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Dirk Blömker

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Amplitude Equations for Locally Cubic Nonautonomous Nonlinearities*

Dirk Blömker†

Abstract. For systems of partial differential equations (PDEs) with locally cubic nonlinearities, which are perturbed by additive noise, we describe the essential dynamics for small solutions. If the system is near a change of stability, then a natural separation of time-scales occurs and the amplitudes of the dominant modes are given on a long time-scale by a stochastic ordinary differential equation. We consider applications to dynamic pitchfork bifurcation, pattern formation below the threshold of stability, and transient dynamics of stochastic PDEs near this deterministic bifurcations.

Key words. amplitude equation, pattern formation, SPDE, slow modes, separation of time-scales, approximate center-manifold, bifurcation, multiple scale analysis

AMS subject classifications. 60H15, 60H10, 37L55, 37L65

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1. Introduction.

Amplitude equations are well known in the physics literature (see, e.g., [H83] or [W97]). They usually describe some order parameter for the system, which evolves on a much slower time-scale. This separation of time-scales occurs, for example, very naturally in a small neighborhood of bifurcations, where a change of stability occurs.

Amplitude equations can be used either for spatially extended systems, where they are stochastic partial differential equations (SPDEs), or for systems on bounded domains, where they are given as stochastic ordinary differential equations (SODEs). This paper will focus on the second case, where an SODE describes the dynamics of the amplitudes of dominant modes evolving on some slow time-scale. On the other hand, all nondominant modes evolve rapidly on a fast time-scale, but they stay much smaller than the dominant ones. The modes in our context are given by the Fourier series expansion with respect to the eigenfunctions of the corresponding linearized operator.

For deterministic systems the theory is rigorously understood even for spatially extended systems (see, e.g., [KSM92, vH91] for the first results). However, there is a lack of results for stochastic systems. The only rigorous example is [BMS01] for a stochastic Swift–Hohenberg equation with periodic boundary conditions on a bounded interval. In this example, a complex-valued SODE was derived describing the amplitude of the dominant mode in a standard complex Fourier series on a very long time-scale.

Our main theorems will extend the results of [BMS01] to a large class of SPDEs and systems of SPDEs. Moreover, our applications will demonstrate the power of this approach, when describing transient dynamics of stochastic equations.

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†Mathematisches Institut, RWTH Aachen, 52052 Aachen, Germany (bloemker@instmath.rwth-aachen.de). Currently a visitor at Mathematics Research Centre, University of Warwick, Coventry CV4 7AL, UK.

We consider the differential equation

$$\partial_t u(t) = L_\varepsilon u(t) + f(\varepsilon, u(t), t) + \varepsilon^2 \xi(t), \quad u(0) = u_0, \tag{1.1}$$

in a real Banach space X with norm $\|\cdot\|_X$. Equations like this arise, for example, when considering some SPDE (or systems of SPDEs) on bounded domains.

The stochastic perturbation is given by the (generalized) stochastic process ξ, which is the derivative of some cylindrical Q-Wiener process W in X. Therefore, we can treat additive noise that is white in time, but we allow correlations in space.

The main assumptions on the operator L_ε will be that it generates a C_0-semigroup $\{e^{tL_\varepsilon}\}_{t\geq 0}$ on X and that the nullspace $\mathcal{N} := N(L_\varepsilon)$ of L_ε is independent of ε and finite-dimensional. In applications this operator is usually some differential operator equipped with suitable boundary conditions.

The nonlinearity f depends not only on $u(t)$, but it could also involve derivatives of u. The important property is that f contains a small part that is linear in u but no quadratic term in u. For example, $f(\varepsilon, u) = \varepsilon^2 u - u^3$ or $f(\varepsilon, u) = \varepsilon^2 u - u(\partial_x u)^2$.

The fact that the linear part is small will reflect that the unperturbed deterministic system is near a change of stability. Moreover, the reason for neglecting the much more complicated quadratic terms is the following. If we want to separate the dynamics of the dominant modes from the other modes, a cubic nonlinearity helps a lot. In contrast to that, quadratic nonlinearities tend to mix the dynamics of different modes much more strongly. In many examples, quadratic nonlinearities map the dominant modes completely onto nondominant ones, allowing complicated interaction of dominant and nondominant modes. This will be the topic of a forthcoming paper [B03].

One typical example we have in mind is an SPDE such that the unperturbed deterministic PDE exhibits a pitchfork-bifurcation. In a specific example, this was already treated in [BMS01]. To be more precise, f should depend on an additional parameter ν such that in the absence of noise there is a bifurcation at $\nu = 0$ in the deterministic equation. One goal of the presented approach is then to describe the dynamics of the SPDE near this deterministic bifurcation, where the time-scales of the dynamics for stable and unstable modes separate. We sketch briefly some results in section 6.3. We can identify regimes of the bifurcation parameter ν, where the transient behavior of the SPDE is, for instance, almost deterministic or independent of ν.

In contrast to the deterministic setting, the stochastic bifurcation is not that well understood. There are at least two main concepts. The first is a phenomenological bifurcation, where a qualitative change in the unique invariant measure of the corresponding Markov semigroup occurs. The second concept of bifurcation considers changes in the number of invariant measures for the corresponding random dynamical system. Moreover, this second concept is also related to structural changes in random attractors. See, for instance, [Ar98, section 9] for precise definitions and discussions of both concepts. One of the major drawbacks of stochastic bifurcation theory is that, only for one-dimensional SODEs (i.e., state space $\mathbb{R}$), there is a complete picture of possible bifurcations (see [CIS99]).

Note that we call the second concept simply bifurcation instead of dynamical bifurcation as in [Ar98]. The reason is that this terminology also exhibits a completely different meaning (see section 6.1). We denote by dynamic bifurcation the situation when the bifurcation parameter is slowly moved through a deterministic bifurcation point.

In our example of a deterministic pitchfork-bifurcation perturbed by additive noise, it is, at least for simple examples, known that there is only a phenomenological bifurcation (see, e.g., [CF98]). In the general SPDE case the shape of the of the invariant measure for the Markov semigroup is in general unknown, and the precise structure will be the topic of forthcoming research.

The approach presented does not describe the stochastic bifurcation, as we characterize only typical transient behavior on long time-scales, in contrast to the behavior for time to infinity, which is usually not accessible in real-world experiments. Our results are on one hand related to phenomenological bifurcations, as we could draw conclusions about the structure of invariant measures, but on the other hand the approach reaches far beyond that, as it is capable of describing the evolution in time of typical trajectories on very long transient time-scales.

For multiplicative noise the picture is much more complicated, as in this case bifurcations can occur easily (see, e.g., [CLR01]). We did not try to apply the machinery of amplitude equations to these equations, but it is possible, at least on a nonrigorous level, to reduce the dynamics to SODEs in some of the typical examples.

Our *main results* can be on a formal level described as follows. We make an ansatz of the type

$$u(t) = \varepsilon a(\varepsilon^2 t) \cdot e + \mathcal{O}(\varepsilon^2), \tag{1.2}$$

where $e = (e_1, \dots, e_n)$ is some basis in $\mathcal{N} = N(L_\varepsilon)$. Then in many physical examples a well-known formal calculation shows that plugging ansatz (1.2) into (1.1) yields a system of SODEs for the amplitude a of the dominant modes corresponding to the basis e. This is the *amplitude equation*, and it will be universal in the sense that it is actually independent of ε. The classical example arising in many applications is the so-called Landau equation $a' = \nu a - ca|a|^2 + \dot{\beta}$, where $\dot{\beta}$ is some noise and c, ν are constant coefficient matrices.

The main theorems of this article are first the *attractivity* (cf. Theorem 3.3) justifying the ansatz (1.2) for some initial time, which is not too big. Second, we obtain the *approximation* (cf. Theorem 4.3), showing by rigorous estimates of the error that (1.2) remains true on a very long time-scale of order $\mathcal{O}(\varepsilon^{-2})$, where a is given by the amplitude equation. In the proofs we will follow the strategy to reduce the probability of events giving approximation and attractivity to large deviation estimates for various random fields like the stochastic convolutions and the amplitude a. These are usually easy to derive and treated in the applications.

Another interpretation of the main results is the following. We describe a deterministic approximate center-manifold given by the vector space $\mathcal{N}$. For small solutions of order ε, where ε^2 denotes the noise strength, we show (up to small errors) that $\mathcal{N}$ locally attracts solutions of (1.1) with high probability. Moreover, in an $\mathcal{O}(\varepsilon^2)$-neighborhood of $\mathcal{N}$, most solutions of (1.1) are described (up to small errors) by an SODE on $\mathcal{N}$. This is in contrast to the concept of random invariant manifolds (cf., e.g., [DLS03]). There one can describe the transient dynamics of all solutions as a flow on the manifold, but the manifold is allowed to move in time. Nevertheless, the presented result can be used to estimate probabilities of how the random invariant manifold evolves in time.

Our theory describes solutions with small noise strength, but in contrast to the well-known Freidlin–Wentzell theory (cf. [FW98]), we consider small coefficients in the equation, too. Therefore, we approximate the solutions of (1.1) by the solutions of an SODE, and not by the solutions of the unperturbed PDE. This is motivated by the fact that we want to describe a perturbed deterministic pitchfork-bifurcation when the noise strength is of comparable order to the distance from the bifurcation.

The paper is organized as follows. In section 2 we give the main standing assumptions valid throughout the whole paper and the formulation of the abstract results. Sections 3 and 4 provide the proofs of our main results, first for the local attractivity of $\mathcal{N}$ and then for the approximation of solutions by the amplitude equation.

