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Fitting Generalized Hyperbolic processes - new insights for generating initial values

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Abstract

The fitting of Lévy processes is an important field of interest in both option pricing and risk management. In literature a large number of fitting methods requiring adequate initial values at the start of the optimization procedure exists. A so-called simplified method of moments (SMoM) generates by assuming a symmetric distribution these initial values for the Variance Gamma process, whereby the idea behind can be easily transferred to the Normal Inverse Gaussian process. However, the characteristics of the Generalized Hyperbolic process prevent such a easy adaption. Therefore, we provide by applying a Taylor series approximation for the modified Bessel function of the third kind, a Tschirnhaus transformation and a symmetric distribution assumption a SMoM for the Generalized Hyperbolic distribution. Our simulation study compares the results of our SMoM with the results of the maximum likelihood estimation. The results show that our proposed approach is an appropriate and useful way for estimating Generalized Hyperbolic process parameters and significantly reduces estimation time.

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1 Introduction

Lévy processes like the Generalized Hyperbolic (GH) process (see Barndorff-Nielsen (1977), Barndorff-Nielsen (1978), Barndorff-Nielsen & Blaesild (1981), Eberlein & Keller (1995) or Eberlein & Prause (2000)), the Normal Inverse Gaussian (NIG) process (see Barndorff-Nielsen (1995), Barndorff-Nielsen (1997a), Barndorff-Nielsen (1997b) or Rydberg (1997)) or the Variance Gamma (VG) process (see Madan & Seneta (1987), Madan & Seneta (1990) or Madan et al. (1998)) are very popular models for handling fat tails or jumps within daily or intraday market data in option pricing. For fitting Lévy processes existing literature proposes a broad range of methods, e.g. the method of moments (MOM), the maximum likelihood estimation (MLE), the empirical characteristic function (ECF) method or the minimum χ^2 (χ^2) method. For a detailed overview of these methods see for example Finlay & Seneta (2008) or Rathgeber et al. (2015). All these proposed methods include optimization problems for which adequate initial values are required ensuring an acceptable optimization time. The initial values also provide a good first guess about the parameter ranges. For the VG process Seneta (2004) develops the so-called simplified method of moments (SMoM) for enabling the generation of initial values. The basic idea regarding the SMoM is the assumption of a symmetric VG distribution. That implies that the symmetry driving parameter is similar to zero and therefore all higher orders of this parameter can be set to zero. Applied to the moments of the distribution, this assumption simplifies the moments in so far that all parameters can be easily obtained by estimating empirically the moments and consequently by solving the equations of the moments. Furthermore, Rathgeber et al. (2015) find out that this method does also provide acceptable fitting results for the VG process for a large empirical data set covering the Dow Jones from 1991 to 2011. While this idea can be transferred to the NIG process, the GH process provides some difficulties due to the modified Bessel function of the third kind within the moments of the distribution. To our best knowledge, there is no appropriate method for generating initial values regarding the fiting of the GH process. We close this gap by developing a SMoM for the GH process. Hereby, we use common methods like the Taylor series approximation for the modified Bessel function of the third kind or the Tschirnhaus transformation for solving a quadric equation. Finally, we conduct a simulation study for evaluating our proposed SMoM for the GH model and compare these results with the MLE method. Based on statistical tests our fidings reveal that the developed SMoM is an appropriate method for generating initial values for the GH process. Furthermore, the results show time of estimation of the MLE combined with the SMoM is significantly lower than the MLE without the SMoM.

The remainder of this paper is structured as follows. The next section provides an overview of the theoretical background of the GH process. In section 3, we develop our SMoM for the GH process. Section 4 describes the simulation study, section 5 compares the simulation results of the SMoM with those of the MLE and section 6 discusses the results and concludes the paper.

