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FAST DIFFUSION LIMIT FOR REACTION-DIFFUSION SYSTEMS WITH STOCHASTIC NEUMANN BOUNDARY CONDITIONS*

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Abstract. We consider a class of reaction-diffusion equations with a stochastic perturbation on the boundary. We show that in the limit of fast diffusion, one can rigorously approximate solutions of the system of PDEs with stochastic Neumann boundary conditions by the solution of a suitable stochastic/deterministic differential equation for the average concentration that involves reactions only. An interesting effect occurs in case the noise on the boundary does not change the averaging concentration but is sufficiently large. Here due to the presence of noise surprising new effective reaction terms may appear in the limit. To study this phenomenon we focus on systems with polynomial nonlinearities and illustrate it with simplified, somewhat artificial, examples, namely, a two-dimensional nonlinear heat equation and the cubic autocatalytic reaction between two chemicals.

Key words. multiscale analysis, SPDEs, stochastic boundary conditions, reaction-diffusion equations, fast diffusion limit

AMS subject classifications. 60H10, 60H15, 35R60, 35K45, 35K57

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1. Introduction.

Stochastic partial differential equations (SPDEs) appear naturally as models for dynamical systems with respect to random influences. Sometimes in a complex physical system the noise has an impact not only on the bulk of the system but on its physical boundary, too. Classical examples are, for instance, heat transfer in a solid in contact with a fluid [14], chemical reactor theory [15], colloid and interface chemistry [23], and the air-sea interactions on the ocean surface [21]. Neumann boundary forcing arises in these examples from random perturbations of flux over the boundary or random forcing on the surface of oceans.

Let us describe mathematically more precise the model we want to study here. Let G be a bounded sufficiently smooth domain in $\mathbb{R}^d$ for $d \geq 1$, which has a smooth boundary ∂G. On the domain G we consider the following system of stochastic reaction-diffusion equations for n species with respect to random Neumann boundary conditions:

$$\begin{aligned} \partial_t u &= \varepsilon^{-2}\mathcal{A}u + \mathcal{F}(u) \quad \text{for } t \geq 0,\ x \in G, \\ \frac{\partial u}{\partial \nu} &= \sigma_\varepsilon \varepsilon^2 \mathbb{D}^{-1}\partial_t W(t) \qquad \text{for } \ t \geq 0,\ x \in \partial G, \\ u(0,x) &= u_0(x) \qquad \text{for } \ x \in G, \end{aligned} \tag{1.1}$$

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with $u = (u_1, \dots, u_n)^t$ is the concentration of the n species,

$$\mathcal{A}u = \mathbb{D} \cdot \begin{pmatrix} \mathcal{A}_1 u_1 \\ \vdots \\ \mathcal{A}_n u_n \end{pmatrix}, \quad \mathcal{F}(u) = \begin{pmatrix} \mathcal{F}_1(u_1, \dots, u_n) \\ \vdots \\ \mathcal{F}_n(u_1, \dots, u_n) \end{pmatrix} \text{ and } W(t) = \begin{pmatrix} W_1(t) \\ \vdots \\ W_n(t) \end{pmatrix},$$

where $\mathcal{A}$ is the diffusion term, $\mathbb{D} = \operatorname{diag}(d_1, \dots, d_n)$ is the diagonal matrix of diffusion constants, and $\frac{\partial u}{\partial \nu}$ is the normal derivative of u on ∂G. The reaction terms $\mathcal{F}_i(u_1, u_2, \dots, u_n)$ are polynomials of degree m_i and the noise is modeled by the derivative of the W_i, which are independent Wiener processes defined on the boundary ∂G. For the noise strength we explicitly stated the factors $\varepsilon^2 \mathbb{D}^{-1}$ that would disappear after changing later to a mild formulation concept of the problem. The assumption of independence of the different noise sources in each species is mainly for convenience of presentation, as now many terms cancel and the technicalities are slightly less involved.

Let us now comment on previously known results for equations of this type. Da Prato and Zabczyk [9, 10] discussed the difference between the problems with Dirichlet and Neumann boundary noises. Sowers [19] investigated multidimensional stochastic reaction diffusion equation with Neumann boundary conditions and showed that there is a unique solution. The publications [1, 5] studied random Dirichlet boundary conditions, where solutions are so irregular that they fail to be a function. Other results are [2, 16].

An interesting result is Schnaubelt and Veraar [22], where the regularity of solutions is studied in detail. Furthermore, mild and weak solutions are shown to coincide.

Recently, Cerrai and Freidlin [6] already considered a similar class of stochastic reaction-diffusion equations with Neumann boundary noise. They studied more general differential operators, but the nonlinearities are globally Lipschitz. They showed in the case $\sigma_\varepsilon = 1$ that when the diffusion rate is much larger than the rate of reaction, it is possible to replace the SPDE by a suitable one-dimensional stochastic differential equation. Also fluctuations around this limit are studied in detail.

Mohammed [17] also considered a class of stochastic Ginzburg–Landau type equation on the slow time-scale with cubic nonlinearities and with additional random Neumann boundary conditions on the interval $[0, 1]$, which applies to the simplest example of the setting in (1.1) with $\sigma_\varepsilon = 1$ where the boundary forcing is given by a single real-valued Brownian motion.

Our aim is to establish rigorously error bounds for the fast diffusion limit for the class of equations given by (1.1). Here in the limit the fast diffusion disappears and the effective dynamic is given by a (stochastic) ordinary differential equation for the average. The error estimates are performed in an $\mathcal{L}^p$-space setting, as we cannot expect solutions to (1.1) to be smooth. Especially, at the forced boundary the solution u is expected to be even unbounded, although it is smoother inside the domain. See [22] for Dirichlet boundary [1].

We consider the following two cases. One case is $\sigma_\varepsilon = \varepsilon^{-1}$ and the other one $\sigma_\varepsilon = 1$. The second one, which is also the scaling of [6], has the relatively simple limit, where the fast diffusion just disappears in the limit. In the first, our main, case we consider large noise, that does not force the global average, but it changes the limiting reaction equation. It introduces new noise induced reaction terms. The physical reason for large noise might be that both diffusion and noise are enhanced by stirring. The mathematical motivation is that in both cases, one considers the noise strength, where the noise has an impact on the limiting equation. Let us also state

that the index c denotes the average (i.e., $v_c = |G|^{-1} \int_G v dx$ which is the projection onto the constants).

First case. If the noise does not directly affect the average but is sufficiently large ($\sigma_\varepsilon = \varepsilon^{-1}$), then the solutions of (1.1) are well approximated by

$$u(t,x) = b(t) + \mathcal{Z}^s(t,x) + \text{error}, \tag{1.2}$$

where $b(t) \in \mathbb{R}^n$ represents the average concentration of the components of u given in general formulation as a solution of

$$\partial_t b(t) = \mathcal{F}(b(t)) + \mathcal{G}(b(t)) \tag{1.3}$$

for some polynomial $\mathcal{G}$ that arises from averaging effects. It is of degree less than or equal to $m-2$ depending on the structure of the noise and will be defined later in (7.2). The stochastic perturbation $\mathcal{Z}^s(t,x)$ is defined later in (4.4). It is an ε-dependent fast Ornstein–Uhlenbeck process (OU-process) corresponding to white noise in the limit $\varepsilon \to 0$. The weak convergence using characteristic functions is straightforward, and we do not state a precise error estimate, as this would be only in a very weak topology. For more details see section 2.1 in [4].

The ODE $\partial_t b(t) = \mathcal{F}(b)$ is the expected result, but due to noise an additional term of noise induced effective reactions appears. We illustrate our results using a relatively simple autocatalytic reaction. For the result presented we always need a square which averages to a constant in the limit $\varepsilon \to 0$. This is mainly because we assumed independent noise terms for each species. In contrast, if the noise terms are dependent, then any reaction term could lead to an additional effective reaction term in the limit. This is a significantly more involved case.

Second case. If $\sigma_\varepsilon = 1$ and no further restriction on the noise, then the solution of (1.1) is well approximated by

$$u(t,x) = b(t) + \text{error}, \tag{1.4}$$

and b is the solution of stochastic ordinary differential equation

$$\partial_t b(t) = \mathcal{F}(b(t)) + \partial_t \tilde{\beta}(t) \tag{1.5}$$

for some Wiener process $\tilde{\beta}$ in $\mathbb{R}^n$, which is essentially the projection of the boundary forcing given by W onto the dominant constant modes, i.e., the direct impact of the noise on the average. This is the somewhat expected result, and in a slightly different setting was already derived in [6] (see also [17] for a special case), where the reaction-diffusion equation under fast diffusion is well approximated by the reaction ODE. As the main work is in the first case, we only give a short proof of the second case here using the technical tools developed for the first case.

As an application of our results, we give some theoretical examples from physics (nonlinear heat equation) and from chemistry (cubic autocatalytic reaction between two chemicals according to the rule $A+B \to 2B$). To illustrate our results let us focus for a moment on the relatively simple two-dimensional nonlinear heat equation (also called Ginzburg–Landau or Allen–Cahn), which is also partly covered by the setting of [6] if one introduces a cut-off in the nonlinearity. A similar case on the real line was treated in detail in [17]. In one dimension the condition of the noise conserving mass would lead to the same noise term on both sides, which seems to be artificial. Let us consider

$$\begin{aligned} &\partial_t u = \varepsilon^{-2}\Delta u + u - u^3 && \text{for } t \geq 0,\ x \in [0,1]^2, \\ &\tfrac{\partial u}{\partial \nu} = \sigma_\varepsilon \varepsilon^2 \partial_t W(t) && \text{for } t \geq 0,\ x \in \partial\,[0,1]^2. \end{aligned} \tag{1.6}$$

For the first case we suppose $\sigma_\varepsilon = \varepsilon^{-1}$ and a noise not acting directly on the dominant modes. A similar much simpler case with body forcing was already studied in [13, 4]. Our main theorem, Theorem 4.2, states that the solution of (1.6) is well approximated by (1.2) and b is the solution of

$$db = [(1 - C_{\alpha,\lambda})b - b^3]dt,$$

where $C_{\alpha,\lambda}$ is a constant depending on the noise intensity parameters $\alpha_{i,k}$ and the eigenvalues of the operator Δ. This additional term appears by the averaging of a term $-3b\mathcal{Z}^2$, where $\mathcal{Z}$ is an OU-process on a fast time-scale.

For the second case $\sigma_\varepsilon = 1$ our main Theorem 4.5 states that the solution of (1.6) is of the form (1.4) and b is the solution of

$$db = [b - b^3]dt + dB,$$

where B is a real-valued standard Brownian motion.

The main novelty of this paper is the observation that large mass-conservative noise has the potential to change effective reaction equations in the limit of large diffusion. We give explicit error estimates in terms of high moments of the error, as in many publications only weak convergence is treated. The technical approach is based on previous works of ours. For example, the averaging lemma for OU-processes based on Itô's formula is similar to results in [3, 4].

The paper is organized as follows. Our assumptions and some definitions are given in the next section. In section 3 we present all details necessary to define the mild solution of our equation. In section 4 we derive the fast diffusion limit with error terms and present the main theorems. Section 5 provides the bounds for high non-dominant modes, while section 6 establishes averaging results with error bounds over the fast OU-process. In section 7, we give the proof of Approximation I, Theorem 4.2, for the first case and some simple examples as applications of our results motivated by physics and chemistry. Finally, in section 8 we prove Approximation II, Theorem 4.5, for the second case and apply this result to the same examples as in section 7, a nonlinear heat equation also known as Allen–Cahn or Ginzburg–Landau, and a cubic autocatalytic reaction between two chemicals.

2. Definition and assumptions. This section states the precise setting for (1.1) and summarizes all assumptions necessary for our results. For the analysis we work in the separable Hilbert space $\mathcal{L}^2(G)$ of square integrable functions, where $G \subset \mathbb{R}^d$ is a bounded domain with sufficiently smooth boundary ∂G (e.g., Lipschitz), equipped with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.

DEFINITION 2.1. *Define for $i = 1, 2, \dots, n$ and diffusion constants $d_i > 0$*

$$\mathcal{A}_i = d_i \Delta \tag{2.1}$$

with domain of definition $D(\mathcal{A}_i) = \left\{ u \in \mathcal{H}^2 : \partial_\nu u|_{\partial G} = 0 \right\}$, where $\partial_\nu u$ is the normal derivative of u on ∂G.

Let $\{g_k\}_{k=1}^\infty$ be an orthonormal basis of eigenfunctions of $\mathcal{A}_i$ in $\mathcal{L}^2(G)$. It is obviously the same basis for all i with corresponding eigenvalues $\{d_i \lambda_k\}_{k=0}^\infty$ depending on i (cf. Courant and Hilbert [7]). Also, let $\{e_k\}_{k=1}^n$ be the standard orthonormal basis of $\mathbb{R}^n$. Hence, $\{g_k e_i\}$ for $k \in \mathbb{N}_0$ and $i = \{1, \dots, n\}$ is an orthonormal basis of $\mathcal{A}$ in $\left[\mathcal{L}^2(G)\right]^n$ such that $\mathcal{A}\,(g_k e_i) = -d_i \lambda_k g_k e_i$.

Assumption 2.2. We assume that for all $k \in \mathbb{N}$

$$\|g_k\|_\infty \leq C\lambda_k^{\gamma_1} \quad \text{for some } \gamma_1 \geq 0\,.$$

This is true in $\mathbb{R}^2$, for instance, on squares, hexagons, and triangles with $\gamma_1 = 0$, while the worst case is $\gamma_1 = (d-1)/2$ realized for balls and spheres. See [11]. This condition might be relaxed, but we focus in examples mainly on cases with $\gamma_1 = 0$.

Define

$$\mathcal{N} := \ker \mathcal{A} = \operatorname{span}\{e_1 g_0, \dots, e_n g_0\},$$

where $g_0 = |G|^{-\frac{1}{2}}$ is a constant and $\lambda_0 = 0$. Define $S = \mathcal{N}^\perp$ to be the orthogonal complement of $\mathcal{N}$ in $\left[\mathcal{L}^2(G)\right]^n$. Denote by $P_c u = \frac{1}{|G|}\int_G u dx$ the projection onto $\mathcal{N}$ and define $P_s u := (\mathcal{I} - P_c)u$ for the projection onto the orthogonal complement, where $\mathcal{I}$ is the identity operator on $\left[\mathcal{L}^2(G)\right]^n$. We define $\mathcal{L}^p_n := [\mathcal{L}^p(G)]^n$.

The operator $\mathcal{A}$ given by Definition 2.1 generates an analytic semigroup $\{e^{t\mathcal{A}}\}_{t\geq 0}$ (cf. Henry [12] or Pazy [20]) on $\mathcal{L}^p_n$ for all $p \geq 2$. It has the following property: There is an $\omega > 0$ and an $M \geq 1$ such that for all $t > 0$ and all $u \in \mathcal{L}^p_n$

$$\left\|e^{t\mathcal{A}} P_s u\right\|_{\mathcal{L}^p_n} \leq M e^{-\omega t} \left\|P_s u\right\|_{\mathcal{L}^p_n}\,, \tag{2.2}$$

where ω depends in general on d_i. Let us remark that results for $p \in [1,2)$ are also possible, but we prefer a setting where later both $\mathcal{L}^p_n$ and $\mathcal{L}^{pm}_n$ are subsets of $\mathcal{L}^2_n$.

Moreover, we obtain

$$\left\|e^{t\mathcal{A}} u\right\|_{\mathcal{L}^p_n} \leq M \left\|u\right\|_{\mathcal{L}^p_n}\,. \tag{2.3}$$

Furthermore, we always suppose as follows.

Assumption 2.3. There is a constant $M \geq 1$ such that for all $t > 0$ and $u \in \mathcal{L}^{mp}_n$

$$\left\|e^{t\mathcal{A}} u\right\|_{\mathcal{L}^{mp}_n} \leq M(1 + t^{-\alpha}) \left\|u\right\|_{\mathcal{L}^p_n} \tag{2.4}$$

with $\alpha = \frac{d}{p}\left(\frac{m-1}{m}\right) \in (0,1)$.