The last two sections are devoted to applications. Section 5 summarizes simple large deviation results necessary to estimate various probabilities occurring in the application of the main results. Finally, in section 6 we discuss examples. First we consider a dynamic pitchfork-bifurcation, which was discussed for a one-dimensional SODE in [BG02]. Note again that this is not the concept introduced in [Ar98]. In our case dynamic bifurcation means that the bifurcation parameter is time-dependent and is moved slowly through the bifurcation. Problems like this are for SPDEs still the topic of active numerical and experimental research (see [MG99, GM03] and the references therein). In an example we carry over the results of [BG02] to some SPDEs, allowing us to describe the transient behavior of solutions very precisely.

Then we focus on pattern formation below the threshold of instability, where, due to the presence of noise, a pattern appears in an otherwise stable deterministic system. This effect is also well known from experiments (see, e.g., [SR94]) and, for instance, in the context of convection problems still subject to recent experimental investigation (see [SA02]). Nevertheless the problem is not fully understood and there is no rigorous mathematical verification of the pattern being present (see, e.g., [HS92] or [SA02] and the references therein). We present a method giving for an example a first step into that direction.

2. Notation and formulation of the problem.

This section summarizes standing assumptions valid throughout the whole article. For the linear operator L_ε in (1.1), we assume the following.

Assumption 2.1. *For all $\varepsilon > 0$ suppose L_ε is some possibly unbounded linear operator on X. The kernel (or nullspace) of L_ε is denoted by $\mathcal{N} := \{v \in D(L_\varepsilon) : L_\varepsilon v = 0\}$, and it is assumed to be independent of ε. We denote a projection onto $\mathcal{N}$ by P_c and define $n := \dim(\mathcal{N})$.*

Later there will be further restrictions on the choice of the projection P_c. The typical example we have in mind is the spectral projection onto $\mathcal{N}$, and many assumptions would be automatically fulfilled in that case.

One complementary projection to P_c is given by $P_s := I - P_c$. As the dimension of $\mathcal{N}$ is finite, it is well known that both P_c and P_s are bounded linear operators (cf. [W80]).

The second assumption on L_ε and P_c is the following.

Assumption 2.2. *We assume that L_ε from Assumption 2.1 generates a strongly continuous semigroup $\{e^{tL_\varepsilon}\}_{t\geq 0}$ of linear operators on X which is exponentially stable on P_sX. To be more precise, there are constants $\omega > 0$ and $M \geq 1$, independent of ε, such that*

$$\|e^{tL_\varepsilon} P_s x\|_X \leq M e^{-t\omega} \|x\|_X \quad \textit{for all } t \geq 0,\ x \in X. \tag{2.1}$$

To deal with the nonlinearity, suppose there is a second Banach space Y, such that X is dense and continuously imbedded into Y. Assume that e^{tL_ε} and P_c can be extended to operators on Y, and for some $\alpha \in [0,1)$ we have

$$\|P_s e^{tL_\varepsilon} y\|_X \le M(1+t^{-\alpha})e^{-t\omega}\|y\|_Y \quad \text{for all } t>0,\ y\in Y. \tag{2.2}$$

Moreover, suppose that P_c, and hence P_s, commutes with e^{tL_ε} on X and Y.

Let us briefly comment on the previous assumption. First, the assumption that P_c commutes with e^{tL_ε} is always true for the spectral projection of L_ε onto $\mathcal{N}$. For self-adjoint operators L_ε in Hilbert spaces, we can, for instance, simply choose the orthogonal projection.

Moreover, under the other assertions of Assumption 2.2 the spectral projection can be extended to a continuous linear operator from Y to $\mathcal{N}$ (i.e., $P_c \in \mathcal{L}(Y,\mathcal{N})$). The main ideas are first to use (2.2) to verify that the residual of L_ε is in $\mathcal{L}(Y,X)$ and second to use the Dunford calculus giving a representation for P_c in terms of the residual (see, e.g., [K95]).

One typical example for Y that we have in mind is an interpolation space between the dual of $D(L)$ and X—for instance, the dual of fractional power spaces in the case when L_ε generates an analytic semigroup.

As $L_\varepsilon \equiv 0$ on $\mathcal{N}$, it is easy to verify that $e^{tL_\varepsilon} = Id$ on $\mathcal{N}$ for all $t \ge 0$. Therefore, we can assume without loss of generality that M is large enough such that

$$\|e^{tL_\varepsilon}x\|_X \le M\|x\|_X \quad \text{for all } t\ge 0,\ x\in X.$$

Moreover, as $\mathcal{N}$ is finite-dimensional, we can also assume that M is sufficiently large such that

$$\|e^{tL_\varepsilon}P_c y\|_X \le M\|P_c y\|_Y \quad \text{for all } t\ge 0,\ y\in Y.$$

For the stochastic perturbation the following assumption is true. For a detailed discussion of Q-Wiener processes and stochastic convolutions, see [dPZ92].

Assumption 2.3. *Suppose that ξ is the generalized derivative of some Q-Wiener process $\{W(t)\}_{t\ge 0}$ on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ such that the stochastic convolution*

$$W_{L_\varepsilon}(t) = \int_0^t e^{(t-\tau)L_\varepsilon}dW(\tau) \tag{2.3}$$

is a well-defined stochastic process with continuous paths in X.

As $P_s e^{tL_\varepsilon} = e^{tL_\varepsilon}P_s$, it is straightforward to verify that

$$P_s[W_{L_\varepsilon}(t)] = \int_0^t e^{(t-\tau)L_\varepsilon}dP_sW(\tau) \quad \text{and} \quad P_c[W_{L_\varepsilon}(t)] = P_cW(t).$$

To give a meaning to (1.1) we will always consider mild solutions.

Assumption 2.4. *We assume that for any (stochastic) initial condition $u_0 \in X$, (1.1) has a mild local solution u. This means we have a stopping time $t^* > 0$ and a stochastic process u such that $u : [0,t^*] \to X$ is $\mathbb{P}$-a.s. a solution of*

$$u(t) = e^{tL_\varepsilon}u_0 + \int_0^t e^{(t-\tau)L_\varepsilon}f(\varepsilon,u(\tau),\tau)d\tau + \varepsilon^2 W_{L_\varepsilon}(t) \qquad \textit{for } t\le t^*. \tag{2.4}$$

The existence of local solutions is standard if we consider locally Lipschitz-continuous nonlinearities (see, e.g., [dPZ92], and for L^p-theory with application to the Navier–Stokes equation see, e.g., [BP99, BP00]).

We can split the variation of constants formula (2.4) into two parts:

$$P_s u(t) = e^{tL_\varepsilon} P_s u_0 + \int_0^t e^{(t-\tau)L_\varepsilon} P_s f(\varepsilon, u(\tau), \tau) d\tau + \varepsilon^2 \int_0^t e^{(t-\tau)L_\varepsilon} dP_s W(t) \tag{2.5}$$

and

$$P_c u(t) = P_c u_0 + \int_0^t P_c f(\varepsilon, u(\tau), \tau) d\tau + \varepsilon^2 P_c W(t). \tag{2.6}$$

We call $u_s(t) = P_s u(t)$ *fast modes*, as they are subject to an exponential decay on a time-scale of order $\mathcal{O}(1)$. Moreover, $u_c(t) = P_c u(t)$ will be the *slow modes*.

For f we suppose the following.

Assumption 2.5. *There is an $\varepsilon_0 > 0$ such that f defines a family of (nonlinear) operators $f(\varepsilon, \cdot, t) : X \to Y$ for all $\varepsilon \in (0, \varepsilon_0]$ and $t > 0$, where Y was defined in Assumption 2.2.*

Suppose we have the following Taylor expansion for f with respect to u:

$$f(\varepsilon, u, t) = \varepsilon^2 f_1(\varepsilon, t)u + f_3(\varepsilon, t)[u]^3 + g(\varepsilon, u, t), \tag{2.7}$$

where $f_1(\varepsilon, t) \in \mathcal{L}(X, Y)$ is a continuous linear map that leaves $\mathcal{N}$ invariant (i.e., $f_1(\varepsilon, t)\mathcal{N} \subset \mathcal{N}$), and $f_3(\varepsilon, t) \in \mathcal{L}_3(X, Y)$ is a continuous trilinear operator.

Suppose there are constants all denoted by C_f such that

$$\sup_{t\in[0,T_0\varepsilon^{-2}]} \|g(\varepsilon, u, t)\|_Y \le C_f \|u\|_X^4 \quad \text{for } \|u\|_X \le \delta_0,\ \varepsilon \in (0, \varepsilon_0], \tag{2.8}$$

$$\sup_{t\in[0,T_0\varepsilon^{-2}]} \|f_1(\varepsilon, t)\|_{\mathcal{L}(X,Y)} \le C_f \quad \text{for } \varepsilon \in (0, \varepsilon_0], \tag{2.9}$$

$$\sup_{t\in[0,T_0\varepsilon^{-2}]} \|f_3(\varepsilon, t)\|_{\mathcal{L}_3(X,Y)} \le C_f \quad \text{for } \varepsilon \in (0, \varepsilon_0]. \tag{2.10}$$

There are functions $\nu : [0, T_0] \to \mathcal{L}(X, Y)$ and $\mu : [0, T_0] \to \mathcal{L}_3(X, Y)$ and an $\eta > 0$ such that

$$\sup_{t\in[0,T_0\varepsilon^{-2}]} \|f_1(\varepsilon, t) - \nu(\varepsilon^2 t)\|_{\mathcal{L}(X,Y)} \le C_f \varepsilon^\eta \quad \text{for } \varepsilon \in (0, \varepsilon_0] \tag{2.11}$$

and

$$\sup_{t\in[0,T_0\varepsilon^{-2}]} \|f_3(\varepsilon, t) - \mu(\varepsilon^2 t)\|_{\mathcal{L}_3(X,Y)} \le C_f \varepsilon^\eta \quad \text{for } \varepsilon \in (0, \varepsilon_0]. \tag{2.12}$$

Example 2.6. *The simplest examples fulfilling Assumption 2.5 are for some given constants $\tilde\nu \in \mathbb{R}$ and $\tau_0 > 0$*

$$f(\varepsilon, u, t) = \tilde\nu \varepsilon^2 u - u^3 \quad \text{or} \quad f(\varepsilon, u, t) = \tilde\nu \varepsilon^4 (t - \tau_0 \varepsilon^{-2}) u - u^3, \tag{2.13}$$

for example, with $X = Y = C^0([a,b])$, which is the space of continuous functions from $[a,b]$ to $\mathbb{R}$. We will use these nonlinearities in our examples in section 6. Note that we can take $\eta > 0$ arbitrarily large and $g \equiv 0$.