2 Theoretical background

The GH process $(\lambda, \beta, \delta, \gamma)$ is defied according to Barndorff-Nielsen (1977), Barndorff-Nielsen (1978), Barndorff-Nielsen & Blaesild (1981), Eberlein & Keller (1995) or Eberlein & Prause (2000) by its characteristic function $\Phi_X(u)$ with

$$\Phi_X(u) = \left(\frac{\gamma^2 - \beta^2}{\gamma^2 - (\beta + iu)^2}\right)^{\frac{\lambda}{2}} \frac{K_\lambda\left(\delta\sqrt{\gamma^2 - (\beta + iu)^2}\right)}{K_\lambda\left(\delta\sqrt{\gamma^2 - \beta^2}\right)}, \tag{1}$$

where $K_\lambda(\cdot)$ is the modified Bessel function of the third kind. For the GH process the following bounds for the parameters are valid:

$$
\begin{aligned}
&\delta \geq 0, \quad |\beta| < \gamma && \text{if} \quad \lambda > 0, \\
&\delta > 0, \quad |\beta| < \gamma && \text{if} \quad \lambda = 0, \\
&\delta > 0, \quad |\beta| \leq \gamma && \text{if} \quad \lambda < 0.
\end{aligned}
$$

While some authors only provide expressions for the first two moments for the GH process Barndorff-Nielsen & Blaesild (1981) and Scott et al. (2011) define all four moments. As the representation of the moments by Barndorff-Nielsen & Blaesild (1981) is complex and uncomfortable, we apply the moments for the GH process by Scott et al. (2011). The moments including a drift parameter respectively a location parameter of the distribution c are defined as follows:

$$M_1 = \frac{\delta^2}{\zeta}\beta\frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)} + c \tag{2}$$

$$M_2 = \frac{\frac{\delta^2}{\zeta}K_{\lambda+1}(\zeta) + \left(\frac{\delta^2}{\zeta}\right)^2\beta^2 K_{\lambda+2}(\zeta)}{K_\lambda(\zeta)} \tag{3}$$

$$M_3 = \frac{3\left(\frac{\delta^2}{\zeta}\right)^2\beta K_{\lambda+2}(\zeta) + \left(\frac{\delta^2}{\zeta}\right)^3\beta^3 K_{\lambda+3}(\zeta)}{K_\lambda(\zeta)} \tag{4}$$

$$M_4 = \frac{3\left(\frac{\delta^2}{\zeta}\right)^2 K_{\lambda+2}(\zeta) + 6\left(\frac{\delta^2}{\zeta}\right)^3\beta^2 K_{\lambda+3}(\zeta) + \left(\frac{\delta^2}{\zeta}\right)^4\beta^4 K_{\lambda+4}(\zeta)}{K_\lambda(\zeta)} \tag{5}$$

with $\zeta = \delta\sqrt{\gamma^2 - \beta^2}$. Furthermore, the skewness (s) and the kurtosis (k) of the GH process are given by

$$s = \frac{M_3}{M_2^{\frac{3}{2}}} \qquad \text{and} \qquad k = \frac{M_4}{M_2^2}. \tag{6}$$

3 Development of the SMoM for the GH process

We assume the symmetric case of the GH distribution implying , $\beta \approx 0$ and $\beta^2 = 0, \beta^3 = 0$, etc. This approach is also used in Seneta (2004) for the SMoM of the VG model. By applying this assumption the moments of the GH process in equation (2) to (5) are simplified as follows

$$M_1 = \kappa\beta \frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)} + c \tag{7}$$

$$M_2 = \kappa \frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)} \tag{8}$$

$$M_3 = 3\kappa^2\beta \frac{K_{\lambda+2}(\zeta)}{K_\lambda(\zeta)} \tag{9}$$

$$M_4 = 3\kappa^2 \frac{K_{\lambda+2}}{K_\lambda(\zeta)} \tag{10}$$

with $\zeta = \delta\gamma$ and $\kappa = \frac{\delta}{\gamma}$. In addition, the moments $M_1, \ldots, M_4$ can be obtained from empirical data, leading to $\hat{M}_1, \ldots, \hat{M}_4$.

