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GALERKIN APPROXIMATIONS FOR THE STOCHASTIC BURGERS EQUATION*

DIRK BLÖMKER† AND ARNULF JENTZEN‡

Abstract. Existence and uniqueness for semilinear stochastic evolution equations with additive noise by means of finite-dimensional Galerkin approximations is established and the convergence rate of the Galerkin approximations to the solution of the stochastic evolution equation is estimated. These abstract results are applied to several examples of stochastic partial differential equations (SPDEs) of evolutionary type including a stochastic heat equation, a stochastic reaction diffusion equation, and a stochastic Burgers equation. The estimated convergence rates are illustrated by numerical simulations. The main novelty in this article is the estimation of the difference of the finite-dimensional Galerkin approximations and of the solution of the infinite-dimensional SPDE uniformly in space, i.e., in the L^∞-topology, instead of the usual Hilbert space estimates in the L^2-topology, that were shown before.

Key words. Galerkin approximations, stochastic partial differential equation, stochastic heat equation, stochastic reaction diffusion equation, stochastic Burgers equation, strong error criteria

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1. Introduction. In this work we present a general abstract result for the spatial approximation of stochastic evolution equations with additive noise via Galerkin methods. This abstract result is applied to several examples of stochastic partial differential equations (SPDEs) of evolutionary type including a stochastic heat equation, a stochastic reaction diffusion equation, and a stochastic Burgers equation. In all examples we need to verify the following conditions. First, we need the rate of approximation of the linear equation obtained by omitting the nonlinear term in the stochastic evolution equation. Then one needs a quite weak Lipschitz condition for the nonlinearity and finally a uniform bound on the sequence of approximations. These results are the key for the main theorem (see Theorem 3.1). The main novelty in this article is the estimation of the difference of the finite-dimensional Galerkin approximations and of the solution of the infinite-dimensional SPDE uniformly in space, i.e., in the L^∞-topology, instead of the usual Hilbert space estimates shown before in the L^2-topology.

Although there are several different methods using finite-dimensional approximations such as spectral Galerkin, finite elements, or wavelets, we focus here on the spectral Galerkin method. Thus the finite-dimensional approximations are given by an expansion in terms of the eigenfunctions of a dominant linear operator. This spectral Galerkin method is one of the key tools in the analysis of stochastic or

deterministic PDEs. For SPDEs see, for example, [16, 9, 17, 2], where the Galerkin method was used to establish the existence of solutions. Moreover, spectral methods are an effective tool for numerical simulations, especially on domains like the interval, where fast Fourier transforms are available. Nevertheless, it is limited on domains where the eigenfunctions of the dominant linear operator are not explicitly known. In recent years there has also been a significant interest in analytic results for the rate of approximation using a spectral Galerkin method as a numerical method for SPDEs; see, for example, [18, 28] for SPDEs with one-dimensional possibly nonadditive noise and globally Lipschitz continuous nonlinearities, [31, 32, 35, 36, 25, 26] for SPDEs with possibly infinite-dimensional additive noise and globally Lipschitz continuous nonlinearities, [30, 23] for SPDEs with possibly infinite-dimensional additive noise and nonglobally Lipschitz continuous nonlinearities, and [20, 21, 34, 33] for SPDEs with possibly infinite-dimensional nonadditive noise and globally Lipschitz continuous nonlinearities. In most of the above-named references the full discretization is also treated, including the time discretization.

In order to illustrate the main result of this article we limit ourself in this introductory section to a stochastic Burgers equation with Dirichlet boundary conditions and refer to section 3 for the general result and to section 4 for further examples. To this end let $T \in (0, \infty)$ be a real number, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space, and let $X \colon [0, T] \times \Omega \to C([0, 1], \mathbb{R})$ be the up-to-indistinguishability unique solution process of the SPDE

(1.1)

$$dX_t(x) = \left[\frac{\partial^2}{\partial x^2} X_t(x) - X_t(x) \cdot \frac{\partial}{\partial x} X_t(x) \right] dt + dW_t(x), \ X_t(0) = X_t(1) = 0, \ X_0 = 0$$

for $t \in [0, T]$ and $x \in (0, 1)$, where W_t, $t \in [0, T]$, is a cylindrical I-Wiener process on $L^2((0, 1), \mathbb{R})$, which models space-time white noise on $(0, 1)$. In this introductory section the initial value $X_0 = 0$ is zero for simplicity of presentation and we refer to section 4.3 below for a more general stochastic Burgers equation with a possibly nonzero initial value. The existence and uniqueness of solutions of the stochastic Burgers equation was, e.g., studied in Da Prato and Gatarek [11] for colored noise and in Da Prato, Debussche, and Temam [10] for space-time white noise (see also Chapter 14 in Da Prato and Zabczyk [14]).

Recently, Alabert and Gyöngy showed the following error estimate for spatial discretizations in the L^2-topology (see Theorem 2.2 in [1]):

(1.2)
$$\mathbb{P}\left[\sup_{0 \le t \le T} \left(\int_0^1 |X_t(x) - X_t^N(x)|^2 \, dx \right)^{1/2} \le C_\varepsilon \cdot N^{\varepsilon - \frac{1}{2}} \right] = 1$$

for every $N \in \mathbb{N} := \{1, 2, \dots\}$ and every arbitrarily small $\varepsilon \in (0, \frac{1}{2})$ with random variables $C_\varepsilon \colon \Omega \to [0, \infty)$, $\varepsilon \in (0, \frac{1}{2})$, where the X^N, $N \in \mathbb{N}$, are given by finite differences approximations. Our results (see Lemma 4.3, Theorem 3.1, and Lemma 4.8) yield the following estimate for the stochastic Burgers equation (1.1) (see section 4.3):

(1.3)
$$\mathbb{P}\left[\sup_{0 \le t \le T} \sup_{0 \le x \le 1} \left| X_t(x) - X_t^N(x) \right| \le C_\varepsilon \cdot N^{\varepsilon - \frac{1}{2}} \right] = 1$$

for every $N \in \mathbb{N}$ and every arbitrarily small $\varepsilon \in (0, \frac{1}{2})$ with random variables $C_\varepsilon \colon \Omega \to [0, \infty)$, $\varepsilon \in (0, \frac{1}{2})$, where X^N, $N \in \mathbb{N}$, are spectral Galerkin approximations. Thus, although the spatial error criterion is estimated in the bigger L^∞-norm instead of the

L^2-norm, the convergence rate remains $\frac{1}{2}-$. This convergence rate with respect to the strong L^∞-norm is also corroborated by a numerical example (see section 4). (For a real number $a \in (0,\infty)$, we write $a-$ for the convergence order, if the convergence order is higher than $a - \varepsilon$ for every arbitrarily small $\varepsilon \in (0,a)$.)

A further instructive related result is given by Liu [30]. He treats stochastic reaction diffusion equations of the Ginzburg–Landau type which fit in the abstract setting in section 2. For such equations he obtained estimates in the H^r-topology with the rate $(\frac{1}{2} - r)-$ for every $r \in (0, \frac{1}{2})$. The convergence rates he obtained in the H^r-topologoy with $r \in (0, \frac{1}{2})$ can, in general, not be improved and, by using Sobolev embeddings, his bounds also yield estimates in the L^p-topology with $p \in (2,\infty)$. Nevertheless, such estimates do not yield convergence in the L^∞-topology since in one dimension H^r is embedded into L^∞ for $r > \frac{1}{2}$ only. Moreover, in contrast to (1.3) this would not give a convergence rate $\frac{1}{2}-$ in any L^p-topology where $p \in (2,\infty]$.

The rest of the paper is organized as follows. Section 2 gives the setting and the assumptions for the main result, which is then presented in section 3. In section 4 we discuss our examples, while in the final section most of the proofs are stated.

Next we add that after the preprint version [3] of this article had appeared, a number of related results appeared in the literature; see, e.g., [6, 19, 29, 8, 7, 15, 4]. In particular, we mention [8, 7] for temporal and spatial discretization estimates in Banach spaces that imply estimates in the L^∞-norm as well as [29] for the analysis of spectral Galerkin methods for semilinear SPDEs with possibly nonadditive noise and globally Lipschitz continuous nonlinearities. We also refer, e.g., to [19] for further spatial approximations of stochastic Burgers equations and, e.g., to [6, 15] for the analysis of spatial and temporal-spatial discretizations of stochastic Navier–Stokes equations. Finally, we would like to point out that parts of this article (see subsection 4.1) appeared in the thesis [24] (see section 2.2.3 in [24]).

2. Setting and assumptions. Throughout this article suppose that the following setting and the following assumptions are fulfilled.

The first assumption is a regularity and approximation condition on the semigroup of the linear operator of the considered SPDE. The second is an appropriate Lipschitz condition on the nonlinearity of the considered SPDE. The third is an assumption on the approximation of the stochastic convolution and the initial value of the considered SPDE, while the final one is a uniform bound on finite-dimensional approximations of the considered SPDE.

Let $T \in (0,\infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two $\mathbb{R}$-Banach spaces. Moreover, let $P_N \colon V \to V$, $N \in \mathbb{N}$, be a sequence of bounded linear operators from V to V.

Assumption 1 (semigroup S). Let $\alpha \in [0,1)$ and $\gamma \in (0,\infty)$ be real numbers and let $S \colon (0,T] \to L(W,V)$ be a strongly continuous mapping which satisfies $\sup_{t \in (0,T]}(t^\alpha \|S_t\|_{L(W,V)}) < \infty$ and $\sup_{N \in \mathbb{N}} \sup_{t \in (0,T]}(t^\alpha N^\gamma \|S_t - P_N S_t\|_{L(W,V)}) < \infty$.

Assumption 2 (nonlinearity F). Let $F \colon V \to W$ be a mapping which satisfies $\sup_{\|v\|_V, \|w\|_V \leq r, v \neq w} \frac{\|F(v) - F(w)\|_W}{\|v - w\|_V} < \infty$ for every $r \in (0,\infty)$.

Assumption 3 (stochastic process O). Let $O \colon [0,T] \times \Omega \to V$ be a stochastic process with continuous sample paths and $\sup_{N \in \mathbb{N}} \sup_{0 \leq t \leq T} N^\gamma \|O_t(\omega) - P_N(O_t(\omega))\|_V < \infty$ for every $\omega \in \Omega$, where $\gamma \in (0,\infty)$ is given in Assumption 1.

Assumption 4 (existence of solutions). Let $X^N \colon [0,T] \times \Omega \to V$, $N \in \mathbb{N}$, be a sequence of stochastic processes with continuous sample paths and with

(2.1)

$$X_t^N(\omega) = \int_0^t P_N \, S_{t-s} \, F(X_s^N(\omega)) \, ds + P_N(O_t(\omega)) \quad \text{and} \quad \sup_{M \in \mathbb{N}} \sup_{s \in [0,T]} \|X_s^M(\omega)\|_V < \infty$$

for every $t \in [0,T]$, $\omega \in \Omega$, and every $N \in \mathbb{N}$.

As usual, we call here a mapping $Y \colon [0,T] \times \Omega \to V$ a stochastic process if for every $t \in [0,T]$ the mapping $Y_t \colon \Omega \ni \omega \mapsto Y_t(\omega) := Y(t,\omega) \in V$ is $\mathcal{F}/\mathcal{B}(V)$-measurable. Additionally, we say that a stochastic process $Y \colon [0,T] \times \Omega \to V$ has continuous sample paths if for every $\omega \in \Omega$ the mapping $[0,T] \ni t \mapsto Y_t(\omega) \in V$ is continuous. Furthermore, we say that a mapping $f \colon (0,T] \to L(W,V)$ is strongly continuous if for every $w \in W$ the mapping $(0,T] \ni t \mapsto f(t)w \in V$ is continuous. Moreover, note that if $Y \colon [0,T] \times \Omega \to V$ is a stochastic process with continuous sample paths, then Assumptions 1 and 2 ensure for every $\omega \in \Omega$, $t \in (0,T]$, and every $N \in \mathbb{N}$ that the mapping $(0,t) \ni s \mapsto P_N \, S_{t-s} \, F(Y_s(\omega)) \in V$ is continuous and, therefore, we obtain for every $\omega \in \Omega$, $t \in [0,T]$, and every $N \in \mathbb{N}$ that the V-valued Bochner integral $\int_0^t P_N \, S_{t-s} \, F(Y_s(\omega)) \, ds \in V$ (see (2.1) in Assumption 4) is well defined.

3. Main result. In this section we state the main approximation result, which is based on the assumptions of the previous section. Its proof is postponed to subsection 5.1.