Remark 2.7. *We will see later in the proof that the assumption on $f_1(\varepsilon, t)$ to leave $\mathcal{N}$ invariant is important to decouple the dynamics of (2.6) for the slow modes from the dynamics of the fast modes. The assumption is true, for example, if f_1 commutes with L_ε, which in turn is obviously true if $f_1(\varepsilon, t)$ is just multiplication by a scalar.*

The time-dependence of ν and μ (cf. (2.11) and (2.12)) reflects the fact that the slow modes should change on a slow time-scale $T = \varepsilon^2 t$.

2.1. The amplitude equation. The amplitude equation is a (system of) SODE that describes the essential dynamics of mild solutions of (1.1) near 0. Some constants in this equation depend heavily on the choice of some basis e in $\mathcal{N}$. One can try to simplify the structure by changing e.

Consider some basis $e = (e_1, \dots, e_n)$ of $\mathcal{N}$ with $\|e_k\|_X = 1$ for all $k = 1, \dots, n$. For $a \in \mathbb{R}^n$ denote $a \cdot e = \sum_{k=1}^n a_k e_k$. Moreover, define the canonical projection $\Pi : X \to \mathbb{R}^n$ by $\Pi(a \cdot e + z) = a$ for all $a \in \mathbb{R}^n$ and all $z \in \text{kernel}(P_c)$. As the spaces $\mathcal{N}$ and $\mathbb{R}^n$ are finite-dimensional, we easily obtain that Π is continuous; i.e., there is a constant $C_\pi > 0$ such that $|\Pi(x)| \le C_\pi \|x\|$ for all $x \in X$.

We define the cubic nonlinearity $\mu_e(T)[\cdot]^3 : \mathbb{R}^n \to \mathbb{R}^n$ for $T \in [0, T_0]$ by

$$\mu_e(T)[a]^3 = \Pi\{\mu(T)[a \cdot e]^3\} = \sum_{i,j,k=1}^n a_i a_j a_k \Pi\{\mu(T)[e_i, e_j, e_k]\} \tag{2.14}$$

and the linearity $\nu_e(T) : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\nu_e(T)a = \Pi\{\nu(T)(a \cdot e)\} = \sum_i^n a_i \Pi\{\nu(T)e_i\}. \tag{2.15}$$

The amplitude equation is now given by

$$a(T) = a_0 + \int_0^T \nu_e(s)a(s)ds + \int_0^T \mu_e(s)[a(s)]^3 ds + \beta(T), \tag{2.16}$$

where $\{\beta(T)\}_{T \ge 0}$ is a Wiener process in $\mathbb{R}^n$ given by $\beta(T) = \varepsilon\Pi(W(\varepsilon^{-2}T))$.

Remark 2.8. *The distribution of β is actually independent of ε due to the scaling properties of a Wiener process. Hence the distribution of solution of (2.16) is independent of ε.*

2.2. Sketch of the results. Our main results are the attractivity (see Theorem 3.3) and the approximation (see Theorem 4.3). In the following we sketch these main results.

For the *attractivity* result assume that the initial condition is of order $\mathcal{O}(\varepsilon)$ and we have large deviation bounds on the stochastic convolution. Then there is a time $t_\varepsilon = \mathcal{O}(\ln(\varepsilon^{-1}))$ such that with high probability for all mild solutions of (1.1) we have

$$u(t_\varepsilon) = \varepsilon a_\varepsilon \cdot e + \varepsilon^2 R_\varepsilon \quad \text{with} \quad |a_\varepsilon|_{\mathbb{R}^n} = \mathcal{O}(1) \text{ and } \|R_\varepsilon\|_X = \mathcal{O}(1).$$

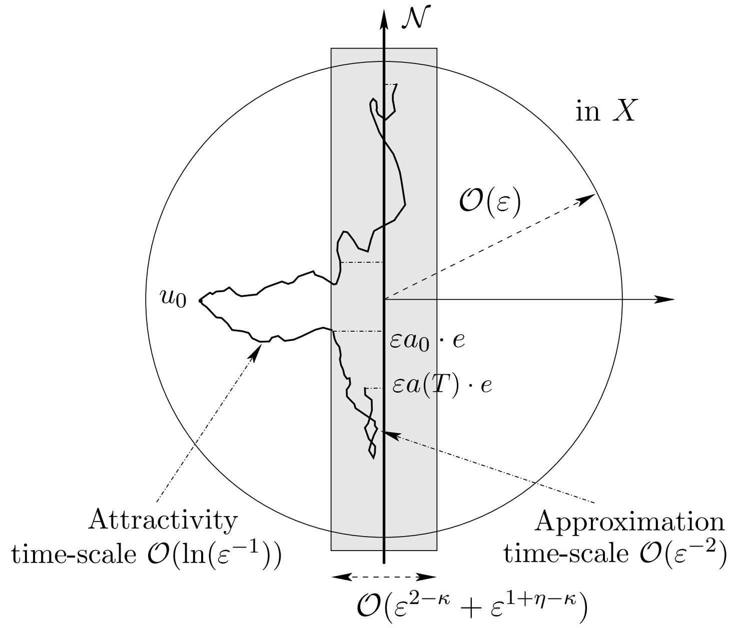

Figure 2.1. *Two typical trajectories of solutions of* (1.1).

For the *approximation* result consider some solution $a(T)$ of the amplitude equation (2.16) and define the approximation $\varepsilon\psi(t) := \varepsilon a(\varepsilon^2 t) \cdot e$. Assume that we have nice large deviation bounds on the stochastic convolution and on the solution $a(T)$, and fix some small $0 < \kappa \ll \eta$. Then we obtain for all mild solutions u of (1.1) that

$$\sup_{t\in[0,T_0\varepsilon^{-2}]} \|u(t) - \varepsilon\psi(t)\|_X < \text{Const} \cdot \left(\varepsilon^{2-\kappa} + \varepsilon^{1+\eta-\kappa}\right) \tag{2.17}$$

with high probability provided $\|u(0) - \varepsilon\psi(0)\|_X = \mathcal{O}(\varepsilon^{2-\kappa} + \varepsilon^{1+\eta-\kappa})$ with high probability. The optimal bound in (2.17) would be $\mathcal{O}(\varepsilon^2)$. Nevertheless, for technical reasons, we are by some κ slightly smaller than that. The constant $\eta > 0$ was defined in (2.11) and (2.12), and we expect $\eta \geq 1$ in many examples.

Combining the attractivity and approximation results, we get a good description of the typical trajectories of (1.1), once we have a good control on various probabilities. A sketch of the typical dynamics is given in Figure 2.1.

Moreover, we can give estimates for the stopping time t^* from Assumption 2.4 like $t^* \geq T_e\varepsilon^{-2}$ with high probability.

The $\mathcal{O}$-notation is used in the following way. A term $G_\varepsilon = \mathcal{O}(g_\varepsilon)$ if and only if there are positive constants ε_0 and C depending only on other constants such that $|G_\varepsilon| \leq C g_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0]$.

3. The attractivity.

First we establish a bound on mild solutions u of (2.4). We show that solutions with initial conditions of order $\mathcal{O}(\varepsilon)$ stay of order $\mathcal{O}(\varepsilon)$ on a large time-scale of order $\mathcal{O}(\varepsilon^{-1})$.

Lemma 3.1. *Suppose all assumptions of section 2 are true. For all times $t_\varepsilon \le \varepsilon^{-1}$ and all constants $0 \le \kappa < 1$, $\delta > 0$, and $D := 2M\delta$, there is an $\varepsilon_0 > 0$ sufficiently small such that for all $\varepsilon \in (0, \varepsilon_0]$ we obtain*

$$\left\{ \sup_{t\in[0,t_\varepsilon]} \|W_{L_\varepsilon}(t)\|_X \le \varepsilon^{-\kappa},\ \|u_0\|_X \le \delta\varepsilon \right\} \Rightarrow \sup_{t\in[0,t_\varepsilon]} \|u(t)\|_X \le D\varepsilon. \tag{3.1}$$

Remark 3.2. *The bound $\varepsilon^{-\kappa}$ on the supremum of the stochastic convolution looks strange at first glance; nevertheless, it is natural. In general we expect (cf. section 5)*

$$\mathbb{P}\left(\sup_{t\in[0,t_\varepsilon]} \|P_s W_{L_\varepsilon}(t)\|_X \le C_w \right) \to 0 \quad \text{for } t_\varepsilon \to \infty.$$

Moreover, for the Brownian motion $P_c W = P_c W_{L_\varepsilon}$

$$\mathbb{P}\left(\sup_{t\in[0,t_\varepsilon]} \|P_c W_{L_\varepsilon}(t)\|_X \le C_w \right) = \mathcal{O}(e^{ct_\varepsilon/C_w^2}) \quad \text{for } t_\varepsilon \text{ and } C_w > 0 \text{ large.}$$

