GARCH processes and the phenomenon of misleading and unambiguous signals

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Abstract

In finance, it is quite usual to assume that a process behaves according to a previously specified target generalized autoregressive conditionally heteroscedastic process. The impact of rumors or other events on this process can be frequently described by an outlier responsible for a short-lived shift in the process mean or by a sustained change in the process variance. This calls for the use of joint schemes for the process mean and variance. Since changes in the mean and in the variance require different actions from the traders/brokers, this paper provides an account on the probabilities of misleading and unambiguous signals of those joint schemes, thus adding insights on their out-of-control performance.

KEYWORDS

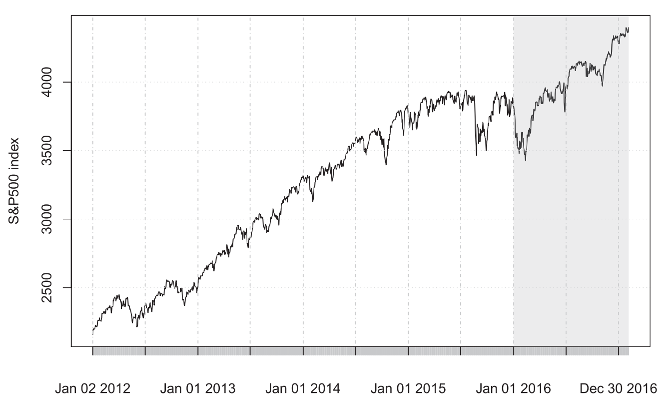
financial time series, joint EWMA schemes, statistical process control, valid signals

1 | VOLATILITY AND GENERALIZED AUTOREGRESSIVE CONDITIONALLY HETEROSCEDASTIC MODELS

The chief cause of the inadequacy of mixed autoregressive moving average (ARMA) models when it comes to financial time series is closely related to several features of this sort of time series.

For instance, many of them seem to meander in the sense that they show no particular tendency to increase or decrease (see pp. 123 and 124 in the work of Enders[1]); others exhibit a clear trend (see p. 121 in the aforementioned work[1]). Most importantly, distorting and aberrant observations can occur and have a major influence on financial time series modeling and forecasting (see p. 20 in the work of Franses[2]). Moreover, these observations tend to appear in clusters and, in some cases, they can be interpreted as resulting, for instance, from the impact of certain news on a stock market (see p. 24 in the aforementioned work[2]).

Figure 1 comprises the time series plot of the Standard & Poor's 500 index (S&P 500) on trading days (P_t), from January 2, 2012 to January 31, 2017. The data was downloaded from Thomson Reuters Datastream and the associated time series plot suggests a typical financial time series with an upward trend. In addition, the time series plot of the associated daily



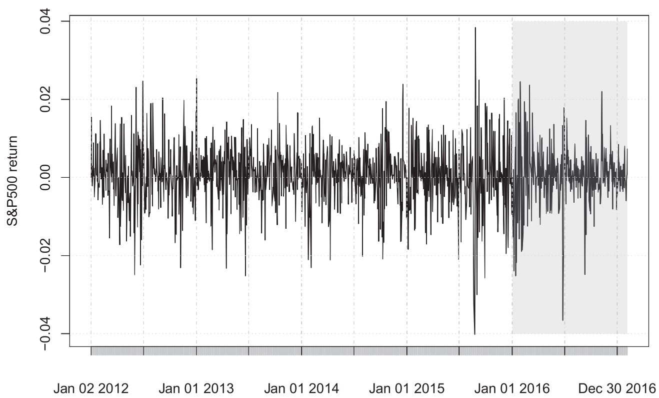


FIGURE 1 S&P 500 index from January 2, 2012 to January 31, 2017 (left); daily log returns of the S&P 500 index from January 3, 2012 to January 31, 2017 (right)

log returns $\log(P_t/P_{t-1})$ in Figure 1 reveals high volatility and leads us to a belief that the associated (short-run) variance is not constant over time, a phenomenon econometricians call *heteroscedasticity*.

While conventional time series operates under the assumption of constant variance, the autoregressive conditionally heteroscedastic (ARCH) model introduced by Engle[3] allows the conditional variance h_t of today's value Y_t to depend on past realizations of the time series, leaving the unconditional variance constant (see the work of Bollerslev[4]).

The simplest example of the ARCH model, ie, the ARCH(1), reads as follows:

$$Y_t = \varepsilon_t h_t^{1/2}, \tag{1}$$

$$h_t = \alpha_0 + \alpha_1 Y_{t-1}^2, \tag{2}$$

where $\{\varepsilon_t : t \in \mathbb{Z}\}$ is a Gaussian white noise (GWN) with zero mean and unit variance, ie, GWN(0, 1); ε_t is independent of h_t, for all t; and $\alpha_0 > 0$ and $\alpha_1 \geq 0$. Furthermore, $Y_t|\psi_{t-1} \sim N(0, h_t)$, that is, Y_t based upon the past information set available at time $t-1$, ie, ψ_{t-1}, is a normally distributed random variable with zero mean and variance equal to h_t.

The ARCH model has been proven useful in modeling several economic phenomena such as the inflation rate, as mentioned by Bollerslev.[4] However, the ARCH(q) model relies on the conditional variance h_t written in terms of a moving average of q previous shocks, ie, $Y_{t-1}^2, \ldots, Y_{t-q}^2$; hence, it is bound to have some limitations and natural extensions.

The extension of the ARCH model bears, as Bollerslev[4] pointed out, much resemblance to the extension of the standard autoregressive (AR) process to the general ARMA process.

This author adds that it seems of *practical interest* to consider that the conditional variance h_t depends not only on those MA terms $(Y_{t-1}^2, \ldots, Y_{t-q}^2)$ but also on AR terms, the lagged conditional variances $(h_{t-1}, \ldots, h_{t-p})$. This leads to the generalized ARCH (GARCH) processes introduced by Bollerslev.[4]

Definition 1. A stochastic process $\{Y_t : t \in \mathbb{Z}\}$ is said to be a GARCH(p, q) process if it satisfies

$$Y_t = \varepsilon_t h_t^{1/2} \tag{3}$$

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i Y_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i}, \tag{4}$$

where $\{\varepsilon_t : t \in \mathbb{Z}\} \sim$ GWN(0, 1); ε_t is independent of h_t, for all $t \in \mathbb{Z}$; $\alpha_0 > 0$; $\alpha_i \geq 0$, $i = 1, \ldots, q (q \geq 0)$; and $\beta_i \geq 0$, $i = 1, \ldots, p (p \geq 0)$. In addition, $Y_t|\psi_{t-1} \sim N(0, h_t)$.

Bollerslev[4] stated in Theorem 1 that the GARCH(p, q) process is weakly stationary if and only if $0 \leq \sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{p} \beta_i < 1$, and in this case $E(Y_t) = 0$, $E(Y_t^2) = \frac{\alpha_0}{1 - \sum_{i=1}^{q} \alpha_i - \sum_{i=1}^{p} \beta_i}$ and $E(Y_t Y_{t-n}) = 0$, for all $t \in \mathbb{Z}$ and $n \in \mathbb{N}$. One is dealing with a stochastic process with zero mean and consisting of uncorrelated r.v.

The ARCH and GWN processes are particular cases of the GARCH processes, ie, when $p = 0$, one is dealing with an ARCH(q) process, whereas $p = q = 0$ leads to $\{Y_t : t \in \mathbb{Z}\} \sim$ GWN$(0, \alpha_0)$. The simplest but frequently useful GARCH process is the GARCH(1, 1).[4]

For a comprehensive review on ARCH/GARCH modeling and on GARCH models and their financial applications, the reader is referred to the work of Giraitis et al[5] and Francq and Zakoïan,[6] respectively.