By using equations (9) and (10) we obtain the first parameter $\hat{\beta}$ of the SMoM

$$\hat{\beta} = \frac{\hat{M}_3}{\hat{M}_4}. \tag{11}$$

$\hat{\zeta}$ is the result of a Taylor series representation of the modified Bessel function of the third kind and a Tschirnhaus transformation.

Hereby, we start using equations (8) and (10)

$$\frac{\hat{M}_2^2}{\hat{M}_4} = \frac{K_{\lambda+1}^2(\zeta)}{3K_\lambda(\zeta)K_{\lambda+2}(\zeta)} \tag{12}$$

and develop the modified Bessel function of the third kind with a Taylor series $f(x)$ (see Abramowitz & Stegun (1972)) of order 2 at the point $x_0 = 1$

$$f(x) = \sum_{n=0}^{2} \frac{f^n(x_0)}{n!}(x - x_0). \tag{13}$$

Hereby, we use the following derivation $\frac{\partial^i B_v(x)}{\partial^i x}$ for $i = 0, \ldots, 2$ of the modified Bessel function $K_v(x)$ of the third kind:

$$B_v(x) = K_v(x) \tag{14}$$
$$\frac{\partial B_v(x)}{\partial x} = -K_{v+1}(x) + v\frac{K_v(x)}{x} \tag{15}$$
$$\frac{\partial^2 B_v(x)}{\partial^2 x} = \frac{K_v(x)x^2 + K_{v+1}(x)x - vK_v(x) + v^2 K_v(x)}{x^2}. \tag{16}$$

After the approximation of the Bessel functions in equation (12), we finally can solve the equation (17) for $\tilde{\zeta}$ with the help of the Tschirnhaus transformation (see King (2008)).

$$\frac{\hat{M}_2^{\,2}}{\hat{M}_4} = \frac{\left(b_o + b_1\tilde{\zeta} + b_2\tilde{\zeta}^2\right)^2}{3(a_o + a_1\tilde{\zeta} + a_2\tilde{\zeta}^2)(c_o + c_1\tilde{\zeta} + c_2\tilde{\zeta}^2)} \tag{17}$$

with $\hat{\zeta} = \tilde{\zeta} + 1$ where

$$a_i = \left.\frac{\partial^i B_\lambda(\tilde{\zeta})}{\partial^i \tilde{\zeta} i!}\right|_1 \tag{18}$$
$$b_i = \left.\frac{\partial^i B_{\lambda+1}(\tilde{\zeta})}{\partial^i \tilde{\zeta} i!}\right|_1 \tag{19}$$
$$c_i = \left.\frac{\partial^i B_{\lambda+2}(\tilde{\zeta})}{\partial^i \tilde{\zeta} i!}\right|_1 . \tag{20}$$

The solution $\hat{\zeta}$ of the quartic equation is

$$\hat{\zeta} = -\frac{B}{4A} + \frac{1}{2}\left[w + \sqrt{-(v_1 + 2x) - 2\left(v_1 + \frac{v_2}{w}\right)}\right] + 1, \tag{21}$$

whereby the variables A, B, w, v, x are the results of simple algebraic or radical equations (for details see appendix). Furthermore, we are able to conclude that there is always a solution. In addition, this solution is unique, if we want to avoid the negativity of the solution. The proofs in connection with the application of the Tschirnhaus transformation can also be found in the appendix.

By rewriting equation (8) for obtaining $\hat{\kappa}$

$$\hat{\kappa} = \hat{M}_2 \frac{K_\lambda(\hat{\zeta})}{K_{\lambda+1}(\hat{\zeta})} \tag{22}$$

and applying the interim results

$$\kappa = \frac{\delta}{\gamma} \qquad \text{and} \qquad \zeta = \delta\gamma \tag{23}$$

we get the last three parameters $\hat{\delta}, \hat{\gamma}$ and $\hat{c}$

$$\hat{\delta} = \sqrt{\frac{\hat{\kappa}}{\hat{\zeta}}}, \qquad \hat{\gamma} = \frac{\hat{\zeta}}{\hat{\delta}} \qquad \text{and} \qquad \hat{c} = M_1 - \hat{\beta}\hat{\kappa}\frac{K_{\lambda+1}(\hat{\zeta})}{K_\lambda(\hat{\zeta})}. \tag{24}$$