Proof. Using the Assumption 2.5 on f for $\varepsilon D \le \delta_0$, we easily show that there is a constant still denoted by C_f such that for $\|v\|_X \le D\varepsilon$ and $\tau \le t_\varepsilon$

$$\|f(\varepsilon, v, \tau)\|_Y \le C_f(\varepsilon^2\|v\|_X + \|v\|_X^3). \tag{3.2}$$

Define the stopping time $\tau_\varepsilon^* := \inf\{\tau > 0 : \|u(\tau)\|_X > D\varepsilon\}$. Hence, as long as $\tau < \tau_\varepsilon^*$, we obtain

$$\|f(\varepsilon, u(\tau), \tau)\|_Y \le C_f \varepsilon^3 (1 + D^2) D. \tag{3.3}$$

Now we derive from (2.4) for $t \le \min\{t_\varepsilon, \tau_\varepsilon^*\}$

$$\begin{aligned}
\|u(t)\|_X &\le M\|u_0\|_X + M\int_0^t (1 + (t-\tau)^{-\alpha})\|f(\varepsilon, u(\tau), \tau)\|_Y d\tau + \varepsilon^2 \|W_{L_\varepsilon}(t)\|_X \\
&\le [M\delta + \varepsilon^{1-\kappa}]\varepsilon + MC_f\varepsilon^3(1+D^2)D\int_0^{t_\varepsilon}(1+\tau^{-\alpha})d\tau \\
&\le [M\delta + \varepsilon^{1-\kappa}]\varepsilon + MC_f(1+D^2)D\frac{2-\alpha}{1-\alpha}\cdot\varepsilon^2 \\
&< D\varepsilon
\end{aligned}$$

for ε sufficiently small. This yields immediately $\tau_\varepsilon^* \ge t_\varepsilon$ on the set on interest, which finishes the proof. ■

Theorem 3.3 (attractivity). *Suppose all assumptions of section 2 are true. Fix the time $t_\varepsilon = \frac{1}{\omega}\ln(\varepsilon^{-2})$ with ω from (2.1) and some $0 \le \kappa < 1$. We can write the mild solution of (1.1) as*

$$u(t_\varepsilon) = \varepsilon a_\varepsilon \cdot e + \varepsilon^2 R_\varepsilon$$

with $a_\varepsilon \in \mathbb{R}^n$ and $R_\varepsilon \in P_s X$ such that for all $\delta > 0$ and $C_w > 0$

$$\left\{\|u_0\| \le \delta\varepsilon, \quad \sup_{t\in[0,t_\varepsilon]} \|W_{L_\varepsilon}(t)\| \le \varepsilon^{-\kappa}, \quad \|P_s W_{L_\varepsilon}(t_\varepsilon)\| \le C_w\right\}$$
$$\Rightarrow \left\{|a_\varepsilon|_{\mathbb{R}^n} \le 2C_\pi M\delta, \quad \|R_\varepsilon\| \le 2C_w\right\}$$

for sufficiently small $\varepsilon > 0$.

Proof. Define $\varepsilon a_\varepsilon = \Pi(u(t_\varepsilon))$ and $\varepsilon^2 R_\varepsilon = P_s u(t_\varepsilon)$. By Lemma 3.1 all we need to show is a bound on $P_s u$, as $|\varepsilon a| = |\Pi(u(t_\varepsilon))| \le C_\pi D\varepsilon$ with C_π from subsection 2.1.

Using (2.5) and (2.1), we obtain

$$\|P_s u(t_\varepsilon)\|_X \le M e^{-\omega t_\varepsilon} \|u_0\|_X + \varepsilon^2 \|P_s W_{L_\varepsilon}(t_\varepsilon)\|_X$$
$$+ M \int_0^{t_\varepsilon} (1 + (t_\varepsilon - \tau)^{-\alpha}) e^{-(t_\varepsilon - \tau)\omega} \|f(\varepsilon, u(\tau), \tau)\|_Y d\tau.$$

As $\tau \le \varepsilon^{-1}$ and $\|u(\tau)\| \le D\varepsilon$ by Lemma 3.1, we use (3.3) to finally end up with

$$\|P_s u(t_\varepsilon)\|_X \le M\delta\varepsilon^3 + MC_f \varepsilon^3 (D + D^3) \int_0^\infty (1 + \tau^{-\alpha}) e^{-\tau\omega} d\tau + C_w \varepsilon^2.$$

This implies the result. ■

4. Approximation.

For a solution a of (2.16) we define the approximation $\varepsilon\psi$ depending on a slow time-scale $T = \varepsilon^2 t$ by

$$\varepsilon\psi(t) := \varepsilon a(\varepsilon^2 t) \cdot e.$$

The residual of $\varepsilon\psi$ is given by

$$\text{Res}(\varepsilon\psi(t)) = -\varepsilon\psi(t) + e^{tL_\varepsilon} \varepsilon\psi(0) + \int_0^t e^{(t-\tau)L_\varepsilon} f(\varepsilon, \varepsilon\psi(\tau), \tau) d\tau + \varepsilon^2 W_{L_\varepsilon}(t). \tag{4.1}$$

In order to show that $\varepsilon\psi$ is a good approximation of a solution u of (2.4), we have to control the residual.

Theorem 4.1 (residual). *Suppose all assumptions of section 2 are true. Fix $0 < \kappa < \eta$ and constants $C_a, C_w > 0$. Then there exists a constant $C_{\text{res}} > C_w$ such that for sufficiently small $\varepsilon > 0$ we obtain for all solutions a of (2.16)*

$$\left\{\sup_{s\in[0,T_0]} |a(s)|_{\mathbb{R}^n} \le C_a \varepsilon^{-\kappa/4}, \quad \sup_{t\in[0,T_0\varepsilon^{-2}]} \|P_s W_{L_\varepsilon}(t)\|_X \le C_w \varepsilon^{-\kappa}\right\}$$
$$\Rightarrow \sup_{t\in[0,T_0\varepsilon^{-2}]} \|\text{Res}(\varepsilon\psi(t))\|_X \le C_{\text{res}}(\varepsilon^{1+\eta-\kappa} + \varepsilon^{2-\kappa}).$$

Remark 4.2. *The results of Theorem 4.1 obviously remain true if we replace $\sup_{s\in[0,T_0]} |a(s)| \le C_a \varepsilon^{-\kappa/4}$ by $\sup_{s\in[0,T_0]} |a(s)| \le C_a$.*

Proof. Let $T = \varepsilon^2 t$ be the slow time. Now (2.7) and $e^{tL_\varepsilon} = Id$ on $\mathcal{N}$ readily imply

$$\text{Res}(\varepsilon\psi(t)) = \varepsilon^2 P_s W_{L_\varepsilon}(t) \tag{4.2}$$
$$- \varepsilon a(T)\cdot e + \varepsilon a(0)\cdot e + \varepsilon^2 P_c W(T\varepsilon^{-2}) \tag{4.3}$$
$$+ \int_0^{T\varepsilon^{-2}} [\varepsilon^2 f_1(\varepsilon,\tau)(\varepsilon a(\varepsilon^2\tau)\cdot e) + P_c f_3(\varepsilon,\tau)[\varepsilon a(\varepsilon^2\tau)\cdot e]^3] d\tau \tag{4.4}$$
$$+ \int_0^t e^{(t-\tau)L_\varepsilon} P_s f_3(\varepsilon,\tau)[\varepsilon a(\varepsilon^2\tau)\cdot e]^3 d\tau \tag{4.5}$$
$$+ \int_0^t e^{(t-\tau)L_\varepsilon} g(\varepsilon, \varepsilon a(\varepsilon^2\tau)\cdot e, \tau) d\tau. \tag{4.6}$$

Now (4.2) is bounded by $C_w\varepsilon^{2-\kappa}$ by assumption, and $C_{\text{res}} > C_w$ is necessary. We choose $C_{\text{res}} > 3C_w$. Using (2.8), we obtain for $t \le T_0\varepsilon^{-2}$

$$\begin{aligned}
\|(4.6)\|_X &\le M\int_0^t (1+(t-\tau)^{-\alpha})\|g(\varepsilon,\varepsilon a(\varepsilon^2\tau)\cdot e,\tau)\|_Y d\tau \\
&\le MC_f\varepsilon^4 \sup_{\tau\in[0,T_0\varepsilon^{-2}]} \|a(\varepsilon^2\tau)\cdot e\|_X^4 \int_0^t (1+(t-\tau)^{-\alpha})d\tau \\
&\le MC_f \sup_{s\in[0,T_0]} |a(s)|^4 n^2 \left(T_0\varepsilon^{-2} + \frac{(T_0\varepsilon^{-2})^{1-\alpha}}{1-\alpha}\right) \\
&\le MC_f C_a^4 n^2 T_0 \left(1 + \frac{T_0^{-\alpha}}{1-\alpha}\right)\cdot\varepsilon^{2-\kappa} < \frac{1}{3}C_{\text{res}}\varepsilon^{2-\kappa},
\end{aligned}$$

which gives us a second condition on C_{res}. Note that for all $a \in \mathbb{R}^n$, $\|a\cdot e\|_X \le \sum_{i=1}^n |a_i| \le n^{1/2}|a|_{\mathbb{R}^n}$ (as $\|e_i\|_X = 1$). Using (2.10) and (2.1), we analogously derive

$$\begin{aligned}
\|(4.5)\|_X &\le MC_f n^{3/2} \sup_{\tau\in[0,t]} |\varepsilon a(\varepsilon^2\tau)|^3 \int_0^t (1+(t-\tau)^{-\alpha})e^{-\tau\omega}d\tau \\
&\le MC_f n^{3/2} C_a^3 \int_0^\infty (1+\tau^{-\alpha})e^{-(t-\tau)\omega}d\tau\cdot\varepsilon^{3-3\kappa/4}.
\end{aligned}$$

This can be also bounded by $\frac{1}{3}C_{\text{res}}\varepsilon^{2-\kappa}$, which implies a third condition on C_{res}.