The remainder of this paper is organized as follows. In Section 2, we define the structural mean/variance breaks of an assumed GARCH(1, 1) model; we briefly describe 4 known joint monitoring schemes for detecting those structural breaks in the process mean and variance; and we provide an instructive example of the use of these joint control schemes in finance. In Section 3, we discuss misleading signals (MS) and unambiguous signals (UNS) and estimate the probabilities of their occurrence via Monte Carlo simulation; tables and plots of these probabilities are supplied to devise their behavior when the 4 joint schemes are used to monitor breaks in the process mean and variance of 2 different GARCH(1, 1) models. A few final thoughts are given in Section 4.

2 | MONITORING STRUCTURAL DEVIATIONS IN FINANCE

While dealing with financial data, Schipper and Schmid[7,8] emphasize the distinction between the target process $\{Y_t : t \in \mathbb{N}\}$ and the observed process $\{X_t : t \in \mathbb{N}\}$ and suggest that the former corresponds to the assumed (or in-control) model and can be obtained, namely, by fitting a suitable model to a starting (or historical) block of data; the latter is associated with the time series itself.

From now on, one considers a GARCH(1, 1) target process

$$Y_t = \mu_0 + \varepsilon_t h_t^{1/2} \tag{5}$$

$$h_t = \alpha_0 + \alpha_1 (Y_{t-1} - \mu_0)^2 + \beta_1 h_{t-1}, \tag{6}$$

with mean μ_0 and variance $\sigma_0^2 = \frac{\alpha_0}{1-\alpha_1-\beta_1}$ (see the work of Schipper and Schmid[7]).

As aptly noted by the aforementioned work,[7] the impact of rumor and other events on the target GARCH process can be frequently described by a *short-lived shift* (see p. 183 in the work of Montgomery[9]) in the process mean or by a *sustained change* in the process variance. Thus, it is reasonable to relate the observed and target processes as follows:

$$X_t = \begin{cases} Y_t, & t < \tau \\ \mu_0 + \theta (Y_t - \mu_0) + \delta \sigma_0, & t = \tau \\ \mu_0 + \theta (Y_t - \mu_0), & t > \tau, \end{cases} \tag{7}$$

where $\tau (\tau \in \mathbb{N})$, $\delta (\delta \in \mathbb{R})$, and $\theta (\theta \in \mathbb{R}^+)$ denote the time epoch of the structural change, the relative size of the short-lived shift in location because of the additive outlier, and the magnitude of the sustained shift in scale, respectively.

Since the mean and variance of X_t are given by[7]

$$E(X_t) = \mu_0 + \delta \sigma_0 \times I_{\{\tau\}}(t) \tag{8}$$

$$V(X_t) = \left[1 - (1 - \theta^2) \times I_{\{k \in \mathbb{N} : k \geq \tau\}}(t)\right] \sigma_0^2, \tag{9}$$

one can write

$$\delta = \frac{E(X_t) - \mu_0}{\sigma_0} \times \mathbb{1}_{\{\tau\}}(t) \tag{10}$$

$$\theta = \mathbb{1}_{\{k \in \mathbb{N} : k < \tau\}}(t) + \frac{\sqrt{V(X_t)}}{\sigma_0} \times \mathbb{1}_{\{k \in \mathbb{N} : k \geq \tau\}}(t). \tag{11}$$

It is also important to note that the observed process $\{X_t : t \in \mathbb{N}\}$ is said to be out-of-control if $\delta \neq 0$ or $\theta \neq 1$ and deemed in control otherwise.

Furthermore, Schipper and Schmid[7] added that control charts should be used to check whether or not the observed process can be reasonably described by the fitted model. Control charts can indeed be used to detect relevant structural deviations in the observed process $\{X_t : t \in \mathbb{N}\}$, underlying a financial time series, from the target process $\{Y_t : t \in \mathbb{N}\}$.

Bear in mind the following.

- The traders need to promptly track structural deviations from that model because they can make a profit when those changes occur, after all, the stock price patterns allow forecasting future stock prices (see p. 7 in the work of Schipper[10]).

- When a control chart triggers a signal and a deviation is found, that information should be considered in the trading strategy, namely, to determine if the trader keeps a security in his portfolio (see p. 9 in the aforementioned work[10]).

2.1 | Joint schemes for the process mean and variance

The first charts specifically designed to monitor the GARCH process were proposed by Severin and Schmid[11]; these control charts were used to monitor the process mean.

Schipper and Schmid[8] studied control charts to detect solely the changes in the process variance; the reader should recall that, in finance, the variance measures the risk of an asset; thus, it is extremely important to detect any changes in this process parameter.

However, in order to efficiently control a process, one needs to jointly monitor its location and scale, thus the use of joint (simultaneous or combined) control schemes. The standard practice is to jointly use 2 individual charts, ie, one for the mean value (μ) and another one for the variance (σ^2). That is the case of the joint exponentially weighted moving average (EWMA) schemes proposed by Schipper[10] (pp. 34 and 35).

In Table 1, one can find the statistics and the control limits of one individual EWMA chart for μ; and 4 individual EWMA charts for σ based on the squared observations (I), the conditional variance (II), the exponentially weighted variance (III), and the logarithm of the squared observations (IV). $\lambda_1, \lambda_2 \in (0,1]$ denote the smoothing parameters of the individual EWMA charts for the process mean and variance, respectively.

Please note that, according to Schipper[10] (pp. 12-14, 35), the conditional variance is given by

$$\hat{\sigma}_t^2 = \begin{cases} \sigma_0^2, & t = 1 \\ \sigma_0^2 + (\alpha_1 + \beta_1)\left[(X_{t-1} - \mu_0)^2 - \sigma_0^2\right] & \\ \quad -\frac{\beta_1}{r_{t-1}}\left[(X_{t-1} - \mu_0)^2 - \hat{\sigma}_{t-1}^2\right], & t \geq 2, \end{cases} \tag{12}$$

where $r_1 = (1 - 2\beta_1\alpha_1 - \beta_1^2)/[1 - (\alpha_1 + \beta_1)^2]$ and $r_t = 1 + \beta_1^2 - \beta_1^2/r_{t-1}$, for $t \geq 2$; and the exponentially weighted variance is taken to be equal to

$$\left(\sigma_t^*\right)^2 = \begin{cases} \sigma_0^2, & t = 0 \\ 0.94\left(\sigma_{t-1}^*\right)^2 + 0.06(X_t - \mu_0)^2, & t \in \mathbb{N}. \end{cases} \tag{13}$$

A control chart should be associated to infrequent false alarms and should trigger a valid signal as quickly as possible when the process is out of control. Expectedly, the performance of any control chart is commonly assessed by making use

TABLE 1 Statistics and control limits of the individual exponentially weighted moving average (EWMA) charts for μ and for σ