THEOREM 3.1. *Let Assumptions* 1–4 *be fulfilled. Then there exists a unique stochastic process* $X \colon [0,T] \times \Omega \to V$ *with continuous sample paths which fulfills*

(3.1)
$$X_t(\omega) = \int_0^t S_{t-s} \, F(X_s(\omega)) \, ds + O_t(\omega)$$

for every $t \in [0,T]$ *and every* $\omega \in \Omega$. *Moreover, there exists an* $\mathcal{F}/\mathcal{B}([0,\infty))$-*measurable mapping* $C \colon \Omega \to [0,\infty)$ *such that*

(3.2)
$$\sup_{0 \le t \le T} \left\| X_t(\omega) - X_t^N(\omega) \right\|_V \le C(\omega) \cdot N^{-\gamma}$$

for every $N \in \mathbb{N}$ *and every* $\omega \in \Omega$, *where* $\gamma \in (0,\infty)$ *is given in Assumption* 1.

Let us add three remarks on Theorem 3.1. First, we would like to point out that the initial value of the stochastic evolution equation (3.1) is incorporated in the driving stochastic process $O \colon [0,T] \times \Omega \to V$ (see also Proposition 4.2 below for more details). Second, we emphasize that the driving stochastic process $O \colon [0,T] \times \Omega \to V$ is not assumed to be a stochastic convolution of the semigroup and a cylindrical Wiener process. In particular, the stochastic evolution equation (3.1) covers SPDEs disturbed by fractional Brownian motions too.

Third, we would like to point out that Theorem 3.1 yields the existence of an $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mapping $C \colon \Omega \to [0,\infty)$ such that (3.2) holds although the $\mathbb{R}$-Banach space $(V, \|\cdot\|_V)$ is not assumed to be separable. The sum and the difference of two $\mathcal{F}/\mathcal{B}(V)$-measurable mappings on the possibly nonseparable $\mathbb{R}$-Banach space V are, in general, not $\mathcal{F}/\mathcal{B}(V)$-measurable anymore. Nonetheless, it is possible to establish the existence of an $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mapping $C \colon \Omega \to [0,\infty)$ such that (3.2) holds by exploiting for every $t \in [0,T]$ and every $N \in \mathbb{N}$ that the difference $O_t - P_N(O_t) = (I - P_N) \, O_t \colon \Omega \to V$ is $\mathcal{F}/\mathcal{B}(V)$-measurable (see (5.1) in the proof of Theorem 3.1 for more details). Note that the composition of two measurable mappings is measurable (on nonseparable $\mathbb{R}$-Banach spaces too). Finally, we note that the error constant $C \colon \Omega \to [0,\infty)$ appearing in (3.2) is described explicitly in the proof of Theorem 3.1 (see definition (5.7) in the proof of Theorem 3.1 for details).

4. Examples. This section presents some examples of the setting in section 2.

4.1. Stochastic heat equation. In this subsection an important example of Assumption 3 is presented. We consider a linear equation with $F = 0$ and thus consider only the approximation of the Ornstein–Uhlenbeck process O.

To this end let $d \in \mathbb{N}$ and let $V = W = C([0,1]^d, \mathbb{R})$ be the $\mathbb{R}$-Banach space of continuous functions from $[0,1]^d$ to $\mathbb{R}$ equipped with the supremum norm $\|\cdot\|_V = \|\cdot\|_W = \|\cdot\|_{C([0,1]^d,\mathbb{R})}$. Moreover, consider the continuous functions $e_i \colon [0,1]^d \to \mathbb{R}$, $i \in \mathbb{N}^d$, and the real numbers λ_i, $i \in \mathbb{N}^d$, defined through

$$(4.1) \qquad e_i(x) := 2^{\frac{d}{2}} \sin(i_1 \pi x_1) \ldots \sin(i_d \pi x_d) \quad \text{and} \quad \lambda_i := \pi^2 \left(|i_1|^2 + \cdots + |i_d|^2 \right)$$

for all $x = (x_1, \ldots, x_d) \in [0,1]^d$ and all $i = (i_1, \ldots, i_d) \in \mathbb{N}^d$. Additionally, suppose that the bounded linear operators $P_N \colon C([0,1]^d, \mathbb{R}) \to C([0,1]^d, \mathbb{R})$, $N \in \mathbb{N}$, are given by

$$(4.2) \qquad (P_N(v))(x) = \sum_{i \in \{1,\ldots,N\}^d} \int_{(0,1)^d} e_i(s)\, v(s)\, ds \cdot e_i(x)$$

for all $x \in [0,1]^d$, $v \in C([0,1]^d, \mathbb{R})$, and all $N \in \mathbb{N}$. The linear operators P_N, $N \in \mathbb{N}$, are projection operators, i.e., they satisfy $P_N(P_N(v)) = P_N(v)$ for all $v \in C([0,1]^d, \mathbb{R})$ and all $N \in \mathbb{N}$ and their images are the finite-dimensional $\mathbb{R}$-vector spaces $P_N\big(C([0,1]^d, \mathbb{R})\big)$, $N \in \mathbb{N}$. The operators P_N, $N \in \mathbb{N}$, are thus compact linear operators and from the Daugavet property of the $\mathbb{R}$-Banach space $C([0,1]^d, \mathbb{R})$ (see, e.g., Definition 2.1 and Example (a) in Werner [40]) we get that $\|I - P_N\|_{L(C([0,1]^d,\mathbb{R}))} = 1 + \|P_N\|_{L(C([0,1]^d,\mathbb{R}))}$ for all $N \in \mathbb{N}$ (see, for instance, Theorem 2.7 in Werner [40]). Next let $S \colon (0,T] \to L(C([0,1]^d, \mathbb{R}))$ be a mapping given by

$$(4.3) \qquad (S_t v)(x) = \sum_{i \in \mathbb{N}^d} e^{-\lambda_i t} \int_{(0,1)^d} e_i(s)\, v(s)\, ds \cdot e_i(x)$$

for all $t \in (0,T]$, $x \in [0,1]^d$, and all $v \in C([0,1]^d, \mathbb{R})$.

LEMMA 4.1. *Let $d \in \{1,2,3\}$. Then the mapping $S \colon (0,T] \to L(C([0,1]^d, \mathbb{R}))$ given by* (4.3) *satisfies Assumption 1 for every $\alpha \in [\frac{d}{4} + \frac{\gamma}{2}, 1)$ and every $\gamma \in (0, 2 - \frac{d}{2})$.*

Clearly, this is simply the semigroup generated by the Laplacian with Dirichlet boundary conditions (see, e.g., section 3.8.1 in [39]). Other boundary conditions such as Neumann or periodic boundary conditions could also be considered here. The proof of Lemma 4.1 is well known and therefore omitted. We also add that the proof of Lemma 4.1 essentially uses a suitable Sobolev embedding and for this the condition $d \leq 3$ is assumed in Lemma 4.1. We now present the promised example of Assumption 3. We consider a stochastic convolution of the semigroup S constructed in (4.3) and a cylindrical Wiener process. The following result provides an appropriate version of such a process, in which the initial value of the stochastic evolution equation (3.1) is additionally incorporated.

PROPOSITION 4.2. *Let $d \in \mathbb{N}$, let $V = C([0,1]^d, \mathbb{R})$ with $\|v\|_V = \|v\|_{C([0,1]^d,\mathbb{R})}$ for every $v \in V$, let $\rho \in (0,\infty)$, let $\beta^i \colon [0,T] \times \Omega \to \mathbb{R}$, $i \in \mathbb{N}^d$, be a family of independent standard Brownian motions with continuous sample paths, and let $b \colon \mathbb{N}^d \to \mathbb{R}$ be a function with $\sum_{i \in \mathbb{N}^d} \left(i_1^2 + \cdots + i_d^2 \right)^{(\rho-1)} |b(i)|^2 < \infty$. Furthermore, suppose that $\xi \colon \Omega \to V$ is an $\mathcal{F}/\mathcal{B}(V)$-measurable mapping with $\sup_{N \in \mathbb{N}} (N^\rho \|\xi(\omega) - P_N(\xi(\omega))\|_V) < \infty$ for*

every $\omega \in \Omega$. Then there exists an up-to-indistinguishability unique stochastic process $O \colon [0,T] \times \Omega \to V$ with continuous sample paths which satisfies

(4.4)

$$\mathbb{P}\left[\lim_{N \to \infty} \sup_{0 < t \leq T} \left\| O_t - S_t \xi - \sum_{i \in \{1,\dots,N\}^d} b(i) \left(-\lambda_i \int_0^t e^{-\lambda_i(t-s)} \beta_s^i \, ds + \beta_t^i \right) e_i \right\|_V = 0 \right] = 1$$

and

(4.5)
$$\sup_{N \in \mathbb{N}} \sup_{0 \leq t \leq T} \left(N^\gamma \| O_t(\omega) - P_N(O_t(\omega)) \|_V \right) < \infty$$

for every $\omega \in \Omega$ and every $\gamma \in (0,\rho)$. In particular, O satisfies Assumption 3 for every $\gamma \in (0,\rho)$. Here the functions $e_i \in V$, $i \in \mathbb{N}^d$, the real numbers λ_i, $i \in \mathbb{N}^d$, and the linear operators $P_N \colon V \to V$, $N \in \mathbb{N}$, are given in (4.1) and (4.2).

Proposition 4.2 follows directly from Lemma 4.3 below. Let us add some remarks concerning Proposition 4.2. Let $L^2((0,1)^d, \mathbb{R})$ be the $\mathbb{R}$-Hilbert space of equivalence classes of $\mathcal{B}((0,1)^d)/\mathcal{B}(\mathbb{R})$-measurable and Lebesgue square integral functions from $(0,1)^d$ to $\mathbb{R}$ and let $B \colon L^2((0,1)^d, \mathbb{R}) \to L^2((0,1)^d, \mathbb{R})$ be a bounded linear operator given by

(4.6)
$$Bv = \sum_{i \in \mathbb{N}^d} b(i) \int_{(0,1)^d} e_i(s) \, v(s) \, ds \cdot e_i$$

for all $v \in L^2((0,1)^d, \mathbb{R})$, where $b \colon \mathbb{N}^d \to \mathbb{R}$ is the function used in Proposition 4.2. Then the stochastic process $O \colon [0,T] \times \Omega \to C([0,1]^d, \mathbb{R})$ in Proposition 4.2 satisfies

(4.7)
$$O_t = S_t \, \xi + \sum_{i \in \mathbb{N}^d} b(i) \int_0^t e^{-\lambda_i(t-s)} \, d\beta_s^i \cdot e_i = S_t \, \xi + \int_0^t S_{t-s} \, B \, dW_s$$

$\mathbb{P}$-a.s. for every $t \in (0,T]$, where W_t, $t \in [0,T]$, is an appropriate cylindrical I-Wiener process on $L^2((0,1)^d, \mathbb{R})$. In particular, $O \colon [0,T] \times \Omega \to C([0,1]^d, \mathbb{R})$ is the up-to-indistinguishability unique mild solution process of the linear SPDE

(4.8)
$$dO_t = \left[\Delta O_t \right] dt + B \, dW_t, \qquad O_t|_{\partial(0,1)^d} \equiv 0, \qquad O_0 = \xi$$

for $t \in [0,T]$ on $C([0,1]^d, \mathbb{R})$. The process O thus includes the initial value and a stochastic convolution of the semigroup generated by the Laplacian with Dirichlet boundary conditions and a cylindrical Wiener process as it is frequently considered in the literature (see, e.g., section 5 in [13]). Note also that the operator B appearing in (4.8) is diagonal with respect to the orthonormal basis $e_i \in L^2((0,1)^d, \mathbb{R})$, $i \in \mathbb{N}^d$, in $L^2((0,1)^d, \mathbb{R})$. This assumption is strongly exploited in Proposition 4.2. However, the abstract setting in section 2 does not need this assumption to be fulfilled and, in principle, linear operators B that are not diagonal with respect to e_i, $i \in \mathbb{N}^d$, could be considered here. The detailed analysis in the nondiagonal case remains an open question for future research. The reader is referred to [4] for first results in that direction.

