Torus actions and positive and non-negative curvature

Habilitationsschrift

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Contents

Some of the chapters presented in this Habilitationsschrift have been published before in modified form. These are:

- *Circle actions and scalar curvature*, first published in Trans. Amer. Math. Soc. in Volume 368, No. 4 (2016), 2939-2966, DOI:10.1090/tran/6666, published by the American Mathematical Society. © 2016 American Mathematical Society.
- *Torus manifolds and non-negative curvature*, J. Lond. Math. Soc., II. Ser. 91 (2015), No. 3, 667-692, DOI:10.1112/jlms/jdv008.
- (with Oliver Goertsches) *Positively curved GKM-manifolds*, Int. Math. Res. Not., 2015 (2015), No. 22, 12015-12041, DOI:10.1093/imrn/rnv046.

Introduction

Michael Wiemeler

During the last 20 years several topological generalizations of non-singular toric varieties were introduced. Among them quasitoric manifolds and torus manifolds are the generalizations which are most studied.

Quasitoric manifolds were introduced by Davis and Januszkiewicz [DJ91]. Whereas torus manifolds were introduced by Hattori and Masuda [HM03].

A torus manifold is a $2n$-dimensional closed connected orientable manifold M on which an n-dimensional torus T acts effectively such that M^T is non-empty. A quasitoric manifold is a torus manifold which satisfies two extra conditions. First it is locally standard. This condition is satisfied if and only if the action is locally modeled on an effective unitary representation of T^n on $\mathbb{C}^n$. It implies that the orbit space of the T-action on M is naturally a manifold with corners. The second condition which is required for a torus manifold to be a quasitoric manifold is that M/T is face preserving homeomorphic to a simple convex polytope.

In this Habilitationsschrift we investigate geometric properties of these manifolds and other manifolds with natural symmetries. For example we discuss the question of which torus manifolds admit invariant metrics of positive/non-negative sectional or scalar curvature. Moreover, we investigate the moduli spaces of invariant positive scalar curvature metrics on quasitoric manifolds.

1. Curvature

Before we describe our results in more detail, we will give a brief overview of the existence of metrics of positive/non-negative curvature on closed manifolds. There are three classical notions of curvature. These are sectional curvature, Ricci curvature and scalar curvature.

Among these the question of which simply connected closed manifolds admit metrics of positive scalar curvature is best understood. For manifolds of dimension $n \geq 5$, there are two types of obstructions against metrics of positive scalar curvature.

The first class of these obstructions comes from the minimal hypersurface method of Schoen and Yau [SY79]. This method says that any minimal hypersurface in a manifold with positive scalar curvature also admits a metric of positive scalar curvature. To apply this method one needs general existence results for minimal hypersurfaces which are at present only available in dimensions less than or equal to eight. Using this method one can prove that tori of dimension less than nine do not admit any metric of positive scalar curvature.

The other class of obstructions comes from the relation between scalar curvature and spin geometry. On the space of sections of the complex spinor bundle associated to a spin structure on a manifold M, one can define an elliptic differential operator, the so-called Dirac operator $\bar{\partial}$. If M is even dimensional, we denote the restriction of $\bar{\partial}$ to the sections of a half spinor bundle by ∂. By the Lichnerowicz–Weizenböck formula [Lic63], the square of the Dirac operator $\bar{\partial}$ on a spin manifold can be expressed as the sum of a non-negative, self-adjoint operator and $\frac{1}{4}$ of the scalar curvature. Therefore there are no harmonic spinors on a spin manifold with a metric of positive scalar curvature, i.e. $\ker \bar{\partial} = 0$. In particular the index $\operatorname{ind} \partial$ of the Dirac operator ∂ vanishes. Here $\operatorname{ind} \partial$ is defined as

$$\operatorname{ind} \partial = \dim \ker \partial - \dim \operatorname{coker} \partial.$$

By the Atiyah–Singer Index Theorem [AS69] $\operatorname{ind} \partial$ is equal to the $\hat{A}$-genus of M, which is a topological invariant of M.

By a similar argument, Hitchin [Hit74] has shown that a KO-theoretic refinement of the $\hat{A}$-genus, the so-called α-invariant, also vanishes for spin manifolds which admit a metric of positive scalar curvature. For simply connected manifolds of high enough dimension the converse also holds. It has been shown by Stolz [Sto92] that a simply connected spin manifold of dimension $n \geq 5$ admits a metric of positive scalar curvature if and only if its α-invariant vanishes. A similar statement for non-spin manifolds was proven previously by Gromov and Lawson: A simply connected non-spin manifold of dimension $n \geq 5$ always admits a metric of positive scalar curvature.

The idea of proof of these two statements is to reduce the existence problem to a bordism problem. This was done by Gromov and Lawson and independently by Schoen and Yau who showed that certain surgeries can be used to construct metrics of positive scalar curvature. Therefore a simply connected manifold of dimension $n \geq 5$ admits a metric of positive scalar curvature if and only if its class in a certain bordism group can be represented by a manifold with a metric of positive scalar curvature. The proofs are then completed by characterizing those classes in the relevant bordism groups which can be represented by manifolds with metrics of positive scalar curvature.

For non-simply connected manifolds it can be shown that certain higher analogues of the α-invariant also vanish [Ros86]. Moreover, index theoretic considerations can also be used to show that manifolds with non-positive sectional curvature do not admit any metric of positive scalar curvature [GL80], [GL83].

After this general overview of obstructions for the existence of metrics of positive scalar curvature, we turn to the relations between symmetries and metrics of positive scalar curvature. We are in particular interested in the question whether a non-trivial action of a compact connected Lie group G on a manifold M implies the existence of an (invariant) metric of positive scalar curvature.

The case where G is non-abelian is completely understood. It has been shown by Lawson and Yau [LY74] that a manifold with an action of such a group admits an invariant metric of positive scalar curvature.

The case where G is a torus is more complicated. The first result which has to be mentioned here is the vanishing of the $\hat{A}$-genus of a spin manifold which admits a non-trivial circle action, proved by Atiyah and Hirzebruch [AH70]. Together with the above mentioned result of Stolz it implies that a simply connected spin manifold of dimension

$4k \geq 8$ which admits a non-trivial circle action also admits a non-invariant metric of positive scalar curvature. Later it has been shown by Ono [Ono91] that the α-invariant of a spin manifold which admits an S^1-action of odd type vanishes. This also gives existence results for non-invariant metrics on simply connected manifolds. Here a S^1-action is of odd type if it does not lift to an action on the spin structure. Otherwise it is of even type

There are examples of even S^1-actions on spin manifolds with non-vanishing α-invariant. Examples of such actions were constructed by Bredon [Bre67], Schultz [Sch75] and Joseph [Jos81] on certain homotopy spheres which do not bound spin manifolds.

But what about S^1-invariant metrics? For the case that the S^1-action is free it has been shown by Bérard Bergery [BB83] that M admits an invariant metric of positive scalar curvature if and only if M/S^1 admits a metric of positive scalar curvature. Therefore principal S^1-bundles over K3-surfaces with simply connected total space admit metrics of positive scalar curvature but no invariant such metric.

Our first result gives the existence of invariant metrics of positive scalar curvature for the case that there are many fixed points.

Theorem A *Let M be a closed connected S^1-manifold such that there is a fixed point component of codimension two. Then there is an invariant metric of positive scalar curvature on M.*[1]

If M^{2n} is a torus manifold, then for some $S^1 \subset T^n$ there is a component of M^{S^1} which has codimension two. Therefore the theorem applies in this case. One can even show that there is a T^n-invariant metric of positive scalar curvature on M.

The idea of the proof of Theorem A is to use equivariant surgery to construct an $S^1 \times S^1$-invariant metric of positive scalar curvature on the total space of a principal S^1-bundle over M. The result then follows from a refinement of Bérard Bergery's result.

The case where there are no codimension two fixed point components is more complicated. We concentrate here on the case where $\dim M \geq 6$ and the maximal stratum of the action is simply connected. Here the maximal stratum $M_{\max}$ is the union of all principal S^1-orbits in M. We show that M admits an invariant metric of positive scalar curvature if and only if its class in a certain S^1-equivariant bordism group can be represented by a manifold which admits an invariant metric of positive scalar curvature.

A construction of generators of the relevant groups leads to the following result.

Theorem B *Let M be a closed S^1-manifold of dimension at least 6 such that the maximal stratum of the S^1-action is simply connected and for each subgroup $H \subset S^1$, M^H is orientable.*

1. *If M_{max} does not admit a spin structure or if M admits a spin structure but the S^1-action does not lift to an action on this structure, then there is an invariant metric of positive scalar curvature on the equivariant connected sum of 2^k copies of M for some $k \in \mathbb{N}$.*

[1] We prove Theorem A in Chapter 2 "Circle actions and scalar curvature".

2. *If M admits a spin structure, if, moreover, the S^1-action lifts to an action on this structure and if a generalized $\hat{A}$-genus of M/S^1 vanishes, then there is a $k \in \mathbb{N}$ such that the equivariant connected sum of 2^k copies of M admits an invariant metric of positive scalar curvature.*[2]

Similarly to our first case, Hanke [Han08] has previously shown that if the S^1-action on M does not have fixed points, M_{max} is non-spin and simply connected, and the normal bundles of the singular strata satisfy some technical Condition C, then there is an invariant metric of positive scalar curvature on M. Our methods for the proof of Theorem B are based in part on refinements of his methods.

If in the second case the action is semi-free, then the generalized $\hat{A}$-genus of M/S^1 from above coincides with a generalized $\hat{A}$-genus introduced by Lott [Lot00]. In general it is the index of a Dirac operator on some submanifold with boundary of M/S^1. It vanishes if M admits an invariant metric of positive scalar curvature. In the case that the S^1-action on M is free, M/S^1 is a spin manifold and its generalized $\hat{A}$-genus coincides with the usual $\hat{A}$-genus of M/S^1.

The above mentioned result of Hanke leads to the question whether taking the connected sum is really necessary to construct an invariant metric of positive scalar curvature on an S^1-manifold. In view of this question we can prove the following results:

1. If M is non-spin and the S^1-action on M is semi-free, then it suffices to take the connected sum of two copies of M. This also holds more generally for S^1-manifolds with non-spin simply connected maximal stratum satisfying Condition C.

2. If M_{max} is non-spin and simply connected, M satisfies Condition C and is equivariantly bordant to a product of two S^1-manifolds M_1, M_2 with non-trivial S^1-actions, then one has a metric of positive scalar curvature on M. One does not have to take the connected sums.

At the moment we do not know whether the first case can be further improved.

The bordism calculations which are used to prove Theorem B are also interesting for other questions. For example, they lead to a new proof of the rigidity of elliptic genera originally proved by Bott and Taubes [BT89] using different methods. Our proof of the rigidity is based on the fact that one only has to check the rigidity of elliptic genera for generators for the S^1-equivariant spin bordism groups. The generators which we obtain in our calculations are of a particularly simple form: They are semi-free S^1-manifolds or so-called generalized Bott manifolds. For the first class of manifolds it has been shown by Ochanine [Och88] that the elliptic genera are rigid. His proof can be modified so that it also works for the manifolds of the second class.

Besides the obstructions to metrics of positive scalar curvature the only known obstruction to a metric of positive Ricci curvature follows from the Theorem of Bonnet–Myers. A consequence of this theorem is that the fundamental group of a closed manifold with

[2] We prove Theorem B in Chapter 3 "S^1-equivariant bordism, invariant metrics of positive scalar curvature and rigidity of elliptic genera.".

a metric of positive Ricci curvature is finite. It has been conjectured by Bazaikin and Matvienko that every quasitoric manifold admits an invariant metric of positive Ricci curvature [BM11], [BM07]. They proved this conjecture in dimension four.

In view of Theorem A one might go even further and conjecture:

Conjecture 1.1 *Every torus manifold with finite fundamental group admits an (invariant) metric of positive Ricci curvature.*[3]

Note that there are torus manifolds with infinite fundamental group. So not all torus manifolds admit metrics of positive Ricci curvature.

By Synge's theorem, the fundamental group of an even dimensional manifold with positive sectional curvature has order at most two. (For results on the fundamental group of odd dimensional positively curved manifolds see [Ron99], [Sha98] and [GS00].) Using this property it is easy to find manifolds with positive Ricci curvature but no metric of positive sectional curvature. An example of such a manifold is given by the product of two copies of $\mathbb{R}P^2$.

Gromov's Betti Number Theorem [Gro81] gives an obstruction against non-negative sectional curvature. It states that the sum of Betti numbers of a closed connected manifold M which admits a metric of non-negative sectional curvature is bounded from above by a constant which only depends on the dimension of M. Since there are metrics of positive Ricci curvature on the connected sum of k copies of $S^n \times S^m$, $n, m > 1$, for every $k \in \mathbb{N}$ [SY91], it follows from the Betti Number Theorem that there are simply connected manifolds which admit metrics of positive Ricci curvature but no metric of non-negative sectional curvature. Because the connected sum of two quasitoric manifolds is again quasitoric the existence of more such examples would follow from Bazaikin's and Matvienko's conjecture.

The orbit space of a non-negatively/positively curved manifold M by an isometric action of a compact Lie group G is a non-negatively/positively curved Alexandrov space. Therefore tools from Alexandrov geometry can be used to study group actions on non-negatively curved manifolds. One important result on non-negatively curved Alexandrov spaces is an analogue of the so-called Soul Theorem of Cheeger and Gromoll [CG72]. It says that if X is a compact non-negatively curved Alexandrov space with non-empty boundary, then there is a totally convex compact subset S without boundary which is a strong deformation retract of X. If X has positive curvature then S is a single point.