4 Simulation and research design

We test our proposed new SMoM on a simulated data set. Hereby, we defie the L évy process X_t with

$$X_t = \beta Z_t + W\sqrt{Z_t}, \tag{25}$$

where $Z \sim \text{GIG}(\lambda, \delta, \frac{1}{\alpha})$ with

$$\frac{1}{\alpha} = \sqrt{\gamma^2 - \beta^2} \tag{26}$$

and W as a standard Brownian motion. Moreover, we simulate the Generalized Inverse Gaussian (GIG) random variates with the non-universal rejection method by Devroye (1986). Finally, we use X_t and integrate the Lévy process in the well-known price process S_t (see

Schoutens (2003)

$$S_t = S_0 e^{ct+X_t}, \tag{27}$$

where c represents the drift of the price process. Finally, the GH($\lambda, \delta, \beta, \gamma, c$) model results. In the simulation, we set $\lambda = 1$ and in doing so we end up with the well-known Hyperbolic model (see Barndorff-Nielsen (1977).

We conduct 1,000 simulation runs according to Finlay & Seneta (2008). Furthermore, we choose six settings of parameters. For getting realistic parameter ranges, we orientate on the fting results of a GH model by Luciano et al. (2014). The parameters chosen can be found in the parameter line in table 1. After the simulation, we reestimate the GH model with the previous developed SMoM as well as with the MLE. In this connection, we use the MLE as a benchmark model. For further estimation methods for Lévy processes we refer to Finlay & Seneta (2008) and Rathgeber et al. (2015). We evaluate the fting quality of the estimation methods according to Finlay & Seneta (2008) with the relative mean absolute deviation (RMAD)

$$RMAD_j = \frac{\frac{1}{m}\sum_{k=1}^{m} \mid \hat{\eta}_{k,j} - \eta_j \mid}{\mid \eta_j \mid}. \tag{28}$$

Note, $\hat{\eta}_{k,j}$ is the reestimated parameter set of the GH process in the k-th simulation run and the j-th parameter setting, η_j the original parameter set and m the number of simulations. The RMADs for all simulated samples and parameters can fially be summed up to the average relative mean absolute deviation (ARMAD) with

$$ARMAD_j = \frac{1}{4}\sum_{j=1}^{4} RMAD_j. \tag{29}$$

Beside the suitability as an appropriate fting method, the time of estimation must be taken into account. For this analysis we run the MLE with random starting parameters drawn from a uniform distribution and compare it with the MLE with the SMoM as starting parameters. For the random starting parameters we take the following intervals: $\delta \in [0, 0.015]$, $\beta \in [-30, 10]$, $\gamma \in [40, 280]$ and $c \in [0, 0.007]$. Computations are done on a Intel (R) Core(TM) i5-4750 CPU @ 3.20 GHz with 8 GB RAM machine.

5 Evaluation of the SMoM

After the simulation of the GH process and reestimation of the GH process parameters', we compare the fiting results of our proposed SMoM with the results of the traditional MLE approach. Table 1 presents the RMAD statistics for both estimation methods. The SMoM works quite well for an initial value generating idea for all settings expect for the fist one compared to the MLE, which is considered the best estimation approach for a simulated data set at least in regards to the Variance Gamma model according to Finlay & Seneta (2008). In addition, we focus on the strength and weaknesses of the SMoM. We see that the SMoM has some problems with the estimation of the kurtosis driving parameter γ (note: a decrease of $\sqrt{\gamma^2 - \beta^2}$ increases the kurtosis), however it outperforms the MLE in estimating the variance responsible parameter δ. Additionally, we have a particular focus on the skewness parameter β. Based on the idea of the SMoM, we assume $\beta \approx 0$ and $\beta^2 = 0, \beta^3 = 0$ etc. and therefore neglect all terms including β with a higher order than one. Although, we choose values for β quite different from zero, we get acceptable estimates of the SMoM compared to the MLE. This fact demonstrates that the assumption taken for developing the SMoM is not too restrictive. Finally, the estimation of the drift c also works well.