Let us remark that (2.4) follows using the Sobolev-embedding of $W^{\alpha,p}$ into $\mathcal{L}^{mp}$. The main assumption here is that $\alpha \in (0,1)$. This is a condition on p being not too small in case m is large. Assumption 2.3 is needed for bounds on the nonlinearity in the mild formulation, which also ensures the existence of the solutions.

An immediate conclusion of Assumption 2.3 and (2.2) is that after possibly changing $M \geq 1$ and $\omega > 0$ we have

$$\left\|e^{t\mathcal{A}} P_s u\right\|_{\mathcal{L}^{mp}_n} \leq M(1 + t^{-\alpha}) e^{-\omega t} \left\|P_s u\right\|_{\mathcal{L}^p_n}\,. \tag{2.5}$$

The key idea of proof for (2.5) is to use first (2.4) for $e^{t\mathcal{A}} P_s u = e^{\frac{1}{2}t\mathcal{A}}[P_s e^{\frac{1}{2}t\mathcal{A}} u]$ and then (2.2) for the remaining $e^{\frac{1}{2}t\mathcal{A}}$. In the final step we need to redefine the constants ω and M so that (2.2)–(2.5) all hold with the same constants for simplicity of presentation.

For the nonlinearity we assume the following.

Assumption 2.4. The nonlinearity $\mathcal{F}$ is a polynomial of at most degree m. Thus for all $p \geq 2$ it is bounded by

$$\left\|\mathcal{F}(u)\right\|_{\mathcal{L}^p_n} \leq C(1 + \|u\|^m_{\mathcal{L}^{pm}_n}) \text{ for all } u \in \mathcal{L}^{pm}_n, \tag{2.6}$$

where $m = \max(m_1, \dots, m_n)$ and the m_i are the degrees of the polynomials $\mathcal{F}_i$.

In the following proof, we also fix $p \geq 2$, although the result will hold for any $p \geq 2$ such that the condition on α from Assumption 2.3 is true. Moreover, we fix a universal $T_0 > 0$ that is the upper bound for all times involved.

For the noise we suppose the following setting.

Assumption 2.5. Let $W = (W_1, \dots, W_n)$ be a collection of n independent Wiener processes on an abstract probability space $(\Omega, F, \mathbb{P})$ with a bounded covariance operator $Q_i : \mathcal{L}^2(\partial G) \to \mathcal{L}^2(\partial G)$ defined by $Q_i f_k = \alpha_{i,k}^2 f_k$ for $i = 1, 2, \dots, n$, where $(\alpha_{i,k})_{k \in \mathsf{N}_0}$ is a bounded sequence of real numbers and $(f_k)_{k \in \mathsf{N}_0}$ is any orthonormal basis on $\mathcal{L}^2(\partial G)$ with $f_0 \equiv$Constant. For $t \geq 0$ we can write $W_i(t)$ (cf. Da Prato and Zabczyk [8]) as

$$W_i(t) = \sum_{k \in \mathsf{N}_0} \alpha_{i,k} \beta_{i,k}(t) f_k \quad \text{for} \quad i = 1, 2, \dots, n, \tag{2.7}$$

where $(\beta_{i,k})_{k \in \mathsf{N}_0}$ are independent, standard Brownian motions in $\mathbb{R}$. Also, we assume by using the orthonormal basis g_k of $\mathcal{A}_i$ in $\mathcal{L}^2(G)$ that for some small $\gamma \in (0, \frac{1}{2})$

$$\sum_{k,\ell=1}^{\infty} (\lambda_k + \lambda_\ell)^{2\gamma + 2\gamma_1 - 1} q_{k,\ell}^{i,i} < \infty \text{ for } \quad i = 1, 2, \dots, n, \tag{2.8}$$

where the covariance $q_{k,\ell}^{i,j}$ between different Fourier modes is defined by

$$q_{j,k}^{i,\ell} = \frac{1}{t} \mathbb{E}\left(\tilde{W}_{i,j}(t) \tilde{W}_{\ell,k}(t) \right) = \begin{cases} 0 & \text{if } i \neq \ell, \\ \langle Q_i g_j, g_k \rangle_{\mathcal{L}^2(\partial G)} & \text{if } i = \ell, \end{cases} \tag{2.9}$$

for the scalar valued Brownian motions $\tilde{W}_{i,j} = \langle W_i, g_j \rangle_{\mathcal{L}^2(\partial G)}$.

Let us remark that (2.8) ensures that the corresponding stochastic convolution is function valued. It measures interplay between decay of correlation given by the $q_{j,k}^{i,\ell}$ and the strength of the linear dissipation given by the eigenvalues λ_k.

The following assumption is used in our first case only. It ensures that the noise is mass-conserving and that various series converge.

Assumption 2.6. Assume for $i = 1, 2, \dots, n$ that

$$\alpha_{i,0} = 0 \,,$$

and for any $N \leq m$ and any $\ell \in \{1, \dots, N\}^n$

$$\sum_{k_1, k_2, \dots, k_N = 1}^{\infty} \left(\sum_{i=1}^{N} d_{\ell_i} \lambda_{k_i} \right)^{-\frac{1}{2}} \left(\prod_{i=1}^{N} \lambda_{k_i}^{2\gamma_1 - 1} q_{k_i, k_i}^{\ell_i, \ell_i} \right)^{\frac{1}{2}} < \infty. \tag{2.10}$$

Remark 2.7. Condition (2.10) is a fairly technical condition. As (2.8) it is a measure of strength of correlations weighted with linear dissipation. But in contrast to (2.8), here we consider higher order correlations and not just second order, which are due to Gaussianity bounded by various second order correlation terms each with its own weight. It is possible to give some simpler conditions. For example, in case $\gamma_1 = 0$, for all $N \geq 1$ (2.10) is implied by the significantly weaker condition

$$\sum_{k=1}^{\infty} \left(q_{k,k}^{i,i} \right)^{\frac{1}{2}} \lambda_k^{-\frac{1}{2} - \frac{1}{2m}} < \infty.$$

The following two assumptions are used in the two cases separately. They are usually lemmas that follow directly from the fact that $\mathcal{F}$ is a polynomial. Note that T_1 in general depends on the initial condition $b(0)$.

Assumption 2.8. Let $b(t)$ in $\mathcal{N}$ be the solution of (1.3). Suppose there is a stopping time $T_1 \leq T_0$ and a constant $C > 0$ such that

$$\sup_{[0,T_1]} |b| \leq C. \tag{2.11}$$

Assumption 2.9. Let $b(t)$ in $\mathcal{N}$ be the solution of (1.5). Suppose there is a stopping time $T_1 \leq T_0$ and $C > 0$ such that for $\delta > 0$ and $\kappa \in (0, \frac{1}{m+1})$ there is a sufficiently large $\zeta \gg 1$ such that

$$\mathbb{P}\left(\sup_{t\in[0,T_1]} |b(t)|^{m-1} \leq C \ln\left(\varepsilon^{-\frac{1}{\zeta}}\right) \right) \geq 1 - \varepsilon^{\delta\kappa}. \tag{2.12}$$

We remark that ζ in applications depends mainly on the time T_0 and κ only (cf. section 8.1). The latter assumption follows usually from the existence of exponential moments for b.

For our result we rely on a cut-off argument of the polynomial nonlinearity, which is implemented by a stopping time that prevents solutions u from being too large, as given by the next definition.

DEFINITION 2.10. *For a mild solution u of* (1.1) *we define for $\kappa \in (0, \frac{1}{m+1})$ and $p \geq 2$ (fixed after the assumption on the nonlinearity) the stopping time τ^* as*

$$\tau^* := T_0 \wedge \inf\left\{ t > 0 : \|u\|_{\mathcal{L}_n^{pm}} > \varepsilon^{-\kappa} \right\}. \tag{2.13}$$

We give error estimates in terms of the following $\mathcal{O}$-notation.

DEFINITION 2.11. *For a real-valued family of processes $\{X_\varepsilon(t)\}_{t\geq 0}$ we say that X_ε is of order f_ε, i.e., $X_\varepsilon = \mathcal{O}(f_\varepsilon)$, if for every $q \geq 1$ there exists a constant C_q such that*

$$\mathbb{E} \sup_{t\in[0,\tau\ ]} |X_\varepsilon(t)|^q \leq C_q f_\varepsilon^q. \tag{2.14}$$

We also use analogous notation for time-independent random variables.

DEFINITION 2.12 (multi-index notation). *Let $\ell \in \mathbb{N}_0^n$, i.e., $\ell = (\ell_1, \ell_2, \ldots, \ell_n)$, be a vector of nonnegative integers, $u = (u_1, u_2, \ldots, u_n)$. Then we define*

$$|\ell| = \sum_{i=1}^n \ell_i, \quad \ell! = \prod_{i=1}^n \ell_1!, \quad u^\ell = \prod_{i=1}^n u_i^{\ell_i}, \quad D^\ell = \partial_{u_1}^{\ell_1}\partial_{u_2}^{\ell_2}\cdots\partial_{u_n}^{\ell_n}.$$

3. Random boundary conditions. Here we present all details necessary to define the mild solution of (1.1).

DEFINITION 3.1 (Neumann map). *The Neumann map $\mathcal{D} : \mathcal{L}^2(\partial G) \to \mathcal{H}^{\frac{3}{2}}(G)$ is a continuous linear operator. It is defined for $f \in \mathcal{L}^2(\partial G)$ as the solution $\mathcal{D}f$ of*

$$(1-\Delta)\mathcal{D}f = 0 \quad \textit{and} \quad \partial_\nu(\mathcal{D}f) = f\ .$$

With a slight abuse of notation, we also denote by $\mathcal{D}$ the extension from $\mathcal{L}_n^2(\partial G)$ to $[\mathcal{H}^{\frac{3}{2}}(G)]^n$. Now we can define as in [10] (see also [6]) the stochastic convolution, which is by definition the mild solution of the linearized problem. Note that as we do not use the same operator $\mathcal{A}$ (at least up to a constant) in the Neumann map and the semigroup, we need a modifying factor in the noise strength.

DEFINITION 3.2. *Define the stochastic convolution* $\mathcal{Z}(t)$ *as*

$$\mathcal{Z}(t) = \sigma_\varepsilon(1-\Delta)\int_0^t e^{\varepsilon^{\ 2}(t-s)\mathcal{A}}\mathcal{D}dW(s). \tag{3.1}$$

The next lemma expands the stochastic convolution $\mathcal{Z}$ as a Fourier series.

LEMMA 3.3. *Under Assumption* 2.5 *let* $\mathcal{Z}$ *be the stochastic convolution defined in* (3.1)*; then (with* $\tilde{W}_{i,j}$ *defined in Assumption* 2.5*)*

$$\mathcal{Z}(t) = \sigma_\varepsilon \sum_{i=1}^{n}\sum_{j=0}^{\infty}\int_0^t e^{-\varepsilon^{\ 2}d_i(t-s)\lambda_j}\, d\tilde{W}_{i,j}(s) g_j \cdot e_i\ . \tag{3.2}$$

Proof. Writing $\mathcal{Z}(t)$ in Fourier expansion yields

$$\mathcal{Z}(t) = \sum_{i=1}^{n}\sum_{j=0}^{\infty}\langle \mathcal{Z}(t), e_i g_j\rangle\, g_j \cdot e_i.$$

Using (3.1)

$$\begin{aligned}
&\langle \mathcal{Z}(t), e_i g_j\rangle_{\mathcal{L}^2_n(G)} = \langle \mathcal{Z}_i(t), g_j\rangle_{\mathcal{L}^2(G)}\\
&= \left\langle \sigma_\varepsilon(1-\Delta)\int_0^t e^{\varepsilon^{\ 2}(t-s)d_i\Delta}\mathcal{D}dW_i(s), g_j\right\rangle_{\mathcal{L}^2(G)}\\
&= \sigma_\varepsilon\int_0^t e^{-\varepsilon^{\ 2}d_i(t-s)\lambda_j}\,\langle \mathcal{D}dW_i(s), (1-\Delta)g_j\rangle_{\mathcal{L}^2(G)}\\
&= \sigma_\varepsilon\int_0^t e^{-\varepsilon^{\ 2}d_i(t-s)\lambda_j}\,\left\{\langle \mathcal{D}dW_i(s), g_j\rangle_{\mathcal{L}^2(G)} - \langle \mathcal{D}dW(s), \Delta g_j\rangle_{\mathcal{L}^2(G)}\right\}\\
&= \sigma_\varepsilon\int_0^t e^{-\varepsilon^{\ 2}d_i(t-s)\lambda_j}\,\left\{\langle (1-\Delta)\mathcal{D}dW_i(s), g_j\rangle_{\mathcal{L}^2(G)} + \langle \partial_\nu \mathcal{D}dW_i(s), g_j\rangle_{\mathcal{L}^2_n(\partial G)}\right\}\\
&= \sigma_\varepsilon\int_0^t e^{-\varepsilon^{\ 2}d_i(t-s)\lambda_j}\,\langle dW_i(s), g_j\rangle_{\mathcal{L}^2(G)}\,,
\end{aligned}$$

where we used the Gauss–Green formula and Definition 3.1. Hence

$$\mathcal{Z}(t) = \sigma_\varepsilon \sum_{i=1}^{n}\sum_{j=0}^{\infty}\int_0^t e^{-\varepsilon^{\ 2}d_i(t-s)\lambda_j}\, d\tilde{W}_{i,j}(s) g_j \cdot e_i.$$

It is easy to check that this series converges in $\mathcal{L}^2_n$, but one can also get higher $\mathcal{L}^q_n$-regularity. ∎

4. Limiting equation and main theorem. In this section we derive formally the limiting equation for (1.1) and we state the main theorem of this paper, which we will prove later. First, recall that we fixed a $p \geq 2$ after Assumption 2.4. Let us define the mild solution of (1.1) according to [9, 10] as follows.

DEFINITION 4.1. *For any fixed* $\varepsilon > 0$*, we call an* $\mathcal{L}^p_n$*-valued stochastic process* u *a mild solution of* (1.1) *in* $\mathcal{L}^p_n$ *if for all* $t > 0$ *up to a positive stopping time*

$$u(t) = e^{\varepsilon^{\ 2}t\mathcal{A}}u(0) + \int_0^t e^{\varepsilon^{\ 2}(t-s)\mathcal{A}}\mathcal{F}(u(s))ds + \mathcal{Z}(t). \tag{4.1}$$

Because we are working with a locally Lipschitz nonlinearity, under Assumption 2.3, the existence and uniqueness of solutions is standard, once $\mathcal{Z}$ is sufficiently regular. See, e.g., [8] and [10]. Moreover, solutions are either global or blow-up in finite time.

We can rewrite (4.1) by Equation (3.2) as

$$u(t) = e^{\varepsilon^{\ 2} t\mathcal{A}} u(0) + \int_0^t e^{\varepsilon^{\ 2}(t-s)\mathcal{A}} \mathcal{F}(u(s))ds + \sigma_\varepsilon \sum_{i=1}^{n} \sum_{j=0}^{\infty} \int_0^t e^{-\varepsilon^{\ 2} d_i (t-s)\lambda_j}\, d\tilde{W}_{i,j}(s) g_j \cdot e_i \tag{4.2}$$

with $\tilde{W}_{i,j}$ defined in Assumption 2.5.

Now, let us discuss our two cases depending on σ_ε and $\alpha_{i,0}$ for $i = 1, \dots, n$.