To illustrate Proposition 4.2 we consider the following simple example. If $d = 2$, $(\xi(\omega))(x) = 0$ for all $x \in [0,1]^2$, $\omega \in \Omega$, and $b((i_1, i_2)) = \frac{1}{(i_1 + i_2)}$ for all $i = (i_1, i_2) \in \mathbb{N}^2$ in Proposition 4.2, then Proposition 4.2 implies the existence of $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mappings $C_\gamma \colon \Omega \to [0,\infty)$, $\gamma \in (0,1)$, such that

(4.9)
$$\sup_{0 \leq t \leq 1} \sup_{x \in [0,1]^2} \left| O_t(\omega, x) - (P_N O_t)(\omega, x) \right| \leq C_\gamma(\omega) \cdot N^{-\gamma}$$

for all $\omega \in \Omega$, $N \in \mathbb{N}$, and all $\gamma \in (0,1)$. Finally, note that Proposition 4.2 follows immediately from the next result (Lemma 4.3), which is also of independent interest. Its proof is postponed to subsection 5.2.1. Estimates related to Lemma 4.3 and its proof can, e.g., be found in section 5.5.1 in Da Prato and Zabczyk [13] and in Propositions 1.1 and 1.2 in Da Prato and Debussche [9]. In particular, the temporal regularity statements in Lemma 4.3 follow, e.g., immediately from Lemma 5.19 and Theorem 5.20 in [13].

LEMMA 4.3. *Let $d \in \mathbb{N}$, let $V = C([0,1]^d, \mathbb{R})$ with $\|v\|_V = \|v\|_{C([0,1]^d, \mathbb{R})}$ for every $v \in V$, let $\rho \in (0, \infty)$, let $\beta^i \colon [0,T] \times \Omega \to \mathbb{R}$, $i \in \mathbb{N}^d$, be a family of independent standard Brownian motions with continuous sample paths, and let $b \colon \mathbb{N}^d \to \mathbb{R}$ be a function with $\sum_{i \in \mathbb{N}^d} \left(i_1^2 + \cdots + i_d^2 \right)^{(\rho - 1)} |b(i)|^2 < \infty$. Then there exists an up-to-indistinguishability unique stochastic process $O \colon [0,T] \times \Omega \to V$ which satisfies*

$$\sup_{N \in \mathbb{N}} \sup_{0 \le t \le T} \left(N^\gamma \|O_t(\omega) - P_N(O_t(\omega))\|_V \right) + \sup_{0 \le t_1 < t_2 \le T} \frac{\|O_{t_2}(\omega) - O_{t_1}(\omega)\|_V}{|t_2 - t_1|^\theta} < \infty$$

for every $\omega \in \Omega$, every $\theta \in (0, \min(\frac{1}{2}, \frac{\rho}{2}))$, every $\gamma \in (0, \rho)$, and which satisfies

$$\sup_{N \in \mathbb{N}} \left\{ N^\gamma \left(\mathbb{E}\left[\sup_{0 \le t \le T} \|O_t - P_N(O_t)\|_V^p \right] \right)^{\frac{1}{p}} \right\} + \sup_{0 \le t_1 < t_2 \le T} \frac{(\mathbb{E}[\|O_{t_2} - O_{t_1}\|_V^p])^{\frac{1}{p}}}{|t_2 - t_1|^\theta} < \infty$$

and

$$\mathbb{P}\left[\lim_{N \to \infty} \sup_{0 \le t \le T} \left\| O_t - \sum_{i \in \{1, \dots, N\}^d} b(i) \left(-\lambda_i \int_0^t e^{-\lambda_i(t-s)} \beta_s^i \, ds + \beta_t^i \right) \cdot e_i \right\|_V = 0 \right] = 1$$

for every $p \in [1, \infty)$, every $\theta \in (0, \frac{\rho}{2}) \cap [0, \frac{1}{2}]$, and every $\gamma \in (0, \rho)$. Here the functions $e_i \in V$, $i \in \mathbb{N}^d$, the real numbers λ_i, $i \in \mathbb{N}^d$, and the linear operators $P_N \colon V \to V$, $N \in \mathbb{N}$, are given in (4.1) and (4.2).

4.2. Stochastic evolution equations with a globally Lipschitz nonlinearity. If the nonlinearity $F \colon V \to W$ given in Assumption 2 is globally Lipschitz continuous from V to W, then Assumption 4 is naturally met.

PROPOSITION 4.4. *Suppose that Assumptions 1–3 are fulfilled. If the nonlinearity $F \colon V \to W$ given in Assumption 2 additionally satisfies $\sup_{v,w \in V, v \ne w} \frac{\|F(v) - F(w)\|_W}{\|v - w\|_V} < \infty$, then Assumption 4 is fulfilled.*

The proof of Proposition 4.4 is straightforward and therefore omitted. In the remainder of this section we illustrate Theorem 3.1 with a stochastic reaction diffusion equation with a globally Lipschitz nonlinearity. The next lemma describes the nonlinearities considered in this subsection. Its proof is clear and hence omitted.

LEMMA 4.5. *Let $d \in \mathbb{N}$ and let $f \colon [0,1]^d \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies*

$$(4.10) \qquad \sup_{x \in [0,1]^d} \sup_{\substack{y_1, y_2 \in \mathbb{R} \\ y_1 \ne y_2}} \frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} < \infty.$$

Then the corresponding Nemytskii operator $F \colon C([0,1]^d, \mathbb{R}) \to C([0,1]^d, \mathbb{R})$ given by $(F(v))(x) = f(x, v(x))$ for every $x \in [0,1]^d$ and every $v \in C([0,1]^d, \mathbb{R})$ satisfies

$$(4.11) \qquad \sup_{v,w \in V, v \ne w} \frac{\|F(v) - F(w)\|_{C([0,1]^d, \mathbb{R})}}{\|v - w\|_{C([0,1]^d, \mathbb{R})}} < \infty.$$

Let $d \in \{1, 2, 3\}$ and $V = W = C([0, 1]^d, \mathbb{R})$ and let $P_N \colon V \to V$, $N \in \mathbb{N}$, $S \colon (0, T] \to L(V)$, $F \colon V \to V$ and $O \colon [0, T] \times \Omega \to V$ be given by (4.2), (4.3), Lemma 4.5, and Proposition 4.2. Then Assumption 4 is fulfilled due to Proposition 4.4 and therefore the assumptions in Theorem 3.1 are fulfilled. In addition, the stochastic evolution equation (3.1) reduces in this case to

$$(4.12) \qquad dX_t = \left[\Delta X_t + f(\cdot, X_t) \right] dt + B \, dW_t, \quad X_t|_{\partial(0,1)^d} \equiv 0, \quad X_0 = \xi$$

for $t \in [0, T]$, where W_t, $t \in [0, T]$, is a cylindrical I-Wiener process on $L^2((0, 1)^d, \mathbb{R})$, where $\xi \colon \Omega \to V$ is used in Proposition 4.2, and where the bounded linear operator $B \colon L^2((0, 1)^d, \mathbb{R}) \to L^2((0, 1)^d, \mathbb{R})$ is given by (4.6) with $b \colon \mathbb{N}^d \to \mathbb{R}$ used in Proposition 4.2. Moreover, the finite-dimensional stochastic ordinary differential equations (SODEs) (2.1) reduce to

$$dX_t^N = \left[\Delta X_t^N + P_N f(\cdot, X_t^N) \right] dt + P_N B \, dW_t, \quad X_t^N|_{\partial(0,1)^d} \equiv 0, \quad X_0^N = P_N(\xi)$$

for $t \in [0, T]$ and $N \in \mathbb{N}$. If $d = 1$ and $b(i) = b(1)$ for all $i \in \mathbb{N}$, then Lemma 4.1, Proposition 4.2, and Theorem 3.1 yield the existence of $\mathcal{F}/\mathcal{B}([0, \infty))$-measurable mappings $C_\gamma \colon \Omega \to [0, \infty)$, $\gamma \in (0, \frac{1}{2})$, such that

$$(4.13) \qquad \sup_{0 \le t \le T} \sup_{0 \le x \le 1} \left| X_t(\omega, x) - X_t^N(\omega, x) \right| \le C_\gamma(\omega) \cdot N^{-\gamma}$$

for every $\omega \in \Omega$, $N \in \mathbb{N}$, and every $\gamma \in (0, \frac{1}{2})$. Hence, in the case $d = 1$ and $b(i) = b(1)$ for all $i \in \mathbb{N}$, we obtain that $X_t^N(\omega, x)$ converges to $X_t(\omega, x)$ uniformly in $t \in [0, T]$ and $x \in [0, 1]$ with the rate $\frac{1}{2}-$ as N goes to infinity for every $\omega \in \Omega$.

4.3. Stochastic Burgers equation. In this subsection a stochastic Burgers equation is formulated in the setting of section 2. For this, a few function spaces from the literature (see, e.g., Chapter 5 in [37]) are presented first. By $(L^2((0, 1), \mathbb{R}), \|\cdot\|_{L^2}, \langle \cdot, \cdot \rangle_{L^2})$ the $\mathbb{R}$-Hilbert space of equivalence classes of $\mathcal{B}(0, 1)/\mathcal{B}(\mathbb{R})$-measurable and Lebesgue square integrable functions from $(0, 1)$ to $\mathbb{R}$ with scalar product $\langle v, w \rangle_{L^2} := \int_0^1 v(x) \, w(x) \, dx$ and norm $\|v\|_{L^2} := (\langle v, v \rangle_{L^2})^{1/2}$ for every $v, w \in L^2((0, 1), \mathbb{R})$ is denoted. In addition, by $H^1((0, 1), \mathbb{R})$ the Sobolev space of weakly differentiable functions from $(0, 1)$ to $\mathbb{R}$ with weak derivatives in $L^2((0, 1), \mathbb{R})$ is denoted. The norm and the scalar product in $H^1((0, 1), \mathbb{R})$ are defined by $\|v\|_{H^1} := (\|v\|_{L^2}^2 + \|v'\|_{L^2}^2)^{1/2}$ and $\langle v, w \rangle_{H^1} := \langle v, w \rangle_{L^2} + \langle v', w' \rangle_{L^2}$, respectively, for every $v, w \in H^1((0, 1), \mathbb{R})$. Additionally, by $H_0^1((0, 1), \mathbb{R})$ the closure of $C_{\mathrm{cpt}}^\infty((0, 1), \mathbb{R})$ in the $\mathbb{R}$-Hilbert space $(H^1((0, 1), \mathbb{R}), \|\cdot\|_{H^1}, \langle \cdot, \cdot \rangle_{H^1})$ is denoted and the norm and the scalar product in $H_0^1((0, 1), \mathbb{R})$ are denoted by $\|v\|_{H_0^1} := \|v'\|_{L^2}$ and $\langle v, w \rangle_{H_0^1} := \langle v', w' \rangle_{L^2}$, respectively, for every $v, w \in H_0^1((0, 1), \mathbb{R})$. The Sobolev space $(H^{-1}((0, 1), \mathbb{R}), \|\cdot\|_{H^{-1}}) := (H_0^1((0, 1), \mathbb{R}), \|\cdot\|_{H_0^1})'$ is also used below and by $\partial \colon L^2((0, 1), \mathbb{R}) \to H^{-1}((0, 1), \mathbb{R})$, the distributional derivative in $L^2((0, 1), \mathbb{R})$ defined by $(\partial v)(\varphi) = (v')(\varphi) := -\langle v, \varphi' \rangle_{L^2}$ for every $\varphi \in H_0^1((0, 1), \mathbb{R})$ and every $v \in L^2((0, 1), \mathbb{R})$ is denoted.

In view of these function spaces, let $W = H^{-1}((0, 1), \mathbb{R})$ with $\|v\|_W := \|v\|_{H^{-1}}$ for all $v \in W$ and let $V = C([0, 1], \mathbb{R})$ with $\|v\|_V := \sup_{x \in [0,1]} |v(x)|$ for all $v \in V$ be the $\mathbb{R}$-Banach space of continuous functions from $[0, 1]$ to $\mathbb{R}$. As in sections 4.1 and 4.2, we use the projection operators $P_N \colon C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$, $N \in \mathbb{N}$, defined by

$$(4.14) \qquad \left(P_N(v) \right)(x) := \sum_{n=1}^N 2 \int_0^1 \sin(n\pi s) \, v(s) \, ds \cdot \sin(n\pi x)$$

for every $x \in [0,1]$, $v \in C([0,1], \mathbb{R})$, and every $N \in \mathbb{N}$. The semigroup is constructed in the following well-known lemma here.