The boundary of M/G is non-empty if for example the G-action is fixed point homogeneous, i.e. there is a component F of M^G such that G acts transitively on the normal sphere at any point of F. If M is a torus manifold, then as observed before there is an $S^1 \subset T$ such that M^{S^1} has codimension two. The action of such an S^1 is fixed point homogeneous. Using this fact and the Soul Theorem it has been shown by Grove

[3] Added in Proof: As pointed out to us by Vitali Kapovitch the finiteness of the fundamental group is not the only obstruction to positive Ricci curvature besides the obstructions to positive scalar curvature: By [KW11], the fundamental group of a positively Ricci curved manifold M contains a nilpotent subgroup whose index is bounded by a constant only depending on the dimension of M. Since every finite presentable group is the fundamental group of some eight-dimensional torus manifold [Wie13], it follows that the conjecture is not true as stated.

and Searle [GS94] that a torus manifold with an invariant metric of positive sectional curvature is diffeomorphic to S^{2n} or $\mathbb{C}P^n$. More recently it has been shown by Spindeler [Spi14] using similar techniques that a fixed point homogeneous manifold of non-negative curvature decomposes as the union of two disc bundles over invariant submanifolds.

One can use Toponogov's Theorem applied to the orbit space of a four-dimensional non-negatively curved torus manifold M with vanishing first Betti number to show that M has at most four fixed points [GGS11]. Together with classification results for simply connected four-dimensional torus manifolds due to Orlik and Raymond [OR70], this leads to a classification of simply connected non-negatively curved four dimensional torus manifolds. Each such manifold is diffeomorphic to S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, $\mathbb{C}P^2 \# \mathbb{C}P^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$. One can also show that these are the only non-negatively curved simply connected four-dimensional manifolds which admit an isometric action of S^1 [Kle90], [SY94].

By a combination of all the results on non-negatively curved torus manifolds mentioned above, we prove the following classification result for non-negatively curved simply connected torus manifolds.

Theorem C *Let M be a simply connected torus manifold which admits an invariant metric of non-negative sectional curvature. Then M is diffeomorphic to a quotient of a free linear torus action on a product of spheres.*[4]

We give a short outline of the proof. At first one uses the results of Spindeler to show that the manifold M is locally standard. This implies that the orbit space M/T is a manifold with corners. Again by an inductive application of Spindeler's result, one then shows that all faces of M/T are diffeomorphic to standard discs. This implies that the diffeomorphism type of M/T is determined by its face poset $\mathcal{P}(M/T)$.

The condition that two-dimensional faces of M/T have at most four vertices has strong implications for the combinatorial type of the face poset of M/T. Indeed one can determine this combinatorial type from this condition. Then one knows that the moment-angle complex associated to this poset is diffeomorphic to a product of spheres. Since M is the quotient of this moment-angle complex by a free torus action, the result follows.

For non-simply connected non-negatively curved torus manifolds we show that their fundamental group is always finite. Therefore their universal coverings are diffeomorphic to manifolds as in Theorem C.

If one considers actions of tori of lower dimension than in the case of torus manifolds, one comes naturally to the class of GKM_k manifolds. These are even-dimensional manifolds with an action of a torus such that:

1. The action has only finitely many fixed points.
2. At each fixed point any k weights of the torus representation at this fixed point are linear independent.

[4] We prove Theorem C in Chapter 4 "Torus manifolds and non-negative curvature".

3. The action is equivariantly formal.

For $k = 2$ one also just speaks of GKM manifolds. GKM manifolds first appeared in the paper [GKM98] by Goresky, Kottwitz and MacPherson. Later on these manifolds have been studied by symplectic geometers (see for example [GM14], [GSZ13], [GHZ06], [GH04] and [GZ99]).

Using a similar combinatorial approach as in the torus manifold case we can compute the real cohomology of GKM_3 manifolds with an invariant metric of positive sectional curvature as follows.

Theorem D *Let M be a compact connected positively curved orientable Riemannian manifold satisfying $H^{odd}(M;\mathbb{R}) = 0$. Assume that M admits an isometric torus action of type GKM_3. Then M has the real cohomology ring of a compact rank one symmetric space.*[5]

Under stronger conditions on the weights at the fixed points, we get a similar result for cohomology with integer coefficients and a finiteness result for the diffeomorphism types of these manifolds.

After solving the existence question for metrics with certain curvature conditions on a manifold, the next question is: How many such metrics exist on the manifold? Or, what is the topology of the space of metrics with this curvature condition?

As far as the question is concerned one should also consider so-called moduli spaces of Riemannian metrics. These are orbit spaces of spaces of Riemannian metrics equipped with the natural action of the diffeomorphism group of M given by pull back of metrics.

Here we focus on the case of quasitoric manifolds and invariant metrics of positive scalar curvature on these manifolds. Our main result for the above questions in this case is the following theorem.

Theorem E *There are quasitoric manifolds M of dimension $2n$ such that for $0 < k < \frac{n}{6} - 7$, n odd and $k \equiv 0 \mod 4$, $\pi_k(\mathcal{M}^+) \otimes \mathbb{Q}$ is non-trivial, where $\mathcal{M}^+$ is some component of the moduli space $\mathcal{M}^+(M; T^n)$ of T^n-invariant metrics of positive scalar curvature on M.*[6]

It might be seen as a first step to understand the topology of the moduli space of metrics of positive scalar curvature on these manifolds. This is because the full moduli space of these metrics is stratified by the rank of the isometry groups of the metrics on M. The above theorem is a non-triviality result for the homotopy type of a minimal stratum of this moduli space. If one has non-triviality results for the homotopy types of all strata, one might expect non-triviality results for the homotopy types of the full moduli space.

The proof of this last theorem is based on an equivariant version of a method used in [BHSW10] to construct non-trivial elements in homotopy groups of observer moduli spaces of positive scalar curvature metrics.

[5] We prove Theorem D in Chapter 5 "Positively curved GKM manifolds".

[6] We prove Theorem A in Chapter 6 "Moduli spaces of invariant metrics of positive scalar curvature on quasitoric manifolds".

2. Organization of the Habilitationsschrift

In the following chapters papers of mine written during the time 2010–2016 are presented.

The first part deals with results on torus actions and positive scalar curvature. In this part we prove Theorems A and B.

The first of these chapters has been published as

1. *Circle actions and scalar curvature*, Trans. Amer. Math. Soc. 368 (2016), No. 4, 2939-2966

The second appeared as

2. *S^1-equivariant bordism, invariant metrics of positive scalar curvature and rigidity of elliptic genera*, Preprint, 2016

on arxiv.org and has been submitted to a journal.

In the next two chapters the results on positive and non-negative sectional curvature and torus manifolds and GKM manifolds are presented in two chapters. Here we prove Theorems C and D. The results of the second of these chapters are joint work with Oliver Goertsches. Modified versions of these chapters have been published as

3. *Torus manifolds and non-negative curvature*, J. Lond. Math. Soc., II. Ser. 91 (2015), No. 3, 667-692,
4. (with Oliver Goertsches) *Positively curved GKM-manifolds*, Int. Math. Res. Not., 2015 (2015), No. 22, 12015-12041.

In the last part we investigate the homotopy types of moduli spaces of (invariant) metrics of positive scalar curvature on quasitoric manifolds and spheres. The chapters of this part were never published before. In this part we prove Theorem E.

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During the academic year 2011–2012 I was visiting the Max Planck Institute for Mathematics in Bonn. I want to thank the MPI for financial support and hospitality during that time. Here I began to work on Part 1 of this Habilitationsschrift.

From October 2012 until September 2014 I was working at the Karlsruhe Institute of Technology. Here I began to work on the second part of this Habilitationsschrift.

I want to thank Fernando Galaz-Garcia, Martin Kerin and Marco Radeschi for comments on this part. Moreover, I would like to thank Wolfgang Spindeler for sharing his results from [Spi14]. They were needed to complete Part 2. I thank Oliver Goertsches for discussions related to GKM theory and positive curvature. I would also like to thank Wilderich Tuschmann for his support during this time.

Since October 2014 I am working at the University of Augsburg. Here I began to work on the last part of this thesis and completed my work on this Habilitationsschrift.

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Part 1

Positive scalar curvature

Circle actions and scalar curvature

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We construct metrics of positive scalar curvature on manifolds with circle actions. One of our main results is that there exist S^1-invariant metrics of positive scalar curvature on every S^1-manifold which has a fixed point component of codimension 2. As a consequence we can prove that there are non-invariant metrics of positive scalar curvature on many manifolds with circle actions. Results from equivariant bordism allow us to show that there is an invariant metric of positive scalar curvature on the connected sum of two copies of a simply connected semi-free S^1-manifold M of dimension at least six provided that M is not spin or that M is spin and the S^1-action is of odd type. If M is spin and the S^1-action of even type then there is a $k > 0$ such that the equivariant connected sum of 2^k copies of M admits an invariant metric of positive scalar curvature if and only if a generalized $\hat{A}$-genus of M/S^1 vanishes.

1. Introduction

In this article we discuss the following questions: Let G be a compact connected Lie-group and M a closed connected effective G-manifold.

1. Is there a G-invariant metric of positive scalar curvature on M?

2. If the answer to the first question is “no”, does there exist a non-invariant metric of positive scalar curvature on M?

It has been shown by Lawson and Yau [LY74] that the answer to the first question is “yes” if G is non-abelian. Therefore we concentrate on the case where G is abelian and especially on the case $G = S^1$.

In this case there are two extreme situations:

1. The S^1-action on M is free.

2. There are “many” S^1-fixed points, i.e. the fixed point set has low codimension.

The first situation was studied by Bérard Bergery [BB83], who showed that a free S^1-manifold M admits an S^1-invariant metric of positive scalar curvature if and only if M/S^1 admits a metric of positive scalar curvature.

For the second case we have the following theorem.

Theorem 1.1 (Theorem 2.4) *Let G be a compact Lie-group. Assume that there is a circle subgroup $S^1 \subset Z(G)$ contained in the center of G. Moreover, let M be a closed connected effective G-manifold such that there is a component F of M^{S^1} with* $\operatorname{codim} F = 2$. *Then there is a G-invariant metric of positive scalar curvature on M.*

A torus manifold M is a closed connected $2n$-dimensional manifold with an effective action of an n-dimensional torus T, such that $M^T \neq \emptyset$. Smooth compact toric varieties are examples of torus manifolds. As a corollary to Theorem 1.1 we prove:

Corollary 1.2 (Corollary 2.6) *Every torus manifold admits an invariant metric of positive scalar curvature.*

We also show that a closed connected semi-free S^1-manifold M of dimension greater than five without fixed point components of codimension less than four admits an invariant metric of positive scalar curvature if and only if the bordism class of M in a certain equivariant bordism group can be represented by an S^1-manifold with an invariant metric of positive scalar curvature (see Theorem 4.5).

If M is simply connected, one has to distinguish between the following three cases:

1. M does not admit a Spin-structure.
2. M admits a Spin-structure and the S^1-action lifts into this structure. In this case it is said that the action is of *even type*.
3. M admits a Spin-structure, but the S^1-action does not lift into it. In this case it is said that the action is of *odd type*.

An investigation of the relevant bordism groups in these cases leads to the following theorems.

Theorem 1.3 (Theorem 4.7) *Let M be a closed simply connected semi-free S^1-manifold of dimension $n > 5$. If M is not spin or spin and the S^1-action is odd, then the equivariant connected sum of two copies of M admits an invariant metric of positive scalar curvature.*

In [Lot00] Lott constructed a generalized $\hat{A}$-genus for orbit spaces of semi-free even S^1-actions on Spin-manifolds. He showed that for such a manifold M, $\hat{A}(M/S^1)$ vanishes if M admits an invariant metric of positive scalar curvature. We prove the following partial converse to his result.

Theorem 1.4 (Theorem 4.11) *Let M be a closed simply connected Spin-manifold of dimension $n > 5$ with even semi-free S^1-action. Then we have $\hat{A}(M/S^1) = 0$ if and only if there is a $k \in \mathbb{N}$ such that the equivariant connected sum of 2^k copies of M admits an invariant metric of positive scalar curvature.*

By using Theorem 1.1, we can show that there are non-invariant metrics of positive scalar curvature on many manifolds with S^1-action. This is the content of the next theorem.

Theorem 1.5 (Theorem 3.3) *Let M be a closed connected effective S^1-manifold of dimension $n \geq 5$ such that the principal orbits in M are null-homotopic. This condition guarantees that the S^1-action on M lifts to an S^1-action on the universal cover $\tilde{M}$ of M. If the universal cover of M is a Spin-manifold assume that the lifted S^1-action on $\tilde{M}$ is of odd type.*

Then M admits a non-invariant metric of positive scalar curvature.

Note that if in the situation of the above theorem $M^{S^1} \neq \emptyset$, then the principal orbits are always null-homotopic. Since a S^1-manifold M with $\chi(M) \neq 0$ has fixed points we get the following corollary.

Corollary 1.6 (Corollary 3.5) *Let M be a closed connected manifold of dimension $n \geq 5$ with non-zero Euler-characteristic such that the universal cover of M is not spin. If M does not admit a metric of positive scalar curvature, then there is no non-trivial S^1-action on M.*

It should be noted that the condition on the principal orbits in the above theorem cannot be omitted. This can be seen by considering a torus $T^n = \prod_{i=1}^n S^1$ on which S^1 acts by multiplication on one of the factors. Then T^n admits a Spin-structure for which the S^1-action is of odd type. But T^n does not admit a metric of positive scalar curvature.

Bredon [Bre67], Schultz [Sch75] and Joseph [Jos81] constructed S^1-actions of even type on homotopy spheres not bounding Spin-manifolds. It is known that these homotopy spheres do not admit metrics of positive scalar curvature. Therefore Theorem 1.5 is not true for S^1-actions of even type on Spin-manifolds.

This paper is organized as follows. In Section 2 we prove Theorem 1.1 and give some applications. Then in Section 3 we prove Theorem 1.5 and give more applications. In Section 4 we discuss the existence of metrics of positive scalar curvature on semi-free S^1-manifolds without fixed point components of codimension two.

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2. Construction of invariant metrics of positive scalar curvature

In this section we construct invariant metrics of positive scalar curvature on S^1-manifolds M such that M^{S^1} has codimension two.