4.1. First case: $\boldsymbol{\sigma_\varepsilon = \varepsilon^{\ 1}}$ and $\boldsymbol{\alpha_{i,0} = 0}$ for $\boldsymbol{i = 1, \dots, n}$. In this case (4.2) takes the form

$$u(t) = e^{\varepsilon^{\ 2} t\mathcal{A}} u(0) + \int_0^t e^{\varepsilon^{\ 2}(t-s)\mathcal{A}} \mathcal{F}(u(s))ds + \mathcal{Z}^s(t), \tag{4.3}$$

where

$$\mathcal{Z}^s(t) = \sum_{i=1}^{n} \mathcal{Z}_i(t) e_i := \sum_{i=1}^{n} \sum_{j=1}^{\infty} \mathcal{Z}_{i,j}(t) g_j \cdot e_i \tag{4.4}$$

is an $\mathcal{S}$-valued process with Fourier series

$$\mathcal{Z}_i(t) = \sum_{j=1}^{\infty} \mathcal{Z}_{i,j}(t) g_j \quad \text{for } i = 1, 2 \dots, n \tag{4.5}$$

and Fourier coefficients

$$\mathcal{Z}_{i,j}(t) = \varepsilon^{-1} \int_0^t e^{-\varepsilon^{\ 2} d_i (t-s)\lambda_j}\, d\tilde{W}_{i,j}(s). \tag{4.6}$$

In order to derive the limiting equation, we split the solution u into

$$u(t,x) = a(t) + \psi(t,x) \tag{4.7}$$

with $a \in \mathcal{N}$ and $\psi \in \mathcal{S}$. Plugging (4.7) into (4.3) and projecting everything onto $\mathcal{N}$ and $\mathcal{S}$ we obtain (with $\mathcal{F}^c = P_c\mathcal{F}$ and $\mathcal{F}^s = P_s\mathcal{F}$)

$$a(t) = a(0) + \int_0^t \mathcal{F}^c(a + \psi)ds \tag{4.8}$$

and

$$\psi(t) = e^{\varepsilon^{\ 2} t\mathcal{A}} \psi(0) + \int_0^t e^{\varepsilon^{\ 2}(t-\tau)\mathcal{A}} \mathcal{F}^s(a + \psi)d\tau + \mathcal{Z}^s(t). \tag{4.9}$$

Formally, we see later (cf. Lemma 5.1) that ψ is well approximated by the fast OU-process $\mathcal{Z}^s$. Thus, we can eliminate ψ in (4.8) by explicitly averaging over the fast modes. This yields the first main result of this paper.

THEOREM 4.2 (Approximation I). *Under Assumptions* 2.2, 2.5, 2.4, 2.6, *and* 2.8, *let* u *be a solution of* (1.1) *with splitting* $u = a + \psi$ *defined in* (4.7) *with the initial condition* $u(0) = a(0) + \psi(0)$ *with* $a(0) \in \mathcal{N}$ *and* $\psi(0) \in S$, *where* $a(0)$ *and* $\psi(0)$ *are of order one, and* b *is a solution of* (1.3) *with* $b(0) = a(0)$. *Then for all* $q \geq 1$ *and all* $\kappa \in (0, \frac{1}{2m+1})$, *there exists a constant* $C > 0$ *such that*

$$\mathbb{P}\left(\sup_{t\in[0,T_1\wedge\tau\ ]} \left\| u(t) - b(t) - \mathcal{Q}(t) \right\|_{\mathcal{L}_n^p} > \varepsilon^{1-2m\kappa-\kappa} \right) \leq C\varepsilon^q, \tag{4.10}$$

where with fast OU-process $\mathcal{Z}^s$ *defined in* (4.4)

$$\mathcal{Q}(t) = e^{\varepsilon^{\ -2}t\mathcal{A}}\psi(0) + \mathcal{Z}^s(t). \tag{4.11}$$

We see that the first part of (4.11) depending on the initial condition decays exponentially fast on the time-scale of order $\mathcal{O}(\varepsilon^2)$ and plays no significant role.

COROLLARY 4.3. *If in the previous theorem we additionally assume that Assumption* 2.3 *holds and* $\|\psi(0)\|_{\mathcal{L}_n^{mp}} \leq C$ *for some* $C > 0$, *then we can replace* $T_1 \wedge \tau^*$ *in* (4.10) *by* T_1.

Remark 4.4. The advantage of Assumption 2.3 is now that we can directly bound the nonlinearity in $\mathcal{L}_n^p$. This is essential, when we want to show that the solution is not too large in $\mathcal{L}_n^{pm}$ up to the stopping time τ^* that bounds the solution in $\mathcal{L}_n^{mp}$. Let us also remark that we can bound the error in that case even in the slightly stronger norm in $\mathcal{L}_n^{pm}$.

4.2. Second case $\boldsymbol{\sigma_\varepsilon = 1}$. This case was already treated in a slightly different [6] or special [17] setting. The mild formulation (4.2) takes the form

$$u(t) = e^{\varepsilon^{\ -2}t\mathcal{A}}u(0) + \int_0^t e^{\varepsilon^{\ -2}(t-s)\mathcal{A}}\mathcal{F}(u(s))ds + \sum_{i=1}^n \sum_{j=0}^\infty \int_0^t e^{-\varepsilon^{\ -2}d_i\,(t-s)\lambda_j}\, d\tilde{W}_{i,j}(s) g_j \cdot e_i. \tag{4.12}$$

Again (cf. (4.7)) we split the solution u into $u(t,x) = a(t) + \varepsilon\psi(t,x)$. But in this scaling, we expect the part in $\mathcal{S}$ to be much smaller, as the strength of the noise is smaller. Plugging our ansatz into (4.12) and projecting onto $\mathcal{N}$ and $\mathcal{S}$ yields

$$a(t) = a(0) + \int_0^t \mathcal{F}^c(a(s) + \varepsilon\psi(s))ds + \sum_{i=1}^n \tilde{W}_{i,0}(t) g_0 \cdot e_i \tag{4.13}$$

and

$$\psi(t) = e^{\varepsilon^{\ -2}t\mathcal{A}}\psi(0) + \frac{1}{\varepsilon}\int_0^t e^{\varepsilon^{\ -2}(t-\tau)\mathcal{A}}\mathcal{F}^s(a + \varepsilon\psi)d\tau + \mathcal{Z}^s(t), \tag{4.14}$$

where $\mathcal{Z}^s(t)$ was defined in (4.4). We write (4.13) as

$$a_i(t) = a_i(0) + \int_0^t \mathcal{F}_i^c(a + \varepsilon\psi)ds + \tilde{W}_{i,0}(t)g_0 \quad \text{for } i = 1, 2 \ldots, n.$$

Now, applying Taylor's expansion to the function $\mathcal{F}_i^c : \mathcal{L}^2(G) \to \mathbb{R}$ yields the following stochastic limiting equation with error:

$$a_i(t) = a_i(0) + \int_0^t \mathcal{F}_i(a)ds + \tilde{W}_{i,0}(t)g_0 + R_i^{(2)}(t), \tag{4.15}$$

where

$$R_i^{(2)}(t) = \sum_{|\ell|\geq 1} P_c \int_0^t \frac{D^\ell \mathcal{F}_i(a)}{\ell!}(\varepsilon\psi)^\ell d\tau = \mathcal{O}(\varepsilon^{1-}). \tag{4.16}$$

The second main result of this paper is the following.

THEOREM 4.5 (Approximation II). *Under Assumptions* 2.2, 2.5, 2.4, *and* 2.9, *let* u *be a solution of* (1.1) *with splitting* $u = a + \varepsilon\psi$ *defined in* (4.7) *with the initial condition* $u(0) = a(0)+\varepsilon\psi(0)$ *with* $a(0) \in \mathcal{N}$ *and* $\psi(0) \in S$, *where* $a(0)$ *and* $\psi(0)$ *are of order one, and* b *is a solution of* (1.5) *with* $b(0) = a(0)$. *Then for* δ *from Assumption* 2.9 *and all* $\kappa \in (0, \frac{1}{m+2})$, *there exists* $C > 0$ *such that*

$$\mathbb{P}\left(\sup_{t\in[0,T_1\wedge\tau\ ]} \|u(t) - b(t)\|_{\mathcal{L}_n^p} > \varepsilon^{1-(m+2)\kappa} \right) \leq C\varepsilon^{\delta\kappa}. \tag{4.17}$$

In our examples if we assume $\mathbb{E}\exp\{c\delta|b(0)|^{m-1}\} \leq C$ for some suitable $c > 0$ and for one $\delta > 0$, then Assumption 2.9 is true. See section 8.1.

COROLLARY 4.6. *If in the previous theorem additionally Assumption* 2.3 *holds and* $\|\psi(0)\|_{\mathcal{L}_n^{mp}} \leq C$ *for some* $C > 0$ *independently of* ε, *then we can replace* $T_1 \wedge \tau^*$ *in* (4.17) *by* T_1.

5. Bounds for the high modes. Let us summarize (4.9) and (4.14) for $\rho \in \{0, 1\}$ by

$$\psi(t) = e^{\varepsilon^{\ -2}t\mathcal{A}}\psi(0) + \varepsilon^{-\rho}\int_0^t e^{\varepsilon^{\ -2}(t-\tau)\mathcal{A}}\mathcal{F}^s(a + \varepsilon^\rho\psi)d\tau + \mathcal{Z}^s(t). \tag{5.1}$$

In the first lemma of this section, we see that ψ is in both cases well approximated by the fast OU-process $\mathcal{Z}^s$ (cf. (4.4)).

LEMMA 5.1. *Under Assumptions* 2.4 *and* 2.5 *for* $\kappa > 0$ *from the definition of* τ^* *there is a constant* $C > 0$ *such*

$$\sup_{t\in[0,\tau\ ]} \left\| \psi(t) - e^{\varepsilon^{\ -2}t\mathcal{A}}\psi(0) - \mathcal{Z}^s(t) \right\|_{\mathcal{L}_n^p} \leq C\varepsilon^{2-\rho-m\kappa}. \tag{5.2}$$

Proof. From (5.1) using semigroup estimates and Assumption 2.4 we obtain

$$\begin{aligned}
\left\| \psi(t) - e^{\varepsilon^{\ -2}t\mathcal{A}}\psi(0) - \mathcal{Z}^s(t) \right\|_{\mathcal{L}_n^p} &= \frac{1}{\varepsilon^\rho}\left\| \int_0^t e^{\varepsilon^{\ -2}\mathcal{A}(T-\tau)}\mathcal{F}^s(u)d\tau \right\|_{\mathcal{L}_n^p} \\
&\leq C\varepsilon^{-\rho} \sup_{\tau\in[0,\tau\ ]} \|\mathcal{F}^s(u)\|_{\mathcal{L}_n^p} \int_0^t e^{-\varepsilon^{\ -2}\omega(t-\tau)}d\tau \\
&\leq C\varepsilon^{2-\rho} \sup_{\tau\in[0,\tau\ ]} (1 + \|u\|_{\mathcal{L}_n^{pm}}^m) \leq C\varepsilon^{2-\rho-m\kappa}.
\end{aligned}$$

The following method of proof based on the factorization method is fairly standard for showing that the stochastic convolution is of order almost 1 uniformly in time. For completeness to see why the argument fails in $\mathcal{L}^\infty$ and where the assumptions (stated in Assumption 2.5) on the correlation do appear, we give a short proof.

LEMMA 5.2. *Under Assumptions* 2.2 *and* 2.5, *for every* $\kappa_0 > 0$ *and* $q \geq 1$ *there is a constant* C, *depending on* q, α_k, λ_k, κ_0, *and* T_0, *such that*

$$\mathbb{E} \sup_{t\in[0,T_0]} \|\mathcal{Z}^s(t)\|^q_{\mathcal{L}^q_n} \leq C\varepsilon^{-\kappa_0}, \tag{5.3}$$

where $\mathcal{Z}^s(t)$ *was defined in* (4.4).

Proof. We use the celebrated factorization method introduced in [8] to prove the bound on $\mathcal{Z}^s(t) = \sum_{i=1}^n \mathcal{Z}_i(t)e_i$. Fix $\gamma \in (0, \frac{1}{2})$. To prove (5.3), it is enough to bound $\mathcal{Z}_i$ for $i = 1, \dots n$. We know from (4.5) that

$$\mathcal{Z}_i(t) = \sum_{j=1}^{\infty} \varepsilon^{-1} \int_0^t e^{-\varepsilon^{\ 2} d_i (t-s)\lambda_j}\, d\tilde{W}_{i,j}(s) g_j = \varepsilon^{-1} \int_0^t e^{\varepsilon^{\ 2}(t-s)\mathcal{A}_i}\, d\tilde{W}_i(s), \tag{5.4}$$

where $\tilde{W}_i(t) = \sum_{j=1}^{\infty} \tilde{W}_{i,j}(s)g_j$ for $i = 1, 2, \dots, n$. Based on a Bessel function and the stochastic Fubini theorem, the standard approach of factorization is the following representation:

$$\mathcal{Z}_i(t) = C_\gamma \varepsilon^{-1} \int_0^t e^{\varepsilon^{\ 2}(t-s)\mathcal{A}_i} (t-s)^{\gamma-1} y_i(s) ds, \tag{5.5}$$

where

$$y_i(s) = \int_0^s e^{\varepsilon^{\ 2}(s-\sigma)\mathcal{A}_i} (s-\sigma)^{-\gamma} d\tilde{W}_i(\sigma) = \sum_{j=1}^{\infty} \int_0^s e^{-\varepsilon^{\ 2} d_i (s-\sigma)\lambda_j} (s-\sigma)^{-\gamma} d\tilde{W}_{i,j}(\sigma) g_j. \tag{5.6}$$

Taking $\|\cdot\|^q_{\mathcal{L}^q_n}$ on both sides of (5.5) and using (2.2), we obtain

$$\|\mathcal{Z}_i(t)\|^q_{\mathcal{L}^q} \leq C^q_\gamma \varepsilon^{-q} \left(\int_0^t e^{-\varepsilon^{\ 2}(t-s)\omega} (t-s)^{\gamma-1} \|y_i(s)\|_{\mathcal{L}^q} ds \right)^q.$$

Using the Hölder inequality with $\frac{1}{q} + \frac{1}{r} = 1$ for sufficiently large q implies

$$\begin{aligned} \|\mathcal{Z}_i(t)\|^q_{\mathcal{L}^q} &\leq C^q_\gamma \varepsilon^{-q} \left(\int_0^t e^{-\varepsilon^{\ 2}(t-s)\omega} (t-s)^{r\gamma-r} ds \right)^{\frac{q}{r}} \cdot \int_0^t \|y_i(s)\|^q_{\mathcal{L}^q} ds \\ &\leq C\varepsilon^{-2+2q(\gamma-\frac{1}{2})} \int_0^t \|y_i(s)\|^q_{\mathcal{L}^q} ds. \end{aligned}$$

Taking supremum after expectation yields

$$\mathbb{E} \sup_{t\in[0,T_0]} \|\mathcal{Z}_i(t)\|^q_{\mathcal{L}^q} \leq C\varepsilon^{-2+2q(\gamma-\frac{1}{2})} \cdot \int_0^{T_0} \mathbb{E}\, \|y_i(s)\|^q_{\mathcal{L}^q} ds. \tag{5.7}$$

Now, we bound $\mathbb{E}\, \|y_i(s)\|^q_{\mathcal{L}^q}$. By Gaussianity

$$\mathbb{E}\, \|y_i(s)\|^q_{\mathcal{L}^q} = \mathbb{E} \int_G |y_i(s,x)|^q dx \leq C_q \left(\int_G \mathbb{E}|y_i(s,x)|^2 \right)^{\frac{q}{2}} dx.$$

Hence by the definition of y_i (5.6)

$$\begin{aligned} \mathbb{E}\, |y_i(s,x)|^2 &= \mathbb{E} \left| \sum_{j=1}^{\infty} \int_0^s e^{-\varepsilon^{\ 2} d_i (s-\sigma)\lambda_j} (s-\sigma)^{-\gamma} d\tilde{W}_{i,j}(\sigma) g_j(x) \right|^2 \\ &= C \sum_{j,k=1}^{\infty} q^{i,i}_{j,k} \int_0^s e^{-\varepsilon^{\ 2} d_i (s-\sigma)(\lambda_j+\lambda_k)} (s-\sigma)^{-2\gamma} d\sigma g_j(x) g_k(x), \end{aligned}$$

where we used the definition of the covariance operator (2.9). Hence, using the bounds on g_j

$$\mathbb{E}\,|y_i(s)|^2 \le C\varepsilon^{2-4\gamma} \sum_{j,k=1}^{\infty} (\lambda_j+\lambda_k)^{2\gamma+2\gamma_1-1} q_{j,k}^{i,i} \le C\varepsilon^{2-4\gamma}, \tag{5.8}$$

where we used (2.8). Thus

$$\sup_{t\in[0,T_0]} \mathbb{E}\,\|y_i(s)\|_{\mathcal{L}^q}^q \le C\varepsilon^{q-2q\gamma}\ . \tag{5.9}$$

Now, returning to (5.7) and using (5.9) yields

$$\mathbb{E} \sup_{t\in[0,T_0]} \|\mathcal{Z}_i(t)\|_{\mathcal{L}_n^q}^q \le C\varepsilon^{-2}.$$

We finish the proof by using the Hölder inequality to derive

$$\mathbb{E} \sup_{t\in[0,T_0]} \|\mathcal{Z}_i(t)\|_{\mathcal{L}_n^q}^q \le \left(\mathbb{E} \sup_{t\in[0,T_0]} \|\mathcal{Z}_i(t)\|_{\mathcal{L}_n^q}^{rq}\right)^{\frac{1}{r}} \le C\left(\mathbb{E} \sup_{t\in[0,T_0]} \|\mathcal{Z}_i(t)\|_{\mathcal{L}_n^{rq}}^{rq}\right)^{\frac{1}{r}} \le C\varepsilon^{-\kappa_0}$$

for all $q>1$ and sufficiently large $r>\frac{2}{\kappa_0}$. ∎

The following corollary states that $\psi(t)$ is with high probability much smaller than $\varepsilon^{-\kappa}$ as assumed in Definition 2.10 for $t\le\tau^*$. Note that this is not sufficient to show $\tau^*\ge T_0$ with high probability.