LEMMA 4.6. *The mapping* $S \colon (0,T] \to L\big(H^{-1}((0,1), \mathbb{R}), C([0,1], \mathbb{R})\big)$ *given by* $\big(S_t(w)\big)(x) = \sum_{n=1}^{\infty} 2 \cdot e^{-n^2 \pi^2 t} \cdot w(\sin(n\pi(\cdot))) \cdot \sin(n\pi x)$ *for every* $x \in [0,1]$, $w \in H^{-1}((0,1), \mathbb{R})$, *and every* $t \in (0,T]$ *is well defined and satisfies Assumption* 1 *for every* $\gamma \in (0, \frac{1}{2})$.

The proof of Lemma 4.6 can be found in subsection 5.3.1. The next well-known lemma describes the nonlinearities for the stochastic Burgers equations considered in this section.

LEMMA 4.7. *Let* $c \in \mathbb{R}$ *be a real number. Then the mapping* $F \colon C([0,1], \mathbb{R}) \to H^{-1}((0,1), \mathbb{R})$ *given by* $F(v) = c \cdot \partial(v^2)$ *for every* $v \in C([0,1], \mathbb{R})$ *satisfies Assumption* 2.

Proof. The estimate $\|\partial v\|_{H^{-1}} \leq \|v\|_{L^2}$ for every $v \in L^2((0,1), \mathbb{R})$ implies

$$(4.15) \quad \|F(v) - F(w)\|_{H^{-1}} = \|c\,\partial(v^2) - c\,\partial(w^2)\|_{H^{-1}} \leq |c| \cdot \|v^2 - w^2\|_{L^2}$$
$$\leq |c| \cdot \big(\|v\|_{C([0,1], \mathbb{R})} + \|w\|_{C([0,1], \mathbb{R})}\big) \cdot \|v - w\|_{C([0,1], \mathbb{R})}$$

for every $v, w \in C([0,1], \mathbb{R})$. This yields $\|F(v) - F(w)\|_{H^{-1}} \leq 2\,r\,|c|\,\|v - w\|_{C([0,1], \mathbb{R})}$ for every $v, w \in C([0,1], \mathbb{R})$ with $\|v\|_{C([0,1], \mathbb{R})}, \|w\|_{C([0,1], \mathbb{R})} \leq r$ and every $r \in (0, \infty)$. The proof of Lemma 4.7 is thus completed. $\qquad \square$

For these types of nonlinearities, Assumption 4 is fulfilled, which can be seen in the following lemma. Its proof is postponed to subsection 5.3.2 below. A related result can be found in Da Prato, Debussche, and Temam [10] (see Lemma 3.1 and Theorem 3.1 in [10]).

LEMMA 4.8. *Let* $V = C([0,1], \mathbb{R})$ *with* $\|v\|_V = \sup_{0 \leq x \leq 1} |v(x)|$ *for all* $v \in V$, *let* $W = H^{-1}((0,1), \mathbb{R})$ *with* $\|v\|_W = \|v\|_{H^{-1}}$ *for all* $v \in W$, *and let* $S \colon (0,T] \to L(W,V)$, $F \colon V \to W$, *and* $P_N \colon V \to V$, $N \in \mathbb{N}$, *be given by Lemmas* 4.6 *and* 4.7 *and* (4.14). *Moreover, let* $O \colon [0,T] \times \Omega \to V$ *be an arbitrary stochastic process with continuous sample paths and with* $\sup_{N \in \mathbb{N}} \sup_{0 \leq t \leq T} \|P_N(O_t(\omega))\|_V < \infty$ *for every* $\omega \in \Omega$. *Then Assumption* 4 *is fulfilled.*

We emphasize that Lemma 4.8 does not assume that the driving noise process $O \colon [0,T] \times \Omega \to V$ is a stochastic convolution involving a cylindrical Wiener process as considered in Proposition 4.2. In particular, Lemma 4.8 covers stochastic Burgers equations driven by fractional Brownian motions. In the next step the consequences of Lemmas 4.6–4.8 and Theorem 3.1 are illustrated by a numerical example.

Numerical Example. We consider the stochastic evolution equation (3.1) with $S \colon (0,T] \to L(W,V)$, $F \colon V \to W$, and $O \colon [0,T] \times \Omega \to V$ given by Lemmas 4.6 and 4.7 and Proposition 4.2 with the parameters $c = -30$, $T = \frac{1}{20}$, $\xi(\omega) = \frac{6}{5} \sin(\pi x)$ for every $\omega \in \Omega$, and $b(i) = \frac{1}{3}$ for every $i \in \mathbb{N}$. The stochastic evolution equation (3.1) then reduces to

$$(4.16) \qquad dX_t(x) = [\Delta X_t - 60 \cdot X_t \cdot X_t'] \, dt + \frac{1}{3} \, dW_t, \qquad X_0(\cdot) = \tfrac{6}{5} \sin(\pi \cdot)$$

with $X_t(0) = X_t(1) = 0$ for $t \in [0, \frac{1}{20}]$ on $C([0,1], \mathbb{R})$ and the finite-dimensional SODEs (2.1) simplify to

$$(4.17) \quad dX_t^N = \big[\Delta X_t^N - 60 \cdot P_N\big(X_t^N \cdot (X_t^N)'\big)\big] \, dt + \tfrac{1}{3} \, P_N \, dW_t, \qquad X_0^N(\cdot) = \tfrac{6}{5} \sin(\pi \cdot)$$

with $X_t^N(0) = X_t^N(1) = 0$ for $t \in [0, \frac{1}{20}]$ and $N \in \mathbb{N}$ on $C([0,1], \mathbb{R})$. Here W_t, $t \in [0, \frac{1}{20}]$, is a cylindrical I-Wiener process on $L^2((0,1), \mathbb{R})$. Combining Proposition 4.2
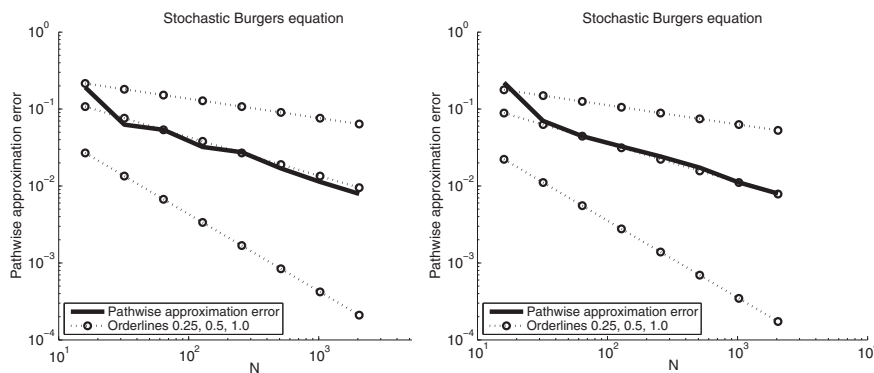
FIG. 4.1. *Pathwise approximation error* (4.20) *against N for* $N \in \{16, 32, 64, \ldots, 1024, 2048\}$ *and two random* $\omega \in \Omega$.

and Lemmas 4.6–4.8 with Theorem 3.1 then yields the existence of a unique solution process $X \colon [0, \frac{1}{20}] \times \Omega \to C([0,1], \mathbb{R})$ with continuous sample paths of the SPDE (4.16). Moreover, Proposition 4.2, Lemmas 4.6–4.8, and Theorem 3.1 imply the existence of $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mappings $C_\gamma \colon \Omega \to [0,\infty)$, $\gamma \in (0, \frac{1}{2})$, such that

$$(4.18) \qquad \sup_{0 \le t \le \frac{1}{20}} \sup_{0 \le x \le 1} \left| X_t(\omega, x) - X_t^N(\omega, x) \right| \le C_\gamma(\omega) \cdot N^{-\gamma}$$

for every $N \in \mathbb{N}$, $\omega \in \Omega$, and every $\gamma \in (0, \frac{1}{2})$. Hence, the solutions $X_t^N(\omega, x)$ of the finite-dimensional SODEs (4.17) converge to the solution $X_t(\omega, x)$ of the stochastic Burgers equation (4.16) with the rate $\frac{1}{2}-$ uniformly in $t \in [0, \frac{1}{20}]$ and $x \in [0,1]$ as N goes to infinity for every $\omega \in \Omega$. In Figure 4.1 the pathwise approximation error

$$(4.19) \qquad \sup_{0 \le t \le \frac{1}{20}} \sup_{0 \le x \le 1} \left| X_t(\omega, x) - X_t^N(\omega, x) \right|$$

is calculated approximatively and plotted against $N \in \{16, 32, 64, \ldots, 1024, 2048\}$ and two random $\omega \in \Omega$. More precisely, in the simulations presented in Figure 4.1, the quantities (4.19) are approximated through the quantities
(4.20)

$$\sup_{t \in \left\{ \frac{m}{4000} \colon m \in \{0,1,\ldots,200\} \right\}} \sup_{x \in \left\{ \frac{k}{16385} \colon k \in \{0,1,\ldots,16385\} \right\}} \left| Y_t^{16384,200}(\omega, x) - Y_t^{N,200}(\omega, x) \right|$$

for $N \in \{16, 32, 64, \ldots, 1024, 2048\}$ and two random $\omega \in \Omega$, where $Y_t^{N,200} \colon \Omega \to P_N C([0,1]^d, \mathbb{R})$ with $Y_t^{N,200} \approx X_t^N$ (N Fourier nodes for the spatial discretization and 200 time steps on the interval $[0, \frac{1}{20}]$ for the temporal discretization) for $N \in \{16, 32, 64, \ldots, 1024, 2048\} \cup \{16384\}$ and $t \in \{0, \frac{1}{4000}, \frac{2}{4000}, \ldots, \frac{199}{4000}, \frac{1}{20}\}$ are suitable accelerated exponential Euler approximations (see section 3 in [25]) for the SPDE (4.16). Figure 4.1 indicates that the quantity (4.19) converges to zero with the (from (4.18)) theoretically predicated order $\frac{1}{2}-$.

5. Proofs. In this section we collect all technical proofs of the previous sections.

5.1. Proof of Theorem 3.1.
Proof. Consider the $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mapping $R \colon \Omega \to [0,\infty)$ defined through

(5.1)

$$R(\omega) := \sup_{N \in \mathbb{N}} \sup_{0 \le t \le T} \left\| F(X_t^N(\omega)) \right\|_W + T + \sup_{N \in \mathbb{N}} \sup_{0 \le t \le T} \left(N^\gamma \left\| O_t(\omega) - P_N(O_t(\omega)) \right\|_V \right)$$

$$+ \frac{1}{(1-\alpha)} + \sup_{N \in \mathbb{N}} \sup_{0 < t \le T} \left(t^\alpha \left\| P_N S_t \right\|_{L(W,V)} \right)$$

$$+ \sup_{N \in \mathbb{N}} \sup_{0 < t \le T} \left(t^\alpha N^\gamma \left\| S_t - P_N S_t \right\|_{L(W,V)} \right)$$

for every $\omega \in \Omega$. Due to Assumptions 1–4, the mapping R is indeed finite. Moreover, note that R is indeed $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable although V is not assumed to be separable. Next consider the $\mathcal{B}([0,\infty))/\mathcal{B}([0,\infty))$-measurable mapping $L\colon [0,\infty) \to [0,\infty)$ given by $L(r) := \sup\left\{ \frac{\|F(v)-F(w)\|_W}{\|v-w\|_V} \colon \|v\|_V \le r,\ \|w\|_V \le r,\ v \ne w \right\}$ for every $r \in [0,\infty)$. Additionally, consider the $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mapping $Z\colon \Omega \to [0,\infty)$ given by $Z(\omega) := L\left(\sup_{N \in \mathbb{N}} \sup_{0 \le t \le T} \|X_t^N(\omega)\|_V \right)$ for every $\omega \in \Omega$. In the next step the definition of R implies

$$(5.2) \quad \|X_t^N - X_t^M\|_V \le \left\| \int_0^t P_N S_{t-s} \left(F(X_s^N) - F(X_s^M) \right) ds \right\|_V$$

$$+ \left\| \int_0^t (P_N - P_M) S_{t-s} F(X_s^M)\, ds \right\|_V + R\left(N^{-\gamma} + M^{-\gamma} \right)$$

for every $N, M \in \mathbb{N}$ and every $t \in [0,T]$, and the estimates $\|(P_N - I) S_{t-s}\|_{L(W,V)} \le RN^{-\gamma}(t-s)^{-\alpha}$ and $\|P_N S_{t-s}\|_{L(W,V)} \le R(t-s)^{-\alpha}$ for every $N, M \in \mathbb{N}$, $s \in [0,t)$, and every $t \in (0,T]$ therefore show