Besides the RMAD statistics, we also consider the ARMAD statistics averaging the estimation errors for one parameter set. Altogether, we see that the SMoM produces approximately a double mistake compared to the MLE, except for the fist parameter combination, where the mistake is remarkably higher. All in all, our developed SMoM is a well, fast and appropriate method for getting initial values for the GH model and its various estimation methods. Furthermore, as the SMoM produces acceptable results, their estimated parameters also offer a very good indication of the GH model parameter ranges.

Finally, we consider a technical component of the SMoM and look at the contribution to the time of estimation of the parameters. It takes about 0.4 seconds to estimate the parameters of the SMoM. This equals about 10% of time of estimation of the MLE without SMoM. The integration of the SMoM as starting parameters in the
MLE optimization reduces estimation time for about 20% to nearly 50% in our simulation setting. Generally it holds, that the better the estimation quality of the parameters (see the ARMAD statistics in table 2) the better the improvement concerning the time of estimation.

6 Conclusion

We closed the existing gap of missing appropriate initial values regarding the fitting of the GH model. Based on the assumption of Seneta (2004), we assumed a symmetric distribution of the GH model and developed a SMoM for the GH model by a Taylor series approximation for the modified Bessel function of the third kind and a Tschirnhaus transformation for solving a quadratic equation. In our simulation study we verified the good quality of the estimated parameters of the SMoM by comparing them to the fitting quality of the MLE method. The RMAD as well as the ARMAD statistics reveal adequate fitting values of the newly developed SMoM. Furthermore, the incorporation of the SMoM in the optimization problem of the MLE significantly reduces the time of estimation. These facts demonstrate that the estimated SMoM parameters can be used as initial values for other estimation methods solving the problem of missing adequate initial values for the GH process.

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Appendix In the following you find a brief summary of solving equation (17) with the Tschirnhaus transformation based scheme described in King (2008). It holds:

$$
\begin{aligned}
0 = &\left(b_2^2 - 3\frac{\hat{M}_2^2}{\hat{M}_4}a_2c_2\right)\tilde{\zeta}^4 + \\
&\left(2b_1b_2 - 3\frac{\hat{M}_2^2}{\hat{M}_4}\left(a_1c_2 + a_2c_1\right)\right)\tilde{\zeta}^3 + \\
&\left(b_1^2 + 2b_0b_2 - 3\frac{\hat{M}_2^2}{\hat{M}_4}\left(a_0c_2 + a_1c_1 + a_2c_0\right)\right)\tilde{\zeta}^2 + \\
&\left(2b_0b_1 - 3\frac{\hat{M}_2^2}{\hat{M}_4}\left(a_0c_1 + c_0a_1\right)\right)\tilde{\zeta} + \\
&b_0^2 - 3\frac{\hat{M}_2^2}{\hat{M}_4}a_0c_0,
\end{aligned}
\tag{A.1}
$$

which can be simplified to the general quarto equation

$$
A\tilde{\zeta}^4 + B\tilde{\zeta}^3 + C\tilde{\zeta}^2 + D\tilde{\zeta} + E = 0. \tag{A.2}
$$

We transform this quarto equation with the help of the Tschirnhaus transformation and the substitution of $\tilde{\zeta} = x - \frac{B}{4A}$ to a monic quartic equation without any cubic term:

$$
x^4 + v_1x^2 + v_2x + v_3 = 0, \tag{A.3}
$$

with

$$
v_1 = -\frac{3B^2}{8A^2} + \frac{C}{A} \tag{A.4}
$$

$$
v_2 = \frac{B^3}{8A^3} - \frac{BC}{2A^2} + \frac{D}{A} \tag{A.5}
$$

$$
v_3 = -\frac{3B^4}{256A^4} + \frac{B^2C}{16A^3} - \frac{BD}{4A^2} + \frac{E}{A}. \tag{A.6}
$$