COROLLARY 5.3. *Under the assumptions of Lemmas* 5.1 *and* 5.2*, if* $\psi(0)=\mathcal{O}(1)$*, then for* $p\ge 1$ *and* $\rho=0$ *or* 1 *there exist a constant* $C>0$ *such that for some* $\kappa_0<\kappa$

$$\mathbb{E} \sup_{t\in[0,\tau\ ]} \|\psi(t)\|_{\mathcal{L}_n^p}^p \le C\varepsilon^{-\kappa_0}\ . \tag{5.10}$$

Proof. By the triangle inequality and Lemma 5.2, we obtain from (5.2)

$$\mathbb{E} \sup_{t\in[0,\tau\ ]} \|\psi(t)\|_{\mathcal{L}_n^p}^p \le C + C\varepsilon^{2p-p\rho-mp\kappa} + C\varepsilon^{-\kappa_0}\ ,$$

which implies (5.10) for $\kappa<\frac{2-\rho}{m}$. ∎

Let us now state a result similar to averaging. When we integrate over the fast decaying contribution of the initial condition in ψ, then this leads to terms of order $\mathcal{O}(\varepsilon^2)$.

LEMMA 5.4. *For* $q\ge 1$ *there exists a constant* $C>0$ *such that*

$$\int_0^t \left\| e^{\tau\varepsilon^{\ -2}\mathcal{A}}\psi(0)\right\|_{\mathcal{L}_n^p}^q d\tau \le C\varepsilon^2 \|\psi(0)\|_{\mathcal{L}_n^p}^q \ \text{ for } \psi(0)\in P_s\mathcal{L}_n^p.$$

Proof. Using (2.2) we obtain

$$\int_0^t \left\| e^{\varepsilon^{\ -2}\mathcal{A}\tau}\psi(0)\right\|_{\mathcal{L}_n^p}^q d\tau \le c\int_0^T e^{-q\varepsilon^{\ -2}\omega\tau} \|\psi(0)\|_{\mathcal{L}_n^p}^q\, d\tau \le \frac{\varepsilon^2}{q\omega}\|\psi(0)\|_{\mathcal{L}_n^p}^q\ .$$ ∎

6. Averaging over the fast OU-process. Here we present explicit averaging results with error bounds based on Itô's formula. We follow ideas developed in [3] and [4].

LEMMA 6.1. *Let Assumption* 2.5 *hold and consider* $\mathcal{Z}_{i,j}(t)$ *as defined in* (4.6). *Then for any small* $\delta_0 \in (0, \frac{1}{2})$ *we obtain*

$$\mathcal{Z}_{i,j}(t) = \lambda_j^{-(\frac{1}{2}-\delta_0)} \left(\mathrm{q}_{j,j}^{i,i}\right)^{\frac{1}{2}} \mathcal{O}(\varepsilon^{-\delta_0}) \tag{6.1}$$

and

$$\mathcal{Z}_{i,j}(t)\mathcal{Z}_{\ell,k}(t) = \left(\lambda_j\lambda_k\right)^{-(\frac{1}{2}-\delta_0)} \left(\mathrm{q}_{j,j}^{i,i}\mathrm{q}_{k,k}^{\ell,\ell}\right)^{\frac{1}{2}} \mathcal{O}(\varepsilon^{-2\delta_0}). \tag{6.2}$$

Moreover, the $\mathcal{O}$*-terms are uniform in* i, j, k, *and* ℓ.

Proof. For the first part, we follow the same steps as in Lemma 5.2 to obtain

$$\mathbb{E} \sup_{t\in[0,T_0]} |\mathcal{Z}_{i,j}(t)|^q \le C\varepsilon^{-2} \left(\lambda_j\right)^{q(\delta_0-\frac{1}{2})} \left(q_{j,j}^{i,i}\right)^{\frac{q}{2}}.$$

Using the Hölder inequality, we derive for sufficiently large r and for a constant independent on i and j

$$\left(\mathbb{E} \sup_{t\in[0,T_0]} |\mathcal{Z}_{i,j}(t)|^q\right)^{1/q} \le C\lambda_j^{\delta_0-\frac{1}{2}} \varepsilon^{\frac{2}{rq}} \left(q_{j,j}^{i,i}\right)^{\frac{1}{2}}.$$

We finish the proof by fixing $\delta_0 = \frac{2}{rq} < \frac{1}{2}$ for large r.

For the second part we use the Cauchy–Schwarz inequality to obtain

$$\mathbb{E} \sup_{[0,T_0]} |\mathcal{Z}_{i,j}\mathcal{Z}_{\ell,k}|^q \le \left(\mathbb{E} \sup_{[0,T_0]} |\mathcal{Z}_{i,j}|^{2q}\right)^{1/2} \left(\mathbb{E} \sup_{[0,T_0]} |\mathcal{Z}_{\ell,k}|^{2q}\right)^{1/2}.$$

Using the first part yields (6.2). ∎

In the next corollary we state without proof the general case of Lemma 6.1. For the proof we can follow the same steps as in the proof above.

COROLLARY 6.2. *Under the assumptions of Lemma* 6.1 *we have*

$$\prod_{j=1}^{N} \mathcal{Z}_{\ell_j,k_j} = \left(\prod_{j=1}^{N} \lambda_{k_j}\right)^{-(\frac{1}{2}-\delta_0)} \left(\prod_{j=1}^{N} \mathrm{q}_{k_j,k_j}^{\ell_j,\ell_j}\right)^{\frac{1}{2}} \mathcal{O}(\varepsilon^{-N\delta_0}). \tag{6.3}$$

We further generalize these bounds to stochastic integrals that we cannot average with error bounds via the Itô formula.

LEMMA 6.3. *Let the assumptions of Lemma* 6.1 *hold and let* X *be a real-valued stochastic process such that for some small* $r \ge 0$ *we have* $X(0) = \mathcal{O}(\varepsilon^{-r})$. *If* $dX = GdT$ *with* $G = \mathcal{O}(\varepsilon^{-r})$, *then*

$$\sup_{t\ge 0} \mathbb{E}|\mathcal{Z}_{i,j}(t)|^2 \le \frac{\mathrm{q}_{j,j}^{i,i}}{2d_i\lambda_j}, \tag{6.4}$$

$$\int_0^t X\mathcal{Z}_{i,j}d\tilde{W}_{k,m} = \left(\frac{\mathrm{q}_{m,m}^{k,k}\mathrm{q}_{j,j}^{i,i}}{\lambda_j}\right)^{\frac{1}{2}} \mathcal{O}(\varepsilon^{-r}), \tag{6.5}$$

and

$$\int_0^t X \prod_{\substack{j=1,\\ j\neq i}}^{N} \mathcal{Z}_{\ell_j,k_j}\, d\tilde{W}_{\ell_i,k_i} = \left(\prod_{\substack{j=1,\\ j\neq i}}^{N} \lambda_{k_j}\right)^{-\frac{1}{2}} \left(\prod_{j=1}^{N} \mathrm{q}_{k_j}^{\ell_j,\ell_j}\right)^{\frac{1}{2}} \mathcal{O}(\varepsilon^{-r}). \tag{6.6}$$

Again all $\mathcal{O}$-terms are uniform in the indices ℓ_j and k_j.

Proof. For the first part, we use Itô isometry to obtain

$$\mathbb{E}\,|\mathcal{Z}_{i,j}|^2 = \frac{1}{\varepsilon^2}\mathbb{E}\Big|\int_0^t e^{-\varepsilon^{\ 2}d_i\,(t-s)\lambda_j}\, d\tilde{W}_{i,j}\Big|^2 = \frac{q_{j,j}^{i,i}}{\varepsilon^2}\int_0^t e^{-2\varepsilon^{\ 2}d_i\,(t-s)\lambda_j}\, ds \le \frac{q_{j,j}^{i,i}}{2d_i\lambda_j}.$$

For the second part, using the Burkholder–Davis–Gundy theorem for the scalar stochastic integral and Hölder inequality yields

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T_0]}\left|\int_0^t X\mathcal{Z}_{i,j}d\tilde{W}_{k,m}\right|^q &\le C_q\left(q_{m,m}^{k,k}\right)^{\frac{q}{2}}\mathbb{E}\left(\int_0^{T_0}|X|^2\left|\mathcal{Z}_{i,j}\right|^2 d\sigma\right)^{\frac{q}{2}}\\
&\le C_q\left(q_{m,m}^{k,k}\right)^{\frac{q}{2}}\mathbb{E}\left(\int_0^{T_0}|X|^\eta\, d\sigma\right)^{\frac{q}{\eta}}\cdot\mathbb{E}\int_0^{T_0}\left|\mathcal{Z}_{i,j}\right|^q d\sigma\\
&\le C_{q,T_0}\varepsilon^{-qr}\left(q_{m,m}^{k,k}\right)^{\frac{q}{2}}\int_0^{T_0}\mathbb{E}|\mathcal{Z}_{i,j}|^q d\sigma.
\end{aligned}$$

By Gaussianity and the first part we obtain

$$\mathbb{E}\sup_{t\in[0,T_0]}\Big|\int_0^t X\mathcal{Z}_{i,j}d\tilde{W}_{k,m}\Big|^q \le C_{q,T_0}\varepsilon^{-qr}\left(\frac{q_{m,m}^{k,k}q_{j,j}^{i,i}}{\lambda_j}\right)^{\frac{q}{2}}.$$

Analogously, for the last term

$$\mathbb{E}\sup_{t\in[0,T_0]}\left|\int_0^t X\prod_{\substack{j=1,\\ j\neq i}}^{N}\mathcal{Z}_{\ell_j,k_j}\, d\tilde{W}_{\ell_i,k_i}\right|^q \le C_{q,T_0}\left(q_{k_i,k_i}^{\ell_i,\ell_i}\right)^{\frac{q}{2}}\mathbb{E}\int_0^{T_0}|X|^q\prod_{\substack{j=1,\\ j\neq i}}^{N}\left|\mathcal{Z}_{\ell_j,k_j}\right|^q d\sigma.$$

Using Hölder, Gaussianity, and the first part, we obtain (6.6). ∎

In the following we state and prove the averaging lemma over the fast OU-process $\mathcal{Z}_{i,j}$ (cf. (4.6)).

LEMMA 6.4. *Under Assumptions* 2.2, 2.5, *and* 2.6, *let X be as in Lemma* 6.3 *and $N \le m$. Then for N odd*

$$\int_0^t X\prod_{i=1}^{N}\mathcal{Z}_{\ell_i,k_i}\, ds = A_{k_1,\cdots,k_N}^{\ell_1,\cdots\ell_N}\,\mathcal{O}(\varepsilon^{1-r}), \tag{6.7}$$

and for N even

$$\begin{aligned}
\int_0^t X\prod_{i=1}^{N}\mathcal{Z}_{\ell_i,k_i}\, ds = \frac{1}{2^{\frac{N}{2}}}\sum_{j\in Per(N)}\prod_{\eta=1}^{N/2}\frac{\mathrm{q}_{k_{j_2}\ \ 1,k_{j_2}}^{\ell_{j_2}\ \ 1,\ell_{j_2}}}{d_{\ell_{j_2}\ \ 1}\,\lambda_{k_{j_2}\ \ 1}+d_{\ell_{j_2}\ 2}\,\lambda_{k_{j_2}\ 2}}\int_0^t X ds\\
+A_{k_1,\cdots,k_N}^{\ell_1,\cdots\ell_N}\,\mathcal{O}(\varepsilon^{1-r}),
\end{aligned} \tag{6.8}$$

where $j \in Per(N)$ if $j = (j_1, \dots, j_N)$ is a permutation of $\{1, \dots, N\}$. The $\mathcal{O}$-terms are again uniform in all indices and we have the summability

$$\sum_{k_1=1}^{\infty} \cdots \sum_{k_N=1}^{\infty} A^{\ell_1,\cdots,\ell_N}_{k_1,\cdots,k_N} \prod_{i=1}^{N} \lambda_{k_i}^{\gamma_1} < \infty. \tag{6.9}$$

Note that in (6.9) we have a weighted summability of the $A^{\ell_1,\cdots,\ell_N}_{k_1,\cdots,k_N}$, which is needed to include the g_k later.

Remark 6.5. The term

$$\sum_{j \in Per(N)} \prod_{\eta=1}^{N/2} \frac{q^{\ell_{j_2}\ \ _1,\ell_{j_2}}_{k_{j_2}\ \ _1,k_{j_2}}}{d_{\ell_{j_2}\ \ _1}\lambda_{k_{j_2}\ \ _1} + d_{\ell_{j_2}\ \ _1}\lambda_{k_{j_2}\ \ _1}}$$

is summable over $k_1, \dots, k_N$ by condition (2.8).

Example 6.6. Let us state explicitly some A's appearing in the proof of the theorem. For $N = 1$ we have $A^{\ell}_{k} = \frac{1}{\lambda_k}(q^{\ell,\ell}_{k,k})^{\frac{1}{2}}$,

$$\text{for } N = 2 \qquad A^{\ell_1,\ell_2}_{k_1,k_2} = \left(\sum_{i=1}^{2} d_{\ell_i}\lambda_{k_i}\right)^{-1/2} \left(\prod_{i=1}^{2} \lambda_{k_i}^{-1} q^{\ell_i,\ell_i}_{k_i,k_i}\right)^{1/2},$$

and for $N = 3$

$$A^{\ell_1,\ell_2,\ell_3}_{k_1,k_2,k_3} = \left(\sum_{i=1}^{3} d_{\ell_i}\lambda_{k_i}\right)^{-\frac{1}{2}} \left(\prod_{i=1}^{3} \lambda_{k_i}^{-1} q^{\ell_i,\ell_i}_{k_i,k_i}\right)^{\frac{1}{2}} + \sum_{\substack{j_1,j_2=1 \\ j_1 \neq j_2}}^{3} \frac{q^{\ell_{j_1},\ell_{j_2}}_{k_{j_1},k_{j_2}}}{d_{\ell_{j_1}}\lambda_{k_{j_1}} + d_{\ell_{j_2}}\lambda_{k_{j_2}}} \frac{\left(q^{i,i}_{j,j}\right)^{\frac{1}{2}}}{\lambda_j}.$$

For larger N the terms have similar structure, but there are about $N/2$ many terms.

Proof. Fix a small $\delta_0 < \frac{1}{N}$ for $N > 1$. First, recall $|X| = \mathcal{O}(\varepsilon^{-r})$. For the first part we treat $N = 1$ and 3. The general case follows by induction.