$$(5.3) \quad \|X_t^N - X_t^M\|_V \le R \int_0^t \frac{\|F(X_s^N) - F(X_s^M)\|_W}{(t-s)^\alpha}\, ds$$

$$+ R\left(N^{-\gamma} + M^{-\gamma} \right) \int_0^t \frac{\|F(X_s^M)\|_W}{(t-s)^\alpha}\, ds + R\left(N^{-\gamma} + M^{-\gamma} \right)$$

for every $N, M \in \mathbb{N}$ and every $t \in [0,T]$. Hence, we have

$$(5.4) \quad \|X_t^N - X_t^M\|_V \le RZ \int_0^t \|X_s^N - X_s^M\|_V\, (t-s)^{-\alpha}\, ds + (R + R^4)(N^{-\gamma} + M^{-\gamma})$$

for every $N, M \in \mathbb{N}$ and every $t \in [0,T]$, where we used the estimate $\frac{T^{(1-\alpha)}}{(1-\alpha)} \le RT^{(1-\alpha)} \le R^{(2-\alpha)} \le R^2$ in the last inequality of (5.4). Lemma 7.1.11 in Henry [22] hence yields

$$(5.5) \quad \left\| X_t^N - X_t^M \right\|_V \le \mathrm{E}_{(1-\alpha)} \left(t \left(RZ\,\Gamma(1-\alpha) \right)^{\frac{1}{(1-\alpha)}} \right) \left(R + R^4 \right) \left(N^{-\gamma} + M^{-\gamma} \right)$$

$$\le \mathrm{E}_{(1-\alpha)} \left(T \left(RZ\,\Gamma(1-\alpha) \right)^{\frac{1}{(1-\alpha)}} \right) \left(2R^4 \right) \left(N^{-\gamma} + M^{-\gamma} \right)$$

for every $N, M \in \mathbb{N}$ and every $t \in [0,T]$. Here and below the functions $\mathrm{E}_r\colon [0,\infty) \to [0,\infty)$, $r \in (0,\infty)$, are defined through $\mathrm{E}_r(x) := \sum_{n=0}^\infty \frac{x^{nr}}{\Gamma(nr+1)}$ for all $x \in [0,\infty)$ and all $r \in (0,\infty)$ (see Lemma 7.1.11 in [22] for details). This shows that $\left(X^N(\omega) \right)_{N \in \mathbb{N}}$ is a Cauchy sequence in $C([0,T],V)$ for every $\omega \in \Omega$. Since $C([0,T],V)$ is complete, we can define the stochastic process $X\colon [0,T] \times \Omega \to V$ with continuous sample paths by $X_t(\omega) := \lim_{N \to \infty} X_t^N(\omega)$ for every $t \in [0,T]$ and every $\omega \in \Omega$. Hence, we obtain

$$X_t(\omega) = \lim_{N \to \infty} X_t^N(\omega) = \lim_{N \to \infty} \left(\int_0^t P_N \, S_{t-s} \, F(X_s^N(\omega)) \, ds + P_N(O_t(\omega)) \right)$$

$$= \lim_{N \to \infty} \left(\int_0^t P_N \, S_{t-s} \, F(X_s^N(\omega)) \, ds \right) + O_t(\omega) = \int_0^t S_{t-s} \, F(X_s(\omega)) \, ds + O_t(\omega)$$

for every $t \in [0, T]$ and every $\omega \in \Omega$. Moreover, if $Y \colon [0, T] \times \Omega \to V$ is a further stochastic process with continuous sample paths and with $Y_t(\omega) = \int_0^t S_{t-s} \, F(Y_s(\omega)) \, ds + O_t(\omega)$ for every $t \in [0, T]$ and every $\omega \in \Omega$, then we obtain

(5.6)

$$\|X_t - Y_t\|_V \leq R \int_0^t (t-s)^{-\alpha} \|F(X_s) - F(Y_s)\|_W \, ds$$

$$\leq R \cdot L \left(\sup_{0 \leq r \leq T} \|X_r\|_V + \sup_{0 \leq r \leq T} \|Y_r\|_V \right) \cdot \int_0^t (t-s)^{-\alpha} \|X_s - Y_s\|_V \, ds$$

for every $t \in [0, T]$. Lemma 7.1.11 in [22] therefore shows that $X \colon [0, T] \times \Omega \to V$ is the pathwise unique stochastic process with continuous sample paths satisfying (3.1). Moreover, (5.5) yields $\sup_{0 \leq t \leq T} \|X_t - X_t^N\|_V \leq C \cdot N^{-\gamma}$ for every $N \in \mathbb{N}$, where the $\mathcal{F}/\mathcal{B}[0, \infty))$-measurable mapping $C \colon \Omega \to [0, \infty)$ is given by

(5.7)
$$C(\omega) := 2 \cdot (R(\omega))^4 \cdot \mathrm{E}_{(1-\alpha)} \left(T \left(R(\omega) Z(\omega) \Gamma(1-\alpha) \right)^{\frac{1}{(1-\alpha)}} \right)$$

for every $\omega \in \Omega$. The proof of Theorem 3.1 is thus completed. $\quad \square$

5.2. Proofs for subsection 4.1.

5.2.1. Proof of Lemma 4.3. Throughout this subsection we use the notation $\|x\|_2 := \left(x_1^2 + \cdots + x_d^2 \right)^{\frac{1}{2}}$ for every $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. We first present three elementary lemmas, which we need in the proof of Lemma 4.3. They are, for example, proved as Lemmas 9, 11, and 12 in [24].

LEMMA 5.1. *It holds that* $\int_{(0,1)^d} \int_{(0,1)^d} \frac{1}{\|x-y\|_2^\alpha} \, dx \, dy \leq \frac{(3d)^d}{(d-\alpha)}$ *for every* $\alpha \in (0, d)$ *and every* $d \in \mathbb{N}$.

LEMMA 5.2. *Let* $d \in \mathbb{N}$ *and let* $e_i \in C([0,1]^d, \mathbb{R})$, $i \in \mathbb{N}^d$, *be given by* (4.1). *Then* $|e_i(x) - e_i(y)| \leq 2^{\frac{d}{2}} \pi \|i\|_2 \|x - y\|_2$ *for every* $x, y \in [0,1]^d$ *and every* $i \in \mathbb{N}^d$.

LEMMA 5.3. *Let* $\beta \colon [0, T] \times \Omega \to \mathbb{R}$ *be a standard Brownian motion. Then* $\mathbb{E}[|\int_0^{t_2} e^{-\lambda(t_2-s)} \, d\beta_s - \int_0^{t_1} e^{-\lambda(t_1-s)} \, d\beta_s|^2] \leq \lambda^{(r-1)} |t_2 - t_1|^r$ *for every* $t_1, t_2 \in [0, T]$, $r \in [0, 1]$, *and every* $\lambda \in (0, \infty)$.

After these three very simple lemmas, we present now two lemmas (Lemmas 5.4 and 5.5), which are the essential constituents in the proof of Lemma 4.3. The first one will ensure the temporal regularity of the processes that are constructed in Lemma 4.3.

LEMMA 5.4. *Let* $d \in \mathbb{N}$, *let* $\beta^i \colon [0, T] \times \Omega \to \mathbb{R}$, $i \in \mathbb{N}^d$, *be a family of independent standard Brownian motions, and let* $b \colon \mathbb{N}^d \to \mathbb{R}$ *be an arbitrary function. Then*
(5.8)
$$\left(\mathbb{E} \left[\sup_{x \in [0,1]^d} |O_{t_2}^N(x) - O_{t_1}^N(x)|^p \right] \right)^{\frac{1}{p}} \leq C_\star \left[\sum_{i \in \{1, \ldots, N\}^d} |b(i)|^2 \|i\|_2^{(4\theta + 4\alpha - 2)} \right]^{\frac{1}{2}} |t_2 - t_1|^\theta$$

for every $t_1, t_2 \in [0, T]$, $N \in \mathbb{N}$, $p \in [1, \infty)$, *and every* $\alpha, \theta \in (0, \frac{1}{2}]$, *where* $C_\star \in [0, \infty)$ *is a constant which depends on* d, p, α, *and* θ *only and where the stochastic process* $O^N \colon [0, T] \times \Omega \to C([0,1]^d, \mathbb{R})$ *is defined through*

$$(5.9) \qquad O_t^N(\omega) := \sum_{i \in \{1,\ldots,N\}^d} b(i) \left(-\lambda_i \int_0^t e^{-\lambda_i(t-s)} \beta_s^i(\omega)\, ds + \beta_t^i(\omega) \right) \cdot e_i$$

for every $t \in [0,T]$, $\omega \in \Omega$, and every $N \in \mathbb{N}$. Here $e_i \in C([0,1]^d, \mathbb{R})$, $i \in \mathbb{N}^d$, and $\lambda_i \in \mathbb{R}$, $i \in \mathbb{N}^d$, are given in (4.1).

Proof. Throughout this proof let $\alpha, \theta \in (0, \frac{1}{2}]$, $p, N \in \mathbb{N}$ with $p > \frac{1}{\alpha}$, and $t_1, t_2 \in [0,T]$ with $t_1 \leq t_2$ be fixed. In addition, let $C = C_{d,p,\alpha,\theta} \in [0,\infty)$ be a constant which changes from line to line but depends on d, p, α, and θ only. We show now inequality (5.8) for these parameters and the case, with a general $p \in [1,\infty)$, then follows from Jensen's inequality. The definition of O^N implies

$$(5.10) \quad (O_{t_2}^N(x) - O_{t_1}^N(x)) - (O_{t_2}^N(y) - O_{t_1}^N(y))$$
$$= \sum_{i \in \{1,\ldots,N\}^d} b(i) \left(\int_0^{t_2} e^{-\lambda_i(t_2-s)}\, d\beta_s^i - \int_0^{t_1} e^{-\lambda_i(t_1-s)}\, d\beta_s^i \right) \cdot (e_i(x) - e_i(y))$$

$\mathbb{P}$-a.s. for every $x, y \in [0,1]^d$. Hence, Lemmas 5.2 and 5.3 yield

(5.11)

$$\mathbb{E}\left[\left| (O_{t_2}^N(x) - O_{t_1}^N(x)) - (O_{t_2}^N(y) - O_{t_1}^N(y)) \right|^2 \right]$$
$$= \sum_{i \in \{1,\ldots,N\}^d} |b(i)|^2 \, \mathbb{E}\left[\left| \int_0^{t_2} e^{-\lambda_i(t_2-s)}\, d\beta_s^i - \int_0^{t_1} e^{-\lambda_i(t_1-s)}\, d\beta_s^i \right|^2 \right] |e_i(x) - e_i(y)|^2$$
$$\leq \sum_{i \in \{1,\ldots,N\}^d} |b(i)|^2 |\lambda_i|^{(2\theta-1)} |t_2 - t_1|^{2\theta}$$
$$\times \left(2^d \pi^2 \|i\|_2^2 \|x-y\|_2^2 \right)^{2\alpha} \left(|e_i(x)| + |e_i(y)| \right)^{2(1-2\alpha)}$$
$$\leq C \, |t_2 - t_1|^{2\theta} \, \|x-y\|_2^{4\alpha} \sum_{i \in \{1,\ldots,N\}^d} |b(i)|^2 \, \|i\|_2^{(4\theta+4\alpha-2)}$$

for every $x, y \in [0,1]^d$. Moreover, Lemma 5.3 gives

$$(5.12) \quad \mathbb{E}\big[|O_{t_2}^N(x) - O_{t_1}^N(x)|^2 \big]$$
$$= \sum_{i \in \{1,\ldots,N\}^d} |b(i)|^2 \, \mathbb{E}\left[\left| \int_0^{t_2} e^{-\lambda_i(t_2-s)}\, d\beta_s^i - \int_0^{t_1} e^{-\lambda_i(t_1-s)}\, d\beta_s^i \right|^2 \right] |e_i(x)|^2$$
$$\leq C \sum_{i \in \{1,\ldots,N\}^d} |b(i)|^2 \, \|i\|_2^{(4\theta-2)} \, |t_2 - t_1|^{2\theta}$$

for every $x \in [0,1]^d$. In the next step the Sobolev embeddings in subsections 2.2.4 and 2.4.4 in [38] yield

$$\mathbb{E}\left[\|O_{t_2}^N - O_{t_1}^N\|_{C([0,1]^d, \mathbb{R})}^p \right]$$
$$\leq C \int_{(0,1)^d} \int_{(0,1)^d} \frac{\mathbb{E}\big[|(O_{t_2}^N(x) - O_{t_1}^N(x)) - (O_{t_2}^N(y) - O_{t_1}^N(y))|^p \big]}{\|x-y\|_2^{d+p\alpha}}\, dx\, dy$$
$$+ C \int_{(0,1)^d} \mathbb{E}\left[|O_{t_2}^N(x) - O_{t_1}^N(x)|^p \right]\, dx$$