For solving (A.3), we apply the Ferrari method, introduce y for writing the equation as the difference of two squares, conduct a coefficient comparison and finally determine the resolvent cubic equation

$$y^3 + a_1 y^2 + a_2 y + a_3 = 0 \tag{A.7}$$

with

$$a_1 = \frac{5}{2}v_1 \qquad a_2 = 2v_1^2 - v_3 \qquad a_3 = \frac{1}{2}v_1(v_1^2 - v_3) - \frac{1}{8}v_2. \tag{A.8}$$

Additionally, for simplifying the terms, later we use $w^2 = v_1 + 2y$, $q^2 = (v_1 + y)^2 - v_3$ and $v_2 = 2wq$.

Again, we use the Tschirnhaus transformation for a cubic equation for getting a cubic equation without any quadratic term ($z^3 + b_2 z + b_3 = 0$) by substituting $y = z - \frac{a_1}{3}$. Subsequently, we solely focus on the main steps. We calculate

$$b_2 = \frac{3a_2 - a_1^2}{3} = -\frac{v_1^2}{12} - v_3 \tag{A.9}$$

$$b_3 = \frac{2a_1^3 - 9a_1 a_2 + 27a_3}{27} = -\frac{v_1^3}{108} + \frac{v_1 v_2}{3} - \frac{v_2^2}{8} \tag{A.10}$$

and defie

$$y_{1,2} = -\frac{5}{6}v_1 + \begin{cases} -\sqrt[3]{b_3} & \text{for} \quad b_2 = 0 \\ u - \frac{b_2}{3u} & \text{for} \quad b_2 \neq 0 \end{cases} \tag{A.11}$$

with

$$u = \sqrt[3]{-\frac{b_3}{2} + \sqrt{\frac{b_3^2}{4} + \frac{b_2^3}{27}}}. \tag{A.12}$$

Looking at the case analysis of $y_{1,2}$ we can show that $b_2 < 0$. To poof this we have to evaluate

$$b_2 = -\frac{v_1^2}{12} - v_3 \neq 0,$$

which is a function of fixed paratemers a_i, b_i, c_i with $i = 0, \ldots, 2$ and the inverse of the kurtosis $\hat{M}_2^2/\hat{M}_4 = k_{inv}$. Hence, we only have to find the roots of a polynomial

$$b_2 = p_1(k_{inv}) = 0$$

of degree 6. The polynomial has only 4 different complex roots and is strictly negative. Altogether, we can conclude that

$$y = -\frac{5}{6}v_1 + u - \frac{b_2}{3u}. \tag{A.13}$$

To proof the existence of $x \in \mathbb{R}$ we have to show that the radicant

$$\frac{b_3^2}{4} + \frac{b_2^3}{27}$$

is non-negative. Again, the radicant is of polynomial form

$$\frac{b_3^2}{4} + \frac{b_2^3}{27} = p_2(k_{inv}) = 0$$

with degree 18 with 12 different complex roots. Hence, it is straightforward that the polynomial has no real root in $\mathbb{R}^+$ and together with a $p_2(k_{inv}^*) > 0$ for a specific $k_{inv}^* > 0$ we can conclude the existence of u.

Remember w and q are interim results of the transformation to ”resolvent cubic”

$$w = \sqrt{v_1 + 2y} \qquad q = \frac{v_2}{2w}. \tag{A.14}$$

To proof the existence of $w \in \mathbb{R}$ we have to analyse the radicant

$$v_1 + 2y.$$

Again it is a polynomial in k_{inv}

$$v_1 + 2y = p_3(k_{inv}) = 0$$

of degree 3 with 2 different roots. Given the fact, that $p_3(k^*_{inv}) > 0$ for a specifi $k^*_{inv} > 0$, we can conclude the positivity of the radicant and the existence of $w \in \mathbb{R}$. Finally, after two resubstitutions, we get the solutions for $\tilde{\zeta}_{1,2,3,4}$