For (4.4) recall that $\mathcal{N}$ is invariant under $f_1(\varepsilon,\tau)$ by Assumption 2.5. Using the substitution $s = \varepsilon^2\tau$ together with (2.11) and (2.12), we obtain

$$\begin{aligned}
(4.4) &= \int_0^T \left[\varepsilon\Pi\left\{f_1(\varepsilon,\varepsilon^{-2}s)a(s)\cdot e + f_3(\varepsilon,\varepsilon^{-2}s)[a(s)\cdot e]^3\right\}\cdot e\right] ds \\
&= \varepsilon\int_0^T \left[\nu_e(s)(a(s)) + \mu_e(s)[a(s)]^3\right]\cdot e\, ds \\
&\quad + \varepsilon T_0 C_\pi C_f \varepsilon^\eta \cdot \left[n^{1/2}C_a\varepsilon^{-\kappa/4} + n^{3/2}C_a^3\varepsilon^{-3\kappa/4}\right] \\
&= \varepsilon\int_0^T \left[\nu_e(s)(a(s)) + \mu_e(s)[a(s)]^3\right] ds\cdot e + \mathcal{O}(\varepsilon^{1+\eta-\kappa}).
\end{aligned}$$

Now we can use the amplitude equation (2.16) to cancel out the remaining terms in (4.3) and (4.4). This yields a fourth condition on C_{res} if we compute the $\mathcal{O}$-terms explicitly.

We finally derive $\|\text{Res}(\varepsilon\psi(t))\| \leq C_{\text{res}}(\varepsilon^{1+\eta-\kappa} + \varepsilon^{2-\kappa})$ for all $t \in [0, T_0\varepsilon^{-2}]$. ∎

Theorem 4.3 (approximation). *Suppose all assumptions of section 2 are true. Fix the constants $C_{\text{res}}, T_0, C_a, \delta > 0$, and $\kappa \in (0, \eta)$. Then there is a constant C_{att} such that for sufficiently small $\varepsilon > 0$ we obtain for all solutions u of (2.4) and all solutions a of (2.16)*

$$\left\{\begin{aligned} &\|u_0 - \varepsilon\psi(0)\|_X \leq \delta\varepsilon^2, \quad \sup_{s\in[0,T_0]} |a(s)|_{\mathbb{R}^n} \leq C_a n^{-1/2}, \\ &\qquad\qquad \sup_{t\in[0,T_0\varepsilon^{-2}]} \|\text{Res}(\varepsilon\psi(t))\|_X \leq C_{\text{res}}(\varepsilon^{2-\kappa} + \varepsilon^{1+\eta-\kappa}) \end{aligned}\right\}$$
$$\Rightarrow \sup_{t\in[0,T_0\varepsilon^{-2}]} \|u(t) - \varepsilon\psi(t)\|_X \leq C_{\text{att}}(\varepsilon^{1+\eta-\kappa} + \varepsilon^{2-\kappa}).$$

Proof. Define $\varepsilon^2 R(t) := u(t) - \varepsilon\psi(t)$. Now (2.4) and (4.1) imply

$$\begin{aligned} R(t) &= e^{tL_\varepsilon}R(0) + \varepsilon^{-2}\text{Res}(\varepsilon\psi(t)) \\ &\quad + \varepsilon^{-2}\int_0^t e^{(t-\tau)L_\varepsilon}[f(\varepsilon, u(\tau), \tau) - f(\varepsilon, \varepsilon\psi(\tau), \tau)]d\tau. \end{aligned} \tag{4.7}$$

The Taylor expansion of f from (2.7) yields

$$\begin{aligned} &f(\varepsilon, u(\tau), \tau) - f(\varepsilon, \varepsilon\psi(\tau), \tau) \\ &= \varepsilon^4 f_1(\varepsilon, \tau)R(\tau) + f_3(\varepsilon, \tau)[\varepsilon\psi(\tau) + \varepsilon^2 R(\tau)]^3 - f_3(\varepsilon, \tau)[\varepsilon\psi(\tau)]^3 \\ &\quad + g(\varepsilon, +\varepsilon^2 R(\tau), \tau) - g(\varepsilon, \varepsilon\psi(\tau), \tau). \end{aligned}$$

First expand the trilinear form to cancel $[\varepsilon\psi(\tau)]^3$. Then using the bound on $a(s)$, it is easy to derive $\|\psi(\tau)\|_X \leq C_a$. Therefore, as long as $\|R(\tau)\|_X \leq \varepsilon^{-1}$,

$$\begin{aligned} &\|f(\varepsilon, u(\tau), \tau) - f(\varepsilon, \varepsilon\psi(\tau), \tau)\|_Y \\ &\leq \varepsilon^4\|R(\tau)\|_X C_f + \varepsilon^4\|R(\tau)\|_X C_f[3C_a^2 + 3C_a + 1] + C_f\varepsilon^4([C_a + 1]^4 + C_a^4) \\ &\leq 3\varepsilon^4\|R(\tau)\|_X C_f[C_a + 1]^2 + 2\varepsilon^4 C_f[C_a + 1]^4. \end{aligned}$$

By assumption $\|R(0)\|_X \leq \delta$. In a first step we prove that $\|R(t)\|_X \leq \varepsilon^{-1}$ for all $t \in [0, T_0\varepsilon^{-2}]$. Therefore, we further split

$$R = R_c + R_s = P_c R + P_s R.$$

Hence, as long as $\|R(t)\|_X \leq \varepsilon^{-1}$ and $t \leq T_0\varepsilon^{-2}$, we obtain from (4.7)

$$\begin{aligned} \|R_s(t)\|_X &\leq M\|R_s(0)\|_X + C_{\text{res}}(\varepsilon^{-\kappa} + \varepsilon^{-1+\eta-\kappa})\|P_s\|_{\mathcal{L}(X)} \\ &\quad + \varepsilon^2 M\int_0^\infty (1 + \tau^{-\alpha})e^{-\tau\omega}d\tau[3\varepsilon^{-1} + 2]C_f[C_a + 1]^4\|P_s\|_{\mathcal{L}(X)} \\ &\leq 2C_{\text{res}}(\varepsilon^{-\kappa} + \varepsilon^{-1+\eta-\kappa})\|P_s\|_{\mathcal{L}(X)} && (4.8) \\ &< \frac{1}{2}\varepsilon^{-1} \qquad \text{for sufficiently small } \varepsilon > 0. && (4.9) \end{aligned}$$

Additionally we obtain

$$\|R_c(t)\|_X \leq M\|R_c(0)\|_X + C_{\text{res}}(\varepsilon^{-\kappa} + \varepsilon^{-1+\eta-\kappa})\|P_c\|_{\mathcal{L}(X)} + 3C_f[C_a+1]^2\|P_c\|_{\mathcal{L}(X)}\varepsilon^2 M\int_0^t \|R(\tau)\|_X d\tau + \mathcal{O}(1).$$

Now we use $\|R(t)\|_X \leq \|R_c(t)\|_X + \mathcal{O}(\varepsilon^{-\kappa} + \varepsilon^{-1+\eta-\kappa})$ by (4.8) and the Gronwall inequality to obtain

$$\|R_c(t)\|_X \leq \mathcal{O}(\varepsilon^{-\kappa} + \varepsilon^{-1+\eta-\kappa}) \cdot \exp\{3C_f[C_a+1]^2 T_0 M\|P_c\|_{\mathcal{L}(X)}\} \tag{4.10}$$

$$< \frac{1}{2}\varepsilon^{-1}. \tag{4.11}$$

Hence, for sufficiently small $\varepsilon > 0$ we obtain from (4.9) and (4.11) first that $\|R(t)\|_X < \varepsilon^{-1}$ for all $t \leq T_0\varepsilon^{-2}$.

Moreover, (4.8) and (4.10) imply $\sup_{t\in[0,T_0\varepsilon^{-2}]}\|R(t)\|_X = \mathcal{O}(\varepsilon^{-\kappa} + \varepsilon^{-1+\eta-\kappa})$. ■

5. Large deviation results.

This section provides large deviation results to control the various probabilities that arise in our application of the abstract result to SPDEs. First we provide estimates for solutions of the amplitude equation (2.16). Then we discuss the stochastic convolution W_L in $C^0([a,b])$, where the operator L is a differential operator.

5.1. Amplitude equation.

Consider any solution $a(T)$ of (2.16). Without the cubic nonlinearity or with Lipschitz-continuous nonlinearities there are numerous results, especially for small noise strength (see, e.g., [FW98]). Nevertheless, for our examples we provide an elementary result, which is based only on a priori estimates and large deviation results for Wiener processes. In our cubic case we have to distinguish between the case of stable or unstable cubic nonlinearities.

