case of potentials giving rise to a self-similar distribution of steps, known as Devil's staircase [26]. In these systems, in the absence of noise the phase locking occurs at every rational value of the driving frequency $\langle v\rangle = \frac{n}{m}\omega$, with n, m being integers. In that case, one has a self-similar distribution of transitions between locked and nonlocked states, suggesting the occurrence of a corresponding cascade of splitting of the peaks in the effective diffusion in the presence of a small amount of noise. In figs. 5 and 6, we have calculated the effective diffusion by solving numerically the model suggested in refs. [27, 28]:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\varepsilon \frac{\mathrm{d}U}{\mathrm{d}x} + A\cos(2\pi t) + \xi, \tag{4}$$

where the amplitude is $\varepsilon = 5$, the periodic potential is $U(x) = [\cos(2\pi x) - 0.5\sin(4\pi x)]/2\pi$, and ξ is a Gaussian white noise with zero mean and $\langle \xi(t)\xi(t')\rangle = 2D\delta(t-t')$. For these values of the parameters, it has been shown in ref. [27] that the model exhibits a Devil's staircase structure in the current behavior, see the $\langle v\rangle$ *vs.* A plots of figs. 5 and 6. Those figures clearly confirm the expected progressive splitting of peaks of the effective diffusion which occurs as the noise intensity is reduced. It reveals the rich structure that the interplay between synchronization and noise generates in this case. The superimposed plots of $\langle v\rangle$ *vs.* A corroborate that the enhancement of the effective diffusion occurs at *every* onset of transition between different locked regimes. The overall behavior for D_{eff} thus assumes a hillcrest-like form, cf. fig. 6. This novel characteristic bearing of D_{eff} *vs.* driving strength A shall be termed a *diffusion-crest*. We remark that in the rather different context of deterministic dynamics in discrete time (chaotic maps) a diffusion coefficient with a Devil's staircase-like behavior has been revealed in [29], while in our case the diffusion coefficient exhibits an even more spectacular behavior, approaching qualitatively the *derivative* of a Devil's staircase for asymptotically weak noise.

In conclusion, we have shown that the cooperation between noise and frequency-locking inherent in the deterministic dynamics of periodically driven systems gives rise to the appearance of a rich transport phenomenology. When the response of a system assumes a staircase structure, the addition of noise leads to the occurrence of counterintuitive phenomena. One then observes the existence of regimes in which the current is suppressed or the presence of resonances or anti-resonances in terms of the noise level. The lowering of the noise level reveals the existence of a very peculiar behavior of the effective diffusion including a strongly multi-enhancement and a crest-like splitting of the peaks which can become self-similar in the case of a locking behavior of the Devil's staircase type. This behavior evidences the increase of the sensitivity of the system to small perturbations at the transition between different locked and nonlocked regimes. The multiple diffusion enhancement is then a general phenomenon occurring in systems for which frequency-synchronization occurs and the response to an input assumes a step-like structure. This new phenomenology implies interesting consequences in transport theory of nonlinear systems and the findings can in principle be applied to actual situations of practical interests. The existence of diffusion-crests provides a promising mechanism for the selective control of diffusion. This mechanism could be useful, for example, to improve and control selectively the release of drugs in biological tissues [30]. For field-responsive systems [31, 32], *i.e.* electro- and magneto-rheological fluids, the giant and controlled enhancement of the diffusion would give rise to a huge increase of the rotational viscosity with potential applications to magnetic dampers.