$$\leq C \int_{(0,1)^d} \int_{(0,1)^d} \frac{\left(\mathbb{E}\left[|(O_{t_2}^N(x) - O_{t_1}^N(x)) - (O_{t_2}^N(y) - O_{t_1}^N(y))|^2]\right)^{\frac{p}{2}}}{\|x - y\|_2^{d+p\alpha}} dx\, dy$$
$$+ C \int_{(0,1)^d} \left(\mathbb{E}\left[|O_{t_2}^N(x) - O_{t_1}^N(x)|^2]\right)^{\frac{p}{2}} dx$$

and (5.11) and (5.12) therefore show

$$\mathbb{E}\|O_{t_2}^N - O_{t_1}^N\|_{C([0,1]^d,\mathbb{R})}^p$$

$$\leq C \int_{(0,1)^d} \int_{(0,1)^d} \frac{|t_2 - t_1|^{p\theta} \|x - y\|_2^{2p\alpha}}{\|x - y\|_2^{d+p\alpha}} dx\, dy \left(\sum_{i \in \{1,\dots,N\}^d} |b(i)|^2 \|i\|_2^{(4\theta+4\alpha-2)}\right)^{\frac{p}{2}}$$

$$+ C \left(\sum_{i \in \{1,\dots,N\}^d} |b(i)|^2 \|i\|_2^{(4\theta-2)} |t_2 - t_1|^{2\theta}\right)^{\frac{p}{2}}$$

$$\leq C \left(1 + \int_{(0,1)^d} \int_{(0,1)^d} \|x - y\|_2^{p\alpha-d} dx\, dy\right) |t_2 - t_1|^{p\theta}$$

$$\times \left(\sum_{i \in \{1,\dots,N\}^d} |b(i)|^2 \|i\|_2^{4\theta+4\alpha-2}\right)^{\frac{p}{2}}.$$

Lemma 5.1 hence gives
(5.13)

$$\left(\mathbb{E}\left[\|O_{t_2}^N - O_{t_1}^N\|_{C([0,1]^d,\mathbb{R})}^p\right]\right)^{\frac{1}{p}} \leq C \left(\sum_{i \in \{1,\dots,N\}^d} |b(i)|^2 \|i\|_2^{(4\theta+4\alpha-2)}\right)^{\frac{1}{2}} |t_2 - t_1|^{\theta}$$

and this completes the proof of Lemma 5.4. $\square$

LEMMA 5.5. *Let $d \in \mathbb{N}$, let $\beta^i\colon [0,T] \times \Omega \to \mathbb{R}$, $i \in \mathbb{N}^d$, be a family of independent standard Brownian motions and let $b\colon \mathbb{N}^d \to \mathbb{R}$ be an arbitrary function. Then*

$$(5.14) \quad \left(\mathbb{E}\left[\sup_{0 \leq t \leq T} \sup_{x \in [0,1]^d} |O_t^N(x) - O_t^M(x)|^p\right]\right)^{\frac{2}{p}} \leq C_\star \sum_{i \in \{1,\dots,N\}^d \setminus \{1,\dots,M\}^d} |b(i)|^2 \|i\|_2^{(8\alpha-2)}$$

for every $N, M \in \mathbb{N}$ with $N \geq M$, every $p \in [1,\infty)$, and every $\alpha \in \left(0,\frac{1}{2}\right)$, where $C_\star \in [0,\infty)$ is a constant which depends on d, p, α, and T only and where $O^N\colon [0,T] \times \Omega \to C([0,1]^d, \mathbb{R})$, $N \in \mathbb{N}$, are stochastic processes defined through (5.9).

Proof. Throughout this proof let $\alpha \in (0,\frac{1}{2})$ and $p, N, M \in \mathbb{N}$ be fixed with $p > \frac{1}{\alpha}$ and $N \geq M$. In addition, let $C = C_{d,p,\alpha,T} \in [0,\infty)$ be a constant, which changes from line to line but which depends on d, p, α, and T only. As in the proof of Lemma 5.4, we show now inequality (5.14) for these parameters, and the case with a general $p \in [1,\infty)$, then follows from Jensen's inequality. We use the factorization method (see [12] and, e.g., section 5.3 in [13] and section 5 in [5]) to show (5.14). For this let $Y^{N,M}\colon [0,T] \times \Omega \to C([0,1]^d, \mathbb{R})$ be a stochastic process with continuous sample paths given by

$$(5.15) \qquad Y_t^{N,M} = \sum_{i \in \{1,\dots,N\}^d \setminus \{1,\dots,M\}^d} b(i) \int_0^t (t-s)^{-\alpha} e^{-\lambda_i(t-s)} d\beta_s^i \cdot e_i$$

$\mathbb{P}$-a.s. for every $t \in [0, T]$. By using Kolmogorov's theorem (see, e.g., Theorem 3.3 in [13]), one can check in a straightforward way that the stochastic processes $\int_0^t (t - s)^{-\alpha} e^{\lambda_i(t-s)} d\beta_s^i$, $t \in [0, T]$, $i \in \mathbb{N}^d$, indeed have modifications with continuous sample paths. The key idea of the factorization method is then to make use of the identity

$$(5.16) \qquad O_t^N - O_t^M = \frac{\sin(\pi\alpha)}{\pi} \int_0^t (t - s)^{(\alpha-1)} S_{t-s} Y_s^{N,M} \, ds$$

$\mathbb{P}$-a.s. for every $t \in [0, T]$ (see, e.g., equation (5.18) in section 5.3 in [13]). More precisely, combining (5.16), the well-known fact $\sup_{0 \le t \le T} \|S_t\|_{L(C([0,1]^d,\mathbb{R}))} \le 1$ (see, e.g., Lemma 6 in [24]), and Hölder's inequality gives

$$(5.17) \qquad \mathbb{E} \sup_{0 \le t \le T} \left\| O_t^N - O_t^M \right\|_{C([0,1]^d,\mathbb{R})}^p$$

$$= \mathbb{E} \sup_{0 \le t \le T} \left\| \frac{\sin(\pi\alpha)}{\pi} \int_0^t (t - s)^{(\alpha-1)} S_{t-s} Y_s^{N,M} \, ds \right\|_{C([0,1]^d,\mathbb{R})}^p$$

$$\le \mathbb{E} \sup_{0 \le t \le T} \left| \int_0^t (t - s)^{(\alpha-1)} \|Y_s^{N,M}\|_{C([0,1]^d,\mathbb{R})} \, ds \right|^p$$

$$\le C \int_0^T \mathbb{E}\left[\|Y_s^{N,M}\|_{C([0,1]^d,\mathbb{R})}^p \right] ds \,.$$

Hence, it remains to bound $\|Y_s^{N,M}\|_{C([0,1]^d,\mathbb{R})}$ in (5.17). For this, denote $\mathcal{I}_N := \{1, 2, \dots, N\}^d$ and $\mathcal{I}_M := \{1, 2, \dots, M\}^d$. Lemma 5.2 then implies

$$(5.18) \quad \mathbb{E}\big[|Y_t^{N,M}(x) - Y_t^{N,M}(y)|^2\big]$$

$$= \mathbb{E}\left[\left| \sum_{i \in \mathcal{I}_N \setminus \mathcal{I}_M} b(i) \int_0^t (t - s)^{-\alpha} e^{-\lambda_i(t-s)} d\beta_s^i \cdot (e_i(x) - e_i(y)) \right|^2 \right]$$

$$= \sum_{i \in \mathcal{I}_N \setminus \mathcal{I}_M} |b(i)|^2 \, \mathbb{E}\left[\left| \int_0^t (t - s)^{-\alpha} e^{-\lambda_i(t-s)} d\beta_s^i \right|^2 \right] \cdot |e_i(x) - e_i(y)|^2$$

$$= \sum_{i \in \mathcal{I}_N \setminus \mathcal{I}_M} |b(i)|^2 \int_0^t s^{-2\alpha} e^{-2\lambda_i s} \, ds \cdot |e_i(x) - e_i(y)|^{4\alpha} \, |e_i(x) - e_i(y)|^{(2-4\alpha)}$$

$$\le C \sum_{i \in \mathcal{I}_N \setminus \mathcal{I}_M} |b(i)|^2 \, \|i\|_2^{(8\alpha-2)} \, \|x - y\|_2^{4\alpha}$$

for every $t \in [0, T]$ and every $x, y \in [0, 1]^d$. In addition, note that

$$(5.19)$$

$$\mathbb{E}\left[|Y_t^{N,M}(x)|^2 \right] = \mathbb{E}\left[\left| \sum_{i \in \mathcal{I}_N \setminus \mathcal{I}_M} b(i) \int_0^t (t - s)^{-\alpha} e^{-\lambda_i(t-s)} d\beta_s^i \cdot e_i(x) \right|^2 \right]$$

$$= \sum_{i \in \mathcal{I}_N \setminus \mathcal{I}_M} |b(i)|^2 \, \mathbb{E}\left[\left| \int_0^t (t - s)^{-\alpha} e^{-\lambda_i(t-s)} d\beta_s^i \right|^2 \right] |e_i(x)|^2$$

$$= \sum_{i \in \mathcal{I}_N \setminus \mathcal{I}_M} |b(i)|^2 \int_0^{2t\lambda_i} s^{-2\alpha} e^{-s} \, ds (2\lambda_i)^{(2\alpha-1)} |e_i(x)|^2 \le C \sum_{i \in \mathcal{I}_N \setminus \mathcal{I}_M} |b(i)|^2 \, \|i\|_2^{(8\alpha-2)}$$

for every $t \in [0, T]$ and every $x, y \in [0, 1]^d$. In the next step the Sobolev embeddings in subsections 2.2.4 and 2.4.4 in [38] give

$$(5.20) \quad \sup_{0 \leq t \leq T} \mathbb{E} \|Y_t^{N,M}\|_{C([0,1]^d, \mathbb{R})}^p$$

$$\leq C \sup_{0 \leq t \leq T} \int_{(0,1)^d} \int_{(0,1)^d} \frac{\left(\mathbb{E} \big[|Y_t^{N,M}(x) - Y_t^{N,M}(y)|^2 \big] \right)^{p/2}}{\|x - y\|_2^{d+p\alpha}} \, dx \, dy$$

$$+ C \sup_{0 \leq t \leq T} \int_{(0,1)^d} \left(\mathbb{E} \big[|Y_t^{N,M}(x)|^2 \big] \right)^{\frac{p}{2}} dx$$

and (5.18), (5.19), and Lemma 5.1 therefore imply

$$(5.21) \quad \sup_{0 \leq t \leq T} \mathbb{E} \left[\|Y_t^{N,M}\|_{C([0,1]^d, \mathbb{R})}^p \right]$$

$$\leq C \int_{(0,1)^d} \int_{(0,1)^d} \frac{\left(\sum_{i \in \mathcal{I}_N \setminus \mathcal{I}_M} |b(i)|^2 \, \|i\|_2^{(8\alpha - 2)} \, \|x - y\|_2^{4\alpha} \right)^{\frac{p}{2}}}{\|x - y\|^{(d+p\alpha)}} \, dx \, dy$$

$$+ C \left(\sum_{i \in \mathcal{I}_N \setminus \mathcal{I}_M} |b(i)|^2 \, \|i\|_2^{(8\alpha - 2)} \right)^{p/2} \leq C \left(\sum_{i \in \mathcal{I}_N \setminus \mathcal{I}_M} |b(i)|^2 \, \|i\|_2^{(8\alpha - 2)} \right)^{p/2}.$$

This and inequality (5.17) then show (5.14). The proof of Lemma 5.5 is thus completed. $\quad \square$