$$\tilde{\zeta}_{1,2,3,4} = -\frac{B}{4A} + \frac{1}{2}\left[sw + r\sqrt{-(v_1 + 2y) - 2\left(v_1 + s\frac{v_2}{w}\right)}\right], \tag{A.15}$$

with $s, r \in \{-1, 1\}$. To show the existence of $\tilde{\zeta}_{1,2,3,4} \in \mathbb{R}$ we have to differentiate between the two cases $s = 1$ and $s = -1$. In both cases the radicant is a polynomial and radical equation

$$-(v_1 + 2y) - 2\left(v_1 + s\frac{v_2}{w}\right) = p_4(k_{inv}) = 0$$

of order 18 (after desolving the radical terms). For neither $s = 1$ nor $s = -1$ it contains real roots. Consequently, $p_4(k_{inv}) \neq 0$. However, in case of $s = -1$ $p_4(k_{inv}) < 0$, whereas in case of $s = 1$ $p_4(k_{inv}) > 0$. Again, both can be proven by evaluating the polynomial for a specifi $k^*_{inv} > 0$. Altogether, there exists only a solution for $s = 1$. However, two possible solutions still remain: $r = 1$ and $r = -1$. To decide, which solution is more appropriate, we have to look at the nature of $\zeta = \delta\sqrt{\gamma^2 - \beta^2}$. Due to the fact that $\delta > 0$, ζ has to be positive and this can only be achieved, if $r = 1$. In case of $r = -1$, there are examples where $\tilde{\zeta} < 0$. To sum up, the fial solution is as follows:

$$\tilde{\zeta} = -\frac{B}{4A} + \frac{1}{2}\left[w + \sqrt{-(v_1 + 2y) - 2\left(v_1 + \frac{v_2}{w}\right)}\right]. \tag{A.16}$$

Table 1: RMAD - fiting results GH model

This table presents the RMAD fiting results for each marginal parameter $(\delta, \beta, \gamma, c)$ of the 6 assets in the GH model. The parameter line describes the input for the simulation. All RMAD quotes are indicated in %.

	1	2	3	4	5	6
δ						
parameter	0.003	0.004	0.005	0.006	0.007	0.008
SMoM	249.88	100.85	26.6	12.14	34.65	46.27
MLE	61.15	72.5	96.41	28.24	45.35	39.58
β						
parameter	-8	-10	-12	-14	-16	-18
SMoM	78.83	48.37	84.22	52.17	56.44	66.9
MLE	30.85	23.85	25.98	38.39	29.55	30.46
γ						
parameter	90	120	150	180	210	240
SMoM	76.75	70.31	74.18	83.77	83.72	70.66
MLE	16.28	16.78	16.29	8.18	9.62	10.48
c						
parameter	0.002	0.0025	0.003	0.0035	0.004	0.0045
SMoM	102.27	34.07	44.38	30.19	12.51	11.03
MLE	28.68	20.97	12.59	7.39	5.67	5.8

Table 2: ARMAD - fting results GH model

This table presents the ARMAD fting results for the 6 assets in the GH model. All ARMAD quotes are indicated in %.

	1	2	3	4	5	6
SMoM	126.93	63.4	57.35	44.57	46.83	48.71
MLE	34.24	33.53	37.82	20.55	22.55	21.58

Table 3: Time for estimation - MLE with SMoM vs. MLE without SMoM

This table presents the effects on the time of estimation for the MLE when it is run without the SMoM and with random starting parameters (MLE without SMoM) and the MLE in combination with the SMoM (MLE with SMoM). The reduction in % includes the time savings from MLE without SMoM compared to the sum of estimation time SMoM and MLE with SMoM. Except the last line all numbers are indicated in seconds.

	1	2	3	4	5	6
MLE without SMoM	4.03	4.22	4.16	3.91	4.13	4.32
MLE with SMoM	2.76	2.38	2.25	1.97	1.94	1.87
SMoM	0.44	0.43	0.42	0.41	0.4	0.41
Reduction in %	-0.21	-0.34	-0.36	-0.39	-0.43	-0.47