EWMA control chart for μ	$LCL_{E-\mu}^{(k)}$	$UCL_{E-\mu}^{(k)}$
$Z_{1,t}^{(k)} = \begin{cases} \mu_0, & t = 0 \\ (1-\lambda_1)Z_{1,t-1}^{(k)} + \lambda_1 X_t, & t \in \mathbb{N} \end{cases}$	$\mu_0 - c_1^{(k)}$	$\mu_0 + c_1^{(k)}$
EWMA control chart for σ based on	$LCL_{E-\sigma}^{(k)}$	$UCL_{E-\sigma}^{(k)}$
X_t^2		
$Z_{2,t}^{(I)} = \begin{cases} \sigma_0^2, & t = 0 \\ (1-\lambda_2)Z_{2,t-1}^{(I)} + \lambda_2(X_t - \mu_0)^2, & t \in \mathbb{N} \end{cases}$	$c_2^{(I)}\alpha_0$	$c_3^{(I)}\alpha_0$
$\hat{\sigma}_t^2$		
$Z_{2,t}^{(II)} = \begin{cases} \sigma_0^2, & t = 0 \\ (1-\lambda_2)Z_{2,t-1}^{(II)} + \lambda_2\hat{\sigma}_t^2, & t \in \mathbb{N} \end{cases}$	$c_2^{(II)}\alpha_0$	$c_3^{(II)}\alpha_0$
$(\sigma_t^*)^2$		
$Z_{2,t}^{(III)} = \begin{cases} \sigma_0^2, & t = 0 \\ (1-\lambda_2)Z_{2,t-1}^{(III)} + \lambda_2(\sigma_t^*)^2, & t \in \mathbb{N} \end{cases}$	$c_2^{(III)}\alpha_0$	$c_3^{(III)}\alpha_0$
$\ln(X_t^2)$		
$Z_{2,t}^{(IV)} = \begin{cases} E(\ln[(Y_t - \mu_0)^2]), & t = 0 \\ (1-\lambda_2)Z_{2,t-1}^{(IV)} + \lambda_2(\ln[(X_t - \mu_0)^2]), & t \in \mathbb{N} \end{cases}$	$c_2^{(IV)}$	$c_3^{(IV)}$

of the run length (RL), which is the number of samples collected before a signal is triggered by the chart (see p. 100 in the work of Wetherill and Brown[12]).

The joint scheme for the process mean and variance gives a signal when at least one of its 2 individual charts does, that is, its RL is defined as

$$RL^{(k)} = \min \left\{ t \in \mathbb{N} : \left| Z_{1,t}^{(k)} - \mu_0 \right| > UCL_{E-\mu}^{(k)} \right. \\ \left. \text{or } Z_{2,t}^{(k)} < LCL_{E-\sigma}^{(k)} \text{ or } Z_{2,t}^{(k)} > UCL_{E-\sigma}^{(k)} \right\}, k \in \{I, II, III, IV\}. \quad (14)$$

In order to set the control limits (or equivalently to compute the associated values), Schipper and Schmid[7] define a system of 3 nonlinear equations to obtain a unique solution satisfying the following conditions.

- The in-control average RL (ARL) of the joint scheme is equal to some desired (and reasonably large) value, say ARL^*.
- $E(RL_1^{\text{upper}}) = E(RL_1^{\text{lower}}) = E(RL_2^{\text{upper}}) = E(RL_2^{\text{lower}})$, where $E(RL_1^{\text{upper}})$ (respectively $E(RL_1^{\text{lower}})$) represents the in-control ARL of an individual upper (respectively lower) one-sided chart for μ, and $E(RL_2^{\text{upper}})$ (respectively $E(RL_2^{\text{lower}})$) denotes the in-control ARL of an individual upper (respectively lower) one-sided chart for σ.

That system can be solved using a 3-dimensional secant rule, having as starting values for the iteration the critical values of the individual one-sided charts (see Schipper and Schmid[7]).

Regretfully, deriving the RL distribution is rather difficult when the target process is governed by a GARCH model. Therefore, one has to rely on Monte Carlo simulations to obtain the critical values and to assess the performance of the joint scheme.

Programs for the R statistical software[13] and the C programming language were written to derive those critical values and to produce all the results and plots in this paper.

2.2 | Illustration

The purpose of this section is to provide an instructive example of the use of joint control schemes in finance.

Example 1. To monitor the mean and variance of the S&P 500 index, the target process was determined from the data of a starting block. In practice, this starting block should be chosen based on the experience of an analyst and it should reflect the typical behavior of the share or index. In this particular case, it consists of the daily log returns of the S&P 500 index dated from January 2, 2012 to December 31, 2015.

One fitted a GARCH(1, 1) model, with target mean μ_0 and variance $\sigma_0^2 = \frac{\alpha_0}{1-\alpha_1-\beta_1}$, to this starting block data and obtained the following estimates: $\hat{\mu}_0 = 0.08046881$; $\hat{\alpha}_0 = 0.07713434$; $\hat{\alpha}_1 = 0.1600751$; and $\hat{\beta}_1 = 0.7177052$.

Based on this fitted model, a joint EWMA scheme with $\lambda_1 = \lambda_2 = 0.1$ was designed to monitor the process mean and variance of the log returns of the S&P 500 index from January 2, 2016 to January 31, 2017.

Considering an in-control ARL of 60 (trading days) led to the following control limits of the 2 individual EWMA charts: $\hat{\mu}_0 - c_1^{(I)} = -0.31482259$ and $\hat{\mu}_0 + c_1^{(I)} = 0.47576021$; and $c_2^{(I)}\hat{\alpha}_0 = 0.2161774$ and $c_3^{(I)}\hat{\alpha}_0 = 1.436697$.

These control limits are represented by horizontal lines, as shown by Figure 2, where one also plotted the observed values of EWMA recursions $Z_{1,t}^{(I)}$ (left) and $Z_{2,t}^{(I)}$ (right).

Let one remind the reader that the aim of this joint EWMA scheme is to detect deviations from the previously fitted model, in particular, because of short-lived shifts in the process mean or sustained changes in the process variance.

This joint scheme triggers signals on several occasions, for instance, on January 7, 2016, June 22, 2016, and December 2, 2016. These and many other signals confirm the occurrence of economic/financial events discussed thoroughly by Schroders[14] and other outlets covering finance.

The signals triggered by the joint scheme on January 7, 8, 11, and 13-29, 2016, reflect the fact that it was a very turbulent month. In the US, the S&P 500 index was negatively affected by fears of slowing global and domestic growth; renewed instability in the Chinese equity market and a further deterioration in the oil price led to steep negative returns in global stock markets.

In a speech to the House of Commons on February 22, 2016, David Cameron announced the date of the referendum on the UK's EU membership, June 23, 2016; on the morning of June 24, the result of this referendum was made public.

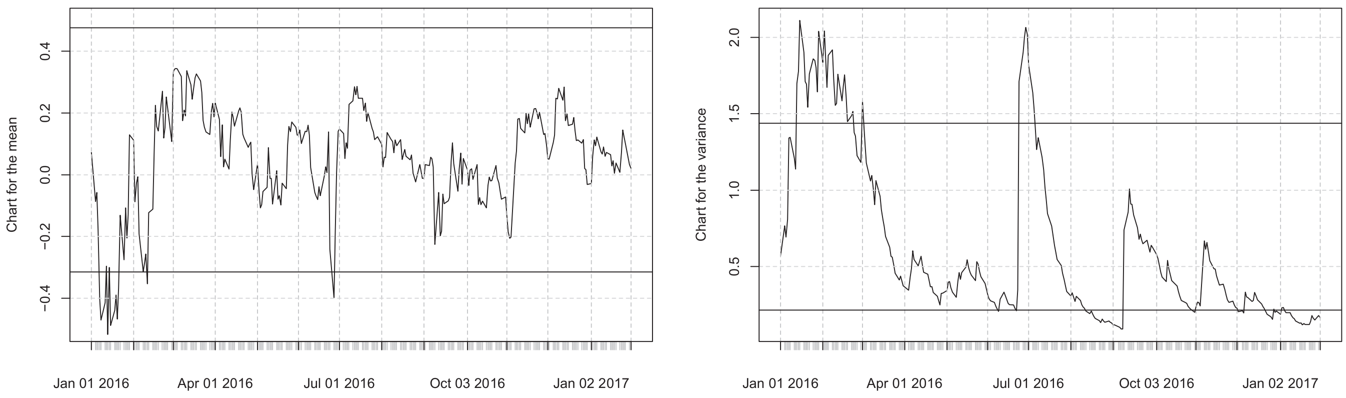


FIGURE 2 Exponentially weighted moving average charts for the process mean (left) and variance (right) of the log returns of the S&P500 index from January 2, 2016 to January 31, 2017

The impact and the anticipation of this news translate into sharp increases in volatility and, consequently, into signals on June 22, 24-30, and early July 2016.