Theorem 5.1. *Suppose Assumption 2.5 is true, and fix some solution $a(T)$ of (2.16). Then there is a constant $c > 0$ depending only on the covariance matrix of β such that the following are true:*

(I) *The unstable case: For all constants $C_a > 0$ and all $T_1 \in (0, T_0]$ with $T_1 < (2C_f(1 + C_a^2))^{-1}$ we obtain*

$$\mathbb{P}\Big(\sup_{s\in[0,T_1]}|a(s)| \geq C_a\Big) \leq \mathbb{P}\Big(|a(0)| \geq C_a/4\Big) + 4ne^{-cC_a^2/T_1 n}.$$

(II) *The stable case: Suppose that $\mu_e(T)[b]^3 \cdot b \leq -C_e|b|^4$ for all $b \in \mathbb{R}^n$. Then there is a constant $C > 0$ depending only on C_f and C_e such that for all constants $T_1 \in (0, T_0]$ and all $C_a > C$ we obtain*

$$\mathbb{P}\Big(\sup_{s\in[0,T_1]}|a(s)| \geq C_a\Big) \leq \mathbb{P}\Big(|a(0)| \geq C_a/2\Big) + 4n\exp\bigg\{-\frac{c}{T_1 n}\sqrt{\frac{C_a^4}{C^4} - 1}\bigg\}.$$

Remark 5.2. *Unfortunately the unstable case has serious drawbacks. High probability is paid by validity of the result only on small time-intervals, although this still gives us for the original equation a time-scale of order $\mathcal{O}(\varepsilon^{-2})$.*

Proof. Using (2.10) and (2.12), we easily obtain that there is a constant also denoted by C_f such that

$$|\mu_e(T)[a]^3| \le C_f|a|^3 \quad \text{for all} \quad a \in \mathbb{R}^n,\ T \in [0, T_0], \tag{5.1}$$

where C_f actually depends only on C_π, n, and the constants in Assumption 2.5. Analogously we obtain

$$|\nu_e(T)[a]| \le C_f|a| \quad \text{for all} \quad a \in \mathbb{R}^n,\ T \in [0, T_0]. \tag{5.2}$$

Using (5.2) and (5.1), we obtain from (2.16)

$$|a(T)| \le |a(0)| + C_f \int_0^T (|a(s)| + |a(s)|^3)ds + |\beta(T)|.$$

Suppose $|a(0)| < C_a/4$ and $\sup_{T\in[0,T_1]} |\beta(T)| < C_a/4$. As long as $|a(T)| < C_a$ we obtain

$$|a(T)| \le C_a/2 + C_f T(C_a + C_a^3) < C_a,$$

as long as $T \le 1/(2C_f(1 + C_a^2))$. Hence

$$\mathbb{P}\left(\sup_{s\in[0,T_1]} |a(s)| < C_a\right) \ge \mathbb{P}\left(|a(0)| < C_a/4, \quad \sup_{s\in[0,T_1]} |\beta(s)| < C_a/4\right).$$

To finish the proof of the unstable case, use, e.g., [DZ98, section 5.2]. As β is a Brownian motion in $\mathbb{R}^d$, we easily obtain the existence of a constant c depending only on the covariance matrix of β such that

$$\mathbb{P}\left(\sup_{s\in[0,T_1]} |\beta(s)| > C_a/4\right) \le 4ne^{-cC_a^2/T_1 n}. \tag{5.3}$$

In the *stable case* define $b = a - \beta$. Hence

$$b \in C^1 \quad \text{with} \quad \partial_T b = \nu_e[b + \beta] + \mu_e[b + \beta]^3.$$

In the following we denote all constants depending only on C_f or C_e simply by C. Using (5.2), the assumption on μ_e, and Young's inequality, we obtain

$$\begin{aligned} \frac{1}{2}\partial_T |b|^2 &\le C_f|b + \beta||b| + \mu_e[b + \beta]^3 \cdot b \\ &\le -\frac{C_e}{2}|b|^4 + C|\beta|^4 + C. \end{aligned}$$

Suppose $|\beta|^4 < R$ and $|a(0)| < \delta$. Then Lemma 5.3 implies

$$|b(T)|^2 < \max\left\{\delta^2,\ C\frac{\sqrt{R^4 + 1}}{C_e^{1/2}}\right\}.$$

Hence

$$|a(T)| < \frac{1}{2}C(R^4+1)^{1/4} + \delta$$

for all $T > 0$.

Define $R^4 = (C_a/C)^4 - 1$ and $\delta = C_a/2$; then for all T_1 and all $C_a > C$

$$\mathbb{P}\left(\sup_{s\in[0,T_1]} |a(s)| \le C_a\right) \ge \mathbb{P}\left(|a(0)| < C_a/2, \quad \sup_{s\in[0,T_2]} |\beta(s)| < \sqrt[4]{(C_a/C)^4 - 1}\right).$$

By (5.3) we easily finish the proof. ■

Lemma 5.3. *Suppose for some constant $d, c > 0$ we have a real-valued function y such that $y(0) \ge -c/d$ and $y' \le -d^2y^2 + c^2$. Then*

$$y \le \max\{y(0),\ c/d\} \quad \textit{for all} \quad t > 0.$$

Proof. A comparison principle (see, e.g., [Ha80]) yields $y \le x$ with $x(0) = y(0)$ and $x' = -d^2x^2 + c^2$. For the proof we just have to use first that $\pm c/d$ are the only stationary solutions for x and then that x is growing if and only if $|x| \le c/d$. ■

5.2. Stochastic convolution. There are many general results for exponential tail estimates for stochastic convolutions in Banach or Hilbert spaces. One of the first results is [CM90]; for recent results, see, for instance, [BP00b] and the references therein. A recent new approach relying on Zygmund's interpolation inequality is [SS03].

For our applications we need estimates for sectorial differential operators in the space of continuous functions. We need especially the dependence of the constants on the time-interval $[0, T]$ which is frequently not covered. For simplicity we will basically apply the results of [P92]. This is not optimal, but it is sufficient for our examples.

Assumption 5.4. *Suppose Assumption 2.2 is true, and let L be some nonpositive self-adjoint differential operator of order $2m$ subject to suitable boundary conditions on some sufficiently smooth bounded domain $G \subset \mathbb{R}^d$ (e.g., $L = p(\Delta)$ for some polynomial p of degree m). Suppose that Assumption 2.1 is true with P_c as the L^2-orthogonal projection onto $\mathcal{N} = N(L)$ and that L generates an analytic semigroup $\{e^{tL}\}_{t\ge0}$ on $L^2(G)$ and therefore also on $H = P_sL^2(G)$, where $P_s = I - P_c$ as before.*

Define $E = P_sC^0(G)$, where $C^0(G)$ is the standard space of continuous functions with sup-norm. Then it is easy to verify that the assumptions (E.1) and (E.2) of [P92] are fulfilled (see, e.g., [L94, section 3]). Also, Assumption 2.2 is true with $X = E = Y$. Note that the L^2-orthogonal projection P_c is still the spectral projection for L defined on E.

Suppose W is a Q-Wiener process as in Assumption 2.3. Then we can write $W(t) = Q^{1/2}\tilde{W}(t)$ with $\tilde{W}(t) = \sum_{k=1}^{\infty} \beta_k(t) f_k$, where $\{\beta_k\}_{k\in\mathbb{N}}$ is some family of independently and identically distributed real-valued Brownian motions, and $\{f_k\}_{k\in\mathbb{N}}$ is an orthonormal basis in $L^2(G)$ of eigenfunctions of Q.

For all $\gamma_0 \in (0, 1/2)$ and $p_0 > 1$ define as in [P92]

$$\kappa_T^{p_0} := \int_0^T t^{(\gamma_0-1)p_0} \|e^{tL}\|^{p_0}_{\mathcal{L}(H,E)} dt, \tag{5.4}$$

$$\eta_T := \sup_{t\in[0,T]} \int_0^t \tau^{-2\gamma_0} \mathrm{tr}_{L^2}\left(Q^{1/2} P_s e^{2\tau L} Q^{1/2}\right) d\tau. \tag{5.5}$$

Provided κ_T and η_T are finite, then [P92, Theorem 1.1] implies

$$\mathbb{P}\left(\sup_{t\in[0,T]} \|P_s W_L(t)\|_{C_0} \geq \delta\right) \leq C(T, p_0) \exp\left\{\frac{-\delta^2}{\kappa_T^2 \eta_T}\right\},$$

with $C(T, p_0) = 4T \exp\{(4Tn_0!)^{1/n_0}\}$ and $n_0 = [p_0(2p_0 - 2)^{-1}] + 1$.

For $s > d/2$ we easily check using Sobolev imbedding of E into $H^s(G)$ that Assumption 2.2 is also true with $X = E$, $Y = H$, $\alpha = s/2m$, and some $\omega > 0$ which is some spectral gap to the first nonzero eigenvalue of L. Now for $u \in H$

$$\|e^{tL}u\|_E = \|P_s e^{tL} u\|_{C^0} \leq M(t^{-s/2m} + 1)e^{-\omega t}\|u\|_{L^2}.$$

We easily obtain $\kappa_T < C_\kappa$ for all $T > 0$ with some constant $C_\kappa > 0$ (depending only on s, m, p_0, γ_0, and ω) if and only if $(1/p_0 - 1 + \gamma_0)2m > s$. If we choose p_0 near 1 and γ_0 near $\frac{1}{2}$, then we will always find such an $s > \frac{d}{2}$ provided $2m > d$. Moreover, if $\gamma_0 = \frac{1}{4}$, we will always find s provided $m > d$.

If $\mathrm{tr}(Q) < \infty$, then $\mathrm{tr}(Q^{1/2} P_s e^{2tL} Q^{1/2}) \leq \mathrm{tr}(Q) M e^{-2\omega\tau}$. This implies the existence of some constant C_η depending only on α_0 and ω such that $\eta_T < C_\eta M \mathrm{tr}(Q)$.