For the construction of our metrics of positive scalar curvature we use a surgery-principle which was first proven independently by Gromov and Lawson [GL80] and Schoen and Yau [SY79]. Later it was noted by Bérard Bergery [BB83] that these constructions also work in the equivariant setting. This gives the following theorem.

Theorem 2.1 ([BB83, Theorem 11.1]) *Let G be a compact Lie-group and M and N be G-manifolds. Assume that N admits an G-invariant metric of positive scalar curvature. If M is obtained from N by equivariant surgery of codimension at least three, then M admits an invariant metric of positive scalar curvature.*

Besides the construction of metrics of positive scalar curvature via surgery, we need the following result which tells us that there are such metrics on certain orbit spaces of free torus actions.

Theorem 2.2 *Let M be a manifold with a free action of a torus T. Assume that there is an action of a compact Lie-group which commutes with the T-action on M. Then there is a G-invariant metric of positive scalar curvature on M/T if and only if there is a $G \times T$-invariant metric of positive scalar curvature on M.*

Proof. In the case that G is the trivial group this theorem is part of Bérard Bergery's Theorem C from [BB83]. So we begin by recalling Bérard Bergery's construction and then indicate what has to be done to get a G-invariant metric.

Bérard Bergery starts with a T-invariant metric g of positive scalar curvature on M. This metric induces a metric g^* on M/T, such that the orbit map $\pi : M \to M/T$ is a Riemannian submersion. The metric g^* is then rescaled by the function $F : M/T \to \mathbb{R}$, $F(x) = f(x)^{2/(\dim M/T - 1)}$, where $f(x)$ is the volume of the T-orbit $\pi^{-1}(x)$. Bérard Bergery proves that the resulting metric $\tilde{g}$ has positive scalar curvature.

If we choose the metric g to be $G \times T$-invariant, then every element of G maps each T-orbit in M isometrically onto another T-orbit because G and T commute. Therefore the function F is G-invariant.

Moreover, if h is an element of G and $x \in M$, then the differential $D_x h$ maps the orthogonal complement of $T_x(Tx)$ in $T_x M$ isometrically onto the orthogonal complement of $T_{hx}(Thx)$ in $T_{hx}M$. Since $\pi : (M, g) \to (M/T, g^*)$ is a Riemannian submersion, it follows that g^* is G-invariant.

Hence, the metric $\tilde{g}$ is also G-invariant.

For the construction of an invariant metric with positive scalar curvature on M from a metric on M/T, just pick a $G \times T$-invariant connection for the principal T-bundle $M \to M/T$ and a flat invariant metric h on T. Then by a result of Vilms [Vil70] there is a unique metric on M such that $M \to M/T$ is a Riemannian submersion with totally geodesic fibers isometric to (T, h) and horizontal distribution associated to the chosen connection. By construction this metric is $G \times T$-invariant. After shrinking the fibers one gets a metric of positive scalar curvature on M. In the following we will call a metric obtained by such a construction a connection metric. $\square$

Now we turn to our construction of invariant metrics of positive scalar curvature. For this we need the following lemma.

Lemma 2.3 *Let G be a compact Lie-group and Z be a compact connected G-manifold with non-empty boundary. We view Z as a bordism between the empty set and ∂Z. Then there is a G-handle decomposition of Z without handles of codimension 0.*

Proof. We choose a special G-Morse function $f : Z \to [0, 1]$ without critical orbits on the boundary of Z, such that $f^{-1}(1) = \partial Z$. For the definition of special G-Morse functions and some of their properties see [Han08] or [May89]. The map f induces a G-handle decomposition of Z such that the handles correspond one-to-one to the critical orbits of f. The codimension of a handle corresponding to a critical orbit is given by the coindex of this orbit. Therefore we have to show that we can change f in such a way that there are no critical orbits of coindex 0. By Lemma 13 of [Han08], the critical orbits of coindex 0 are principal orbits.

Therefore, as in the proof of Theorem 15 of [Han08], non-equivariant handle cancellation on the orbit space can be used to remove all handles of codimension 0. Note here that we only work with handles of codimension 0. Therefore we do not need the dimension assumption from Hanke's theorem. □

Now we can prove the first of our main theorems.

Theorem 2.4 *Let G be a compact Lie-group. Assume that there is a circle subgroup $S^1_0 \subset Z(G)$ contained in the center of G. Moreover, let M be a closed connected effective G-manifold such that there is a component F of $M^{S^1_0}$ with* $\operatorname{codim} F = 2$. *Then there is an G-invariant metric of positive scalar curvature on M.*

Proof. Let Z be M with an open tubular neighborhood of F removed. Then Z is a G-manifold with boundary SF, the normal sphere bundle of F. Let S^1_1 act on $D^2 \subset \mathbb{C}$ via multiplication with the inverse. Then we have a $G \times S^1_1$-manifold $X = Z \times D^2$ (with corners equivariantly smoothed). A G-handle decomposition of Z induces a $G \times S^1_1$-handle decomposition of X (both viewed as bordisms between the empty set and their boundaries) such that

1. the G-handles of Z are one-to-one to the $G \times S^1_1$-handles of X.
2. the codimension of a handle of X is given by the codimension of the corresponding handle of Z plus two.

By Lemma 2.3, we may assume that there is no handle of codimension 0 in the decomposition of Z. Therefore in the decomposition of X all handles have codimension at least three. Hence, it follows from Theorem 2.1 that ∂X admits an $G \times S^1_1$-invariant metric of positive scalar curvature.

Now we have

$$\partial X = SF \times D^2 \cup_F Z \times S^1,$$

where the gluing map $F : \partial(SF \times D^2) = SF \times S^1 \to SF \times S^1 = \partial(Z \times S^1)$ is given by $f \times g^{-1}$. Here $f : SF \hookrightarrow Z$ and $g : S^1 \hookrightarrow D^2$ are the natural inclusions.

Note that SF is a principal S^1-bundle over F and the action of S^1_0 on this bundle is given by multiplication on the fibers. Therefore the orbit space of the free $\operatorname{diag}(S^1_0 \times S^1_1)$-action on $SF \times D^2$ is the normal disc bundle of F in M. A diffeomorphism is induced by the map

$$SF \times D^2 \to N(F, M) \qquad\qquad (x, \lambda) \mapsto \lambda x,$$

where λx is given by complex multiplication of $\lambda \in D^2 \subset \mathbb{C}$ and $x \in SF$ with respect to the complex structure on $N(F, M)$ induced by the action of S^1_0.

Moreover, the orbit space of the free $\operatorname{diag}(S^1_0 \times S^1_1)$-action on $Z \times S^1$ is diffeomorphic to Z. A diffeomorphism is induced by the map

$$Z \times S^1 \to Z \qquad\qquad (x, \lambda) \mapsto \lambda \cdot x,$$

where $\cdot$ denotes the action of S^1_0 on Z.

Hence, it follows from the special form of F described above that F induces the natural inclusion $SF \hookrightarrow Z$ on the orbit space.

Therefore the quotient of the free $\operatorname{diag}(S^1_0 \times S^1_1)$-action on ∂X is G-equivariantly diffeomorphic to M. Since the action of G on ∂X commutes with the action of $\operatorname{diag}(S^1_0 \times S^1_1)$, it follows from Theorem 2.2 that M admits a G-invariant metric of positive scalar curvature. □

Note that if a torus T acts effectively on a manifold M, then all fixed point components have codimension greater or equal to $2 \dim T$. For the case of equality we have the following corollary.

Corollary 2.5 *Let T be a torus that acts effectively on the closed connected manifold M. If there is a component of M^T of codimension $2 \dim T$, then there is a T-invariant metric of positive scalar curvature on M.*

Proof. Let F be a component of M^T of codimension $2 \dim T$ and $x \in F$. Then up to an automorphism of T the T-representation on the normal space $N_x(F, M)$ is given by the standard T-representation. Therefore there is a codimension-two subspace of $N_x(F, M)$ which is fixed pointwise by a circle subgroup $S^1 \subset T$. Hence, it follows that F is contained in a codimension-two submanifold of M which is fixed pointwise by S^1. Now the statement follows from Theorem 2.4. □

A torus manifold is a closed connected $2n$-dimensional manifold M with an effective action of an n-dimensional torus T, such that $M^T \neq \emptyset$. Smooth compact toric varieties are examples of torus manifolds. As a consequence of Corollary 2.5 we get the following corollary.

Corollary 2.6 *Every torus manifold admits an invariant metric of positive scalar curvature.*

Proof. The fixed point set of a torus manifold consists of isolated points. Therefore the statement follows from Corollary 2.5. □

As an application of Corollary 2.6 we can improve the upper bound for the degree of symmetry of manifolds which do not admit metrics of positive scalar curvature given by Lawson and Yau in [LY74].

Corollary 2.7 *Let G be a compact connected Lie-group which acts effectively on a closed connected manifold M which does not admit a metric of positive scalar curvature and has non-zero Euler-characteristic. Then G is a torus of dimension less than $\frac{1}{2}\dim M$.*

Proof. By a result of Lawson and Yau [LY74], G must be a torus. Because the Euler-characteristic of M is non-zero, there are G-fixed points in M. Therefore we have $2\dim G \leq \dim M$. If equality holds in this inequality, then M is a torus manifold. Therefore, by Corollary 2.6, we have $2\dim G < \dim M$. □

By [ABP67, Corollary 2.7], a homotopy sphere of dimension greater than one bounds a Spin-manifold if and only if it has vanishing α-invariant. Moreover, homotopy spheres with non-vanishing α-invariant only exist in dimensions greater or equal to 9 congruent to 1 or 2 mod 8. In these dimensions they constitute half of all homotopy spheres (see [Mil65, Proof of Theorem 2] and [Ada66, Theorem 1.2]). Since the α-invariant vanishes for spin-manifolds which admit metrics of positive scalar curvature, such homotopy spheres do not admit a metric of positive scalar curvature.

Hence, even-dimensional exotic spheres Σ which do not bound Spin-manifolds are examples of manifolds for which the assumptions on M from the above corollary hold. Other examples of manifolds M can be constructed as follows. Let N be a torus manifold which is spin with $10 \leq \dim N \equiv 2 \mod 8$. Then the α-invariant of $M = N\#\Sigma$ does not vanish. Therefore M does not admit a metric of positive scalar curvature and satisfies the assumptions of the corollary. Therefore M does not admit a smooth action of a torus of dimension $\frac{1}{2}\dim M = \frac{1}{2}\dim N$.

For four-dimensional S^1-manifolds we also have the following two corollaries.

Corollary 2.8 *Let M be a closed connected effective S^1-manifold with $\dim M = 4$ and $\chi(M) < 0$. Then there is an invariant metric of positive scalar curvature on M.*

Proof. Since $\chi(M^{S^1}) = \chi(M) < 0$, there must be a fixed point component of dimension two. Hence, the corollary follows from Theorem 2.4. □

Corollary 2.9 *Let M be a closed connected oriented semi-free S^1-manifold with $\dim M = 4$ and non-vanishing signature. Then there is an invariant metric of positive scalar curvature on M.*

Proof. By Corollary 6.24 of [Kaw91], there is a fixed point component of dimension two in M. Hence, the corollary follows from Theorem 2.4. □

3. Constructions of non-invariant metrics of positive scalar curvature

In this section we construct non-invariant metrics of positive scalar curvature on many manifolds with circle actions. For this construction we need the following lemma.

Lemma 3.1 *Let M be a closed connected effective S^1-manifold of dimension $n > 1$. If $n > 2$, then M is equivariantly bordant to a closed connected effective S^1-manifold N such that there is a component of N^{S^1} of codimension two. If $n = 2$, then M is equivariantly bordant to a closed effective S^1-manifold with at most two components such that each component contains an isolated S^1-fixed point. Moreover, we have:*

1. *If M is oriented, then N is also oriented and they are equivariantly bordant as oriented manifolds.*
2. *If M is spin and the S^1-action on M is of odd type, then the same holds for N and they are equivariantly bordant as Spin-manifolds.*

Proof. Let $S^1_0 \hookrightarrow M$ be the inclusion of a principal orbit in M. Then the equivariant normal bundle of S^1_0 is given by $S^1_0 \times \mathbb{R}^{n-1}$, where S^1 acts trivially on the $\mathbb{R}^{n-1}$-factor. Therefore we can do equivariant surgery on S^1_0 to obtain an S^1-manifold N. If $n > 2$, then N is connected and has a fixed point component of codimension two. If $n = 2$, then N might have two components both containing an isolated S^1-fixed point. If we let W be the trace of this surgery we see that (1) holds.

For (2) we need an extra argument to show that W is spin. This is the case if the inclusions

$$S^1_0 \times D^{n-1} \hookrightarrow M \qquad \text{and} \qquad S^1_0 \times D^{n-1} \hookrightarrow D^2 \times D^{n-1}$$

induce the same Spin-structure on $S^1_0 \times D^{n-1}$.

Let $\hat{S}^1$ be the connected double cover of S^1. Then the S^1-action on M induces an action of $\hat{S}^1$ on M. This action of $\hat{S}^1$ lifts into the Spin-structure on M. Let $\mathbb{Z}_2 \subset \hat{S}^1$ be the kernel of $\hat{S}^1 \to S^1$. If the S^1-action on M is of odd type, then $\mathbb{Z}_2$ acts non-trivially on each fiber of the principal Spin-bundle over M. If the S^1-action on M is of even type, then $\mathbb{Z}_2$ acts trivially on this bundle.

On $S^1_0 \times D^{n-1}$ there are two Spin-structures. For one of them the S^1-action on $S^1_0 \times D^{n-1}$ is of even type. For the other it is of odd type. By the above remark about the action of $\mathbb{Z}_2$, it follows that the inclusion $S^1_0 \times D^{n-1} \hookrightarrow M$, induces the second Spin-structure on $S^1_0 \times D^{n-1}$. This is also the Spin-structure which is induced by the inclusion of $S^1_0 \times D^{n-1} \hookrightarrow D^2 \times D^{n-1}$. Therefore W is an equivariant Spin-cobordism. □

By combining Theorem 2.4 and Lemma 3.1 we recover the following result of Ono.