For $N = 1$ we apply the Itô formula to the term $X\mathcal{Z}_{i,j}$ to obtain

$$\int_0^t X\mathcal{Z}_{i,j}ds = -\frac{\varepsilon^2}{d_i\lambda_j}X(t)\mathcal{Z}_{i,j}(t) + \frac{\varepsilon^2}{d_i\lambda_j}\int_0^t G\mathcal{Z}_{i,j}ds + \frac{\varepsilon}{d_i\lambda_j}\int_0^t X d\tilde{W}_{i,j}.$$

Using Lemmas 6.1 and the Burkholder–Davis–Gundy theorem yields

$$\begin{aligned}\int_0^t X\mathcal{Z}_{i,j}ds &= \left(q^{i,i}_{j,j}\right)^{\frac{1}{2}} \left[\frac{1}{(d_i\lambda_j)\,\lambda_j^{\frac{1}{2}-\frac{1}{2}\delta_0}}\mathcal{O}(\varepsilon^{2-r-\delta_0}) + \frac{1}{d_i\lambda_j}\mathcal{O}(\varepsilon^{1-r})\right] \\ &= \frac{1}{\lambda_j}\left(q^{i,i}_{j,j}\right)^{\frac{1}{2}}\mathcal{O}(\varepsilon^{1-r}). \end{aligned} \tag{6.10}$$

For $N \in \{3,5,\dots\}$ we apply the Itô formula to the term $X \prod_{i=1}^{N} \mathcal{Z}_{\ell_i,k_i}$ to obtain

$$\int_0^t X \prod_{i=1}^{N} \mathcal{Z}_{\ell_i,k_i}\,ds = \frac{1}{\sum_{i=1}^{N} d_{\ell_i}\lambda_{k_i}} \Bigg\{ \varepsilon^2 X \prod_{i=1}^{N} \mathcal{Z}_{\ell_i,k_i} + \varepsilon^2 \int_0^t G \prod_{i=1}^{N} \mathcal{Z}_{\ell_i,k_i}\,ds + \varepsilon \sum_{j=1}^{N} \int_0^t X \prod_{\substack{i=1,\\ i\neq j}}^{N} \mathcal{Z}_{\ell_i,k_i}\,d\tilde{W}_{\ell_j,k_j} + \sum_{j_1\neq j_2=1}^{N} \int_0^t X \prod_{\substack{i=1,\\ i\notin\{j_1;j_2\}}}^{N} \mathcal{Z}_{\ell_i,k_i}\,d\tilde{W}_{\ell_{j_1},k_{j_1}}\,d\tilde{W}_{\ell_{j_2},k_{j_2}} \Bigg\}.$$

Using Corollary 6.2 and Lemma 6.3 we obtain

$$\int_0^t X \prod_{i=1}^{N} \mathcal{Z}_{\ell_i,k_i}\,ds = \frac{\left(\prod_{i=1}^{N} \lambda_{k_i}^{-1} q_{k_i,k_i}^{\ell_i,\ell_i}\right)^{\frac{1}{2}}}{\sum_{i=1}^{N} d_{\ell_i}\lambda_{k_i}} \left\{ \prod_{i=1}^{N} \lambda_{k_i}^{\frac{1}{2}\delta_0} \mathcal{O}(\varepsilon^{2-r-N\delta_0}) + \sum_{i=1}^{N} \lambda_{k_i}^{\frac{1}{2}} \mathcal{O}(\varepsilon^{1-r}) \right\} + \frac{1}{\sum_{i=1}^{N} d_{\ell_i}\lambda_{k_i}} \sum_{\substack{j_1,j_2=1,\\ j_1\neq j_2}}^{N} q_{k_{j_1},k_{j_2}}^{\ell_{j_1},\ell_{j_2}} \int_0^t X \prod_{\substack{i=1,\\ i\notin\{j_1;j_2\}}}^{N} \mathcal{Z}_{\ell_i,k_i}\,ds.$$

We now use $\sum_{i=1}^{N} d_{\ell_i}\lambda_{k_i} \geq c \prod_{i=1}^{N} \lambda_{k_i}^{1/N}$ with $c = \prod_{i=1}^{N} d_{\ell_j i}^{1/N}$ and the equivalence of norms in $\mathbb{R}^N$ which implies for $C_1, C_2 > 0$

$$C_1 \left(\sum_{i=1}^{N} \lambda_{k_i}\right)^{\frac{1}{2}} \leq \sum_{i=1}^{N} \lambda_{k_i}^{\frac{1}{2}} \leq C_2 \left(\sum_{i=1}^{N} \lambda_{k_i}\right)^{\frac{1}{2}}. \tag{6.11}$$

Hence,

$$\int_0^t X \prod_{i=1}^{N} \mathcal{Z}_{\ell_i,k_i}\,ds = \left(\frac{1}{\sum_{i=1}^{N} d_{\ell_i}\lambda_{k_i}} \prod_{i=1}^{N} \lambda_{k_i}^{-1} q_{k_i,k_i}^{\ell_i,\ell_i} \right)^{\frac{1}{2}} \mathcal{O}(\varepsilon^{1-r}) + \frac{1}{\sum_{i=1}^{N} d_{\ell_i}\lambda_{k_i}} \sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}}^{N} q_{k_{j_1},k_{j_2}}^{\ell_{j_1},\ell_{j_2}} \int_0^t X \prod_{\substack{i=1,\\ i\notin\{j_1;j_2\}}}^{N} \mathcal{Z}_{\ell_i,k_i}\,ds. \tag{6.12}$$

In the case $N = 3$, for example, (6.12) takes the form

$$\int_0^t X \prod_{i=1}^{3} \mathcal{Z}_{\ell_i,k_i}\,ds = \left(\frac{1}{\sum_{i=1}^{3} d_{\ell_i}\lambda_{k_i}} \prod_{i=1}^{3} \lambda_{k_i}^{-1} q_{k_i,k_i}^{\ell_i,\ell_i} \right)^{\frac{1}{2}} \mathcal{O}(\varepsilon^{1-r}) + \sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}}^{3} \frac{q_{k_{j_1},k_{j_2}}^{\ell_{j_1},\ell_{j_2}}}{d_{\ell_{j_1}}\lambda_{k_{j_1}} + d_{\ell_{j_2}}\lambda_{k_{j_2}}} \frac{\left(q_{j,j}^{i,i}\right)^{\frac{1}{2}}}{\lambda_j} \mathcal{O}(\varepsilon^{1-r}),$$

where we used (6.10) and $d_{\ell_{j_1}}\lambda_{k_{j_1}}+d_{\ell_{j_2}}\lambda_{k_{j_2}}\le\sum_{i=1}^{3}d_{\ell_i}\lambda_{k_i}$ for $j_1,j_2\in\{1,\ 2,\ 3\}$. The general case for $N\in\{5,7,\dots\}$ follows similarly.

We prove the second part only for $N=2$ and we can proceed by induction. Applying the Itô formula to $X\cdot\prod_{i=1}^{2}\mathcal{Z}_{\ell_i,k_i}$ and integrating from 0 to t, we obtain

$$\int_0^t X\prod_{i=1}^{2}\mathcal{Z}_{\ell_i,k_i}\,ds=\frac{1}{\sum\limits_{i=1}^{2}d_{\ell_i}\lambda_{k_i}}\left\{-\varepsilon^2X(t)\prod_{i=1}^{2}\mathcal{Z}_{\ell_i,k_i}(t)+\varepsilon^2\int_0^t G\prod_{i=1}^{2}\mathcal{Z}_{\ell_i,k_i}\,ds\right.$$
$$\left.+\ \varepsilon\sum_{j=1}^{2}\int_0^t X\prod_{\substack{i=1\\ i\neq j}}^{2}\mathcal{Z}_{\ell_i,k_i}\,d\tilde{W}_{\ell_j,k_j}+\int_0^t X d\tilde{W}_{\ell_{j_1},k_{j_1}}d\tilde{W}_{\ell_{j_2},k_{j_2}}\right\}.$$

Using Corollary 6.2 and (6.6) from Lemma 6.3 we obtain

$$\int_0^t X\prod_{i=1}^{2}\mathcal{Z}_{\ell_i,k_i}\,ds=\frac{1}{\sum\limits_{i=1}^{2}d_{\ell_i}\lambda_{k_i}}\left(\prod_{i=1}^{2}\lambda_{k_i}^{-1}q_{k_i,k_i}^{\ell_i,\ell_i}\right)^{\frac12}\left\{\mathcal{O}(\varepsilon^{1-r-2\delta_0})\prod_{i=1}^{2}\lambda_{k_i}^{\frac12\delta_0}\right.$$
$$\left.+\,\mathcal{O}(\varepsilon^{1-r})\left(\sum_{i=1}^{2}d_{\ell_i}\lambda_{k_i}\right)^{1/2}\right\}+\frac{q_{k_1,k_2}^{\ell_1,\ell_2}}{\sum\limits_{i=1}^{2}d_{\ell_i}\lambda_{k_i}}\int_0^t X ds.$$

Note that we get from (6.6) the factor $(\lambda_{k_1}+\lambda_{k_2})^{-1/2}\le C(d_{\ell_1}\lambda_{k_1}+d_{\ell_2}\lambda_{k_2})^{-1/2}$. Using (6.11) with $(\sum_{i=1}^{2}d_{\ell_i}\lambda_{k_i})^{\frac12}\ge c\prod_{i=1}^{2}\lambda_{k_i}^{\frac14}$ we obtain for small $\delta_0<\frac12$

$$\int_0^t X\prod_{i=1}^{2}\mathcal{Z}_{\ell_i,k_i}\,ds=\left(\frac{1}{\sum\limits_{i=1}^{2}d_{\ell_i}\lambda_{k_i}}\prod_{i=1}^{2}\lambda_{k_i}^{-1}q_{k_i,k_i}^{\ell_i,\ell_i}\right)^{\frac12}\mathcal{O}(\varepsilon^{1-r})+\frac{q_{k_1,k_2}^{\ell_1,\ell_2}}{\sum\limits_{i=1}^{2}d_{\ell_i}\lambda_{k_i}}\int_0^t X ds.$$

For $N\in\{4,6,\dots\}$ we apply Itô formula to the term $X\prod_{i=1}^{N}\mathcal{Z}_{\ell_i,k_i}$ to obtain

$$\int_0^t X\prod_{i=1}^{N}\mathcal{Z}_{\ell_i,k_i}\,ds=\frac{1}{\sum\limits_{i=1}^{N}d_{\ell_i}\lambda_{k_i}}\left\{-\varepsilon^2X(t)\prod_{i=1}^{N}\mathcal{Z}_{\ell_i,k_i}(t)+\varepsilon^2\int_0^t G\prod_{i=1}^{N}\mathcal{Z}_{\ell_i,k_i}\,ds\right.$$
$$+\varepsilon\sum_{j=1}^{N}\int_0^t X\prod_{\substack{i=1,\\ i\neq j}}^{N}\mathcal{Z}_{\ell_i,k_i}\,d\tilde{W}_{\ell_j,k_j}$$
$$\left.+\sum_{j_1\neq j_2=1}^{N}\int_0^t X\prod_{\substack{i=1,\\ i\neq j_1\neq j_2}}^{N}\mathcal{Z}_{\ell_i,k_i}\,d\tilde{W}_{\ell_{j_1},k_{j_1}}d\tilde{W}_{\ell_{j_2},k_{j_2}}\right\}.$$

We use Corollary 6.2 and Lemma 6.3 to obtain as in the odd case before

$$\int_0^t X \prod_{i=1}^N \mathcal{Z}_{\ell_i,k_i}\,ds = \left(\frac{1}{\sum_{i=1}^N d_{\ell_i}\lambda_{k_i}} \prod_{i=1}^N \lambda_{k_i}^{-1} q_{k_i,k_i}^{\ell_i,\ell_i}\right)^{\frac{1}{2}} \mathcal{O}(\varepsilon^{1-r}) + \sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}}^N \frac{q_{k_{j_1},k_{j_2}}^{\ell_{j_1},\ell_{j_2}}}{\sum_{i=1}^N d_{\ell_i}\lambda_{k_i}} \cdot \int_0^t X \prod_{\substack{i=1,\\ i\notin\{j_1,j_2\}}}^N \mathcal{Z}_{\ell_i,k_i}\,ds.$$

The first factor in the sum is summable over j_1 and j_2 by condition (2.8). Now, we can proceed by induction and apply the assertion for $N-2$ to obtain (6.8). ∎

LEMMA 6.7. *Under Assumptions* 2.2, 2.5, *and* 2.6 *let* X *be as in Lemma* 6.4. *Then, for* $\ell \in \mathbb{N}_0^n$ *with* $m \geq |\ell| \geq 1$, *we obtain the following:*

1. *If one of the* ℓ_i *is odd, then*

$$P_c \int_0^t X(\mathcal{Z}^s)^\ell d\tau = \mathcal{O}(\varepsilon^{1-r}). \tag{6.13}$$

2. *If all* ℓ_i *are even, then there is a constant* C_ℓ *such that*

$$P_c \int_0^t X(\mathcal{Z}^s)^\ell d\tau = C_\ell \int_0^t X d\tau + \mathcal{O}(\varepsilon^{1-r}), \tag{6.14}$$

where C_ℓ *is given by*

$$C_\ell = \prod_{i=1}^n \left(\frac{1}{2^{\ell_i/2} d_i^{\ell_i/2}} \sum_{k_{1,.},k_{\cdot i}=1}^\infty \sum_{j\in Per(\ell_i)} \prod_{\eta=1}^{\ell_i/2} \frac{\mathrm{q}_{k_{j_{2\ 1}},k_{j_2}}^{i,i}}{\lambda_{k_{j_{2\ 1}}} + \lambda_{k_{j_2}}} P_c \prod_{\eta=1}^{\ell_i} g_k \right). \tag{6.15}$$

Proof. From the definition of $\mathcal{Z}^s$ (cf. (4.4)), we obtain

$$(\mathcal{Z}^s)^\ell = \prod_{i=1}^n \mathcal{Z}_i^{\ell_i} = \prod_{i=1}^n \left(\sum_{j_1,\dots,j_{\cdot i}=1}^\infty \prod_{k=1}^{\ell_i} \mathcal{Z}_{i,j_k} g_{j_k} \right). \tag{6.16}$$

We focus in the proof on the cases $n=1$ and $n=2$ as they are needed for our applications. The general case follows similarly but it is technically more involved. For $n=1$ we have $\ell = \ell_1$ and

$$P_c \int_0^t X(\mathcal{Z}^s)^\ell d\tau = P_c \sum_{j_1,\dots,j_{\cdot}=1}^\infty \prod_{k=1}^\ell g_{j_k} \int_0^t X \prod_{k=1}^\ell \mathcal{Z}_{1,j_k}\,d\tau.$$

Now we consider two cases. First if $|\ell|$ is odd, then Lemma 6.4 with $N = |\ell|$ yields

$$\int_0^t X(\mathcal{Z}^s)^\ell d\tau = \sum_{k_1=1}^\infty \cdots \sum_{k_{\cdot}=1}^\infty A_{k_1,\dots,k_{\cdot}}^{1,\dots,1} \prod_{j=1}^\ell g_{k_j} \cdot \mathcal{O}(\varepsilon^{1-r}).$$

As the A's are summable with weights (see (6.9))

$$P_c \int_0^t X(\mathcal{Z}^s)^\ell d\tau = \mathcal{O}(\varepsilon^{1-r}).$$

Second, if $|\ell|$ is even, then Lemma 6.4 implies

$$\int_0^t X(\mathcal{Z}^s)^\ell d\tau = \sum_{k_1,\dots,k_{\grave{\cdot}}=1}^{\infty} \frac{1}{2^{\grave{\frac{\cdot}{2}}} d_1^{\grave{\frac{\cdot}{2}}}} \sum_{j\in Per(\ell)} \prod_{\eta=1}^{|\ell|/2} \frac{q^{1,1}_{k_{j_{2\eta}}\ \ _1,k_{j_{2\eta}}}}{\lambda_{k_{j_{2\eta}}\ \ _1}+\lambda_{k_{j_{2\eta}}}} \prod_{\eta=1}^{\ell} g_k \int_0^t X ds + \sum_{k_1=1}^{\infty}\cdots\sum_{k_{\cdot}=1}^{\infty} A^{1,\dots,1}_{k_1,\dots,k_{\cdot}} \prod_{\eta=1}^{\ell} g_k\ \mathcal{O}(\varepsilon^{1-r}).$$