The build-up to the US presidential election and the subsequent victory for Donald Trump dominated the news flow in November 2016. The impact of Donald Trump's electoral college vote win on markets was short lived because investors rapidly embraced the idea of a Republican-controlled Congress being a game changer by implementing fiscal stimulus and tax cuts. As a consequence, one detects a modest increase in volatility in the beginning of November, followed by downward shifts in the volatility, indicating some *relaxation* of the financial markets. These shifts are responsible for a first signal on December 2, 2016, several signals later that year, and on most of the trading days of January 2017.

For an illustration of a joint Shewhart scheme for the mean and variance of the returns of the Deutsche Bank share, the reader is referred to pp. 49-56 in the work of Schipper.[10] This author adds that, when the joint scheme signals, the fund manager should decide to acquire or retain a riskier security; thus, the joint scheme can be used as a hedge tool against structural deviations. It is worth mentioning that section 3.5 in the aforementioned work[10] continued to explore this example, namely, by describing an objective trading strategy that combines the developments in statistical process control with trading techniques for options (see p. 57 in the work of Schipper[10]).

3 | MISLEADING AND UNAMBIGUOUS SIGNALS

The concept of misleading signal was firstly introduced by St. John and Bragg,[15] who identified the 3 following types of MS.

- **I.** The process mean increases but the signal is triggered by the chart for the variance or on the negative side of the chart for the mean.
- **II.** The process mean decreases but the signal is triggered by the chart for the variance or on the positive side of the chart for the mean.
- **III.** The process variance has shifted up but the signal is triggered by the chart for the mean.

Posteriorly, a fourth type was defined by Morais and Pacheco.[16]

- **IV.** The process mean shifts but the signal is triggered by the chart for the variance.

Moreover, it seemed to Morais and Pacheco[16] that it is more relevant to focus on valid signals corresponding to the misinterpretation of a shift on the process mean (respectively variance) as a shift in the process variance (respectively mean) when one makes use of a joint scheme for these 2 parameters.

Additionally, Ramos[17] (p. 112) identified another class of valid signals, ie, the unambiguous signals that were divided into 2 types by Ralha et al.[18]

- **III.** The process variance is out of control and the first chart that triggers a signal is the chart for the variance.
- **IV.** The process mean is out of control and the corresponding chart is the first to signal.

Regarding the MS and UNS, a joint scheme should trigger MS (respectively UNS) as infrequently (respectively often) as possible, ie, the probability of the MS (PMS) and the probability of UNS (PUNS) should be small (respectively large).

These probabilities have been used to assess the performance of several joint schemes. For instance, the PMS have been computed and analyzed by several authors for univariate normal i.i.d. output (eg, see related works[16,18,19]); univariate stationary Gaussian processes (see other works[20-22]); bivariate normal i.i.d. output (eg, see the works of Ramos et al[17,23]); and multivariate normal i.i.d. processes.[24] As far as one has investigated, the PUNS have only been addressed by Ralha et al[18,25] for univariate and bivariate normal i.i.d. output.

Since the assignable causes on charts for μ can be different of those on charts for σ, the appropriate trading strategy that follows a signal can differ depending on whether the signal is triggered by the chart for μ or by the one for σ. Therefore, an MS can lead to a reduction in the profit made by the traders/brokers if it led them to use an inappropriate trading strategy, say not buying a highly profitable asset or not selling an unprofitable security.

Moreover, PMS and PUNS should be indeed used to evaluate the performance of joint schemes that are used to monitor changes in the mean and variance of financial time series, such as the 4 joint EWMA schemes introduced by Schipper[10] and briefly described in the previous section.

In this paper, the attention is restricted to the misleading and unambiguous signals of types III and IV systematized as follows.

Valid Signal	μ	σ	First Chart to Signal
MS_{III}	In control	Out of control	Chart for μ
MS_{IV}	Out of control	In control	Chart for σ
UNS_{III}	In control	Out of control	Chart for σ
UNS_{IV}	Out of control	In control	Chart for μ

Unexpectedly, the PMS and PUNS types III and IV are given by

$$\mathrm{PMS}_{III}(\theta) = P[RL_\sigma(0,\theta) > RL_\mu(0,\theta)],\ \theta \neq 1 \tag{15}$$

$$\mathrm{PMS}_{IV}(\delta) = P[RL_\mu(\delta,1) > RL_\sigma(\delta,1)],\ \delta \neq 0 \tag{16}$$

$$\mathrm{PUNS}_{III}(\theta) = P[RL_\mu(0,\theta) > RL_\sigma(0,\theta)],\ \theta \neq 1 \tag{17}$$

$$\mathrm{PUNS}_{IV}(\delta) = P[RL_\sigma(\delta,1) > RL_\mu(\delta,1)],\ \delta \neq 0, \tag{18}$$

where RL_μ and RL_σ are the RLs of the individual charts for μ and for σ.

Regretfully, one is unable to derive the joint distributions regarding $\{Y_t : t \in \mathbb{N}\}$ needed to determine the distribution of the RL of the individual control charts and the joint distribution of those 2 dependent RLs when the target process is governed by a GARCH model. Consequently, one cannot simplify the expressions of the probabilities of misleading and unambiguous signals of types III and IV and one has to rely on Monte Carlo simulations to estimate $\mathrm{PMS}_{III}(\theta)$, $\mathrm{PMS}_{IV}(\delta)$, $\mathrm{PUNS}_{III}(\theta)$, and $\mathrm{PUNS}_{IV}(\delta)$.

3.1 | Estimating PMS and PUNS via Monte Carlo simulation

One considered the 2 following target processes in p. 39 in the work of Schipper[10]:

- Process I: $\alpha_0 = 0.1\alpha_1 = 0.05\beta_1 = 0.9$; and
- Process II: $\alpha_0 = 1.0\alpha_1 = 0.25\beta_1 = 0.7$.

Let one remind the reader that, in finance, the value of α_1 is usually small and the one of β_1 is typically large.

The smoothing parameters λ_1 and λ_2 of the individual EWMA charts are in Table 2 and coincide with the ones considered by Schipper[10] (p. 48) and Schipper and Schmid.[7]

These schemes were calibrated in such a way that ARL^* is approximately equal to 60. According to Schipper and Schmid,[8] the value 60 roughly reflects 3 months at the stock exchange. Moreover, the 3 critical values for each joint scheme and for (λ_1, λ_2) can be found in the work of Schipper[10] (tables A.3 to A.6, pp. 109 and 110) and have been reproduced by Sousa[26] (tables A.1 to A.4, p. 89).