Corollary 3.2 ([Ono91]) *Let M be a closed connected Spin-manifold with an S^1-action of odd type. Then the α-invariant of M vanishes.*

Proof. The α-invariant is a Spin-bordism invariant and vanishes for Spin-manifolds which admit metrics of positive scalar curvature. Since an action of odd type is always non-trivial, we may assume that the S^1-action on M is effective. Because by Theorem 2.4 and Lemma 3.1, M is Spin-bordant to a manifold with a metric of positive scalar curvature the statement follows. □

As another application of Theorem 2.4 we get:

Theorem 3.3 *Let M be a closed connected effective S^1-manifold of dimension $n \geq 5$ such that the principal orbits in M are null-homotopic. This condition guarantees that the S^1-action on M lifts to an S^1-action on the universal cover $\tilde{M}$ of M. If the universal cover of M is a Spin-manifold assume that the lifted S^1-action on $\tilde{M}$ is of odd type.*

Then M admits a non-invariant metric of positive scalar curvature.

Proof. At first do an equivariant surgery on a principal orbit of the S^1-action on M as in the proof of Lemma 3.1. By Theorem 2.4, the resulting manifold has an invariant metric of positive scalar curvature. Moreover, since the principal orbits are null-homotopic, it follows from the assumption on the lifted action to the universal cover, that N is diffeomorphic to the connected sum of M and $S^2 \times S^{n-2}$.

Indeed, the result of a surgery on a null-homotopic circle S in M is diffeomorphic to the connected sum of M and $S^2 \times S^{n-2}$ or the connected sum of M and the non-trivial S^{n-2}-bundle over S^2, depending on the choice of the framing of the circle, for which there are two choices. If the universal cover of M is not spin, then these two manifolds are diffeomorphic. To see this take an embedding of $S^2 \hookrightarrow M$ with non-trivial normal bundle. We may assume that S is contained in S^2 as a small circle around the north pole. We fix a framing of S in M. We may move S along the meridians of S^2 to the south pole and then rotate S^2 so that the north and south pole are interchanged. During the way of S in S^2 its framing changes, so that the result of a surgery on S is independent of the choice of the framing.

If the universal cover of M is spin, then the assumption on the lifted action and the argument from the proof of Lemma 3.1 imply that the universal cover of the result of the surgery is also spin. Therefore N is the connected sum of M and $S^2 \times S^{n-2}$. Hence, by surgery on the S^2-factor we recover M. Since this surgery is of codimension at least three, it follows that M admits a metric of positive scalar curvature. □

We give an example which shows that the assumptions of the above theorem are not sufficient for the existence of S^1-invariant metrics. Let $X = (\mathbb{C}P^2 \times \mathbb{C}P^1)\#T^6$ and M be the principal S^1-bundle over X with first Chern class a generator of $H^2(\mathbb{C}P^2;\mathbb{Z}) \subset H^2(X;\mathbb{Z})$. Then it follows from an inspection of a Mayer-Vietoris-sequence that M is a Spin-manifold. Moreover, the S^1-action on M is of odd type because X is not a Spin-manifold. The S^1-orbits in M are null-homotopic because $M|_{\mathbb{C}P^2 \subset X} = S^5$ is simply connected. But there is a degree-one map $X \to T^6$. Hence, X does not admit a metric of positive scalar curvature by [SY79, Corollary 2]. Therefore, by Theorem 2.2, M does not admit a S^1-invariant metric of positive scalar curvature.

As a consequence of the proof of Theorem 3.3 we get the following corollary.

Corollary 3.4 *Let M be a manifold of dimension $n \geq 4$ with effective S^1-action and null-homotopic principal orbits. Denote by N the non-trivial S^{n-2}-bundle over S^2. Then the following holds:*

1. *One of the manifolds $M\#(S^2 \times S^{n-2})$ or $M\#N$ admits an S^1-action with an invariant metric of positive scalar curvature.*
2. *$M\#N\#(S^2\times S^{n-2})$ admits an S^1-action with an invariant metric of positive scalar curvature.*

Proof. Let M' be the result of an equivariant surgery on a principal orbit in M. Then the S^1-action on M' has a fixed point component of codimension two. Therefore M' has an invariant metric of positive scalar curvature. Since the principal orbits in M are null-homotopic, M' is diffeomorphic to $M\#(S^2 \times S^{n-2})$ or $M\#N$. Therefore the first claim follows.

The structure group of $N \to S^2$ reduces to S^1. Let S^1 act on S^2 by rotation. Then, by results of Hattori and Yoshida [HY76], the S^1-action on S^2 lifts to an action on N such that the action on the fiber over one of the fixed points in S^2 is trivial. Therefore the lifted action on N has a fixed point component of codimension two. A similar statement holds for an S^1-action on $S^2 \times S^{n-2}$.

Then we can form the connected sum of M' and N (and also of M' and $S^2 \times S^{n-2}$) in an equivariant way. Therefore $M'\#N$ and $M'\#(S^2 \times S^{n-2})$ admit an S^1-action with an invariant metrics of positive scalar curvature. This proves the second claim. □

Since the principal orbits of an S^1-action are always null-homotopic if the S^1-action has fixed points, we get the following corollary.

Corollary 3.5 *Let M be a closed connected manifold of dimension $n \geq 5$ with non-zero Euler-characteristic such that the universal cover of M is not spin. If M does not admit a metric of positive scalar curvature, then there is no non-trivial S^1-action on M.*

Proof. Since $\chi(M) \neq 0$ every S^1-action on M must have fixed points. Therefore it follows from Theorem 3.3 that there is no non-trivial S^1-action on M. □

It follows from Corollary 3.5, that the manifold X from the example after Theorem 3.3 does not admit any non-trivial circle action.

It was an idea of Bernhard Hanke to combine Corollary 3.5 with ideas of Schick to construct new obstructions to S^1-actions on manifolds with non-spin universal cover of dimension greater than four. In the remainder of this section we describe what grew out of this idea.

For these manifolds, there is only one known obstruction to a metric of positive scalar curvature, namely the minimal hypersurface method of Schoen and Yau [SY79]. Using this method the following theorem was proved by Joachim and Schick.

Theorem 3.6 ([JS00]) *Let G be a discrete group and $[M \to BG] \in \Omega_n^{SO}(BG)$ with $2 \leq n \leq 8$ and M connected. If there are $\alpha_1, \dots, \alpha_{n-2} \in H^1(BG;\mathbb{Z})$ such that*

$$\alpha_1 \cap (\cdots \cap (\alpha_{n-2} \cap [M]) \dots) \neq 0 \in \Omega_2^{SO}(BG),$$

then M does not admit a metric of positive scalar curvature.

Here, the map

$$\cap : H^1(BG;\mathbb{Z}) \times \Omega_n^{SO}(BG) \to \Omega_{n-1}^{SO}(BG), \quad (\alpha,[M]) \mapsto \alpha \cap [M]$$

is defined as follows. Represent an element $\alpha \in H^1(BG;\mathbb{Z})$ by a map $f : BG \to S^1$ and let $[\phi : M \to BG] \in \Omega_n^{SO}(BG)$. Let $\psi : M \to S^1$ be a differentiable map homotopic to $f \circ \phi$ and $x \in S^1$ a regular value of ψ. Then $\alpha \cap [M]$ is represented by the restriction of ϕ to the hypersurface $\psi^{-1}(x) \subset M$.

So, by combining Theorem 3.3 and Theorem 3.6, one gets the following Theorem 3.7 for certain manifolds of dimension greater than four and smaller than nine. But there is also a direct proof for this theorem which does not use the scalar curvature of metrics on the manifolds involved. We give this proof below.

Theorem 3.7 *Let G be a discrete group and $[\phi : M \to BG] \in \Omega_n^{SO}(BG)$ with $n \geq 2$ and M connected. If there are $\alpha_1, \dots, \alpha_{n-2} \in H^1(BG;\mathbb{Z})$ such that*

$$\alpha_1 \cap (\cdots \cap (\alpha_{n-2} \cap [M]) \dots) \neq 0 \in \Omega_2^{SO}(BG),$$

then M does not admit an effective S^1-action with null-homotopic principal orbits.

Proof. Assume that there is such an action on M. At first note that the map ψ from the above construction can be assumed to be S^1-equivariant with S^1 acting trivially on S^1. This is because, by [DS82, Theorem 1.11], $\pi_1(M/S^1) = \pi_1(M)/H$ where H, is generated by elements of finite order. Therefore up to homotopy every map $M \to S^1$ factors through M/S^1. Note also, that if $N \subset M$ is an invariant submanifold, then the restriction of $f \circ \phi$ to N factors up to homotopy through M/S^1 and therefore also through N/S^1.

Hence, by applying the construction $\cap$ described above several times, we get an orientable invariant two-dimensional submanifold N' of M with equivariantly trivial normal bundle. Therefore the S^1-action on N' is effective. From the classification of orientable S^1-manifolds with one-dimensional orbit space it follows that N' is equivariantly diffeomorphic to a sphere with S^1 acting by rotation or to a torus $S^1 \times S^1$ with S^1 acting by multiplication on the first factor. In the first case $[N'] = 0 \in \Omega_2^{SO}(BG)$ because $\pi_2(BG) = 0$. In the second case the map $N' = S^1 \times S^1 \to BG$ extends to a map $D^2 \times S^1 \to BG$ because the principal orbits in M are null-homotopic. Hence, N' is a boundary. Therefore the statement follows. □

It follows from the above theorem that Schick's five-dimensional counterexample [Sch98] to the Gromov–Lawson–Rosenberg Conjecture does not admit any S^1-action with fixed points. If one does Schick's construction in dimension six, then one gets a manifold with non-trivial Euler characteristic. Therefore this manifold does not admit any S^1-action.

4. Semi-free circle actions

In this section we discuss the question when a semi-free S^1-manifold without fixed point components of codimension less than four admits an invariant metric of positive scalar curvature. For this discussion we need some notations and results of Stolz [Sto].

A supergroup γ is a triple $(\pi, w, \hat{\pi})$, where π is a group, $\hat{\pi} \to \pi$ is an extension of π such that $\ker(\hat{\pi} \to \pi)$ has order less than three and $w : \pi \to \mathbb{Z}_2$ is a homomorphism. We call a supergroup discrete if π is a discrete group.

For a supergroup $\gamma = (\pi, w, \hat{\pi})$ we define the even part of γ as the group $\ker w \circ \varphi$.

For a discrete supergroup γ one defines a Lie group $G(n, \gamma)$ as the even part of the superproduct $\mathrm{Pin}(n) \hat{\times} \gamma$. There is a homomorphism $G(n, \gamma) \to O(n)$, which is surjective if $w \neq 0$ and has image $SO(n)$ if $w = 0$. In both cases $G(n, \gamma)$ is a covering of its image under this homomorphism.

A γ-structure on a vector bundle $E \to X$ equipped with an inner product and with $\dim E \geq 3$, is a reduction of structure group of E through the homomorphism $G(n, \gamma) \to O(n)$, i.e. a γ-structure is a principal $G(n, \gamma)$-bundle $P_{G(n,\gamma)}(E)$ over X together with an isomorphism of principal $O(n)$-bundles $\xi : P_{G(n,\gamma)}(E) \times_{G(n,\gamma)} O(n) \to P_{O(n)}(E)$, where $P_{O(n)}(E)$ is the orthogonal frame bundle of E. If M is a manifold, then a γ-structure on M is a γ-structure on its tangent bundle. A manifold with a γ-structure is called γ-manifold.

It can be shown that there is a natural bijection between the γ-structures on E and $E \oplus \mathbb{R}$. Moreover, in the case that $w = 0$, a γ-structure induces an orientation on E.

For each vector bundle $E \to X$ over a connected space X there is a supergroup $\gamma(E)$, which encodes the information contained in the fundamental group of X and the first two Stiefel-Whitney classes of E. It is defined as follows:

1. $\pi = \pi_1(X)$,
2. $w : \pi \to \mathbb{Z}_2$ is the orientation character of E,
3. $\hat{\pi} \to \pi$ is the extension of π induced by the projection map $P_{O(n)}(E)/\langle r \rangle \to X$, where $r \in O(n)$ is the reflection in the hyperplane perpendicular to $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$.

A vector bundle together with a choice of a base point for its orthogonal frame bundle is called a pointed vector bundle. Stolz shows that for every pointed vector bundle E there is a canonical $\gamma(E)$-structure on E. We denote this $\gamma(E)$-structure of E by $P_{\gamma(E)}(E)$.

Now let X be a S^1-space. Then we call a γ-structure on an S^1-vector bundle $E \to X$ equivariant if the S^1-action on $P_{O(n)}(E)$ lifts to an S^1-action on $P_{G(n,\gamma)}(E)$ in such a way that the S^1-action commutes with the $G(n, \gamma)$-action. If the S^1-action on X is free, then there is a natural isomorphism $E \cong p^*(E/S^1)$, where $p : X \to X/S^1$ is the orbit map. In this case we have the following lemma.

Lemma 4.1 *Let $E \to X$ be a S^1-vector bundle over a free S^1-space with $\dim E \geq 3$ and γ a discrete supergroup. Then the following two statements hold:*

1. *There is a bijection between the γ-structures on E/S^1 and the S^1-equivariant γ-structures on E, induced by pullback.*

2. *Equip $P_{O(n)}(E)$ and $P_{O(n)}(E/S^1) = P_{O(n)}(E)/S^1$ with basepoints x and S^1x. Then there are homomorphisms*

$$\gamma(E) \to \gamma(E/S^1) \qquad \text{and} \qquad G(n,\gamma(E)) \to G(n,\gamma(E/S^1))$$

and an isomorphism of $\gamma(E/S^1)$-structures

$$P_{\gamma(E)}(E) \times_{G(n,\gamma(E))} G(n,\gamma(E/S^1)) \to p^*P_{\gamma(E/S^1)}(E/S^1).$$

Proof. At first we prove (1). If $P_\gamma(E)$ is an equivariant γ-structure on E, then the isomorphism $P_\gamma(E) \times_{G(n,\gamma)} O(n) \to P_{O(n)}(E)$ induces an isomorphism

$$P_\gamma(E)/S^1 \times_{G(n,\gamma)} O(n) \to P_{O(n)}(E)/S^1 = P_{O(n)}(E/S^1).$$

Hence, $P_\gamma(E)/S^1$ is a γ-structure for E/S^1.