As the A's are summable with weights (see (6.9))

$$P_c\int_0^t X(\mathcal{Z}^s)^\ell d\tau = \sum_{k_1,\dots,k_{\grave{\cdot}}=1}^{\infty} \frac{1}{2^{\grave{\frac{\cdot}{2}}} d_1^{\grave{\frac{\cdot}{2}}}} \sum_{j\in Per(\ell)} \prod_{\eta=1}^{|\ell|/2} \frac{q^{1,1}_{k_{j_{2\eta}}\ \ _1,k_{j_{2\eta}}}}{\lambda_{k_{j_{2\eta}}\ \ _1}+\lambda_{k_{j_{2\eta}}}} P_c \prod_{\eta=1}^{\ell} g_k \int_0^t X ds + \mathcal{O}(\varepsilon^{1-r}).$$

For $n=2$, we have $N=|\ell|=\ell_1+\ell_2$ and from (6.16)

$$\int_0^t X(\mathcal{Z}^s)^\ell d\tau = \sum_{j_1=1}^{\infty}\cdots\sum_{j_{\grave{\mathrm{j}}}=1}^{\infty}\int_0^t X \prod_{k=1}^{|\ell|} \mathcal{Z}_{i_k,j_k}\, d\tau \prod_{k=1}^{|\ell|} g_{j_k}$$

with $i_1=\cdots=i_{\ell_1}=1$ and $i_{\ell_1+1}=\cdots=i_{|\ell|}=2$. Similarly to the first part, we consider two cases. First if $|\ell|$ is odd, then we apply Lemma 6.4 to obtain

$$\int_0^t X(\mathcal{Z}^s)^\ell d\tau = \sum_{j_1=1}^{\infty}\cdots\sum_{j_{\grave{\mathrm{j}}}=1}^{\infty} A^{i_1,\cdots,i_{\grave{\mathrm{j}}}}_{k_{j_1},\cdots,k_{j_{\grave{\mathrm{j}}}}} \prod_{k=1}^{|\ell|} g_{j_k}\,\mathcal{O}(\varepsilon^{1-r}).$$

As the A's are summable with weights (see (6.9))

$$P_c\int_0^t X(\mathcal{Z}^s)^\ell d\tau = \mathcal{O}(\varepsilon^{1-r}).$$

In the second case, when $|\ell|$ is even, we apply Lemma 6.4 and analogously to the first case we obtain

$$\int_0^t X(\mathcal{Z}^s)^\ell d\tau = \sum_{j_1,\dots,j_{\grave{\mathrm{j}}}=1}^{\infty} \frac{1}{2^{\grave{\frac{\mathrm{j}}{2}}}} \sum_{j\in Per(|\ell|)} \prod_{\eta=1}^{|\ell|/2} \frac{q^{i_{k_{2\eta}}\ \ _1,i_{k_{2\eta}}}_{j_{k_{2\eta}}\ \ _1,j_{k_{2\eta}}}}{d_{i_{k_{2\eta}}\ \ _1}\lambda_{j_{k_{2\eta}}\ \ _1}+d_{i_{k_{2\eta}}}\lambda_{j_{k_{2\eta}}}} \prod_{k=1}^{|\ell|} g_{j_k} \int_0^t X ds + \sum_{j_1=1}^{\infty}\cdots\sum_{j_{\grave{\mathrm{j}}}=1}^{\infty} A^{i_1,\cdots,i_{\grave{\mathrm{j}}}}_{k_{j_1},\cdots,k_{j_{\grave{\mathrm{j}}}}} \prod_{k=1}^{|\ell|} g_{j_k}\,\mathcal{O}(\varepsilon^{1-r}).$$

We obtain

$$P_c\int_0^t X(\mathcal{Z}^s)^\ell d\tau = \sum_{j_1,\dots,j_{\grave{\mathrm{j}}}=1}^{\infty} \frac{1}{2^{\grave{\frac{\mathrm{j}}{2}}}} \sum_{j\in Per(|\ell|)} \prod_{\eta=1}^{|\ell|/2} \frac{q^{i_{k_{2\eta}}\ \ _1,i_{k_{2\eta}}}_{j_{k_{2\eta}}\ \ _1,j_{k_{2\eta}}}}{d_{i_{k_{2\eta}}\ \ _1}\lambda_{j_{k_{2\eta}}\ \ _1}+d_{i_{k_{2\eta}}}\lambda_{j_{k_{2\eta}}}} \times P_c\left(\prod_{k=1}^{|\ell|} g_{j_k}\right)\int_0^t X ds + \mathcal{O}(\varepsilon^{1-r}).$$

We can distinguish between two cases when $|\ell|$ is even. First one of ℓ_1 and ℓ_2 is odd. Here $q^{i_{j_{1}},i_{j_{1}+1}}_{j_{1},j_{1}+1}=0$, where $i_{\ell_1}=1$ and $i_{\ell_1+1}=2$. Thus

$$P_c\int_0^t X(\mathcal{Z}^s)^\ell d\tau=\mathcal{O}(\varepsilon^{1-r}).$$

In the second case when ℓ_1 and ℓ_2 are both even, we have

$$\begin{aligned}
&P_c\int_0^t X(\mathcal{Z}^s)^\ell d\tau\\
&=P_c\times\sum_{j_1=1}^{\infty}\cdots\sum_{j_{1}=1}^{\infty}\frac{1}{(2d_1)^{\frac{1}{2}}}\sum_{j\in Per(\ell_1)}\prod_{\eta=1}^{\ell_1/2}\frac{q^{1,1}_{j_{k_2}\ \ 1,j_{k_2}}}{\lambda_{j_{k_2}\ \ 1}+\lambda_{j_{k_2}}}\prod_{k=1}^{\ell_1}g_{j_k}\\
&\quad\times\sum_{j_1=1}^{\infty}\cdots\sum_{j_{2}=1}^{\infty}\frac{1}{(2d_2)^{\frac{2}{2}}}\sum_{j\in Per(\ell_2)}\prod_{\eta=1}^{\ell_2/2}\frac{q^{2,2}_{j_{k_2}\ \ 1,j_{k_2}}}{\lambda_{j_{k_2}\ \ 1}+\lambda_{j_{k_2}}}\prod_{k=1}^{\ell_2}g_{j_k}\int_0^t X ds\\
&\quad+\mathcal{O}(\varepsilon^{1-r})\\
&=\prod_{i=1}^{2}\sum_{j_1=1}^{\infty}\cdots\sum_{j_{i}=1}^{\infty}\frac{1}{(2d_i)^{\frac{i}{2}}}\sum_{j\in Per(\ell_1)}\prod_{\eta=1}^{\ell_i/2}\frac{q^{2,2}_{j_{k_2}\ \ 1,j_{k_2}}}{\lambda_{j_{k_2}\ \ 1}+\lambda_{j_{k_2}}}\int_0^t X ds\cdot P_c\prod_{k=1}^{|\ell|}g_{j_k}\\
&\quad+\mathcal{O}(\varepsilon^{1-r}).
\end{aligned}$$

The general case for $n>2$ follows in a similar way, as $(\sum_{j_1,\cdots,j_{i}=1}^{\infty}\prod_{k=1}^{\ell_i}\mathcal{Z}_{i,j_k}g_{j_k})_{i=1,2,\ldots,n}$ are independent random variables, and we can thus glue together the individual averaging results as above. ∎

7. Proof of Approximation I, Theorem 4.2. First we establish the limiting equation with residual.

LEMMA 7.1. *Let Assumptions* 2.2, 2.5, *and* 2.4 *hold. Then*

$$a_i(t)=a_i(0)+\int_0^t\mathcal{F}_i(a)d\tau+\int_0^t\mathcal{G}(a(\tau))d\tau+\tilde{R}(t), \tag{7.1}$$

where

$$\mathcal{G}(a(t))=\sum_{|\ell|=2,4,\ldots}\frac{C_\ell}{\ell!}D^\ell\mathcal{F}_i(a), \tag{7.2}$$

where C_ℓ *was defined in* (6.15) *and the error is bounded by* $\tilde{R}=\mathcal{O}(\varepsilon^{1-2m\kappa-\kappa_0})$.

Proof. The mild formulation from (4.9) and Lemma 5.1 with $\rho=0$ yields

$$\psi(t)=\mathcal{Z}^s(t)+e^{\varepsilon^{-2}t\mathcal{A}}\psi(0)+\mathcal{O}(\varepsilon^{2-m\kappa})=:\mathcal{Z}^s(t)+y(t)+R(t), \tag{7.3}$$

where

$$y(t)=e^{\varepsilon^{-2}t\mathcal{A}}\psi(0)\text{ and }R(t)=\mathcal{O}(\varepsilon^{2-m\kappa}).$$

Substituting from (7.3) into (4.8) yields

$$a_i(t)=a_i(0)+P_c\int_0^t\mathcal{F}_i(a+\mathcal{Z}^s+y+R)(\tau)d\tau. \tag{7.4}$$

Taylor's expansion for the polynomial $\mathcal{F}_i : \mathcal{L}^p_n \to \mathbb{R}$ yields

$$a_i(t) = a_i(0) + P_c \int_0^t \mathcal{F}_i(a + \mathcal{Z}^s)(\tau) d\tau + R_i^{(1)}(t), \tag{7.5}$$

where $R^{(1)}(t)$ is given by

$$R_i^{(1)}(t) = \sum_{|\ell|\geq 1} P_c \int_0^t \frac{D^\ell \mathcal{F}_i(a + \mathcal{Z}^s)}{\ell!} (y + R)^\ell d\tau.$$

We see later that $R^{(1)}$ is small, as all terms contain at least one $R = \mathcal{O}(\varepsilon^{2-m\kappa})$. Taylor's expansion again for the polynomial $\mathcal{F}_i : \mathcal{L}^p_n \to \mathbb{R}$, yields

$$a_i(t) = a_i(0) + \sum_{|\ell|\geq 0} P_c \int_0^t \frac{D^\ell \mathcal{F}_i(a)}{\ell!} (\mathcal{Z}^s)^\ell d\tau + R_1(t).$$

Applying the averaging lemma, Lemma 6.7, yields

$$a_i(t) = a_i(0) + \sum_{|\ell|\geq 0} \frac{C_\ell}{\ell!} \int_0^t D^\ell \mathcal{F}_i(a) d\tau + \mathcal{O}(\varepsilon^{1-m_i\kappa}) + R_i^{(1)}(t),$$

where $C_0 = 1$ and $C_\ell = 0$ if one ℓ_i is odd. Thus

$$a_i(t) = a_i(0) + \sum_{|\ell|=0,2,4,\dots} \frac{C_\ell}{\ell!} \int_0^t D^\ell \mathcal{F}_i(a) d\tau + \tilde{R}_i(t),$$

where $\tilde{R}(t) = R^{(1)}(t) + \mathcal{O}(\varepsilon^{1-m\kappa})$.

To bound $\tilde{R}$ we use Lemmas 5.4 and 5.2 and Assumption 2.4. ∎

Definition 7.2. *Define the set $\Omega^* \subset \Omega$ such that all the following estimates hold on Ω^*:*

$$\sup_{[0,\tau\ ]} \|\psi - \mathcal{Q}\|_{\mathcal{L}^p_n} < C\varepsilon^{2-m\kappa-\kappa}\ , \tag{7.6}$$

$$\sup_{[0,\tau\ ]} \|\psi\|_{\mathcal{L}^p_n} < C\varepsilon^{-\frac{3}{2}\kappa_0}\ , \tag{7.7}$$

$$\sup_{[0,\tau\ ]} \|\tilde{R}\|_{\mathcal{L}^p_n} < C\varepsilon^{1-2m\kappa-\kappa}\ , \tag{7.8}$$

and

$$\sup_{[0,T_1]} |b| \leq \tilde{C}_0\ . \tag{7.9}$$

Proposition 7.3. *The set Ω^* has approximately probability* 1.

Proof.

$$\mathbb{P}(\Omega^*) \geq 1 - \mathbb{P}\left(\sup_{[0,\tau\ ]} \|\psi - \mathcal{Q}\|_{\mathcal{L}^p_n} \geq C\varepsilon^{2-m\kappa-\kappa}\right) - \mathbb{P}\left(\sup_{[0,\tau\ ]} \|\psi\|_{\mathcal{L}^p_n} \geq C\varepsilon^{-\frac{3}{2}\kappa_0}\right)$$
$$-\mathbb{P}\left(\sup_{[0,\tau\ ]} \|\tilde{R}\|_{\mathcal{L}^p_n} \geq C\varepsilon^{1-2m\kappa-\kappa}\right) - \mathbb{P}\left(\sup_{[0,T_1]} |b| > \tilde{C}_0\right).$$

Using Chebychev inequality, Lemmas 5.1 and 7.1, and Corollary 5.3, we obtain for $\kappa > \kappa_0$ and sufficiently large $q > \frac{2p}{(\kappa-\kappa_0)} > 0$

$$\begin{aligned}\mathbb{P}(\Omega^*) &\geq 1 - C[\varepsilon^{q\kappa} + \varepsilon^{\frac{1}{2}q\kappa} + \varepsilon^{q(\kappa-\kappa_0)}] - \mathbb{P}\left(\sup_{[0,T_1]} |b| > \tilde{C}_0\right) \\ &\geq 1 - C\varepsilon^{\frac{1}{2}q(\kappa-\kappa_0)} - \mathbb{P}\left(\sup_{[0,T_1]} |b| > \tilde{C}_0\right) \\ &\geq 1 - C\varepsilon^p, \end{aligned} \tag{7.10}$$

where $\tilde{C}_0$ is chosen sufficiently large ($\sup_{[0,T_1]} |b| \leq C$ by Assumption 2.8). ∎

THEOREM 7.4. *Assume that Assumptions* 2.4 *and* 2.8 *hold. Suppose* $a(0) = \mathcal{O}(1)$ *and* $\psi(0) = \mathcal{O}(1)$. *Let* b *be a solution of* (1.3) *and* a *as defined in* (7.1). *If the initial conditions satisfy* $a(0) = b(0)$, *then for* $\kappa < \frac{1}{2m+1}$ *we obtain*

$$\sup_{t\in[0,T_1\wedge\tau\ ]} |a(t) - b(t)| \leq C\varepsilon^{1-2m\kappa-\kappa} \quad \text{on } \Omega^* \tag{7.11}$$

and

$$\sup_{t\in[0,T_1\wedge\tau\ ]} |a(t)| \leq C \quad \text{on } \Omega^*. \tag{7.12}$$

We note that all norms in a finite dimensional space are equivalent. Thus for simplicity of notation we always use the standard Euclidean norm.

Proof. Subtracting (1.3) from (7.1) and defining

$$h := a - b, \tag{7.13}$$

we obtain

$$h(t) = \sum_{|\ell|=0,2,4,\dots} \frac{C_\ell}{\ell!} \int_0^t [D^\ell \mathcal{F}_i(h+b) - D^\ell \mathcal{F}_i(b)]d\tau + \tilde{R}(t), \tag{7.14}$$

where the error $\tilde{R}$ is bounded by $\tilde{R} = \mathcal{O}(\varepsilon^{1-2m\kappa})$.