TABLE 2 Pairs of smoothing parameters (λ_1, λ_2)–joint exponentially weighted moving average schemes

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
λ_1	0.10	0.25	0.50	0.75	1.00	0.10	1.00	1.00	1.00
λ_2	0.10	0.25	0.50	0.75	1.00	1.00	0.10	0.25	0.50

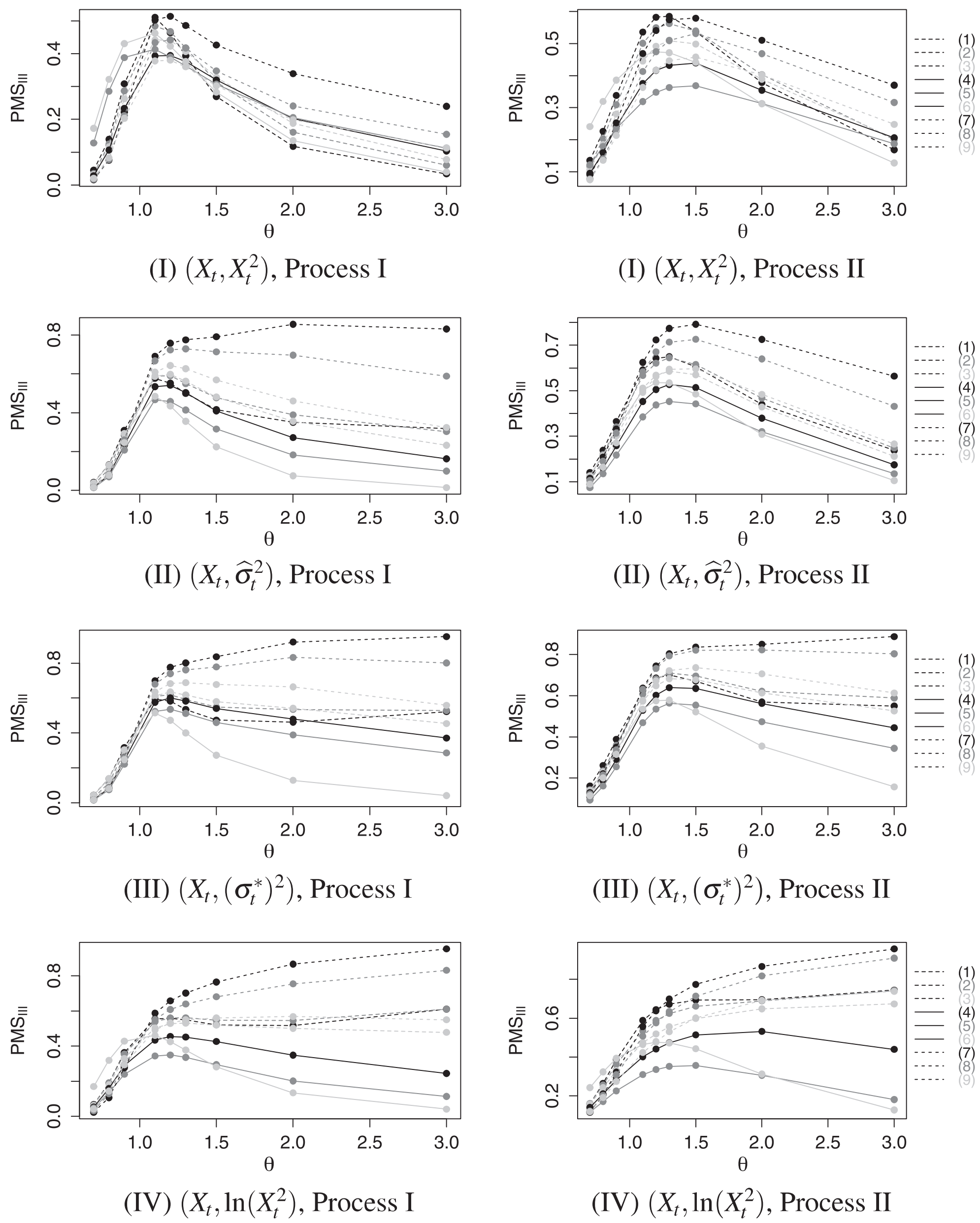


FIGURE 3 Estimated $\text{PMS}_{III}(\theta)$ for all pairs (λ_1, λ_2)

As far as the shifts in the parameters are concerned, 2 different out-of-control scenarios have been considered to estimate the PMS and PUNS of types III and IV.

- In the first case, the process standard deviation suffers a sustained shift of magnitude $\theta = 0.7, 0.8, 0.9, 1.1, 1.2, 1.3, 1.5, 2.0, 3.0$ and the process mean stays on-target ($\delta = 0$).
- In the second situation, the process mean suffers a short-lived shift of magnitude $\delta = 0.5, 1.0, 1.5, 2.0, 2.5, 3.0$ and the process variance is on-target ($\theta = 1$).

TABLE 3 Minimum $PMS_{III}(\theta)$ and maximum $PUNS_{III}(\theta)$, processes I and II–joint exponentially weighted moving average scheme with chart for σ listed in order corresponding to squared observations (I), conditional variance (II), exponentially weighted variance (III), and logarithm of squared observations (IV)

					θ				
Process I	**0.7**	**0.8**	**0.9**	**1.1**	**1.2**	**1.3**	**1.5**	**2.0**	**3.0**
min(PMS_{III})	0.016 (7)	0.075 (7)	**0.203** (9)	0.378 (9)	0.380 (9)	0.360 (9)	0.269 (1)	0.118 (1)	0.034 (1)
	0.014 (5)	**0.070** (5)	0.208 (5)	0.467 (5)	0.435 (6)	0.356 (6)	**0.224** (6)	**0.075** (6)	**0.015** (6)
	0.016 (5)	0.074 (5)	0.221 (5)	0.515 (6)	0.473 (6)	0.400 (6)	0.273 (6)	0.129 (6)	0.042 (6)
	0.023 (7)	0.107 (7)	0.240 (5)	**0.343** (5)	**0.349** (5)	**0.336** (5)	0.281 (6)	0.134 (6)	0.041 (6)
max($PUNS_{III}$)	0.981 (7)	0.909 (7)	0.717 (7)	0.474 (6)	**0.491** (6)	0.522 (6)	0.582 (6)	0.670 (6)	0.632 (6)
	0.984 (8)	**0.918** (7)	**0.737** (7)	0.447 (6)	0.472 (6)	**0.528** (6)	**0.632** (6)	**0.730** 6)	**0.681** (6)
	0.983 (8)	0.915 (8)	0.736 (7)	0.419 (6)	0.433 (6)	0.486 (6)	0.571 (6)	0.635 (6)	0.578 (6)
	0.976 (7)	0.889 (7)	0.709 (7)	**0.475** (6)	0.490 (6)	0.521 (6)	0.585 (6)	0.671 (6)	0.632 (6)
					θ				
Process II	**0.7**	**0.8**	**0.9**	**1.1**	**1.2**	**1.3**	**1.5**	**2.0**	**3.0**
min(PMS_{III})	0.075 (9)	**0.135** (9)	**0.213** (9)	0.319 (5)	0.348 (5)	0.363 (5)	0.369 (5)	0.312 (6)	0.127 (6)
	0.074 (5)	0.136 (5)	0.218 (5)	0.385 (5)	0.437 (5)	0.453 (5)	0.442 (5)	0.308 (6)	**0.105** (6)
	0.094 (5)	0.161 (5)	0.255 (5)	0.469 (5)	0.535 (5)	0.563 (5)	0.521 (6)	0.355 (6)	0.157 (6)
	0.115 (7)	0.171 (5)	0.2251 (5)	**0.309** (5)	**0.335** (5)	**0.351** (5)	**0.357** (5)	**0.306** (5)	0.127 (6)
max($PUNS_{III}$)	0.869 (7)	0.769 (7)	0.629 (7)	**0.455** (6)	0.428 (6)	**0.419** (6)	**0.427** (6)	0.500 (6)	0.556 (6)
	0.876 (9)	**0.774** (8)	0.637 (7)	0.413 (6)	0.364 (6)	0.353 (6)	0.384 (6)	**0.518** (6)	**0.620** (6)
	0.871 (5)	0.774 (5)	0.643 (8)	0.391 (6)	0.332 (6)	0.313 (6)	0.342 (6)	0.437 (6)	0.485 (6)
	0.872 (7)	0.772 (7)	**0.643** (7)	0.452 (6)	**0.428** (6)	0.418 (6)	0.425 (6)	0.500 (6)	0.557 (6)

When it comes to the simulations, it is important to refer that the *rugarch* package was used instead of *fGarch*.* Furthermore, for each joint scheme and tuple $(\alpha_0, \alpha_1, \beta_1, \lambda_1, \lambda_2, \delta, \theta)$, one:

- simulated the time series $\{y_{i,1}, \dots, y_{i,N}\}$ for each $i(i = 1, \dots, rep)$, drawn from the target process, where the number of replications is equal to $rep = 10^5$ and the number of observations is equal to $N = 1000$;†
- obtained the corresponding realization of the observed process $\{x_{i,1}, \dots, x_{i,N}\}$ for each $i(i = 1, \dots, rep)$;
- computed the associated values of the control statistics for the charts, for the mean, and for the variance; and compared them with the associated control limits; and
- counted the number of misleading signals, the number of unambiguous signals, and the number of signals triggered by the joint schemes; and estimated the corresponding PMS and PUNS.