Otherwise, if $P_\gamma(E/S^1)$ is a γ-structure for E/S^1, then the isomorphism

$$P_\gamma(E/S^1) \times_{G(n,\gamma)} O(n) \to P_{O(n)}(E/S^1),$$

induces an S^1-equivariant isomorphism $p^*P_\gamma(E/S^1) \times_{G(n,\gamma)} O(n) \to p^*P_{O(n)}(E/S^1)$. Therefore $p^*P_\gamma(E/S^1)$ is an equivariant γ-structure for $p^*(E/S^1)$.

Since, for an equivariant γ-structure $P_\gamma(E)$, $p^*(P_\gamma(E)/S^1)$ is naturally isomorphic to $P_\gamma(E)$ and $p^*(E/S^1)$ is naturally isomorphic to E, these two operations are inverse to each other. Hence, (1) is proved.

Now we prove (2). The existence of the homomorphisms $\gamma(E) \to \gamma(E/S^1)$ and $G(n,\gamma(E)) \to G(n,\gamma(E/S^1))$ follows from the definitions of $\gamma(E)$ and $G(n,\gamma(E))$ and the choices of the basepoints. Therefore we only have to prove the existence of the isomorphism of the $\gamma(E/S^1)$-structures.

Such an isomorphism exists if and only if there is a $G(n,\gamma(E))$-equivariant map ϕ such that the following diagram commutes

$$\begin{tikzcd}
P_{\gamma(E)}(E) \arrow[r] \arrow[d, "\phi"'] & P_{\gamma(E)}(E) \times_{G(n,\gamma(E))} O(n) \arrow[r, "\cong"] & P_{O(n)}(E) \\
p^*P_{\gamma(E/S^1)}(E/S^1) \arrow[r] & p^*P_{\gamma(E)}(E/S^1) \times_{G(n,\gamma(E/S^1))} O(n) \arrow[r, "\cong"] & p^*P_{O(n)}(E/S^1) \arrow[u, "\cong"']
\end{tikzcd}$$

Now, by the proof of [Sto, Proposition 2.12], $P_{\gamma(E)}(E)$ can be identified with

$$\{a : [0,1] \to P_{O(n)}(E);\ a(0) = x\}/\sim .$$

Here two paths are identified if they are homotopic relative endpoints. Moreover, the map $P_{\gamma(E)}(E) \to P_{O(n)}(E)$ is given by $[a] \mapsto a(1)$.

Clearly, this map factors through

$$\begin{aligned}
\phi : P_{\gamma(E)}(E) &\to p^*P_{\gamma(E/S^1)}(E/S^1) = \{(y,e) \in X \times E/S^1;\ p(y) = \psi'(e)\} \\
[a] &\mapsto (\psi(a(1)), [\bar{p} \circ a])
\end{aligned}$$

Here $\psi : P_{O(n)}(E) \to X$ and $\psi' : P_{O(n)}(E/S^1) \to X/S^1$ denote the bundle projections and

$$\bar{p} : P_{O(n)}(E) \to P_{O(n)}(E)/S^1 = P_{O(n)}(E/S^1)$$

is the orbit map. It follows from the description of the $G(n, \gamma(E))$-action on $P_{\gamma(E)}(E)$ given in the proof of [Sto, Proposition 2.12] that ϕ is $G(n, \gamma(E))$-equivariant. Hence, the lemma is proved. □

Construction 4.2 *If M is a connected free S^1-manifold, then there is an isomorphism $TM \to p^*(T(M/S^1) \oplus \mathbb{R})$. Hence, by Lemma 4.1, one gets a equivariant $\gamma(M/S^1)$-structure on M from the canonical $\gamma(M/S^1)$-structure on M/S^1.*

We want to extend this construction to connected semi-free S^1-manifolds M with $\operatorname{codim} M^{S^1} \geq 4$. *In this case we let $\gamma(M/S^1) = \gamma((M - M^{S^1})/S^1)$. By the assumption on the codimension of the fixed point set, the inclusion $M - M^{S^1} \to M$ is three-connected.*

In particular, $\pi_i(M - M^{S^1}) \cong \pi_i(M)$, for $i = 1, 2$. Hence, it follows from comparing the exact homotopy sequences for the fibrations $P_{O(n)}(T(M - M^{S^1}))/\langle r \rangle \to M - M^{S^1}$ and $P_{O(n)}(TM)/\langle r \rangle \to M$ that the group extensions $\pi_1(P_{O(n)}(T(M - M^{S^1}))/\langle r \rangle) \to \pi_1(M - M^{S^1})$ and $\pi_1(P_{O(n)}(TM)/\langle r \rangle) \to \pi_1(M)$ are isomorphic. Moreover, since $M - M^{S^1} \subset M$ is open it follows that the orientation character of $M - M^{S^1}$ factors through the orientation character of M. Therefore we have $\gamma(M) = \gamma(M - M^{S^1})$.

By the case of a free action, we have a S^1-equivariant $\gamma(M/S^1)$-structure on $M - M^{S^1}$. By part (2) of Lemma 4.1, this $\gamma(M/S^1)$-structure extends to an $\gamma(M/S^1)$-structure $P_{\gamma(M/S^1)}$ on all of M. We will show that the S^1-action on M lifts into this $\gamma(M/S^1)$-structure on M. Since $\gamma(M/S^1)$ is discrete, $P_{\gamma(M/S^1)}$ is a covering of $P_{O(n)}(TM)$ if $w \neq 0$ or $P_{SO(n)}(TM)$ if $w = 0$. Therefore the S^1-action on $P_{O(n)}(TM)$ or on $P_{SO(n)}(TM)$ induces an $\mathbb{R}$-action on $P_{\gamma(M/S^1)}$. By construction, the restriction of this $\mathbb{R}$-action to $P_{\gamma(M/S^1)}|_{M-M^{S^1}}$ factors through S^1. Since $M - M^{S^1}$ is dense in M, this also holds for the $\mathbb{R}$-action on $P_{\gamma(M/S^1)}$. Hence, the S^1-action lifts into $P_{\gamma(M/S^1)}$.

So on every connected semi-free S^1-manifold M with $\operatorname{codim} M^{S^1} \geq 4$ *there is a preferred equivariant $\gamma(M/S^1)$-structure.*

We need one more definition from [Sto].

Definition 4.3 *For $n \geq 0$ and a discrete supergroup γ let $R_n(\gamma)$ be the bordism group of n-dimensional γ-manifolds with positive scalar curvature metrics on their boundary, i.e. the objects of this bordism groups are pairs (M, h) where M is an n-dimensional γ-manifold possibly with boundary and h a metric of positive scalar curvature on ∂M. Two pairs (M, h) and (M', h') are identified if*

1. *there is an n-dimensional oriented manifold V with $\partial V = -\partial M \amalg \partial M'$ and a positive scalar curvature metric on V which restricts to h and h' on the boundary respectively, and*
2. *there is an $(n+1)$-dimensional γ-manifold W with $\partial W = M \cup_{\partial M} V \cup_{\partial M'} -M'$.*

In the above definition and the following we assume that all metrics on a manifold with boundary are product metrics near the boundary.

We also define equivariant bordism groups.

Definition 4.4 *For $n \geq 0$ and a discrete supergroup γ let $\Omega^{SF}_{n,\geq 4}(\gamma)$ be the bordism groups of closed n-dimensional semi-free S^1-manifolds equipped with equivariant γ-structures and without fixed point components of codimension less than four. Here we identify two manifolds M_1 and M_2 if there is a semi-free S^1-manifold with boundary, equivariant γ-structure and without fixed point components of codimension less than four such that $\partial W = M_1 \amalg -M_2$.*

Now we want to define a homomorphism $\phi : \Omega^{SF}_{n,\geq 4}(\gamma) \to R_{n-1}(\gamma)$.

Let M be a S^1-manifold as in Definition 4.4. Then we can construct from M an $(n-1)$-dimensional manifold N with boundary by removing from M an open tubular neighborhood of the fixed point set M^{S^1} and taking the quotient of the free S^1-action on this complement. The boundary of this quotient is a disjoint union of $\mathbb{C}P^k$-bundles, $k \geq 1$, over the components of M^{S^1}. There is a natural γ-structure on this quotient because $(TM|_{M-M^{S^1}})/S^1 = T((M - M^{S^1})/S^1) \oplus \mathbb{R}$.

We may choose a metric on M^{S^1} and a connection for the $\mathbb{C}P^k$-bundles over M^{S^1}. With these choices one can construct a connection metric h on ∂N with positive scalar curvature, such that the fibers of the bundle are up to scaling with a constant isometric to $\mathbb{C}P^k$ with standard metric. Note that the isotopy class of h does not depend on the above choices. Therefore (N, h) defines a well defined element of $R_{n-1}(\gamma)$.

If W is an equivariant bordism without fixed point components of codimension less than four between M and another S^1-manifold M', then the quotient $\tilde{W}$ of the free S^1-action on the complement of an open tubular neighborhood of the fixed point set in W gives a bordism between N and N'. The part of the boundary of $\tilde{W}$ which does not belong to N or N' can be equipped with a metric of positive scalar curvature by the same argument as above. Therefore the bordism class of (N, h) depends only on the bordism class of M. Hence, we get the desired homomorphism $\phi : \Omega^{SF}_{n,\geq 4}(\gamma) \to R_{n-1}(\gamma)$.

It has been shown by Stolz [Sto] that a connected manifold M of dimension $n \geq 5$ with boundary and a given metric h of positive scalar curvature on ∂M admits a metric of positive scalar curvature, which extends h and is a product metric near the boundary, if and only if (M, h) equipped with the canonical $\gamma(M)$-structure represents zero in $R_n(\gamma(M))$.

From this fact we get the following theorem.

Theorem 4.5 *Let M be closed connected semi-free S^1-manifold M of dimension $n \geq 6$ and* $\operatorname{codim} M^{S^1} \geq 4$. *We equip M with the equivariant $\gamma(M/S^1)$-structure from Construction 4.2. Then M admits an invariant metric of positive scalar curvature if and only if the class $[M] \in \Omega^{SF}_{n,\geq 4}(\gamma(M/S^1))$ can be represented by an S^1-manifold N which admits an invariant metric of positive scalar curvature.*

Proof. We will show that M admits an invariant metric of positive scalar curvature if and only if $\phi([M]) = 0 \in R_{n-1}(\gamma(M/S^1))$. From this the statement follows.

At first assume that M admits an invariant metric of positive scalar curvature. By the proof of Gromov's and Lawson's surgery theorem [GL80] (see also [Gaj87]), this is the case if and only if it admits an invariant metric of positive scalar curvature which is a connection metric on a tubular neighborhood N of M^{S^1}. Here the metrics on the fibers of $N \to M^{S^1}$ are so called "torpedo metrics".

Therefore after removing a small open tubular neighborhood N' of M^{S^1}, the metric on $M - N'$ is a product metric near the boundary. Its restriction to the boundary is a connection metric with fibers isometric to a round sphere. Hence, the volume of the S^1-orbits is constant in a small neighborhood of the boundary. Therefore, by Theorem 2.2 and the definition of ϕ, we have $\phi([M]) = 0 \in R_{n-1}(\gamma(M/S^1))$.

Now assume that $\phi([M]) = 0$. Then, by the result of Stolz mentioned above, there is a metric of positive scalar curvature on $(M - N')/S^1$ whose restriction to the boundary is a connection metric with fibers isometric to $\mathbb{C}P^k$, $k > 1$. By Theorem 2.2, there is an invariant metric on $M - N'$ whose restriction to the boundary is a connection metric for the bundle $\partial(M-N') \to \partial(M-N')/S^1$. Since the fibers of $\partial(M-N')/S^1 \to M^{S^1}$ are (up to scaling) isometric to $\mathbb{C}P^k$ with standard metric, the fibers of $\partial(M - N') \to M^{S^1}$ are isometric to spheres (S^{2k+1}, g), where the metric g can be constructed from the standard round metric by shrinking the orbits of the free linear S^1-action on S^{2k+1}. Moreover, the metric on $\partial(M-N')$ is a connection metric for the bundle $\partial(M-N') \to M^{S^1}$. Therefore the metric on $\partial(M-N')$ is isotopic to a connection metric with fibers isometric to round spheres. Hence, we can glue in the normal disc bundle of $M^{S^1} \subset M$ equipped with an appropriate metric to get an invariant metric of positive scalar curvature on M. $\square$

Now we want to discuss the special case, where M is a simply connected semi-free S^1-manifold with $\operatorname{codim} M^{S^1} \geq 4$. In this case M/S^1 and $(M - M^{S^1})/S^1$ are also simply connected (see [Bre72, Corollary 6.3, p. 91]). Hence, a $\gamma(M/S^1)$-structure on M/S^1 is a Spin-structure on $(M - M^{S^1})/S^1$ if $(M - M^{S^1})/S^1$ admits a Spin-structure or an orientation on $(M - M^{S^1})/S^1$, otherwise.

Note that $(M - M^{S^1})/S^1$ admits a Spin-structure if and only if M admits a Spin-structure and the S^1-action on M is of even type.