Define Q as

$$Q := h - \tilde{R}. \tag{7.15}$$

From (7.14) we obtain

$$\partial_t Q = \sum_{|\ell|=0,2,4,\dots} \frac{C_\ell}{\ell!} D^\ell [\mathcal{F}_i(Q + \tilde{R} + b) - D^\ell \mathcal{F}_i(b)].$$

Taking the scalar product $\langle Q, \cdot \rangle$ on both sides yields

$$\frac{1}{2}\partial_t |Q|^2 = \sum_{|\ell|=0,2,4,\dots} \frac{C_\ell}{\ell!} \left\langle D^\ell \mathcal{F}_i(Q + \tilde{R} + b) - D^\ell \mathcal{F}_i(b), Q \right\rangle.$$

Using the Young and Cauchy–Schwarz inequalities, where $\mathcal{F}$ is a polynomial of degree m, we obtain

$$\frac{1}{2}\partial_t |Q|^2 \leq C\left(1 + |Q|^{m-1} + |\tilde{R}|^{m-1} + |b|^{m-1}\right)\left(|Q|^2 + |\tilde{R}|^2\right). \tag{7.16}$$

Now we will work for times up to a stopping time where $|Q| < 1$ fails to be true or $\tau^* \wedge T_1 \leq T_0$ is reached. Up to that stopping time we show that $|Q| \leq \frac{1}{2}$, and thus $|Q| < 1$ is true for all times up to $\tau^* \wedge T_1$.

Using (7.8) and (7.9), we obtain for $\kappa < \frac{1}{2m+1}$ up to the stopping time where $|Q| \leq 1$

$$\frac{1}{2}\partial_t |Q|^2 \leq c\,|Q|^2 + C\varepsilon^{2-2(2m+1)\kappa} \text{ on } \Omega^*.$$

Using Gronwall's lemma, we obtain up to the stopping time

$$|Q(t)| \leq C\varepsilon^{1-(2m+1)\kappa} e^{cT_0}\,.$$

Thus for $\kappa < \frac{1}{2m+1}$ and $\varepsilon > 0$ sufficiently small we first obtain $|Q(t)| \leq \frac{1}{2}$ for $t \leq \tau^* \wedge T_1$. Second, taking the supremum on $[0, \tau^* \wedge T_1]$ yields

$$\sup_{t\in[0,\tau\ \wedge T_1]} |Q(t)|^2 \leq C\varepsilon^{1-(2m+1)\kappa} \text{ on } \Omega^*.$$

Hence,

$$\begin{aligned}\sup_{[0,\tau\ \wedge T_1]} |a-b| = \sup_{[0,\tau\ \wedge T_1]} |Q-\tilde{R}| &\leq \sup_{[0,\tau\ \wedge T_1]} |Q(t)| + \sup_{[0,\tau\ \wedge T_1]} |\tilde{R}| \\ &\leq C\varepsilon^{1-(2m+1)\kappa} \text{ on } \Omega^*. \end{aligned} \tag{7.17}$$

We finish the proof by using (7.13), (7.15), and

$$\sup_{[0,\tau\ \wedge T_1]} |a| \leq \sup_{[0,\tau\ \wedge T_1]} |a-b| + \sup_{[0,\tau\ \wedge T_1]} |b| \leq C. \qquad \square$$

Now we can collect the results obtained previously to prove the main result of Theorem 4.2 and Corollary 4.3 for the system of SPDE (1.1).

Proof of Theorem 4.2. Using (4.7) and the triangle inequality, we obtain

$$\begin{aligned}\sup_{t\in[0,T_1\wedge\tau\ ]} \|u(t)-b(t)-\mathcal{Q}(t)\|_{\mathcal{L}^p_n} &\leq \sup_{[0,T_1\wedge\tau\ ]} \|a-b\|_{\mathcal{L}^p_n} + \sup_{[0,T_1\wedge\tau\ ]} \|\psi-\mathcal{Q}\|_{\mathcal{L}^p_n} \\ &\leq C \sup_{[0,T_1\wedge\tau\ ]} |a-b| + \sup_{[0,\tau\ ]} \|\psi-\mathcal{Q}\|_{\mathcal{L}^p_n}\,.\end{aligned}$$

From (7.6) and (7.11), we obtain

$$\sup_{t\in[0,T_1\wedge\tau\ ]} \|u(t)-b(t)-\mathcal{Q}(t)\|_{\mathcal{L}^p_n} \leq C\varepsilon^{1-(2m+1)\kappa} \text{ on } \Omega^*.$$

Hence,

$$\mathbb{P}\left(\sup_{t\in[0,T_1\wedge\tau\ ]} \|u(t)-b(t)-\mathcal{Q}(t)\|_{\mathcal{L}^p_n} > C\varepsilon^{1-(2m+1)\kappa} \right) \leq 1-\mathbb{P}(\Omega^*)\,.$$

Using (7.10) yields (4.10). $\square$

Proof of Corollary 4.3. We note that by the semigroup estimate based on Assumption 2.3 and (2.2)

$$\begin{aligned}\|\psi(t)\|_{\mathcal{L}^{mp}_n} &\leq \|e^{\varepsilon^{\ 2}t\mathcal{A}}\psi(0)\|_{\mathcal{L}^{mp}_n} + \|\mathcal{Z}^s(t)\|_{\mathcal{L}^{mp}_n} + \frac{1}{\varepsilon^\rho}\left\| \int_0^t e^{\varepsilon^{\ 2}\mathcal{A}(t-\tau)}\mathcal{F}^s(u)d\tau \right\|_{\mathcal{L}^{mp}_n} \\ &\leq e^{-\varepsilon^{\ 2}t\omega}\|\psi(0)\|_{\mathcal{L}^{mp}_n} + \|\mathcal{Z}^s(t)\|_{\mathcal{L}^{mp}_n} \\ &\quad + \frac{M}{\varepsilon^\rho}\int_0^t (1+\varepsilon^{2\alpha}(t-\tau)^{-\alpha}) e^{\varepsilon^{\ 2}(t-\tau)\omega}\|\mathcal{F}^s(u)\|_{\mathcal{L}^{mp}_n}\,d\tau \\ &\leq \|\psi(0)\|_{\mathcal{L}^{mp}_n} + \|\mathcal{Z}^s(t)\|_{\mathcal{L}^{mp}_n} + C\varepsilon^{2-\rho} \sup_{\tau\in[0,\tau\ ]} (1+\|u\|^m_{\mathcal{L}^{pm}_n}),\end{aligned}$$

where we used Assumption 2.4 to bound $\mathcal{F}$. Thus by the definition of τ^* and the bounds on $\mathcal{Z}^s$ by $\varepsilon^{-\kappa_0}$ for some $0 < \kappa_0 < \kappa$ (cf. (5.3)) we obtain on Ω^*

$$\sup_{t\in[0,\tau\ ]} \|\psi(t)\|_{\mathcal{L}_n^{mp}} \le C\varepsilon^{-\kappa_0}.$$

Additionally from Theorem 7.4 we have a bound $a = \mathcal{O}(1)$. Thus we derive that $u = a + \psi$ is bounded by $\mathcal{O}(\varepsilon^{-\kappa_0})$ up to $\tau^* \wedge T_1$, at least on Ω^*. Or, to be more precise,

$$\Omega \supset \{\tau^* > T_1\} \supseteq \left\{ \sup_{[0,T_1\wedge\tau\ ]} \|u\|_{\mathcal{L}_n^{mp}} < \varepsilon^{-\kappa} \right\} \supseteq \Omega^*.$$

Hence, $\tau^* \wedge T_1 = T_1$ on Ω^*, and moreover

$$\begin{aligned}\sup_{t\in[0,T_1]} \|u(t) - b(t) - \mathcal{Q}(t)\|_{\mathcal{L}_n^{mp}} &\le \sup_{[0,T_1]} \|a - b\|_{\mathcal{L}_n^{mp}} + \sup_{[0,T_1]} \|\psi - \mathcal{Q}\|_{\mathcal{L}_n^{mp}} \\ &\le C \sup_{[0,T_1]} |a - b| + \sup_{[0,\tau\ ]} \|\psi - \mathcal{Q}\|_{\mathcal{L}_n^{mp}}.\end{aligned}$$

Proceeding as in the proof of Theorem 4.2 we bound the error in $\mathcal{L}_n^{mp}$. ∎

7.1. Application of Approximation I, Theorem 4.2. In this subsection we consider all examples with nonhomogeneous Neumann boundary condition on $[0,1]^2$. The simpler case of the one-dimensional interval $[0,1]$ does not seem to be that reasonable, as the conservation of mass would imply that the same Brownian motion forces the system on both sides just with opposite signs.

In two spatial dimensions the eigenfunctions are for $k = (k_1, k_2) \in (\mathbb{N}_0)^2$

$$g_{k_1,k_2} = c_k \cos(\pi k_1 x)\cos(\pi k_2 y)$$

with normalization constant $c_k = 2$ if both k_i are nonzero, $\sqrt{2}$ if only one k_i is nonzero, and when $k = 0$ we have $c_0 = 1$.

The eigenvalues of the operator $-\mathcal{A}_i = -\Delta = -(\partial_x^2 + \partial_y^2)$ are $\lambda_{k_1,k_2} = \pi^2(k_1^2 + k_2^2)$, and its nullspace therefore is $\mathcal{N} = \{1\}$. Define the orthonormal basis for the boundary by using $f_\ell(z)$ for $\ell \in \mathbb{N}_0$ given as

$$f_0(z) = 1 \quad \text{and} \quad f_\ell(z) = \sqrt{2}\cos(\pi\ell z) \text{ for } \ell > 0.$$

We now also have $g_k(x,y) = f_{k_1}(x) f_{k_2}(y)$ for $k \in \mathbb{N}_0^2$.

7.1.1. Nonlinear heat equation. The heat equation plays a significant role in several areas of science including mathematics, probability theory, and financial mathematics. Here we consider a nonlinear variant, which is known as the Allen–Cahn or the Ginzburg–Landau equation.

We apply our main theorem, Theorem 4.2, to the following partial differential equation with stochastic Neumann boundary condition:

$$\partial_t u = \varepsilon^{-2}\left(\partial_x^2 + \partial_y^2\right) u + u - u^3 \qquad \text{for } 0 \le x \le 1,\ 0 \le y \le 1, \tag{7.18}$$

$$\begin{aligned}&\partial_x u(t,x,0) = \sigma_\varepsilon \varepsilon^2 \partial_t W_1(t,x), \quad \partial_x u(t,x,1) = \sigma_\varepsilon \varepsilon^2 \partial_t W_2(t,x) \quad \text{for } x \in (0,1),\\ &\partial_y u(t,0,y) = \sigma_\varepsilon \varepsilon^2 \partial_t W_3(t,x), \quad \partial_y u(t,1,y) = \sigma_\varepsilon \varepsilon^2 \partial_t W_4(t,y) \quad \text{for } y \in (0,1).\end{aligned}$$

Define the noise on the boundary in terms of the Wiener processes $W_i(t)$ for $i = 1,2,3,4$ given in Fourier series expansion as $W_i(t) = \sum_{j=1}^\infty \alpha_{i,j}\beta_{i,j}(t) f_j$ with a family

of independent Brownian motions $\beta_{i,j}$ and bounded weights $\alpha_{i,j}$. Later we need to assume that the weights have some decay to 0. Note that for simplicity we assume that the average of the noise on each side is already zero.

Our main theorem, Theorem 4.2, states that the solution of the nonlinear heat equation (7.18) with $\sigma_\varepsilon = \varepsilon^{-1}$ is well approximated by

$$u(t,x,y) = b(t) + \mathcal{Z}^s(t,x,y) + \mathcal{O}(\varepsilon^{1-}),$$

where the error is bounded with high probability uniformly in time on a fixed interval and spatially in any L^p-norm. The average value b is the solution of

$$\partial_t b = (1 - 3C_2)b - b^3, \tag{7.19}$$

and C_2 is the constant from (6.15) (case $n = 1$, $\ell = 2$) given by

$$C_2 = \sum_{k,j \in \mathsf{N}^2} \frac{q_{k,j}}{\lambda_k + \lambda_j} P_c\left(g_k g_j\right) \ .$$

It arises in the averaging of the term $-3b(\mathcal{Z}^s)^2$. All other $\mathcal{Z}^s$-dependent terms in the expansion of $-(b + \mathcal{Z}^s)^3$ average to 0 as they contain odd powers.

In order to calculate C_2 consider first

$$P_c\left(g_k g_j\right) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{otherwise,} \end{cases}$$

while for the correlations we obtain

$$\begin{aligned} q_{k,j} = {} & \delta_{k_1,j_1}\alpha_{1,k_1}^2 f_{k_2}(0) f_{j_2}(0) + \delta_{k_1,j_1}\alpha_{2,k_1}^2 f_{k_2}(1) f_{j_2}(1) \\ & + \delta_{k_2,j_2}\alpha_{3,k_2}^2 f_{k_1}(0) f_{j_1}(0) + \delta_{k_2,j_2}\alpha_{4,k_1}^2 f_{k_1}(1) f_{j_1}(1). \end{aligned}$$

Thus

$$C_2 = \frac{1}{\pi^2} \sum_{k_1,k_2=1}^{\infty} \frac{\alpha_{1,k_1}^2 + \alpha_{2,k_1}^2 + \alpha_{3,k_2}^2 + \alpha_{4,k_2}^2}{k_1^2 + k_2^2}.$$

If we choose for any $\mu > 0$ that $\alpha_{i,k}^2 \le C|k|^{-2\mu}$ for $i = 1, \dots, 4$ and all $k \in \mathbb{N}$, then C_2 is finite, and furthermore, all summability conditions are satisfied.

Let us finally check that Assumption 2.8 is always true for deterministic initial conditions if we choose the constant there sufficiently large. Taking the product with b on both sides of (7.19) yields

$$\partial_t |b|^2 = C|b|^2 - 2|b|^4 \le C|b|^2.$$

Using Gronwall's lemma, we obtain for $0 \le t \le T_0$ that $\sup_{[0,T_0]} |b|^2 \le |b(0)|^2 e^{CT_0}$.

7.1.2. Cubic autocatalytic reaction. A simple archetypical example for a reaction-diffusion system is a cubic autocatalytic reaction between two chemicals according to the rule $A + B \to 2B$ with rate $r = \rho u_1 u_2^2$. As the setting for the example is similar to the previous case, the presentation will be brief.

Denote by u_1 and u_2 the concentration of A and B, respectively. The two species satisfy the equations

$$\partial_t u_1 = \frac{1}{\varepsilon^2}\Delta u_1 - \rho u_1 u_2^2 \quad \text{and} \quad \partial_t u_2 = \frac{1}{\varepsilon^2}\Delta u_2 + \rho u_1 u_2^2 \tag{7.20}$$

with respect to stochastic boundary conditions for $i = 1, 2$,

$$\partial_x u_i(t,x,0) = \sigma_\varepsilon \varepsilon^2 \partial_t W_1^{(i)}(t,x), \quad \partial_x u_i(t,x,1) = \sigma_\varepsilon \varepsilon^2 \partial_t W_2^{(i)}(t,x) \qquad \text{for } x \in (0,1),$$
$$\partial_y u_i(t,0,y) = \sigma_\varepsilon \varepsilon^2 \partial_t W_3^{(i)}(t,x), \quad \partial_y u_i(t,1,y) = \sigma_\varepsilon \varepsilon^2 \partial_t W_4^{(i)}(t,y) \qquad \text{for } y \in (0,1),$$

where $W_j^{(i)}(t) = \sum_{k=1}^\infty \alpha_{j,k}^{(i)} \beta_{j,k}^{(i)}(t) f_k$ for $j = 1, \dots, 4,$, with independent Brownian motions, bounded weights, and f_k defined as before. We define the nullspace $\mathcal{N} = \{(1,0)^t,\ (0,1)^t\}$ and take $\sigma_\varepsilon = \varepsilon^{-1}$.

Then our main theorem states that

$$u(t) = b(t) + \mathcal{Z}^s(t) + \mathcal{O}(\varepsilon^{1-})$$

$$\text{with} \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \text{ and } \mathcal{Z}^s = \begin{pmatrix} \mathcal{Z}_1^s \\ \mathcal{Z}_2^s \end{pmatrix},$$

where b_1 and b_2 are the solutions of

$$\partial_t b_1 = -\rho b_1 b_2^2 - \rho C_2 b_1, \qquad \partial_t b_2 = \rho b_1 b_2^2 + \rho C_2 b_1$$

with constant C_2 from the previous example arising in the averaging of the term $-3b_1(\mathcal{Z}_2^s)^2$. Using (6.15) in the case $n = 2$ and $\ell = (2,0)$ or $(0,2)$, a short calculation shows that in our setting $C_2 = C_{(0,2)} = C_{(2,0)}$.