Proof of Lemma 4.3. Throughout this proof let $O^N \colon [0, T] \times \Omega \to C([0, 1]^d, \mathbb{R})$, $N \in \mathbb{N}$, be a sequence of stochastic processes defined through (5.9). Next note that Lemma 5.5 implies

$$(5.22) \quad \left(\mathbb{E} \left[\sup_{0 \leq t \leq T} \|O_t^N - O_t^M\|_{C([0,1]^d, \mathbb{R})}^p \right] \right)^{\frac{1}{p}} \leq C \left(\sum_{i \in \mathbb{N}^d \setminus \{1, \dots, M\}^d} |b(i)|^2 \, \|i\|_2^{(8\alpha - 2)} \right)^{\frac{1}{2}}$$

$$\leq C \left(\sum_{i \in \mathbb{N}^d} |b(i)|^2 \, \|i\|_2^{(2\rho - 2)} \right)^{\frac{1}{2}} M^{(4\alpha - \rho)}$$

for every $N, M \in \mathbb{N}$ with $N \geq M$, every $p \in [1, \infty)$, and every $\alpha \in (0, \min(\frac{1}{2}, \frac{\rho}{4}))$, where $C \in [0, \infty)$ is a constant which depends on d, p, α, and T only. This, in particular, gives that $O^N \colon [0, T] \times \Omega \to C([0, 1]^d, \mathbb{R})$, $N \in \mathbb{N}$, is a Cauchy sequence in $L^p(\Omega; C([0, T], C([0, 1]^d, \mathbb{R})))$. Hence, there exists a stochastic process $\tilde{O} \colon [0, T] \times \Omega \to C([0, 1]^d, \mathbb{R})$ with continuous sample paths which satisfies

$$(5.23) \quad \left(\mathbb{E} \left[\sup_{0 \leq t \leq T} \|\tilde{O}_t - O_t^N\|_{C([0,1]^d, \mathbb{R})}^p \right] \right)^{\frac{1}{p}} \leq C \left(\sum_{i \in \mathbb{N}^d} |b(i)|^2 \, \|i\|_2^{(2\rho - 2)} \right)^{\frac{1}{2}} N^{(4\alpha - \rho)}$$

for every $N \in \mathbb{N}$, every $p \in [1, \infty)$, and every $\alpha \in (0, \min(\frac{1}{2}, \frac{\rho}{4}))$. Therefore, we have

$$(5.24) \quad \sup_{N \in \mathbb{N}} \left\{ N^\gamma \left(\mathbb{E} \left[\sup_{0 \leq t \leq T} \|\tilde{O}_t - O_t^N\|_{C([0,1]^d, \mathbb{R})}^p \right] \right)^{\frac{1}{p}} \right\} < \infty$$

for every $\gamma \in (0, \rho)$ and every $p \in [1, \infty)$. This implies

$$(5.25) \qquad \mathbb{P}\left[\sup_{N \in \mathbb{N}} \left(N^\gamma \sup_{0 \le t \le T} \|\tilde{O}_t - O_t^N\|_{C([0,1]^d, \mathbb{R})} \right) < \infty \right] = 1$$

for every $\gamma \in (0, \rho)$ (due to Lemma 2.1 in [27]). This yields

$$(5.26) \qquad \mathbb{P}\left[\forall\, \gamma \in (0, \rho) \colon \sup_{N \in \mathbb{N}} \sup_{0 \le t \le T} \left(N^\gamma \|\tilde{O}_t - O_t^N\|_{C([0,1]^d, \mathbb{R})} \right) < \infty \right] = 1$$

and hence, we obtain that

$$(5.27) \qquad \mathbb{P}\left[\lim_{N \to \infty} \sup_{0 \le t \le T} \|\tilde{O}_t - O_t^N\|_{C([0,1]^d, \mathbb{R})} = 0 \right] = 1 \qquad \text{and}$$

$$(5.28) \qquad \mathbb{P}\left[\forall\, \gamma \in (0, \rho) \colon \sup_{N \in \mathbb{N}} \sup_{0 \le t \le T} \left(N^\gamma \|\tilde{O}_t - P_N(\tilde{O}_t)\|_{C([0,1]^d, \mathbb{R})} \right) < \infty \right] = 1.$$

In addition, Lemma 5.4 gives

$$\left(\mathbb{E}\left[\|O_{t_2}^N - O_{t_1}^N\|_{C([0,1]^d, \mathbb{R})}^p \right] \right)^{\frac{1}{p}} \le \tilde{C}_{d,p,\rho,\theta} \left(\sum_{i \in \{1,\dots,N\}^d} |b(i)|^2 \|i\|_2^{4\theta + 4(\frac{\rho}{2} - \theta) - 2} \right)^{\frac{1}{2}} |t_2 - t_1|^\theta$$

$$\le \tilde{C}_{d,p,\rho,\theta} \left(\sum_{i \in \mathbb{N}^d} |b(i)|^2 \|i\|_2^{(2\rho-2)} \right)^{\frac{1}{2}} |t_2 - t_1|^\theta$$

for every $t_1, t_2 \in [0, T]$, $N \in \mathbb{N}$, $p \in [1, \infty)$, and every $\theta \in (0, \frac{\rho}{2}) \cap [0, \frac{1}{2}]$, where $\tilde{C}_{d,p,\rho,\theta} \in [0, \infty)$ is a constant which depends on d, p, ρ, and θ only. This shows

$$(5.29) \qquad \left(\mathbb{E}\left[\|\tilde{O}_{t_2} - \tilde{O}_{t_1}\|_{C([0,1]^d, \mathbb{R})}^p \right] \right)^{\frac{1}{p}} \le \tilde{C}_{d,p,\rho,\theta} \left(\sum_{i \in \mathbb{N}^d} |b(i)|^2 \|i\|_2^{2\rho-2} \right)^{\frac{1}{2}} |t_2 - t_1|^\theta$$

for every $t_1, t_2 \in [0, T]$, $p \in [1, \infty)$, and every $\theta \in \left(0, \frac{\rho}{2}\right)$, $\theta \le \frac{1}{2}$. Kolmogorov's theorem (see, e.g., Theorem 3.3 in [13]) hence yields

$$(5.30) \qquad \mathbb{P}\left[\sup_{0 \le t_1 < t_2 \le T} \frac{\|\tilde{O}_{t_2} - \tilde{O}_{t_1}\|_{C([0,1]^d, \mathbb{R})}}{|t_2 - t_1|^\theta} < \infty \right] = 1$$

for every $\theta \in \left(0, \min\left(\frac{1}{2}, \frac{\rho}{2}\right)\right)$. This implies

$$(5.31) \qquad \mathbb{P}\left[\forall\, \theta \in (0, \min\{\tfrac{1}{2}, \tfrac{\rho}{2}\}) \colon \sup_{0 \le t_1 < t_2 \le T} \frac{\|\tilde{O}_{t_2} - \tilde{O}_{t_1}\|_{C([0,1]^d, \mathbb{R})}}{|t_2 - t_1|^\theta} < \infty \right] = 1.$$

Combining (5.28) and (5.31) shows the existence of a stochastic process $O \colon [0, T] \times \Omega \to C([0,1]^d, \mathbb{R})$ with continuous sample paths which is indistinguishable from $\tilde{O}$, i.e., $\mathbb{P}\left[\forall\, t \in [0, T] \colon O_t = \tilde{O}_t \right] = 1$ and which satisfies $\sup_{0 \le t_1 < t_2 \le T} \frac{\|O_{t_2}(\omega) - O_{t_1}(\omega)\|_{C([0,1]^d, \mathbb{R})}}{|t_2 - t_1|^\theta}$ $< \infty$ and $\sup_{N \in \mathbb{N}} \sup_{0 \le t \le T} (N^\gamma \|O_t(\omega) - P_N(O_t(\omega))\|_{C([0,1]^d, \mathbb{R})}) < \infty$ for every $\omega \in \Omega$, every $\theta \in \left(0, \min\left(\frac{1}{2}, \frac{\rho}{2}\right)\right)$, and every $\gamma \in (0, \rho)$. The proof of Lemma 4.3 is thus completed. $\qquad \square$

5.3. Proofs for subsection 4.3.

5.3.1. Proof of Lemma 4.6.

Proof. First, note that

$$
\sum_{n=1}^{\infty} n^{2+2\gamma} e^{-2n^2\pi^2 t} \leq \int_0^{\infty} (x+1)^{2+2\gamma} e^{-2x^2\pi^2 t} \, dx
$$

$$
\leq \int_0^{\infty} 8 \big(x^{2+2\gamma} + 1 \big) e^{-2x^2\pi^2 t} \, dx = \frac{1}{2\pi\sqrt{t}} \int_0^{\infty} 8 \left(\frac{x^{2+2\gamma}}{(2\pi\sqrt{t})^{2+2\gamma}} + 1 \right) e^{-\frac{x^2}{2}} \, dx
$$

$$
\leq \frac{4}{\pi\sqrt{t}} \int_0^{\infty} \left(\frac{x^{2+2\gamma}}{t^{1+\gamma}} + 1 \right) e^{-\frac{x^2}{2}} \, dx \leq \frac{4}{\pi\sqrt{t}} \int_0^{\infty} \left(\frac{x^4 + 1}{t^{1+\gamma}} + \frac{T^{1+\gamma}}{t^{1+\gamma}} \right) e^{-\frac{x^2}{2}} \, dx
$$

$$
\leq \frac{4\sqrt{2\pi}}{t^{\frac{3}{2}+\gamma}\pi} \int_{\mathbb{R}} \frac{x^4 + T^2 + 2}{\sqrt{2\pi}} \, e^{-\frac{x^2}{2}} \, dx \leq \frac{4(T^2+5)}{t^{\frac{3}{2}+\gamma}}
$$

for every $t \in (0,T]$ and every $\gamma \in [0, \frac{1}{2})$.

The identity $\|w\|_{H^{-1}} = \sum_{n=1}^{\infty} n^{-2}\pi^{-2} \, |w(\sqrt{2}\sin(n\pi\cdot))|^2$ for every $w \in H^{-1}((0,1),\mathbb{R})$ hence gives

$$
\sup_{0 \leq x \leq 1} \left(\sum_{n=N}^{\infty} 2 \cdot e^{-n^2\pi^2 t} \cdot |w(\sin(n\pi(\cdot)))| \cdot |\sin(n\pi x)| \right)
$$

$$
\leq \pi\sqrt{2} \sum_{n=N}^{\infty} n \, e^{-n^2\pi^2 t} \, \frac{|w(\sqrt{2}\sin(n\pi(\cdot)))|}{n\pi}
$$

$$
\leq \pi\sqrt{2} \left(\sum_{n=N}^{\infty} n^2 \, e^{-2n^2\pi^2 t} \right)^{\frac{1}{2}} \|w\|_{H^{-1}} \leq \pi\sqrt{2} \, N^{-\gamma} \left(\sum_{n=N}^{\infty} n^{(2+2\gamma)} \, e^{-2n^2\pi^2 t} \right)^{\frac{1}{2}} \|w\|_{H^{-1}}
$$

$$
\leq \pi\sqrt{2} \, N^{-\gamma} \left(4 \left(T^2 + 5 \right) t^{-\left(\frac{3}{2}+\gamma \right)} \right)^{\frac{1}{2}} \|w\|_{H^{-1}} \leq 10 \, (T+3) \, t^{-\left(\frac{3}{4}+\frac{\gamma}{2} \right)} N^{-\gamma} \|w\|_{H^{-1}}
$$

for every $w \in H^{-1}((0,1),\mathbb{R})$, $N \in \mathbb{N}$, $\gamma \in [0, \frac{1}{2})$, and every $t \in (0,T]$. This implies $\|S_t(w)\|_{C([0,1],\mathbb{R})} \leq 10 \, (T+3) \, t^{-\frac{3}{4}} \|w\|_{H^{-1}}$ and

$$
(5.32) \quad \|S_t(w) - P_N(S_t(w))\|_{C([0,1],\mathbb{R})} \leq \frac{10 \, (T+3) \, \|w\|_{H^{-1}}}{t^{\left(\frac{3}{4}+\frac{\gamma}{2} \right)} (N+1)^{\gamma}} \leq \frac{10 \, (T+3) \, \|w\|_{H^{-1}}}{t^{\left(\frac{3}{4}+\frac{\gamma}{2} \right)} N^{\gamma}}
$$

for every $t \in (0,T]$, $w \in H^{-1}((0,1),\mathbb{R})$, $\gamma \in [0, \frac{1}{2})$, and every $N \in \mathbb{N}$. Therefore, we finally obtain $\sup_{0<t\leq T} \left(t^{\frac{3}{4}} \|S_t\|_{L(H^{-1}((0,1),\mathbb{R}), C([0,1],\mathbb{R}))} \right) < \infty$ and

$$
(5.33) \qquad \sup_{N\in\mathbb{N}} \sup_{0<t\leq T} \left(t^{\left(\frac{3}{4}+\frac{\gamma}{2} \right)} N^{\gamma} \|S_t - P_N S_t\|_{L(H^{-1}((0,1),\mathbb{R}), C([0,1],\mathbb{R}))} \right) < \infty
$$

for every $\gamma \in [0, \frac{1}{2})$. The proof of Lemma 4.6 is thus completed. $\square$

5.3.2. Proof of Lemma 4.8. In the proof of Lemma 4.8 the following well-known estimates for the analytic semigroup generated by the Laplacian are used. Their proofs can, e.g., be found in Lemma 5.8 in [3].