Denote by $\Omega_n^{SO,SF}$ the bordism group of closed oriented semi-free S^1-manifolds and by $\Omega_n^{\mathrm{Spin,even},SF}$ the bordism group of closed semi-free S^1-manifolds with Spin-structure and an S^1-action of even type. Then we have:

Theorem 4.6 *Let M be a closed simply connected semi-free oriented S^1-manifold M of dimension $n \geq 6$ and $\operatorname{codim} M^{S^1} \geq 4$. If M is not spin or spin with the S^1-action of odd type, then M admits an invariant metric of positive scalar curvature if and only if the class $[M] \in \Omega_n^{SO,SF}$ can be represented by an S^1-manifold N with $\operatorname{codim} N^{S^1} \geq 4$ and an invariant metric of positive scalar curvature. If M is spin and the S^1-action is of even type then the same holds with the oriented equivariant bordism ring replaced by the spin equivariant bordism ring.*

Proof. At first assume that M is spin and the S^1-action on M is of even type. Since on a Spin-manifold with even semi-free S^1-action there are no fixed point components of codimension two, we have $\Omega^{SF}_{n,\geq 4}(\gamma(M/S^1)) = \Omega_n^{\mathrm{Spin,even},SF}$. Therefore the theorem follows from Theorem 4.5 in this case.

Next assume that we are in the other case. Then $\Omega_n^{SF}(\gamma(M/S^1))$ is just the bordism group $\Omega^{SO,SF}_{n,\geq 4}$ of closed oriented semi-free S^1-manifolds without fixed point components of codimension less than four. By the proof of Theorem 4.5, M admits an invariant metric of positive scalar curvature if and only if $\phi([M]) = 0$. Therefore it is sufficient to show that $\phi([M])$ depends only on the image of $[M]$ under the natural map $\Omega^{SO,SF}_{n,\geq 4} \to \Omega_n^{SO,SF}$. This can be shown as follows.

Let W be a bordism between the semi-free S^1-manifolds M_1 and M_2 such that all fixed point components of W of codimension two do not meet the boundary. Then we can cut out these components to get a bordism without fixed point components of codimension two between M_1 and $M_2 \amalg N_1 \amalg \cdots \amalg N_k$, where the N_i are free S^1-manifolds. The claim follows if we show that $\phi([N_i])$ vanishes for all i. The orbit spaces of the N_i are closed manifolds. Moreover, every class in Ω^{SO}_{n-1} can be represented by a manifold which admits a metric of positive scalar curvature (see [GL80, Proof of Theorem C]). Therefore it follows from the definition of the groups $R_{n-1}(\gamma)$ that $\phi([N_i]) = [N_i/S^1] = 0$. $\square$

4.1. The non-spin case

Our next goal is to prove the following theorem.

Theorem 4.7 *Let M be a closed simply connected semi-free S^1-manifold of dimension $n > 5$. If M is not spin or spin and the S^1-action is odd, then the equivariant connected sum of two copies of M admits an invariant metric of positive scalar curvature.*

The proof of this theorem is divided into two cases:

1. M has a fixed point component of codimension two.
2. All fixed point components of M have codimension at least four.

In the first case, the theorem follows from Theorem 2.4. In the second case, the idea for the proof of Theorem 4.7 is to show that the class of $2M$ in $\Omega_*^{SO,SF}$ can be represented by a manifold which admits an invariant metric of positive scalar curvature and does not have fixed point components of codimension less than four. Then it follows from Theorem 4.6.

We prepare the proof by describing some structure results about the ring $\Omega_*^{SO,SF}$. For background information on these results see for example [CF64], [May96, Chapter XV] or [Uch70]. At first there is an exact sequence

$$0 \longrightarrow \Omega_*^{SO,SF} \xrightarrow{\lambda} F_* \xrightarrow{\mu} \Omega^{SO}_{*-2}(BU(1)) \longrightarrow 0.$$

Here, $F_* = \bigoplus_{n\geq 0} \Omega^{SO}_{*-2n}(BU(n))$ is the bordism ring of complex vector bundles over some base spaces. The map λ sends a semi-free S^1-manifold M to the normal bundle of its fixed point set. Note that this bundle is naturally isomorphic to a complex S^1-vector bundle over M^{S^1} of the form $V \otimes \rho$, where V is a complex vector bundle over M^{S^1} with trivial S^1-action and ρ is the standard one-dimensional complex S^1-representation. Therefore F_* might we viewed as the bordism group of fixed point data of semi-free oriented S^1-manifolds.

In this picture the summand F_0 is isomorphic to the subgroup of $\Omega_*^{SO,SF}$ consisting of bordism classes of manifolds with trivial S^1-action. An isomorphism is induced by λ.

The map μ can be described as follows. The restriction of the multiplication with elements of $S^1 \subset \mathbb{C}$ on a complex vector bundle to its sphere bundle defines a free S^1-action. So we get a map from F_* to the bordism group of free S^1-manifolds. Since every free S^1-manifold is a principal S^1-bundle over its orbit space, we may identify the bordism group of free S^1-actions with $\Omega_*^{SO}(BU(1))$. The map μ is the composition of the above map and this identification.

$\Omega_*^{SO,SF}$ and F_* are actually algebras over Ω_*^{SO}, with multiplication given by direct products. Hence, λ is an algebra homomorphism. Moreover, F_* is isomorphic to $\Omega_*^{SO}[X_i; i \geq 0]$, where X_i can be identified with the dual Hopf bundle over the i-dimensional complex projective space. Note that, by our grading of F_*, X_i has degree $2i+2$.

We denote by $\mathbb{C}P^n(\rho)$ the n-dimensional complex projective space equipped with the S^1-action induced by the representation $\rho \oplus \mathbb{C}^n$. Here S^1 acts trivially on the $\mathbb{C}^n$ summands. Then we have

$$\lambda(\mathbb{C}P^n(\rho)) = X_{n-1} + (-1)^n X_0^n.$$

Hence, F_* is isomorphic to $\Omega_*^{SO}[X_0, \lambda(\mathbb{C}P^n(\rho)), n \geq 2]$.

For the proof of Theorem 4.7 we need the following two lemmas. To motivated our first lemma, consider the situation where M is a $2n$-dimensional semi-free S^1-manifold with isolated fixed points. Then the normal bundle of the fixed point set in M is trivial. Moreover, $\lambda(M) = X_0^n \lambda(S)$, where S is a zero dimensional S^1-manifold with trivial action. By [Kaw91, Corollar 6.24], the signature of M vanishes. Therefore, by [Kaw91, Corollary 6.23], S represents zero in Ω_0^{SO}.

The following lemma is a generalization of this fact to semi-free S^1-manifolds M such that the normal bundle of the fixed point set M^{S^1} has a section which is nowhere zero.

Lemma 4.8 *Let $L \in F_*$. If $X_0 L \in \ker\mu = \Omega_*^{SO,SF}$, then the same holds for L. In other words, L is the fixed point data of some semi-free S^1-manifold $\tilde{L}$. Moreover, $\tilde{L}$ is mapped to zero by the forgetful map $\Omega_*^{SO,SF} \to \Omega_*^{SO}$.*

Proof. By Theorem 17.5 of [CF64, p. 49], a class $[\phi : L' \to BU(1)] \in \Omega_*^{SO}(BU(1))$ represents zero if and only if its Stiefel-Whitney and Pontrjagin numbers vanish, i.e. the characteristic numbers of the form

$$\langle \phi^*(x)^l p_I(L'), [L'] \rangle \qquad \text{and} \qquad \langle \phi^*(x)^l w_I(L'), [L'] \rangle$$

vanish, where x is a generator of $H^2(BU(1))$ and $p_I(L')$ and $w_I(L')$ are products of the Pontrjagin classes and Stiefel–Whitney classes of L', respectively. Now note that $\mu(L)$ is represented by a sum of tautological bundles over the projectivizations $P(E_i)$ of complex vector bundles E_i. The class $\mu(X_0L)$ is represented by the sum of tautological bundles over the projectivizations $P(E_i \oplus \mathbb{C})$ of the sum $E_i \oplus \mathbb{C}$ of these complex vector bundles with a trivial line bundle.

Let E be a complex vector bundle of dimension n over a connected manifold B. Then for $K = \mathbb{Z}_2$ or $K = \mathbb{Q}$, we have

$$H^*(P(E); K) = H^*(B; K)[x]/(\sum_{i=0}^{n} c_i(E)x^{n-i})$$

and

$$H^*(P(E \oplus \mathbb{C}); K) = H^*(B; K)[x]/((\sum_{i=0}^{n} c_i(E)x^{n-i})x),$$

where x has degree two and is minus the first Chern-class of the tautological bundle over $P(E)$ and $P(E \oplus \mathbb{C})$.

Let y be a generator of the top cohomology group of B. Then we can orient $P(E)$ and $P(E \oplus \mathbb{C})$ in such a way that

$$\langle yx^{n-1}, [P(E)]\rangle = 1 = \langle yx^n, [P(E \oplus \mathbb{C})]\rangle.$$

Hence, if $f(x)$ is a power series with coefficients in $H^*(B; K)$, then we have

$$\langle f(x), [P(E)]\rangle = \langle xf(x), [P(E \oplus \mathbb{C})]\rangle.$$

The total Pontrjagin and Stiefel-Whitney classes of $P(E)$ are given by

$$p(P(E)) = p(B)\prod_i (1 + (a_i + x)^2), \qquad w(P(E)) = w(B)\prod_i (1 + a_i + x),$$

where the a_i are the formal roots of the Chern classes of E.

Therefore there are the following relations between the total Pontrjagin-classes and Stiefel-Whitney-classes of $P(E)$ and $P(E \oplus \mathbb{C})$ (both viewed as power series in x with coefficients in $H^*(B; K)$)

$$p(P(E \oplus \mathbb{C})) = p(P(E))(1 + x^2), \qquad w(P(E \oplus \mathbb{C})) = w(E)(1 + x).$$

This implies

$$p(P(E)) = p(P(E \oplus \mathbb{C}))(\sum_{i=0}^{\infty}(-x^2)^i), \qquad w(P(E)) = w(E \oplus \mathbb{C})(\sum_{i=0}^{\infty}(-x)^i).$$

For a power series $f(x)$ and a finite sequence I of positive integers let $f_I = \prod_{i \in I} f_i$, where f_i denotes the degree i part of f. With this notation we have

$$\begin{aligned}
\langle x^l p_I(P(E)), [P(E)]\rangle &= \langle x^{l+1} p_I(P(E)), [P(E \oplus \mathbb{C})]\rangle \\
&= \langle x^{l+1}(p(P(E \oplus \mathbb{C}))(\sum_{i=0}^{\infty}(-x^2)^i))_I, [P(E \oplus \mathbb{C})]\rangle \\
&= \sum_J a_J \langle x^{l+1+d_J} p_J(P(E \oplus \mathbb{C})), [P(E \oplus \mathbb{C})]\rangle,
\end{aligned}$$

where $a_J, d_J \in \mathbb{Z}$ only depend on I but not on E. A similar calculation shows

$$\langle x^l w_I(P(E)), [P(E)]\rangle = \sum_J b_J \langle x^{l+1+d'_J} w_J(P(E\oplus\mathbb{C})), [P(E\oplus\mathbb{C})]\rangle.$$

Since $\mu(L)$ and $\mu(X_0L)$ are linear combinations of some projectivizations of complex vector bundles, it follows that $\mu(L)=0$ if $\mu(X_0L)=0$. This proves that there is a $\tilde{L}$ with $L=\lambda(\tilde{L})$.

Now let E be the principal S^1-bundle associated to the tautological bundle over $\mathbb{C}P^1(\rho)$. Then the S^1-action on $\mathbb{C}P^1(\rho)$ lifts into E in such a way that the action on the fiber over one of the fixed points in $\mathbb{C}P^1(\rho)$ is trivial and multiplication on the fiber over the other fixed point. This action induces an S^1-action on $E\times_{S^1}\tilde{L}$, for this action we have

$$\lambda(E\times_{S^1}\tilde{L}) = X_0L - X_0\tilde{L}',$$

where $\tilde{L}'$ is $\tilde{L}$ with trivial S^1-action. Therefore, we have

$$\mu(X_0)\tilde{L}' = \mu(X_0L) - \mu(\lambda(E\times_{S^1}\tilde{L})) = 0.$$

But $\mu(X_0)$ is part of a Ω_*^{SO}-basis of $\Omega_*^{SO}(BU(1))$. Hence $\tilde{L}'$ represents zero in Ω_*^{SO} and the lemma is proved. □

On $\Omega_*^{SO,SF}$ there is an involution ι which sends a semi-free S^1-manifold M to itself equipped with the inverse S^1-action. Similarly there is an involution on F_*, which sends a complex vector bundle to its dual and changes the orientation of the base space if the fiber dimension is odd. Since λ is compatible with these two involutions, we denote the involution on F_* also by ι.

Lemma 4.9 *Let $[M]\in\Omega_*^{SO,SF}$. Then we have*

$$\iota([M]) = \begin{cases} [M] & \text{if } \dim M \equiv 0 \mod 4 \\ -[M] & \text{if } \dim M \equiv 2 \mod 4. \end{cases}$$

Proof. At first note that there is an equivariant diffeomorphism between $\mathbb{C}P^n(\rho)$ and $\iota(\mathbb{C}P^n(\rho))$ given by complex conjugation. It is orientation preserving if and only if n is even. Moreover, $\iota(X_0) = -X_0$.

As noted before we can write $\lambda([M])$ as a linear combination of products

$$X_0^k\lambda(\prod_i \mathbb{C}P^{n_i}(\rho)) \times \beta = P\times\beta$$

with $\beta\in\Omega_*^{SO}$. By the above remark we have $\iota(P\times\beta) = (-1)^{k+l}P\times\beta$, where l is the number of odd n_i appearing in the product. If $\dim\beta \not\equiv 0 \mod 4$, then β is of order two. Therefore we have $\iota(P\times\beta) = -P\times\beta = P\times\beta$ in this case.

If $\dim\beta\equiv 0 \mod 4$, we have $\dim M - 2(k+l) \equiv \dim\beta \equiv 0 \mod 4$. Therefore the statement follows. □

The following construction provides examples of semi-free S^1-manifolds with invariant metrics of positive scalar curvature and without fixed point components of dimension less than four.