3.2 | Summary of findings

The following tables and plots prove to be essential to devise the behavior of the estimated probabilities of misleading and unambiguous signals of types III and IV.

The reader should bear something important in mind, one desires that a joint scheme triggers misleading signals sporadically and triggers unambiguous signals as often as possible. Thus, when several joint control schemes are compared, one ought to choose the one associated with the minimum PMS and/or the maximum PUNS.

Before making any comments on the estimates of PMS_{III}, the reader should also be reminded that Knoth et al[20] (respectively p. 27 in the work of Ramos[17]) have previously established that the PMS_{III} does not depend on the parameters that translate the autocorrelation structure of the target process when a joint residual scheme is used to control both increases and decreases (respectively increases) in the process mean and solely increases in the process variance of an AR(1) (respectively stationary Gaussian) process. Moreover, these authors have reported a decreasing behavior of $PMS_{III}(\theta)$ in this specific case.

*Note that this last package can lead to simulation times up to nine times larger than the ones associated with *rugarch*.
†One was forced to simulate a fixed number of observations N because the software does not allow to sequentially simulate a GARCH process, say until a signal is triggered.

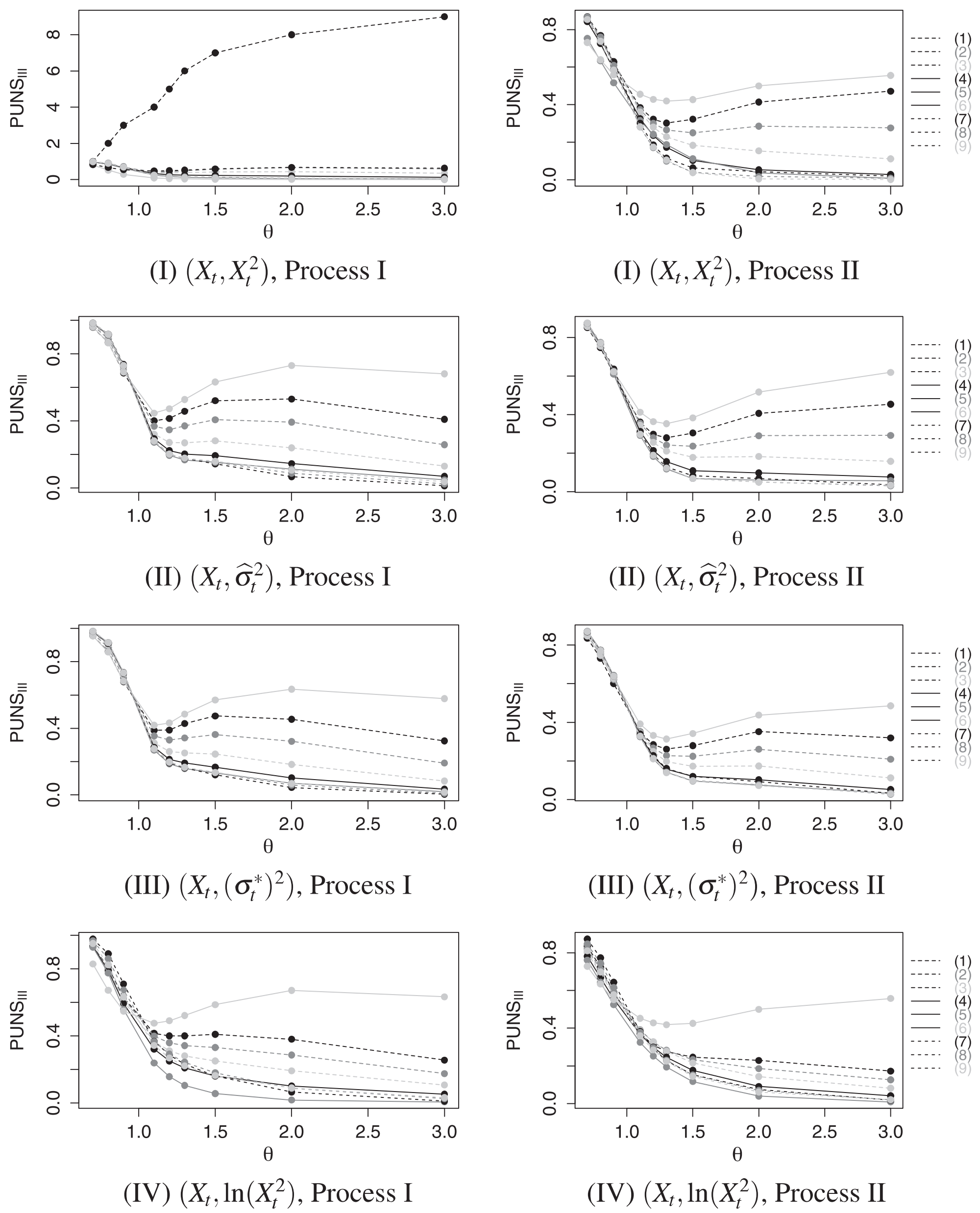


FIGURE 4 Estimated $\text{PUNS}_{III}(\theta)$ for all pairs (λ_1, λ_2)

This is obviously not the case when one is leading with a GARCH(1, 1) target process and other types of joint schemes. Judging by the plots in Figure 3, the estimates of PMS_{III} can increase with the magnitude $\theta > 1$ (an undesirable behavior); on top of that, they can be larger than 0.8 for large shifts in the process variance. Additionally, the general behavior of the estimates of PMS_{III} can vary not only with the type of joint scheme and the pair (λ_1, λ_2) but also with the target process.

The estimates of the minimum PMS_{III} and the maximum PUNS_{III} for processes I and II can be found in Table 3 and are followed by the associated pair (λ_1, λ_2). These estimates suggest that, when there is a sustained shift in the process variance (and the process mean is on-target), the joint scheme with the minimum PMS_{III} in **bold** were attained, in most cases, when one uses joint scheme (II) (respectively (IV)), with $\lambda_2 = 1.0$, for Process I (respectively II).

The plots of the estimates of PUNS_{III} in Figure 4 give no clue as to which of the joint schemes should be used to maximize the PUNS_{III} under the presence of decreases in the process variance. However, we can add that the joint scheme (IV), with a chart for σ based on $\ln X_t^2$, should be avoided when one is dealing with Process II and anticipating moderate or large increases in the process variance; the estimates of PUNS_{III} are indeed rather small for all pairs (λ_1, λ_2), with the honorable exception of $(\lambda_1, \lambda_2) = (0.10, 1.00)$.