For $Q = I$ we obtain

$$\eta_T = \sup_{t\in[0,T]} \int_0^t \tau^{-2\gamma_0} \sum_{\lambda_k \neq 0} e^{2\tau\lambda_k} d\tau \leq \sum_{\lambda_k\neq 0} \frac{1}{|\lambda_k|^{1-2\gamma_0}} \int_0^\infty s^{-2\gamma_0} e^{-2s} ds. \tag{5.6}$$

As λ_k is proportional to $-k^{2m/d}$ (cf., e.g., [EE87]), we obtain that (5.6) is finite if and only if $2m(1 - 2\gamma_0) > d$, which in turn is true for $\gamma_0 = \frac{1}{4}$ and $m > d$.

Applying the results of [P92], we immediately prove the following.

Theorem 5.5. *Let Assumption 5.4 be true, and let W be a Q-Wiener process as in Assumption 2.3. Fix some arbitrary small $\zeta > 0$, and suppose either* $\mathrm{tr}(Q) < \infty$ *and $2m > d$ or $Q = I$ and $m > d$.*

Then there are constants $c_1, c_2 > 0$ such that for all $T, \delta > 0$ we have

$$\mathbb{P}\left(\sup_{t\in[0,T]} \|P_s W_L(t)\|_{C^0} > \delta\right) \leq \exp\{c_1 T^\zeta - c_2\delta^2\}.$$

We remark without proof that the condition $m > d$ is not optimal for $Q = I$; here $2m > d$ should be also true.

Using a result similar to (5.3) for P_cW, we obtain the following.

Corollary 5.6. *Let the assumptions of the previous theorem be true. Then there are constants $c_i > 0$ such that for all $T, C_w > 0$, we obtain*

$$\mathbb{P}\Big(\sup_{t\in[0,T]} \|W_L(t)\|_{C^0} > C_w\Big) \le c_3 \exp\{-c_4 C_w^2/T\} + \exp\{c_1 T^\zeta - c_2 C_w^2\}.$$

For sufficiently large $t > 0$ and all $\delta > 0$ it is well known that

$$\mathbb{P}\Big(\|P_s W_{L_\varepsilon}(t)\|_{C^0} > C_w\Big) \le c_6 \exp\{-c_5 C_w^2\}. \tag{5.7}$$

This is, for example, easily proven by using the imbedding of C^0 into H^s for $s > d/2$ and by calculating that for some small $h > 0$ the exponential moment $\mathbb{E}\exp\{h\|P_s W_L(t)\|_{H^s}^2\}$ is uniformly bounded with respect to t.

6. Applications.

In our applications we consider for simplicity of presentation only examples which are restricted to scalar SPDEs in one space dimension $d = 1$. Moreover, we will consider only examples where the amplitude equation is only an SODE in $\mathbb{R}$. We could treat systems of SPDEs in higher dimension or higher-dimensional SODEs as amplitude equations, but we want to keep the notation as simple as possible in order to demonstrate the main ideas of our applications.

Let us fix some notation needed in what follows. Suppose W is some Q-Wiener process with either $\mathrm{tr}(Q) < \infty$ or $Q = I$. Define $L = -(1 + \partial_x^2)^2$, which is a self-adjoint operator on $L^2([0, \pi])$ subject to zero Dirichlet boundary conditions for u and $\partial_x^2 u$.

It is well known that the fourth order differential operator L generates a bounded semigroup on $Y = X = C^0([0, \pi])$, which fulfills Assumptions 2.1 and 2.2 with $\mathcal{N} = \mathrm{span}\{\sin\}$ and $\alpha = 0$ in (2.2). Therefore, we fix P_c to be the L^2-orthogonal projection onto $\mathcal{N}$, which coincides in this case with the spectral projection of L defined on X. It is also easy to check that Assumption 2.3 is true.

Moreover, we have the L^2-orthogonal basis of eigenfunctions $e_k(x) = \sin(kx)$ in X with $\|e_k\|_X = 1$. Note that Assumption 5.4 is true with $m = 2$ and $p(z) = -(1 + z^2)^2$. Therefore, we can apply all large deviation results of the previous subsection.

6.1. Dynamic pitchfork-bifurcation.

The dynamic pitchfork-bifurcation is a well-studied experimental effect (see, e.g., [BK99, ME87, GM03] and the references therein). It refers to a system which is moved slowly though some deterministic bifurcation point by slowly changing the bifurcation parameter with time. This leads to hysteresis-type effects, where the solution stays near the unstable equilibrium after passing through the deterministic bifurcation point. This result was studied rigorously for the deterministic and the stochastic equations (see [BG02] and the references therein), both in an ODE setting. We will describe how the latter result can be immediately carried over to SPDEs by the results presented in the previous sections.

Consider as an example the scalar SPDE

$$\partial_t u(t) = Lu(t) + \tilde{\nu}\varepsilon^4(t - \tau_0\varepsilon^{-2})u(t) - u^3(t) + \sigma\varepsilon^2\partial_t W(t) \quad \text{for} \quad t > 0, \tag{6.1}$$

subject to zero Dirichlet boundary conditions on $[0,\pi]$ and initial condition $u(0)=u_0$. Here $\tilde{\nu}$ and τ_0 are some constants fixed later on. It was already discussed in Example 2.6 that Assumption 2.5 is true with $X=Y=C^0([0,\pi])$.

As obviously $\mathcal{N}=\operatorname{span}\{\sin\}$, the corresponding amplitude equation describes the amplitude $a\in\mathbb{R}$ of the sine. It is given by an easy calculation (cf. section 2.1):

$$a'(T)=(s-\tau_0)\tilde{\nu}a(T)-\frac{3\pi}{8}a^3(T)+\sigma\beta'(T)\quad\text{for}\quad T>0, \tag{6.2}$$

where $\beta(T)=\langle\varepsilon W(T\varepsilon^{-2}),\sin\rangle_{L^2}$ is a real-valued Brownian motion.

We will first state rigorous results that verify $u(t)\approx a(\varepsilon^2 t)\sin$. After that we briefly comment on the dynamic pitchfork-bifurcation. Note that (6.2) exhibits exactly the same structure as the equations discussed in [BG02]. Therefore, an analogue of their result for (6.1) follows immediately.

Theorem 6.1 (attractivity). *There are constants $c_i>0$ such that for any choice of $\tau_0>0$ and $\tilde{\nu},\sigma\in[-1,1]$, all mild solutions u of (6.1), and all solutions a of (6.2), we obtain the following.*

For $t_\varepsilon=\ln(\varepsilon^{-1})$ we can write $u(t_\varepsilon)=\varepsilon a_\varepsilon\sin+\varepsilon^2 R_\varepsilon$ with

$$\begin{aligned}&\mathbb{P}\Big(|a_\varepsilon|\le 2C_\pi M\delta,\quad \|R_\varepsilon\|_{C^0}\le 2C_w\Big)\\ &\ge\mathbb{P}\Big(\|u_0\|_{C^0}\le\delta\varepsilon\Big)-c_1e^{-c_2\varepsilon^{-1}}-c_3e^{-c_4C_w^2}\end{aligned}$$

for all $C_w,\delta>0$, and sufficiently small $\varepsilon>0$.

Note that the probability bound on the right-hand side is in general only positive for ε small and C_w large.

Proof. The proof is by straightforward application of Theorem 3.3 together with Corollary 5.6 and (5.7) to bound the probabilities. ■

Theorem 6.2 (approximation). *For all $T_0>\tau_0>0$ and $0<\kappa<1$ there are constants $c_i>0$ such that the following is true.*

Given $\delta>0$, $C_a>0$, $C_w>0$, there is a constant $C_{\text{att}}>0$ such that for any choice of $\tilde{\nu},\sigma\in[-1,1]$, all mild solutions u of (6.1), and all solutions a of (6.2),

$$\begin{aligned}&\mathbb{P}\Bigg(\sup_{t\in[0,T_0\varepsilon^{-2}]}\|u(t)-\varepsilon a(\varepsilon^2 t)\sin\|_{C^0}\le C_{\text{att}}\varepsilon^{2-\kappa}\Bigg)\\ &\ge 1-\mathbb{P}\Big(\|u_0-\varepsilon a(0)\sin\|_{C^0}>C_w\varepsilon\Big)\\ &\qquad -\mathbb{P}\Big(|a(0)|>C_a\Big)-c_1e^{-c_2\varepsilon^{-\kappa}}-c_3e^{-c_4C_a^2/T_0}\end{aligned} \tag{6.3}$$

for sufficiently small $\varepsilon>0$.

Proof. The proof is by application of Theorems 4.3 and 4.1 together with Theorems 5.5 and 5.1 to bound probabilities. ■

We can further bound (6.3) in Theorem 6.2 by Theorem 6.1. Summarizing both theorems, we can write with high probability all mild solutions u of (6.1) with $\|u(0)\|=\mathcal{O}(\varepsilon)$ as

$$u(t)=\varepsilon a(\varepsilon^2 t)\sin+\mathcal{O}(\varepsilon^{2-\kappa}) \tag{6.4}$$

for all $t \in \ln(\varepsilon^{-1}) + [0, T_0\varepsilon^{-2}]$, where a is a solution of the amplitude equation (6.2) with initial condition $a_0 = \varepsilon^{-1}\Pi u(t_\varepsilon)$, where the projection Π was defined in section 2.1.

Now we can immediately apply the results of [BG02] first to (6.2) and then via (6.4) to solutions of (6.1). Hence, for ε sufficiently small, (6.1) exhibits the same dynamics as the one-dimensional dynamic pitchfork-bifurcation, but on a much slower time-scale. To keep the presentation short, we refrain from restating the elaborate description of the transient dynamics from [BG02]. Note, finally, that it is essential to derive a one-dimensional kernel $\mathcal{N}$, as the results of [BG02] are only valid for one-dimensional amplitude equations.