Construction 4.10 *Let γ be a complex line bundle over an oriented manifold M_0. Let also M be an oriented $S^1 \times S^1$-manifold such that the S^1-actions induced by the inclusions of the S^1-factors in $S^1 \times S^1$ are semi-free. Denote M equipped with the first (resp. second) of these actions by M_1 (resp. M_2). Then the multiplication on γ induces an S^1-action on the projectivization $P(\gamma \oplus \mathbb{C})$.*

This action can be lifted in two different ways in the tautological bundle

$$\gamma'' = \{(x, v) \in P(\gamma \oplus \mathbb{C}) \times (\gamma \oplus \mathbb{C});\ v \in x\}.$$

The first action is given by $g(x, v) = (g \cdot x, g \cdot v)$, for $g \in S^1$ and $(x, v) \in \gamma''$. Here $\cdot$ denotes the S^1-action induced by multiplication on γ.

The second action is given by $g(x, v) = (g \cdot x, g \cdot (\lambda(g)v))$. Here $\cdot$ is a defined as above and $\lambda(g)v$ is given by complex multiplication of $v \in \gamma \oplus \mathbb{C}$ with $\lambda(g) = g^{-1}$.

By dualizing these two actions we get two S^1-actions on the dual γ' of γ''.

Let E be the principal $S^1 \times S^1$-bundle associated to $\gamma' \oplus \gamma'$. Then from the two S^1-actions on γ', we see that the S^1-action on $P(\gamma \oplus \mathbb{C})$ lifts into E in such a way that the weights of the restriction of the lifted S^1-action to a fiber over the two fixed point components in $P(\gamma \oplus \mathbb{C})$ are given by $(1, 0)$ and $(0, -1)$, respectively.

The S^1-action on E induces a semi-free S^1-action on $\Gamma(\gamma, M) = E \times_{S^1 \times S^1} M$ and we have

$$\lambda(\Gamma(\gamma, M)) = [\gamma]\lambda(M_1) - [\gamma]\lambda(\iota(M_2)),$$

where $[\gamma] \in \Omega_*^{SO}(BU(1))$ is represented by the bundle γ. If the S^1-action on all components of M_i, $i = 1, 2$, is non-trivial, then $\Gamma(\gamma, M)$ does not have any fixed point components of codimension less than four.

Moreover, $\Gamma(\gamma, M)$ admits an invariant metric of positive scalar curvature. This can be seen as follows. Since there is an invariant metric of positive scalar curvature on the fibers $\mathbb{C}P^1(\rho)$ of $P(\gamma \oplus \mathbb{C})$, it follows that there is an S^1-invariant connection metric of positive scalar curvature on $P(\gamma \oplus \mathbb{C})$. Therefore it follows from Theorem 2.2 that E admits an $S^1 \times S^1 \times S^1$-invariant metric of positive scalar curvature. Hence, $E \times M$ admits an $S^1 \times S^1 \times S^1$-invariant metric of positive scalar curvature. Now it follows, again from Theorem 2.2, that $\Gamma(\gamma, M) = E \times_{S^1 \times S^1} M$ admits an S^1-invariant metric of positive scalar curvature.

Let γ and M_0 as in the above construction and M a semi-free S^1-manifold. We equip M with a $S^1 \times S^1$-action induced by the homomorphism $S^1 \times S^1 \to S^1$, $(z_1, z_2) \mapsto z_1 z_2$ and define $\Delta(\gamma, M) = \Gamma(\gamma, M)$. Then we have

$$\lambda(\Delta(\gamma, M)) = [\gamma]\lambda(M) - [\gamma]\lambda(\iota(M)).$$

Now we can prove Theorem 4.7.

Proof of Theorem 4.7. Let M be a semi-free S^1-manifold. By Theorem 2.4, we may assume that M does not have fixed point components of codimension less than four. Denote by n the dimension of M and assume $n \geq 6$.

If $n \equiv 1, 3 \mod 4$, then the class $[M] \in \Omega_*^{SO,SF}$ is of order two, because all torsion elements of Ω_*^{SO} have order two, Ω_*^{SO}/torsion is concentrated in degrees divisible by four and the generators of F_* all have even degrees. Therefore, by Theorem 4.6, the theorem follows in this case.

Next assume that $n \equiv 0 \mod 4$. Then there are an L in the augmentation ideal of $F_* = \Omega_*^{SO}[X_0, \lambda(\mathbb{C}P^n(\rho)); n \geq 2]$ and $\beta_J \in \Omega_*^{SO}$ such that

$$\lambda(M) = \sum_J \beta_J \lambda(\prod_{i \in J} \mathbb{C}P^i(\rho)) + X_0 L.$$

Here, the sum is taken over all finite sequences J with at least two elements, i.e. the products $\prod_{i \in J} \mathbb{C}P^i(\rho)$ consist out of at least two factors. Since all these products admit invariant metrics of positive scalar curvature, by Theorem 4.6, we only have to deal with the case $\lambda(M) = X_0 L$. It follows from Lemma 4.8 that there is a S^1-manifold $\tilde{L}$ of dimension $n-2$ with $\lambda(\tilde{L}) = L$. Since L is contained in the augmentation ideal of F_*, we may assume that $\tilde{L}$ does not have any components with trivial S^1-action. Hence, by Lemma 4.9, we have $2\lambda(M) = \lambda(\Delta(X_0, \tilde{L}))$. Therefore the statement follows from Theorem 4.6.

Now assume that $n \equiv 2 \mod 4$. Then as in the previous case, me may assume that

$$\lambda(M) = X_0 \lambda(\tilde{L}),$$

where $\tilde{L}$ is a semi-free S^1-manifold without components on which S^1 acts trivially. By a similar argument as above, we have

$$\lambda(\tilde{L}) = \sum_J \beta_J \lambda(\prod_{i \in J} \mathbb{C}P^i(\rho)) + X_0 \lambda(L')$$

with $L' \in \Omega_{n-4}^{SO,SF}$, $\beta_J \in \Omega_*^{SO}$. Here the sum is taken over all finite sequences J with at least one element. At first we show that we may assume $L' = 0$. Let L'' be the union of those components of L' on which the S^1-action is non-trivial. Then we have, for

$$N = 2\mathbb{C}P^2(\rho) \times L'' - \Delta(X_1, L''),$$

$\lambda(N) = 2X_0^2 \lambda(L'') = 2X_0^2 \lambda(L')$ because $L' - L''$ has order two in Ω_*^{SO}. By a similar construction, one sees that we may assume that the products $\prod_{i \in J} \mathbb{C}P^i(\rho)$ do not have factors of odd complex dimension.

The next step is to show that we may assume that all these products consist out of exactly one factor.

To see this note that one can equip $\prod_{i \in J} \mathbb{C}P^i$ with two semi-free commuting S^1-actions, namely $\prod_{i \in J} \mathbb{C}P^i(\rho)$ and $\prod_{i \in J - \{i_0\}} \mathbb{C}P^i \times \mathbb{C}P^{i_0}(\rho)$. Therefore, by Lemma 4.9, we have

$$\lambda(\Gamma(X_0, \prod_{i \in J} \mathbb{C}P^i)) = X_0 \lambda(\prod_{i \in J} \mathbb{C}P^i(\rho)) - X_0 \lambda(\mathbb{C}P^{i_0}(\rho)) \times \prod_{i \in J - \{i_0\}} \mathbb{C}P^i.$$

Hence, after adding several multiples of $\Gamma(X_0, \prod_{i\in J}\mathbb{C}P^i)$, we may assume that

$$\lambda(2M) = X_0 \sum_{k=1}^{(n-2)/4} 2\beta_k \lambda(\mathbb{C}P^{2k}(\rho)),$$

with $\beta_k \in \Omega_*^{SO}$. Since all torsion elements in Ω_*^{SO} are of order two, this sum depends only on the equivalence classes of the β_k in $\Omega_*^{SO}/\text{torsion}$.

$\Omega_*^{SO}/\text{torsion}$ is a polynomial ring over $\mathbb{Z}$ with one generator Y_{4k} in each dimension divisible by four. We will construct a non-trivial semi-free S^1-action on each of the Y_{4k}. The Y_{4k} can be chosen in such a way that they admit stably complex structures, that is they belong to the image of the natural map $\Omega_*^U \to \Omega_*^{SO}$. It has been shown by Buchstaber and Ray [BR01], [BR98] that Ω_*^U is generated by certain projectivizations of sums of complex line bundles over bounded flag manifolds. These generators are toric manifolds and admit a non-trivial semi-free S^1-action induced by multiplication on one of the line bundles. Hence, we may assume that each Y_{4k} admits a non-trivial semi-free S^1-action. We denote the resulting S^1-manifold by Y'_{4k}.

Next we show that we may assume

$$\lambda(2M) = X_0 \left(\sum_{k=1}^{(n-2)/4-1} \sum_{J_k} 2a_{k,J_k} \prod_{i\in J_k} Y_{4i}\lambda(Y'_{4k}) + 2b\lambda(\mathbb{C}P^{(n-2)/2}(\rho)) \right).$$

Here the second sum is over all finite sequences J_k of elements of the set $\{1, \dots, k\}$ and $a_{k,J_k}, b \in \mathbb{Z}$.

Since each β_k, $k < (n-2)/4$, is a linear combination of products of the Y_{4k}, this can be achieved by adding to $2M$ multiples of manifolds of the form $\Gamma(X_0, Y_{4i} \times \mathbb{C}P^{2k})$ and $\Gamma(X_0, Y_{4i_1} \times Y_{4i_2})$, where $S^1 \times S^1$ acts factorwise on the product, i.e. the i-th S^1-factor acts on the i-th factor of the product.

The manifold $\mathbb{C}P^{(n-2)/2}$ is indecomposable in $\Omega_*^{SO}/\text{torsion}$. Therefore it follows from the above structure results on $\Omega_*^{SO}/\text{torsion}$ and Lemma 4.8 that $2M = 0$ in $\Omega_*^{SO,SF}$. This proves the theorem. $\square$

4.2. The spin case

In [Lot00] Lott constructed a generalized $\hat{A}$-genus for orbit spaces of semi-free even S^1-actions on Spin-manifolds. If M/S^1 has dimension divisible by 4, then $\hat{A}(M/S^1)$ is an integer. In all other dimensions it vanishes. Moreover, if M/S^1 is a manifold without boundary, i.e. the S^1-action on M is free or trivial, Lott's generalized $\hat{A}$-genus of M/S^1 coincides with the usual $\hat{A}$-genus of M/S^1.

He showed that for a semi-free even S^1-manifold M, $\hat{A}(M/S^1)$ vanishes if M admits an invariant metric of positive scalar curvature. In this subsection we prove the following partial converse to his result.

Theorem 4.11 *Let M be a closed simply connected Spin-manifold of dimension $n > 5$ with an even semi-free S^1-action. Then $\hat{A}(M/S^1) = 0$ if and only if there is a $k \in \mathbb{N}$ such that the equivariant connected sum of 2^k copies of M admits an invariant metric of positive scalar curvature.*

The strategy for the proof of Theorem 4.11 is the same as in the proof of Theorem 4.7. That means that we construct generators for the kernel of the $\hat{A}$-genus which admit invariant metrics of positive scalar curvature. To do so we first review some results about the ring $\Omega_*^{\mathrm{Spin,even},SF} \otimes \mathbb{Z}[\frac{1}{2}]$ for proofs of these results with $\mathbb{Z}[\frac{1}{2}]$ replaced by the rationals see [Bor87]. The proofs there also work in our case since $\Omega_*^{\mathrm{Spin}} \otimes \mathbb{Z}[\frac{1}{2}] \cong \Omega_*^{SO} \otimes \mathbb{Z}[\frac{1}{2}]$. There are exact sequences

$$\begin{tikzcd}
\Omega_{*-1}^{\mathrm{Spin}}(BU(1)) \otimes \mathbb{Z}[\frac{1}{2}] \arrow[r] & \Omega_*^{\mathrm{Spin,even},SF} \otimes \mathbb{Z}[\frac{1}{2}] \arrow[d, "\lambda"] & \\
& F_*^{\mathrm{even}} \otimes \mathbb{Z}[\frac{1}{2}] \arrow[r, "\mu"] & \Omega_{*-2}^{SO}(BU(1)) \otimes \mathbb{Z}[\frac{1}{2}]
\end{tikzcd}$$

and

$$0 \longrightarrow \Omega_*^{\mathrm{Spin,odd},SF} \otimes \mathbb{Z}[\tfrac{1}{2}] \xrightarrow{\lambda} F_*^{\mathrm{odd}} \otimes \mathbb{Z}[\tfrac{1}{2}] \xrightarrow{\mu} \Omega_{*-2}^{SO}(BU(1)) \otimes \mathbb{Z}[\tfrac{1}{2}] \longrightarrow 0$$

Here the first map is the natural map from the bordism group of Spin-manifolds with free even action into the bordism group of semi-free actions and

$$\begin{aligned}
F_*^{\mathrm{even}} &= \bigoplus_{k\geq 0} \Omega_{*-4k}^{SO}(BU(2k)) \subset F_*, \\
F_*^{\mathrm{odd}} &= \bigoplus_{k\geq 0} \Omega_{*-4k-2}^{SO}(BU(2k+1)) \subset F_*.
\end{aligned}$$

Moreover, the maps λ and μ are the same as in the oriented case.

At first we describe another set of generators of $F_* \otimes \mathbb{Z}[\frac{1}{2}]$ which are more suitable for the discussion of the spin case than the generators described in the previous section. Denote by $\tilde{X}_{2k+1}$, $k > 1$, the normal bundle of $P(\mathbb{C}^2 \oplus \bigoplus_{i=1}^{2k-1} \gamma\otimes\gamma)$ in $P(\mathbb{C}^2 \oplus \bigoplus_{i=1}^{2k} \gamma\otimes\gamma)$, where γ is the dual Hopf bundle over $\mathbb{C}P^1$ and $P(E)$ denotes the projectivization of the vector bundle E. Then we have

$$\langle c_1(\tilde{X}_{2k+1})^{2k+1}, [P(\mathbb{C}^2 \oplus \bigoplus_{i=1}^{2k-1} \gamma \otimes \gamma)] \rangle = 4.$$

Therefore, by [CF64, Theorem 18.1], the $\tilde{X}_{2k+1}$, $k > 0$, together with the X_{2k}, $k \geq 0$ and X_1 are a basis of $\Omega_*^{SO}(BU(1)) \otimes \mathbb{Z}[\frac{1}{2}]$ as a $\Omega_*^{SO} \otimes \mathbb{Z}[\frac{1}{2}]$-module. Hence, we have $F_* \otimes \mathbb{Z}[\frac{1}{2}] = \Omega_*^{SO}[\frac{1}{2}, X_0, X_1, X_{2k}, \tilde{X}_{2k+1}; k \geq 1]$.