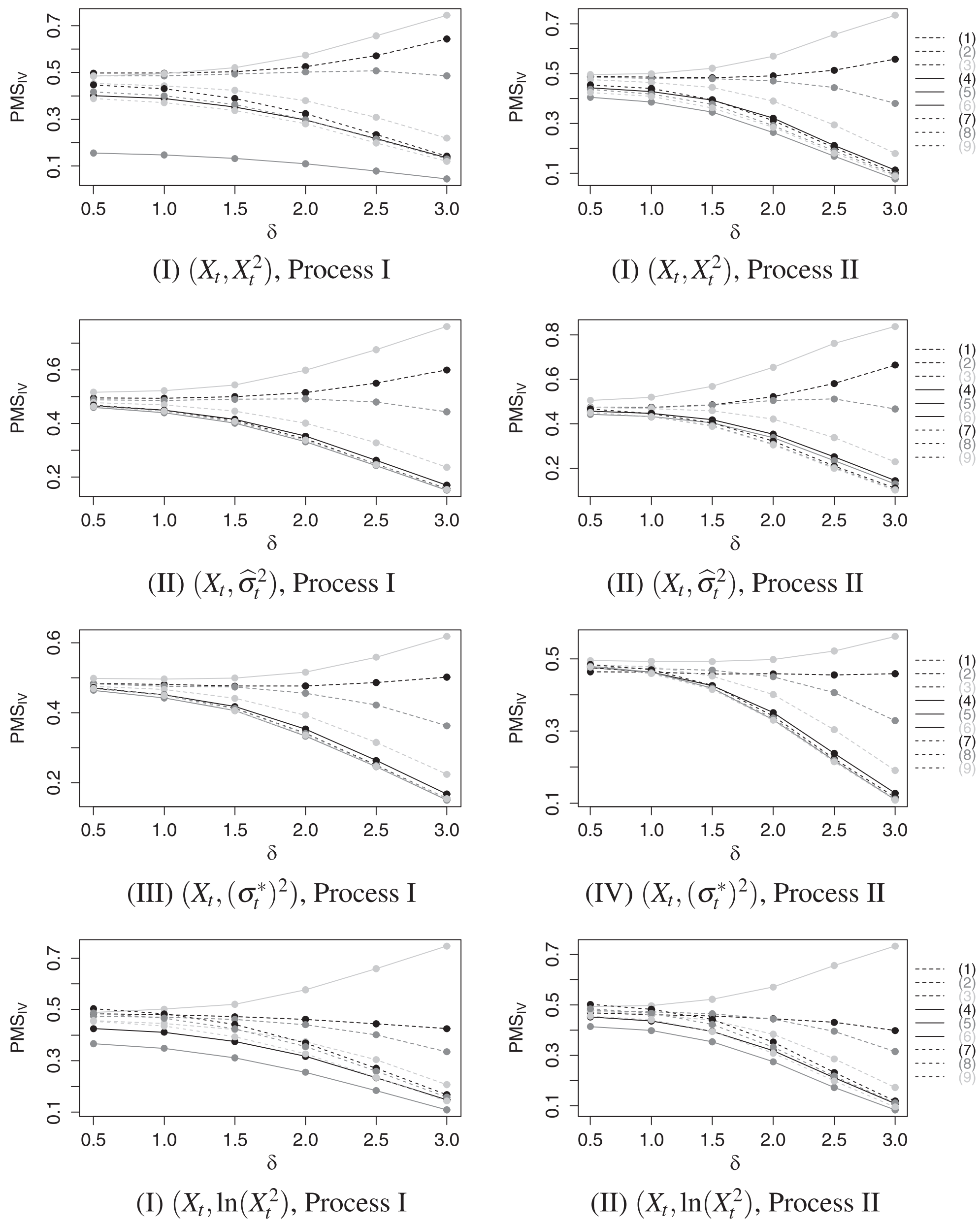


FIGURE 5 Estimated $\text{PMS}_{IV}(\delta)$ for all pairs (λ_1, λ_2)

Moreover, the PUNS_{III} estimates decreases with $\theta > 0$ in several cases, contrary to what one would expect and strongly wishes for, ie, an increasing behavior with $\theta > 1$ and a decreasing behavior with $\theta < 1$.

It is worth mentioning that one got the estimates of the PUNS_{III} much smaller than 0.1 for large shifts in the process variance. This is not a surprising fact given the large values of the PMS_{III} under these circumstances.

According to Table 3, the largest PUNS_{III} estimates are usually observed, for both processes, when one adopts the joint scheme (II) with a chart for σ based on the conditional variance. It is worth mentioning that, for some small increases in σ, the maximum values of PUNS_{III} in **bold** are regretfully smaller than 0.5.

In the presence of a short-lived shift in the process mean (with the process variance on target), the examination of the PMS_{IV} profiles in Figure 5 suggests that the joint scheme (III), with a chart for σ based on $(\sigma_t^*)^2$, should be used if one intends to avoid PMS_{IV} values larger than 0.5 as much as possible.

Furthermore, for the 2 GARCH processes one considered, the PMS_{IV} estimates usually decrease with δ, as desired; regretfully, for some joint schemes and pairs $(\lambda_!, \lambda_2)$, those estimates may increase and exceed 0.6 for large shifts in the process mean. One also realized that the behavior of the estimates of PMS_{IV} depends on the autocorrelation structure

TABLE 4 Minimum $PMS_{IV}(\delta)$ and maximum $PUNS_{IV}(\delta)$, processes I and II–joint exponentially weighted moving average scheme with chart for σ listed in order corresponding to squared observations (I), conditional variance (II), exponentially weighted variance (III), and logarithm of squared observations (IV)

	δ					
Process I	**0.5**	**1.0**	**1.5**	**2.0**	**2.5**	**3.0**
min(PMS_{IV})	**0.155** (5)	**0.148** (5)	**0.132** (5)	**0.110** (5)	**0.079** (5)	**0.045** (5)
	0.460 (5)	0.440 (5)	0.401 (5)	0.331 (5)	0.243 (5)	0.151 (5)
	0.464 (5)	0.442 (5)	0.406 (5)	0.333 (5)	0.246 (5)	0.150 (5)
	0.366 (5)	0.348 (5)	0.311 (5)	0.255 (5)	0.184 (5)	0.109 (5)
max($PUNS_{IV}$)	0.472 (6)	0.462 (1)	0.454 (1)	0.465 (7)	0.462 (7)	0.405 (7)
	0.516 (7)	0.536 (7)	0.574 (7)	0.644 (7)	0.741 (7)	0.835 (7)
	0.516 (7)	**0.538** (7)	**0.577** (7)	**0.649** (7)	**0.742** (7)	**0.841** (7)
	0.497 (1)	0.501 (1)	0.530 (7)	0.607 (7)	0.711 (7)	0.821 (8)
	δ					
Process II	**0.5**	**1.0**	**1.5**	**2.0**	**2.5**	**3.0**
min(PMS_{IV})	**0.405**(5)	**0.386** (5)	**0.345** (5)	**0.264** (5)	**0.168** (5)	**0.077** (5)
	0.442 (5)	0.431 (8)	0.388 (9)	0.305 (9)	0.199 (9)	0.101 (9)
	0.464 (1)	0.459 (9)	0.415 (9)	0.330 (9)	0.215 (9)	0.109 (9)
	0.414 (5)	0.398 (5)	0.354 (5)	0.275 (5)	0.172 (5)	0.084 (5)
max($PUNS_{IV}$)	0.454 (1)	0.453 (1)	0.455 (1)	0.486 (7)	0.506 (7)	0.439 (7)
	0.491 (1)	0.503 (7)	0.536 (7)	0.588 (7)	0.611 (7)	0.546 (7)
	0.518(1)	**0.519**(1)	**0.558**(7)	**0.645**(7)	**0.767**(7)	**0.875**(7)
	0.500 (1)	0.509 (1)	0.515 (7)	0.613 (7)	0.744 (7)	0.868 (7)

of the target process, as reported by Ramos[17] (p. 27) and Knoth et al,[20] for joint residual schemes to monitor the process mean and variance of other target processes.