6.2. Pattern formation. The formation of a pattern below the threshold of a change of stability is a well-known experimental phenomenon. See, for example, [SA02] or [SR94] for noise-induced convection rolls below the onset of convection in Bénard's problem. In this case the system is slightly below a change of stability in the unperturbed (deterministic) system, which undergoes a pitchfork-bifurcation.

Unfortunately, this problem is out of reach for the present approach, as it consists of a three-dimensional Navier–Stokes equation which is coupled to a heat equation. We could treat systems of SPDEs on three-dimensional domains, but we cannot treat the quadratic nonlinearity with the method presented in this article. This will be a topic of further research (see [B03]).

We sketch a simple problem, which exhibits pattern formation below threshold of stability. Let us consider the well-known Swift–Hohenberg equation, which is in the theory of convection frequently used as a simplified model for the first convective instability. It is similar to the equations of the previous section,

$$\partial_t u(t) = Lu(t) + \nu\varepsilon^2 u(t) - u^3(t) + \varepsilon^2\partial_t W(t) \quad \text{for} \quad t > 0, \tag{6.5}$$

subject to zero Dirichlet-type boundary conditions on $[0, \pi]$ and initial condition $u_0 = 0$, where L and W are given in the beginning of section 6 and ν is some parameter.

The pattern in our toy-model us just the sine representing the convection roll in the full problem. Due to the special size of the domain, we have only one period of the sine. If we would consider large domains, we would get several periods of the pattern depending on the size of the domain. Nevertheless, for sake of simplicity we stay with this very simple model.

For the unperturbed deterministic equation it is well known that it undergoes a pitchfork-bifurcation at $\nu = 0$. There $\sin \in N(L)$ becomes unstable. For $\nu < 0$ the homogeneous solution $u = 0$ is the only stable solution, and for $\nu > 0$ we end up with a stable pattern that is a small deformation of the sine. To verify this result is a lot of work but is standard, using, for instance, the celebrated theory of Crandel and Rabinowitz. In contrast to that, we will see that also in the case of $\nu < 0$ due to additive noise the pattern appears and sustains for long times, although it should decay due to the stability of the homogeneous solution $u = 0$. In the following we will verify a result that the probability $\mathbb{P}$(pattern visible for "most" $t \in [0, T_0\varepsilon^{-2}]$) is near 1, where T_0 is just some arbitrary constant.

Let us first apply the main results of this paper to (6.5). Due to $u(0) = 0$ it is obvious that the assertion of the attractivity result readily holds with $t_\varepsilon = 0$, $a_\varepsilon = 0$, and $R_\varepsilon = 0$.

Moreover, an easy calculation (see section 2.1) establishes that the corresponding amplitude equation is

$$a' = \nu a - \frac{3\pi}{8}a^3 + \beta', \tag{6.6}$$

where $\beta(T) = \langle \varepsilon W(T\varepsilon^{-2}), \sin\rangle_{L^2}$ as before in section 6.1. Now we can easily verify an approximation result as in Theorem 6.2 to obtain

$$u(t) = \varepsilon a(\varepsilon^2 t)\sin + \mathcal{O}(\varepsilon^{2-\kappa}) \qquad \text{uniformly for all} \quad t \in [0, T_e\varepsilon^{-2}] \tag{6.7}$$

with high probability for all $\varepsilon \in [0, \varepsilon_0]$. We are free to choose any $T_0 > 0$ we want, but we have to pay for that with $\varepsilon_0 > 0$ small. We refrain from restating the precise result, as it is completely analogous to Theorem 6.2.

To prove a pattern result, we can, for example, verify that $|a(T)| \geq C\varepsilon^{1/2}$ for "a lot of" times $T \in [0, T_0]$. In what follows we give a short argument for this.

First define

$$l_\varepsilon(T) := |\{s \in [0,T] \ : \ |a(s)| \leq \varepsilon^{1/2}\}|.$$

This is the "bad" set of times, where we possibly do not see the pattern. However, we will definitely see the pattern for all times in $[0, T_0\varepsilon^{-2}] - l_\varepsilon(T_0)$. The following remark summarizes the result, which is now possible to verify. We refrain from stating an abstract theorem.

Remark 6.3. *For a main result on pattern formation, we will verify*

$$\mathbb{P}(l_\varepsilon(T_0) \geq \varepsilon^{1/4}) \leq C\varepsilon^{1/4}T_0 \tag{6.8}$$

for T_0 large and ε small enough.

Hence the probability is high to see the pattern on a set of times with measure $(T_0 - C\varepsilon^{1/4})\varepsilon^{-2}$ for any choice of the bifurcation parameter $\varepsilon^2\nu$, provided, for example, $|\nu| \leq 1$.

Note finally that there is nothing special about dimension one. Similar results will apply in case $n = \dim(\mathcal{N}) > 1$, where $\mathcal{N}$ is then some space of pattern. The only thing we rely on are some technical assumptions for the existence of an invariant Markov measure for the amplitude equation.

To establish (6.8) first recall that the distribution of a is independent of ε by Remark 2.8. Moreover, it is well known under certain assumptions on the noise (e.g., full rank of the covariance matrix) and the stability of the cubic nonlinearity that there exists a unique invariant Markov measure $\mathbb{P}^*$ for the amplitude equation. Moreover, the Lebesgue-density $p^* := d\mathbb{P}^*/d\lambda$ of this measure is continuous.

By the definition of l_ε

$$\mathbb{E}l_\varepsilon(T) = \int_0^T \mathbb{P}(|a(s)| \leq \varepsilon^{1/2})ds.$$

Now the celebrated Birkhoff ergodic theorem (cf., e.g., [dPZ96]) for invariant Markov measures implies

$$\frac{1}{T}\mathbb{E}l_\varepsilon(T) \to \mathbb{P}^*([-\varepsilon^{1/2}, \varepsilon^{1/2}]) \quad \text{for} \quad T \to \infty.$$

Moreover, we obtain $\frac{1}{2}\varepsilon^{-1/2}\mathbb{P}^*([-\varepsilon^{1/2},\varepsilon^{1/2}]) \to p^*(0)$ by the continuity of the density. Furthermore, by the Chebyshev inequality,

$$\mathbb{P}(l_\varepsilon(T) \geq \varepsilon^{1/4}) \leq \varepsilon^{-1/4}\mathbb{E}l_\varepsilon(T) \approx 2\varepsilon^{1/4}Tp^*(0),$$

and it is straightforward to establish (6.8).

6.3. Bifurcation. In this section we briefly sketch transient stochastic dynamics near a deterministic pitchfork-bifurcation. As mentioned in the introduction, we will not describe the whole bifurcation but rather will focus on examples of parameter regimes, where we can establish the amplitude equation describing finite time behavior of solutions.

As we will see, there are different scenarios depending on the ratio between noise-strength and bifurcation parameter. This will give another indication to the well-known fact that a stochastically perturbed bifurcation leads to a soft transition of the transient dynamics (cf., e.g., [Ar98]), in contrast to the sharp separation in case of a deterministic bifurcation.

Consider the same type of equation as in the previous section, which is for $\sigma = 0$ a classical example of a pitchfork-bifurcation in a PDE:

$$\partial_t u = Lu + \mu u - u^3 + \sigma^2 \partial_t W, \tag{6.9}$$

subject to zero Dirichlet-type boundary conditions on $[0,\pi]$.

We can distinguish between three different regimes, and in the following we just give the corresponding amplitude equation and discuss the transient dynamics that we expect. Moreover, in the end we sketch the basic ideas of how to modify our results to derive the amplitude equations for the different cases. Note that we do not specify the constant c appearing in the equations, as it depends on the normalization of the function e spanning $\mathcal{N}$.

1. Case $|\mu| \approx \sigma^2 \approx \varepsilon^2$.
   Fix $\sigma^2 = \varepsilon^2$ and $\mu = \nu\varepsilon^2$. The amplitude equation is $a' = \nu a - ca^3 + \beta'$.
2. Case $|\mu| \ll \sigma^2 \approx \varepsilon^2$.
   Fix $\sigma^2 = \varepsilon^2$ and $|\mu| \leq \varepsilon^3$. The amplitude equation is $a' = -ca^3 + \beta'$.
3. Case $\sigma^2 \ll |\mu| \approx \varepsilon^2$.
   Fix $\varepsilon^2 = |\mu|$ and $\sigma \leq \varepsilon^3$. The amplitude equation is $a' = \text{sgn}(\mu)a - ca^3$.

Case 2 corresponds to the case when we are very near to the deterministic bifurcation point. The amplitude equation and the dynamics are independent of the bifurcation parameter in this case. Hence the deterministic bifurcation point is widened to a longer interval.

Case 3 corresponds to the case when we are far away from the bifurcation. Here the dynamics is essentially given by the stable equilibria of the deterministic equation. The stochastic nature of the original SPDE is only seen in small fluctuations around these fixed points.

In Case 1, we have an intermediate regime, when we are of order noise-strength away from the bifurcation. Here the amplitude equation is stochastic, and the dynamical behavior interpolates between the deterministic behavior (Case 3) and the bifurcation regime (Case 2).

Let us finally comment on how to rigorously derive the amplitude equations in the previous statements. Case 1 was already discussed in the previous sections. The second case is rather easy. We can, for instance, follow our proofs of the approximation result to see that all ν-dependent terms are now of lower order. Hence they disappear into the error terms, and there will be no contribution to the amplitude equation. For Case 3 we can also follow our approach. This would give an amplitude equation of the type $a' = \mathrm{sgn}(\mu)a - ca^3 + \sigma^2\varepsilon^{-2}\beta'$. Then we can add a Freidlin–Wentzell-type argument to eliminate the noise term, as $\sigma^2\varepsilon^{-2} \to 0$.

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