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REFERENCES

[1] Sherwin M. S. and Zettl A., *Phys. Rev. B*, **32** (1985) 5536.
[2] Petrov V., Ouyang Q. and Swinney H. L., *Nature (London)*, **388** (1997) 655.
[3] Shapiro S., *Phys. Rev. Lett.*, **11** (1963) 80.
[4] Barone A. and Paternó G., *Physics and Applications of the Josephson Effect* (Wiley, New York) 1982.
[5] Gruner G., Zawadowski A. and Chaikin P. M., *Phys. Rev. Lett.*, **46** (1981) 511.
[6] Fulde P., Pietronero L., Schneider W. R. and Strässler S., *Phys. Rev. Lett.*, **35** (1975) 1776.
[7] Reguera D., Rubí J. M. and Pérez-Madrid A., *Phys. Rev. E*, **62** (2000) 5313.
[8] Lindsey W. C., *Synchronization Systems in Communication and Control* (Prentice-Hall, Englewood Cliffs, NJ) 1972.
[9] Frenken J. W. M. and Van der Veen J. F., *Phys. Rev. Lett.*, **54** (1985) 34.
[10] Adjari A. and Prost J., *Proc. Natl. Acad. Sci. USA*, **88** (1992) 4468.
[11] More examples can be found in: Risken H., *The Fokker-Planck Equation* (Springer, Berlin) 1984; Reimann P., to be published in *Phys. Rep.*; see also cond-mat/0010237.
[12] Borromeo M., Constantini G. and Marchesoni F., *Phys. Rev. Lett.*, **82** (1999) 2820.
[13] Gammaitoni L., Hänggi P., Jung P. and Marchesoni F., *Rev. Mod. Phys.*, **70** (1998) 223.
[14] Yeh W. J. and Kao Y. H., *Phys. Rev. Lett.*, **49** (1982) 1888; *Appl. Phys. Lett.*, **42** (1983) 299.
[15] Kautz R. L., *J. Appl. Phys.*, **52** (198) 3528.
[16] Kautz R. L., *Rep. Prog. Phys.*, **59** (1996) 935.
[17] Stratonovich R. L., *Oscillator synchronization in the presence of noise*, *Radiotekhn. elektron.*, **3** (1958) 497. English translation in *Non-linear Transformations of Stochastic Processes*, edited by Kuznetsov P. I., Stratonovich R. L. and Tikhonov V. I. (Pergamon, Oxford) 1965.
[18] Coffey W. T., Déjardin J. L. and Kalmykov Y. P., *Phys. Rev. E*, **61** (2000) 4599.
[19] Schleich W., Cha C.-S. and Cresser J. D., *Phys. Rev. A*, **29** (1984) 230.
[20] Jung P. and Hänggi P., *Ber. Bersenges. Phys. Chem.*, **95** (1991) 311.
[21] Wiesenfeld K. and Satija I., *Phys. Rev. B*, **36** (1987) 2483; Crommie M. F., Craig K., Sherwing M. S. and Zettl A., *Phys. Rev. B*, **43** (1991) 13699.
[22] Reimann P., Van den Broeck C., Linke H., Hänggi P., Rubí J. M. and Pérez-Madrid A., *Phys. Rev. Lett.*, **87** (2001) 010602; submitted to *Phys. Rev. E.*
[23] Constantini G. and Marchesoni F., *Europhys. Lett.*, **48** (1999) 491.
[24] Gang H., Daffertshofer A. and Haken H., *Phys. Rev. Lett.*, **76** (1996) 4874.
[25] Schreier M., Reimann P., Hänggi P. and Pollak E., *Europhys. Lett.*, **44** (1998) 416.
[26] Bak P., *Phys. Today*, **86** (1986) 38.
[27] Ajdari A., Mukamel D., Peliti L. and Prost J., *J. Phys. I*, **4** (1994) 1551; see also Hänggi P. and Bartussek R., *Lect. Notes Phys.*, **476** (1996) 294.
[28] Bartussek R., Hänggi P. and Kissner J. G., *Europhys. Lett.*, **28** (1994) 459.
[29] Klages R. and Dorfman J. R., *Phys. Rev. Lett.*, **74** (1995) 387.
[30] Ruuge E. K. and Rusetski A. N., *J. Magn. & Magn. Mater.*, **122** (1993) 335.
[31] Rosensweig R. E., *Science*, **271** (1996) 614.
[32] Rub J. M. and Vilar J. M. G., *J. Phys. Condens. Matter*, **12** (2000) A75.