The estimates found in Table 4 lead one to believe that the smallest estimates of PMS_{IV} were obtained, for both processes, when one uses the joint Shewhart scheme (I), with $(\lambda_1, \lambda_2) = (1.00, 1.00)$ and a chart for σ based on the squared observations.

The plots in Figure 6 refer to the estimates PUNS of Type IV as function of δ and suggest that this performance measure seems to behave heterogeneously, can sadly decrease, depends on the parameters of the target process, and can be smaller than 0.20.

The estimates of the probability of unambiguous signals of Type IV in Table 4 lead one to conclude that the use of the joint scheme (III), with a chart for σ based on $(\sigma_t^*)^2$, leads in general to the largest values of $PUNS_{IV}$ for both GARCH processes; moreover, the maximum $PUNS_{IV}$ estimates were indeed attained, in nearly all the cases, when $\lambda_2 = 0.1$.

4 | FINAL THOUGHTS

The main purpose of this paper is to assess the phenomenon of misleading and unambiguous signals, in joint schemes, for monitoring changes in the mean and variance of GARCH processes.

The reader should be reminded that MS can result in money and time spent in attempting to identify inexistent causes of unnatural variation or misapplying a trading strategy.

We wrap up with a few recommendations for future work to complement the findings of this paper.

The PMS and PUNS of types III and IV of the 4 joint EWMA schemes introduced by Schipper[10] were estimated only for 2 different GARCH(1, 1) target processes. Consequently, further results on the influence of the model structure on PMS and PUNS should be obtained to shed more light on the phenomenon of misleading and unambiguous signals.

In addition, we should also compute the probability of a simultaneous signal to provide further insights on the out-of-control performance of the 4 joint control schemes. The reader should bear in mind that a simultaneous signal occurs when both μ and σ are out of control and both individual charts of the joint scheme, trigger a signal (see Ralha et al.[18]).

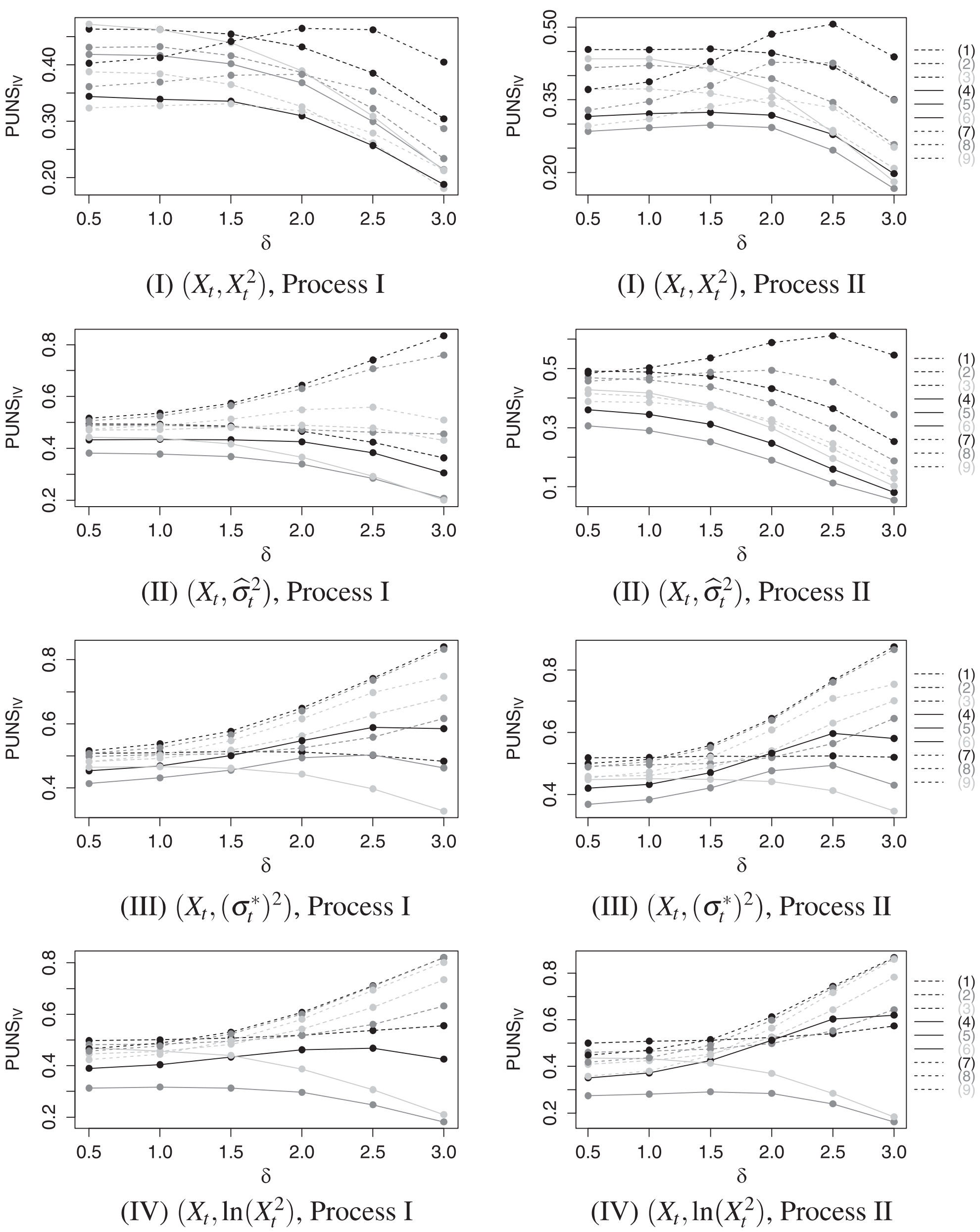


FIGURE 6 Estimated $\text{PUNS}_{IV}(\delta)$ for all pairs (λ_1, λ_2)

Other joint control schemes can be considered, namely, a different chart to monitor the variance, such as the one used by Schipper and Schmid[7] and based on the squared residuals $(X_t - \mu_0)^2/\widehat{\sigma}_t^2$ or a chart for μ based on a more complex function of X_t.

It is important to refer that the simulation study carried out in this paper ought to be extended to GARCH processes of higher order and for the univariate and multivariate extensions of these models, namely, for the following.

- The multivariate GARCH(p, q).[27] Okhrin and Schmid[28] analyzed surveillance procedures for the variance (respectively covariance matrix) of univariate (respectively multivariate) GARCH processes, therefore adding charts to control the process mean (respectively mean vector) would lead to joint schemes whose ARL, PMS, and PUNS performances ought be evaluated.
- The threshold GARCH models.[29] As far as we have investigated, the in-control ARL of a modified Shewhart chart to control the mean of a threshold GARCH process was studied by Gonçalves et al;[30] thus, if joint schemes are to be proposed to monitor this process mean and variance, then their ARL, PMS, and PUNS performances should be assessed.
- The constant conditional correlation model (see p. 167 in the work of Enders[1]). Garthoff et al[31] introduced control charts to monitor jointly the means and variances of CCC processes but only assessed their ARL performance.

We strongly believe that proposing joint schemes to monitor the mean and variance of intraday data and assessing their ARL, PMS, and PUNS performances also deserves some consideration, after all, intraday price movements are particularly important to short-term traders looking to make many trades over the course of a single trading session.

Finally, in practical applications, μ_0, σ_0, θ, δ, and τ are often unknown and have to be estimated from a starting (or historical) block of data; thus, it is crucial to investigate the effect of estimation on the ARL and the probabilities of misleading and unambiguous signals of these joint schemes. This topic is far from trivial and is certainly worthy of future research.

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