LEMMA 5.6. *Let* $S \colon (0,T] \to L\big(H^{-1}((0,1),\mathbb{R}), C([0,1],\mathbb{R}) \big)$ *be given by Lemma* 4.6 *and let* $P_N \colon C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$, $N \in \mathbb{N}$, *be given by* (4.14). *Then*

$$\|P_N S_t\|_{L(L^2((0,1),\mathbb{R}),C([0,1],\mathbb{R}))} \leq t^{-\frac{1}{4}}, \quad \|P_N S_t\|_{L(L^2((0,1),\mathbb{R}),L^4((0,1),\mathbb{R}))} \leq t^{-\frac{1}{8}},$$

$$\|S_t\|_{L(H^{-1}((0,1),\mathbb{R}),L^2((0,1),\mathbb{R}))} \leq t^{-\frac{1}{2}}, \quad \text{and} \quad \|S_t(v')\|_{L^2} \leq 4\,(t+1)\,t^{-\frac{3}{4}}\,\|v\|_{L^1}$$

for every $t \in (0,T]$, $N \in \mathbb{N}$, and every $v \in C^1([0,1],\mathbb{R})$.

In addition to Lemma 5.6, the following elementary global coercivity estimate for Burgers equation is used in the proof of Lemma 4.8 (see also Lemma 3.1 in [10]).

LEMMA 5.7. *Let* $F\colon C([0,1],\mathbb{R}) \to H^{-1}((0,1),\mathbb{R})$ *be given by Lemma* 4.7. *Then*

$$(5.34) \qquad \langle v, v'' + F(v+w)\rangle_{L^2} \leq 2\,c^2\|v\|_{L^2}^2\|w\|_{C([0,1],\mathbb{R})}^2 + 2\,c^2\|w\|_{C([0,1],\mathbb{R})}^4$$

for all twice continuously differentiable functions $v\colon [0,1] \to \mathbb{R}$ *and* $w\colon [0,1] \to \mathbb{R}$ *with* $v(0) = v(1) = 0$ *and where* $c \in \mathbb{R}$ *is used in Lemma 4.7.*

Proof. Integration by parts and the identity $\int_0^1 v'(x)\,|v(x)|^2\,dx = 0$ imply

$$(5.35) \quad \langle v, F(v+w)\rangle_{L^2} = -2c\int_0^1 v'(x)\cdot v(x)\cdot w(x)\,dx - c\int_0^1 v'(x)\cdot|w(x)|^2\,dx$$

$$\leq 2|c|\left(\|v\|_{L^2}\cdot\|w\|_{C([0,1],\mathbb{R})} + \|w\|_{C([0,1],\mathbb{R})}^2\right)\cdot\|v'\|_{L^2}$$

$$\leq 2c^2\|v\|_{L^2}^2\|w\|_{C([0,1],\mathbb{R})}^2 + 2c^2\|w\|_{C([0,1],\mathbb{R})}^4 + \|v'\|_{L^2}^2$$

and therefore using integration by parts

$$\langle v, v'' + F(v+w)\rangle_{L^2} \leq 2c^2\|v\|_{L^2}^2\|w\|_{C([0,1],\mathbb{R})}^2 + 2c^2\|w\|_{C([0,1],\mathbb{R})}^4$$

for all twice continuously differentiable functions $v, w\colon [0,1] \to \mathbb{R}$ with $v(0) = v(1) = 0$. The proof of Lemma 5.7 is thus completed. $\quad\square$

In the next step note that Lemma 4.8 follows by combining the next lemma (see also Lemma 3.1 in [10] for a related result) and a standard fix point argument (see also Theorem 3.2 in [10]).

LEMMA 5.8. *Let* $\tau \in (0,T]$, $N \in \mathbb{N}$ *and let* $x_N\colon [0,\tau] \to P_N(C([0,1],\mathbb{R}))$ *and* $o_N\colon [0,\tau] \to P_N(C([0,1],\mathbb{R}))$ *be two continuous functions which satisfy* $x_N(t) = \int_0^t P_N S_{t-s} F(x_N(s))\,ds + o_N(t)$ *for every* $t \in [0,\tau]$. *Then*

$$(5.36)$$

$$\sup_{0 \leq t \leq \tau} \|x_N(t)\|_{C([0,1],\mathbb{R})} \leq \exp\left(24(c^2+1)(T+1)\left(\sup_{0 \leq t \leq \tau}\|o_N(t)\|_{C([0,1],\mathbb{R})}^2 + 1\right)\right),$$

where $c \in \mathbb{R}$ *is used in Lemma* 4.7.

Proof. First, note that the definition of $S\colon (0,T] \to L(W,V)$ in Lemma 4.6 implies

$$(5.37) \quad S_t v - v = \int_0^t S_s(v'')\,ds \quad \text{and} \quad \|(S_t - I)v\|_{C([0,1],\mathbb{R})} \leq t\cdot\|v''\|_{C([0,1],\mathbb{R})}$$

for every $t \in (0,T]$ and every $v \in P_N(C([0,1],\mathbb{R}))$. In the next step define the continuous function $y_N\colon [0,\tau] \to P_N(C([0,1],\mathbb{R}))$ by

$$(5.38) \quad y_N(t) := x_N(t) - o_N(t) = \int_0^t P_N S_{t-s} F(x_N(s))\,ds = \int_0^t S_{t-s} P_N F(x_N(s))\,ds$$

for every $t \in [0,\tau]$. Here the $\mathbb{R}$-vector space $P_N(C([0,1],\mathbb{R}))$ is equipped with the supremum norm $\|v\|_V = \sup_{0 \leq x \leq 1}|v(x)|$ for every $v \in P_N(C([0,1],\mathbb{R}))$. Furthermore, observe that equation (5.37) implies

(5.39)
$$\frac{y_N(t_2) - y_N(t_1)}{t_2 - t_1} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} S_{t_2-s} \, P_N \, F(x_N(s)) \, ds + \frac{(S_{t_2-t_1} - I) \, y_N(t_1)}{t_2 - t_1}$$

$$= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} S_{t_2-s} \, P_N \, F(x_N(s)) \, ds + \frac{1}{t_2 - t_1} \int_0^{t_2-t_1} S_s \, \Delta y_N(t_1) \, ds$$

for all $t_1, t_2 \in [0, \tau]$ with $t_1 < t_2$. Here, and below, $\Delta y_N(t)$ is the second derivative of $y_N(t)$ in the spatial variable, i.e., $(\Delta y_N(t))(x) = \left(\frac{\partial^2 y_N}{\partial x^2}\right)(t, x)$ for all $t \in [0, \tau]$ and all $x \in [0, 1]$. Next, again (5.37) implies

(5.40)
$$\lim_{\substack{t_1 \nearrow t, \, t_2 \searrow t \\ 0 \le t_1 < t_2 \le \tau}} \frac{1}{t_2 - t_1} \int_0^{t_2-t_1} S_s \, \Delta y_N(t_1) \, ds = \Delta y_N(t) \qquad \text{and}$$

(5.41)
$$\lim_{\substack{t_1 \nearrow t, \, t_2 \searrow t \\ 0 \le t_1 < t_2 \le \tau}} \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} S_{t_2-s} \, P_N \, F(x_N(s)) \, ds = P_N \, F(x_N(t))$$

for every $t \in [0, \tau]$. Combining (5.39)–(5.41) then results in

(5.42)
$$\tfrac{\partial}{\partial t} \, y_N(t) = \Delta y_N(t) + P_N \, F(x_N(t)) = \Delta y_N(t) + P_N \, F\big(y_N(t) + o_N(t)\big)$$

and Lemma 5.7 hence gives

$$\tfrac{\partial}{\partial t} \|y_N(t)\|_{L^2}^2 = 2 \, \langle y_N(t), \Delta y_N(t) + F(y_N(t) + o_N(t)) \rangle_{L^2}$$

$$\le 4c^2 \|y_N(t)\|_{L^2}^2 \sup_{0 \le s \le \tau} \|o_N(s)\|_{C([0,1],\mathbb{R})}^2 + 4c^2 \sup_{0 \le s \le \tau} \|o_N(s)\|_{C([0,1],\mathbb{R})}^4$$

for every $t \in [0, \tau]$, Gronwall's lemma and the estimates $x^2 \le e^x$ and $x \le e^x$ for all $x \in [0, \infty)$ therefore yield

(5.43)
$$\|y_N(t)\|_{L^2}^2 \le \exp\big(4c^2 z^2 T\big) \, 4c^2 z^4 T \le e^{4c^2 z^2 T + 2|c|z^2 + T} \le e^{4(c^2+1)(T+1)z^2}$$

for every $t \in [0, \tau]$, where here and below $z := \max(1, \sup_{0 \le s \le \tau} \|o_N(s)\|_{C([0,1],\mathbb{R})})$. In the next step Lemma 5.6 shows

(5.44)
$$\|y_N(t)\|_{L^4} = \int_0^t \|P_N \, S_{t-s} \, F(x_N(s))\|_{L^4} \, ds$$

$$\le 2^{\frac{1}{8}} |c| \int_0^t (t-s)^{-\frac{1}{8}} \left\| S_{\frac{(t-s)}{2}} \Big(\big((x_N(s))^2\big)' \Big) \right\|_{L^2} ds$$

$$\le 2^{\frac{1}{8}} |c| \int_0^t (t-s)^{-\frac{1}{8}} \cdot 4 \cdot (T+1) \cdot \left(\tfrac{t-s}{2}\right)^{-\frac{3}{4}} \cdot \|(x_N(s))^2\|_{L^1} \, ds$$

$$\le 64 \, (T+1) \, T^{\frac{1}{8}} \, |c| \left(\sup_{0 \le s \le \tau} \|x_N(s)\|_{L^2}^2 \right)$$

for every $t \in [0, \tau]$. Additionally, again Lemma 5.6 gives

(5.45)
$$\|y_N(t)\|_{C([0,1],\mathbb{R})} = \left\| \int_0^t P_N \, S_{t-s} \, F(x^N(s)) \, ds \right\|_{C([0,1],\mathbb{R})}$$

$$\le 2 \int_0^t (t-s)^{-\frac{3}{4}} \|F(x^N(s))\|_{H^{-1}} \, ds \le 8 \, |c| \, T^{\frac{1}{4}} \left(\sup_{0 \le s \le \tau} \|x_N(s)\|_{L^4}^2 \right)$$

for every $t \in [0, \tau]$. Combining (5.44) and (5.45) then yields

$$
\begin{aligned}
\sup_{0 \le t \le \tau} \|y^N(t)\|_{C([0,1],\mathbb{R})} &\le 8 \, |c| \, T^{\frac{1}{4}} \left(\sup_{0 \le t \le \tau} \|y^N(t)\|_{L^4} + \sup_{0 \le t \le \tau} \|o^N(t)\|_{L^4} \right)^2 \\
&\le 8 \, |c| \, T^{\frac{1}{4}} \left(64 \, (T+1) \, T^{\frac{1}{8}} \, |c| \sup_{0 \le t \le \tau} \|x_N(t)\|_{L^2}^2 + z \right)^2 \\
&\le 2^{16} \, (|c|^3 + 1) \, (T+1)^3 \left(\left(\sup_{0 \le t \le \tau} \|y_N(t)\|_{L^2} + z \right)^4 + z^2 \right).
\end{aligned}
$$

Inequality (5.43) therefore shows

$$
\begin{aligned}
\sup_{0 \le t \le \tau} \|y_N(t)\|_{C([0,1],\mathbb{R})} &\le 2^{19} \, (|c|^3 + 1) \, (T+1)^3 \left(\sup_{0 \le t \le \tau} \|y_N(t)\|_{L^2}^4 + 2z^4 \right) \\
&\le 2^{19} \, (|c|^3 + 1) \, (T+1)^3 \left(e^{8(c^2+1)(T+1)z^2} + 2z^4 \right) \\
&\le \left(2^7 \, (c^2 + 1) \, (T+1) \right)^3 e^{8(c^2+1)(T+1)z^2} \le e^{23(c^2+1)(T+1)z^2}
\end{aligned}
$$

and this completes the proof of Lemma 5.8. $\square$

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