We note that high fluctuations in combination with fast diffusion lead to effective new terms describing the transformation of b_1 to b_2. Although both terms individually do not change the average $\int u_i dx = b_i$, their nonlinear combination does.

It remains to check the bound on b from Assumption 2.8. We note that

$$\sum_{i=1}^2 \partial_t b_i = 0 \quad \text{and thus} \quad \sum_{i=1}^2 b_i(t) = \sum_{i=1}^2 b_i(0) = C_0.$$

As $b_1(t) \geq 0$ and thus $b_2(t) \geq b_2(0) \geq 0$, we have $0 \leq b_i(t) \leq \sum_{i=1}^2 b_i(t) \leq C_0$.

Hence, for all times $t > 0$ we obtain $\|b(t)\| = (\sum_{i=1}^2 b_i^2(t))^{1/2} \leq C_0\sqrt{2}$.

8. Proof of Approximation II, Theorem 4.5. In this section, we use many ideas and lemmas of the previous sections, as the main ideas are similar.

LEMMA 8.1. *Let Assumption* 2.4 *hold. Then for* $R^{(2)}$ *defined in* (4.16) *we have*

$$R_i^{(2)} = \sum_{|\ell| \geq 1} \int_0^t \frac{D^\ell \mathcal{F}_i(a)}{\ell!} P_c(\varepsilon\psi)^\ell d\tau = \mathcal{O}(\varepsilon^{1-m\kappa}) \,.$$

Proof. Using Assumption 2.4

$$\mathbb{E} \sup_{[0,\tau\ ]} |R_i^{(2)}|^p \leq C \sum_{|\ell| \geq 1} \frac{1}{\ell!} \mathbb{E} \sup_{[0,\tau\ ]} \int_0^t |D^\ell \mathcal{F}_i(a)| \|\varepsilon\psi\|_{\mathcal{L}_n^{\mathrm{i}\cdot\mathrm{j}}}^{|\ell|p} d\tau$$
$$\leq C \sum_{|\ell| \geq 1} \frac{1}{\ell!} [1 + \varepsilon^{(\ell-m)p\kappa}] \varepsilon^{\ell p(1-\kappa)} \quad \leq C\varepsilon^{1-m\kappa}. \qquad \square$$

DEFINITION 8.2. *Define the set* $\overset{**}{\Omega} \subset \Omega$ *such that for sufficiently large* $\zeta \gg 1$ *all the following estimates hold on* $\overset{**}{\Omega}$*:*

$$\sup_{[0,\tau\ ]} \|\psi - \mathcal{Q}\|_{\mathcal{L}^p_n} < C\varepsilon^{1-m\kappa-\kappa}\,, \tag{8.1}$$

$$\sup_{[0,\tau\ ]} \|\psi\|_{\mathcal{L}^p_n} < C\varepsilon^{-\frac{3}{2}\kappa_0}\,, \tag{8.2}$$

$$\sup_{[0,\tau\ ]} |R^{(2)}| < C\varepsilon^{1-m\kappa-\kappa}\,, \tag{8.3}$$

and

$$\sup_{[0,T_1]} |b|^{m-1} \leq \ln(\varepsilon^{-\frac{1}{}})\,. \tag{8.4}$$

PROPOSITION 8.3. *The set* $\overset{**}{\Omega}$ *has approximately probability* 1.

Proof.

$$\mathbb{P}(\overset{**}{\Omega}) \geq 1 - \mathbb{P}\left(\sup_{[0,\tau\ ]} \|\psi - \mathcal{Q}\|_{\mathcal{L}^p_n} \geq C\varepsilon^{1-m\kappa-\kappa}\right) - \mathbb{P}\left(\sup_{[0,\tau\ ]} \|\psi\|_{\mathcal{L}^p_n} \geq C\varepsilon^{-\frac{3}{2}\kappa_0}\right)$$

$$-\mathbb{P}\left(\sup_{[0,\tau\ ]} |R^{(2)}| \geq C\varepsilon^{1-m\kappa-\kappa}\right) - \mathbb{P}\left(\sup_{[0,T_1]} |b|^{m-1} > \ln\left(\varepsilon^{-\frac{1}{}}\right)\right).$$

Using the Chebychev inequality, Lemmas 5.1 and 8.1, and Corollary 5.3, we obtain for $\kappa > \kappa_0$ and sufficiently large $q > \frac{2p}{(\kappa-\kappa_0)} > 0$

$$\begin{aligned}\mathbb{P}(\overset{**}{\Omega}) &\geq 1 - C\left[\varepsilon^{q\kappa} + \varepsilon^{\frac{1}{2}q\kappa} + \varepsilon^{q(\kappa-\kappa_0)}\right] - \mathbb{P}\left(\sup_{[0,T_1]} |b|^{m-1} > \ln\left(\varepsilon^{-\frac{1}{}}\right)\right) \\ &\geq 1 - C\varepsilon^{\frac{1}{2}q(\kappa-\kappa_0)} - \mathbb{P}\left(\sup_{[0,T_1]} |b|^{m-1} > \ln\left(\varepsilon^{-\frac{1}{}}\right)\right) \quad \geq 1 - C\varepsilon^{\delta\kappa}\,, \end{aligned} \tag{8.5}$$

where we used Assumption 2.9. ∎

THEOREM 8.4. *Assume that Assumptions* 2.5, 2.4, *and* 2.9 *hold. Suppose* $a(0) = \mathcal{O}(1)$ *and* $\psi(0) = \mathcal{O}(1)$*. Let* $b \in \mathcal{N}$ *be a solution of* (1.5) *and* $a \in \mathcal{N}$ *as defined in* (4.15)*. If the initial conditions satisfy* $a(0) = b(0)$*, then for* $\kappa < \frac{1}{m+2}$ *we obtain*

$$\sup_{t\in[0,\tau\ \wedge T_1]} |a(t) - b(t)| \leq C\varepsilon^{1-(m+2)\kappa} \quad \text{on } \overset{**}{\Omega}. \tag{8.6}$$

Proof. We follow the same steps as in the proof of Lemma 7.4 until (7.16) to obtain

$$\frac{1}{2}\partial_t|Q|^2 \leq C\left(1 + |Q|^{m-1} + |R^{(2)}|^{m-1} + |b|^{m-1}\right)\left(|Q|^2 + |R^{(2)}|^2\right).$$

Again we use a stopping time argument as after (7.16). As long as $|Q| < 1$, using (8.3) and (8.4), we obtain

$$\frac{1}{2}\partial_t |Q(t)|^2 \le c(1+\ln(\varepsilon^{-1}))\,|Q(t)|^2 + C\varepsilon^{2-2(m+1)\kappa} \quad \text{on} \quad \overset{**}{\Omega}.$$

Using Gronwall's lemma, we obtain for $t \le \tau^* \wedge T_1 \le T_0$

$$|Q(t)|^2 \le C\varepsilon^{2-2(m+1)\kappa}\exp(2c(1+\ln(\varepsilon^{-1}))T_0) \ \le Ce^{2cT_0}\varepsilon^{2-2(m+1)\kappa-2\tilde{\kappa}},$$

where $\tilde{\kappa} = \frac{cT_0}{\zeta}$. If we choose $\tilde{\kappa} \le \kappa$ for sufficiently large ζ, then for $\kappa < \frac{1}{m+2}$ and small ε first $|Q(t)| \le \frac{1}{2}$. Second, taking the supremum on $[0, \tau^* \wedge T_1]$ yields

$$\sup_{t\in[0,\tau\ \wedge T_1]} |Q(t)| \le C\varepsilon^{1-(m+2)\kappa} \quad \text{on} \quad \overset{**}{\Omega}. \tag{8.7}$$

Hence, on $\overset{**}{\Omega}$

$$\sup_{[0,\tau\ \wedge T_1]} |a-b| = \sup_{[0,\tau\ \wedge T_1]} |Q - R^{(2)}| \le \sup_{[0,\tau\ \wedge T_1]} |Q| + \sup_{[0,\tau\ \wedge T_1]} |R^{(2)}| \le C\varepsilon^{1-(m+2)\kappa}.$$

□

Now we can use the results obtained previously to prove the main result of Theorem 4.5 and Corollary 4.6 for the SPDE (1.1).

Proof of Theorem 4.5. The steps are similar to the proof of Theorem 4.2. □

Proof of Corollary 4.6. The steps are similar to the proof of Corollary 4.3. □

8.1. Application of Approximation II, Theorem 4.5. In this subsection we apply our main theorem, Theorem 4.5, to the same examples as before, the nonlinear heat equation (7.18) and the cubic autocatalytic reaction (7.20), but now with $\sigma_\varepsilon = 1$ and nonzero $\alpha_{k,0}$. The final results are similar to [6]. One new problem arises here when we try to establish bounds on b, as the nonlinearity of the limiting equation is not Lipschitz.

8.1.1. Nonlinear heat equation. Our main theorem, Theorem 4.5, in this case states that the solution of (7.18) takes the form

$$u(t) = b(t) + \mathcal{O}(\varepsilon^{1-}),$$

where b is the solution of stochastic ordinary differential equation

$$db = [b - b^3]dt + dB, \tag{8.8}$$

and B is an $\mathbb{R}$-valued standard Brownian motion given by the OU-process projected onto the constant. Here it is given as

$$B(t) = \alpha_{1,0}\beta_{1,0}(t) + \alpha_{2,0}\beta_{2,0}(t) + \alpha_{3,0}\beta_{3,0}(t) + \alpha_{4,0}\beta_{4,0}(t).$$

To check the bound on b consider $\exp\{\delta|b|^2\}$. We note that

$$d\exp\{\delta|b|^2\} = \delta\exp\{\delta|b|^2\}d|b|^2 + \delta^2\exp\{\delta|b|^2\}(d|b|^2)^2, \tag{8.9}$$

and $d|b|^2 = 2b\cdot db + db\cdot db$. From (8.8) we obtain for some constant $c > 0$

$$\begin{aligned} d|b|^2 &= 2|b|^2dt - 2|b|^4dt + 2b\cdot dB + dB\cdot dB \\ &= (C + 2|b|^2 - 2|b|^4)dt + 2b\cdot dB. \end{aligned}$$

Substituting this into (8.9) yields

$$\begin{aligned} d\exp\{\delta|b|^2\} &= \delta(C+(2+4\delta)|b|^2-2|b|^4)\exp\{\delta|b|^2\}dt + 2\delta\exp\{\delta|b|^2\}b\cdot dB \\ &\le c_\delta \exp\{\delta|b|^2\}dt + 2\delta\exp\{\delta|b|^2\}b\cdot dB. \end{aligned} \tag{8.10}$$

Integrating from 0 to t and taking expectation yields

$$\mathbb{E}\exp\{\delta|b(t)|^2\} \le \mathbb{E}\exp\{\delta|b(0)|^2\} + c_\delta\int_0^t \mathbb{E}\exp\{\delta|b|^2\}dt.$$

As $\mathbb{E}\exp\{3\delta|b(0)|^2\} \le C$ and applying Gronwall's lemma yields for $t \le T_1$

$$\sup_{[0,T_1]} \mathbb{E}\exp\{\delta\,|b|^2\} \le C.$$

With 3δ instead of δ, we have

$$\sup_{[0,T_1]} \mathbb{E}\exp\{3\delta\,|b|^2\} \le C. \tag{8.11}$$

Taking expectation after supremum on both sides of (8.10) we obtain

$$\begin{aligned} &\mathbb{E}\sup_{t\in[0,T_1]}\exp\{\delta|b(t)|^2\} \\ &\le \mathbb{E}\exp\{\delta|b(0)|^2\} + c_\delta\mathbb{E}\sup_{t\in[0,T_1]}\int_0^t \exp\{\delta|b(s)|^2\}ds \\ &\qquad + 2\delta\mathbb{E}\sup_{t\in[0,T_1]}\int_0^t b(s)\exp\{\delta|b(s)|^2\}dB(s) \\ &\le C + c_\delta\mathbb{E}\int_0^{T_1}\exp\{\delta|b(s)|^2\}ds + 2\delta\mathbb{E}\left(\int_0^{T_1} b(s)^2\exp\{2\delta|b(s)|^2\}ds\right)^{1/2}. \end{aligned}$$

Using (8.11) together with $xe^{2\delta x} \le Ce^{3\delta x}$ for all $x > 0$ yields

$$\mathbb{E}\sup_{t\in[0,T_1]}\exp\{\delta|b(t)|^2\} \le C.$$

Now, using the Chebychev inequality

$$\mathbb{P}(\sup_{[0,T_1]}|b(t)|^2 > \ln(\varepsilon^{-\kappa})) \le \frac{\mathbb{E}\sup_{t\in[0,T_1]}\exp\left(\delta|b(t)|^2\right)}{\exp\left(\delta\ln(\varepsilon^{-\kappa})\right)} \le C\varepsilon^{\delta\kappa}.$$

8.1.2. Cubic autocatalytic reaction. Our main theorem states that the solution of (7.20) takes the form

$$u(t) = b(t) + \mathcal{O}(\varepsilon^{1-}) \qquad \text{with} \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{and } b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

In this case b_1 and b_2 are the solutions of

$$db_1 = -\rho b_1 b_2^2 dt + dB_1(t) \qquad \text{and} \qquad db_2 = \rho b_1 b_2^2 dt + dB_2(t), \tag{8.12}$$

where the projected OU-process is

$$B_i(t) = \alpha_{i_1,0}\beta^{(i)}_{1,0} + \alpha^{(i)}_{2,0}\beta^{(i)}_{2,0}(t) + \alpha^{(i)}_{3,0}\beta^{(i)}_{3,0}(t) + \alpha^{(i)}_{4,0}\beta^{(i)}_{4,0}(t) \quad \text{for } i = 1, 2.$$

To verify the bound on b define first the stopping T_1 as

$$T_1 = T_0 \wedge \inf\{t > 0 : \exists\, i \in \{1, 2\} : b_i(t) < 0\}.$$

This means that our approximation result is true only as long as the concentrations b_i are nonnegative. To bound the b_i's from above, we note that $\sum_{i=1}^2 db_i = \sum_{i=1}^2 dB_i$. Integrating this from 0 to t yields

$$\sum_{i=1}^{2} b_i(t) = \sum_{i=1}^{2} b_i(0) + \sum_{i=1}^{2} B_i(t). \tag{8.13}$$

Hence, up to T_1 we obtain

$$|b(t)| \le \sum_{i=1}^{2} b_i(t) = \sum_{i=1}^{2} b_i(0) + \sum_{i=1}^{2} B_i(t) \le \sqrt{2}|B(t)| + \sqrt{2}|b(0)|,$$

where we used $\left(x^2 + y^2\right)^{1/2} \le |x| + |y| \le \sqrt{2}(x^2 + y^2)^{1/2}$. Moreover,

$$|b(t)|^2 \le 4|B(t)|^2 + 4|b(0)|^2.$$

Thus

$$\mathbb{E} \sup_{[0,T_1]} \exp\{\delta|b|^2\} \le \mathbb{E} \sup_{[0,T_1]} \exp\{4\delta|B|^2\} \cdot \exp\{4\delta|b(0)|^2\} \le C,$$

but only for sufficiently small δ. Using the Chebychev inequality

$$\mathbb{P}(\sup_{[0,T_1]} |b(t)|^2 > \ln(\varepsilon^{-\kappa})) \le \frac{\mathbb{E}\sup_{t\in[0,T_1]}\left(\exp\left(\delta|b(t)|^2\right)\right)}{\exp\left(\delta\ln(\varepsilon^{-\kappa})\right)} \le C\varepsilon^{\delta\kappa}.$$

So the probability is close, but not very close to 1, as δ cannot be arbitrarily large.

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