For $k \geq 1$, let $M_{2k+1} = \mathbb{C}P^{2k+1}(\rho)$ and $M_{2k+2} = P(\mathbb{C}^2 \oplus \bigoplus_{i=1}^{2k} \gamma \otimes \gamma)$, where S^1 acts by multiplication on one of the $\gamma \otimes \gamma$ summands. Then we have

$$\lambda(M_{2k+2}) = \tilde{X}_{2k+1} - 4X_1X_0^{2k}.$$

Moreover, let M_0 be $\mathbb{H}P^2 = Sp(3)/(Sp(2)\times Sp(1))$ equipped with the semi-free S^1-action induced by the embedding

$$S^1 \hookrightarrow \{I\} \times Sp(1) \hookrightarrow Sp(2) \times Sp(1) \hookrightarrow Sp(3).$$

This action has two fixed point components, namely $\mathbb{H}P^1$ and an isolated fixed point. The Chern classes of the normal bundle of $\mathbb{H}P^1$ in $\mathbb{H}P^2$ are given by

$$c_1 = 0 \qquad\qquad c_2 = -u,$$

where u is a generator of $H^4(\mathbb{H}P^1; \mathbb{Z})$. Hence, it follows from a calculation of characteristic numbers (cf. Theorem 17.5 of [CF64, p. 49]) that

$$\lambda(M_0) = -X_1^2 + 2\lambda(M_3)X_0 - \frac{1}{8}[K]X_0^2 + X_0^4,$$

where K is the Kummer surface. Therefore we have

$$F_* \otimes \mathbb{Z}[\frac{1}{2}] = \Omega_*^{SO}[\frac{1}{2}, X_0, X_1, \lambda(M_k); k \geq 0]/(R - X_1^2),$$

where $R - X_1^2$ is the relation described above.

The following lemma shows that the manifolds M_i defined above are Spin-manifolds. The S^1-action on M_i is even if and only if $i = 0$.

Lemma 4.12 *Let N be a Spin-manifold with an action of a torus $T = S_1^1 \times \cdots \times S_k^1$ such that each factor S_i^1 has a fixed point in N and $f : E \to M$ be a principal T-bundle. Then we have*

$$w_2(E \times_T N) = f^*(w_2(M) + \sum_{i=1}^{k} \epsilon_i w_2(E_i)),$$

where E_i is the principal S_i^1-bundle

$$E/(S_1^1 \times \cdots \times S_{i-1}^1 \times \{e\} \times S_{i+1} \times \cdots \times S_k^1) \to M$$

and ϵ_i is one or zero if the S_i^1-action on N is odd or even, respectively.

Proof. The tangent bundle of $E \times_T N$ is the sum of the pullback of the tangent bundle of M and the bundle TF along the fibers. Let $\phi : M \to BT$ be the classifying map of the principal bundle E. Then there is a pullback diagram.

$$\begin{tikzcd}
TF \arrow[r] \arrow[d] & TN_T \arrow[d] \\
E \times_T N \arrow[r] \arrow[d, "f"] & N_T \arrow[d, "\pi"] \\
M \arrow[r, "\phi"] & BT
\end{tikzcd}$$

Here TN_T and N_T denote the Borel constructions of TN and N, respectively. Therefore it is sufficient to show that $w_2(TN_T) = \pi^*(\sum_{i=1}^k \epsilon_i x_i)$, where x_i is the generator of $H^2(BS_i^1;\mathbb{Z}_2) \subset H^2(BT;\mathbb{Z}_2)$.

There is an exact sequence

$$0 \longrightarrow H^2(BT;\mathbb{Z}_2) \xrightarrow{\pi^*} H^2(N_T;\mathbb{Z}_2) \longrightarrow H^2(N;\mathbb{Z}_2)$$

Hence, we have $w_2(TN_T) \in \operatorname{im} \pi^*$. Let $\iota : BS_i^1 \to BT$ be the inclusion. Then we have $\iota^* N_T = N_{S_i^1}$. Since there are S_i^1-fixed points in N there is a section to $N_{S_i^1} \to BS_i^1$ induced by the inclusion of a fixed point pt. Therefore we have $w_2(TN_{S_i^1}) = \pi^* \sum a_j x_i$, where the a_j are the weights of the S_i^1-representation $T_{pt}N$. Now the S_i^1-action on N is even if and only if $\operatorname{codim} N^{\mathbb{Z}_2} \equiv 0 \mod 4$. That is if and only if there is an even number of odd a_j. Therefore the statement follows. $\square$

It follows from this lemma, that, for a complex line bundle γ over an oriented manifold N and a Spin-manifold M with semi-free S^1-action, $\Delta(\gamma, M)$ is spin if and only if $w_2(N) \equiv c_1(\gamma) \mod 2$. Moreover, if M' is a Spin-manifold with two commuting semi-free S^1-actions, then $\Gamma(\gamma, M')$ is spin if and only if $w_2(N) \equiv c_1(\gamma) \mod 2$ and both S^1-actions are even or both S^1-actions are odd.

Proof of Theorem 4.11. We will show that for every $[M] \in \Omega_n^{\text{Spin,even},SF}$, $n \geq 6$, there is a $[M'] \in \Omega_n^{\text{Spin,even},SF}$ such that M' admits an invariant metric of positive scalar curvature and $\lambda(2^k[M]) = \lambda([M'])$ in $F_*^{\text{even}} \otimes \mathbb{Z}[\frac{1}{2}]$. Hence, $2^{k+k'}[M] - 2^{k'}[M']$ is bordant to a free S^1-manifold N. After doing some surgeries we may assume that N/S^1 is simply connected. Because $\hat{A}(M/S^1)$ is an invariant of the equivariant bordism type of M and agrees with the usual $\hat{A}$-genus if the S^1-action is free, it follows from Theorems 4.6 and 2.2 that $2^{k+k'+1}M$ admits an invariant metric of positive scalar curvature if and only if $2^{k+k'+1}\hat{A}(M/S^1) = 2\hat{A}(N/S^1) = 0$. Hence, Theorem 4.11 follows.

Now we turn to the construction of M'. At first assume that n is odd. Then $F_n^{\text{even}} \otimes \mathbb{Z}[\frac{1}{2}]$ vanishes as in the non-spin case. Therefore we may assume that $[M'] = 0$.

Next assume that $n \equiv 0 \mod 4$. Then there are $\alpha_J, \beta_J \in \Omega_*^{\text{Spin}} \otimes \mathbb{Z}[\frac{1}{2}]$ and an $L \in F_*^{\text{odd}} \otimes \mathbb{Z}[\frac{1}{2}]$ such that

$$\lambda(M) = \sum_J \alpha_J \lambda(\prod_{i\in J} M_i) + X_1 \sum_{J'} \beta_{J'} \lambda(\prod_{i \in J'} M_i) + X_0 L.$$

Here the sums are taken over all finite sequences J and J' with at least two elements or one element, respectively. Since all M_i admit invariant metrics of positive scalar curvature we may assume that $\alpha_J = 0$ for all J.

The S^1-action on $\sum_{J'} \beta_{J'} \prod_{i \in J'} M_i$ is of odd type. Therefore each product $\prod_{i \in J'} M_i$ contains a factor M_{i_0} with $i_0 > 0$.

Let E be the principal S^1-bundle associated to the dual of the tautological bundle γ over $\mathbb{C}P^2(\rho)$. The S^1-action on $\mathbb{C}P^2(\rho)$ lifts into E such that the action on the fiber over the isolated fixed point is trivial. Since the S^1-action on M_{i_0} is odd, $E \times_{S^1} M_{i_0}$ is a Spin-manifold by Lemma 4.12. Moreover, it follows from a calculation of characteristic numbers that

$$\lambda(E \times_{S^1} M_{i_0}) = X_1 \lambda(M_{i_0}) + X_0 L'$$

with some $L' \in F_*^{\mathrm{odd}} \otimes \mathbb{Z}[\frac{1}{2}]$.

Indeed, there are three fixed point components in $E \times_{S^1} M_{i_0}$, namely the fiber over the isolated fixed point in $\mathbb{C}P^2(\rho)$, and two components which are bundles over the two-dimensional fixed point component of $\mathbb{C}P^2(\rho)$ with fibers the fixed point components of M_{i_0}. The normal bundle of the first fixed point component is trivial.

The other fixed point components are diffeomorphic to cartesian products $\mathbb{C}P^1 \times F_i$, where F_i, $i = 1, 2$, are the fixed point components in M_{i_0}. Hence, their cohomology with coefficients in $\mathbb{Q}$ or $\mathbb{Z}_2$ is isomorphic to $H^*(\mathbb{C}P^1) \otimes H^*(F_i)$. Moreover, there are $k_i \in \mathbb{Z}$, $i = 1, 2$, such that the first two Chern classes of their normal bundles are given by

$$c_1 = k_i c_1(\gamma) + c_1(N(F_i, M_{i_0})) \qquad\qquad c_2 = c_1(\gamma) c_1(N(F_i, M_{i_0})).$$

The other Chern classes vanish because these fixed point components have codimension or dimension four, respectively.

Therefore, all characteristic numbers of $\lambda(E \times_{S^1} M_{i_0}) - X_1 \lambda(M_{i_0})$ involving Chern classes c_i, $i > 1$, of the normal bundle of the fixed point components vanish (cf. Theorem 17.5 of [CF64, p. 49]). Since these normal bundles have complex dimension greater than one, it follows that $\lambda(E \times_{S^1} M_{i_0}) - X_1 \lambda(M_{i_0})$ is contained in the ideal of $F_* \otimes \mathbb{Z}[\frac{1}{2}]$ generated by X_0.

By the same argument as in Construction 4.10, $E \times_{S^1} M_{i_0}$ admits an invariant metric of positive scalar curvature. Therefore we may assume that all $\beta_{J'}$ are zero.

Hence, by the same argument as in the proof of Lemma 4.8, there is a Spin-manifold $\tilde{L}$ with semi-free S^1-action of odd type such that $\lambda(\tilde{L}) = 2^k L$. Then $\Delta(X_0, \tilde{L})$ is a Spin-manifold such that

$$\lambda(\Delta(X_0, \tilde{L})) = 2^k (X_0 L - X_0 \iota(L)) = 2^{k+1} X_0 L.$$

Hence, we may assume that $L = 0$. Therefore the claim follows in this case.

Next assume that $n \equiv 2 \mod 4$. Then there are $\alpha_{k,l,J} \in \Omega_*^{\mathrm{Spin}} \otimes \mathbb{Z}[\frac{1}{2}]$ such that

$$\lambda(M) = \sum_{k=0}^{n/2} \sum_{l=0}^{1} \sum_{J} \alpha_{k,l,J} \lambda(\prod_{i \in J} M_i) X_0^k X_1^l.$$

We will show that we may assume that all $\alpha_{k,l,J}$ vanish after adding Spin-manifolds with even semi-free S^1-actions which admit invariant metrics of positive scalar curvature. In the case that $k = l = 0$ there is nothing to show.

Next assume that $k = 0$ and $l = 1$. Then the dimension of $\prod_{i\in J} M_i$ with $\alpha_{k,l,J} \neq 0$ is congruent to $2 \mod 4$. Moreover, $\Delta(X_1 \otimes X_1, \prod_{i\in J} M_i)$ is a Spin-manifold with semi-free S^1-action such that

$$\lambda(\Delta(X_1 \otimes X_1, \prod_{i\in J} M_i)) = 4X_1\lambda(\prod_{i\in J} M_i).$$

Therefore we may assume that $\alpha_{0,1,J}$ vanishes for all J.

Next assume that $k > 0$ is odd and $l = 0$. Then the dimension of $\prod_{i\in J} M_i$ with $\alpha_{k,l,J} \neq 0$ is divisible by four and the action on this product is of odd type. Therefore such a product contains at least one factor M_{i_0} with $i_0 > 0$.

At first assume that i_0 is odd. Then for the Spin-manifold

$$N = 2\mathbb{C}P^k(\rho) \times M_{i_0} - \Delta(X_{k-1}, M_{i_0}).$$

we have $\lambda(N) = -2X_0^k\lambda(M_{i_0})$. Therefore we may assume that all $\alpha_{k,l,J}$ with $k > 0$ odd and $l = 0$ vanish if J contains an odd number.

Now we turn to the case where i_0 is even. Then there is a second semi-free S^1-action on M_{i_0} induced from a lift of the S^1-action on $\mathbb{C}P^1(\rho)$ to $\gamma \otimes \gamma$. The two S^1-actions commute and are both of odd type. Let M'_{i_0} be M_{i_0} equipped with the second S^1-action.

Let

$$N = \mathbb{C}P^k(\rho) \times (M_{i_0} - M'_{i_0}) - \Gamma(X_{k-1}, M_{i_0}).$$

Then we have, by Lemma 4.9,

$$\lambda(N) = -X_0^k\lambda(M_{i_0}) + X_0^{k+1}\lambda(L),$$

where L is $\mathbb{C}P^{i_0-1}$ equipped with some semi-free S^1-action. Hence, we may assume that all $\alpha_{k,l,J}$ with $k > 0$ odd and $l = 0$ vanish.

Next assume that $k > 0$ is even and $l = 1$. Then the dimension of $\prod_{i\in J} M_i$ with $\alpha_{k,l,J} \neq 0$ is congruent to $2 \mod 4$ and the action on this product is of odd type.

Hence,

$$N = 2M_{k+2} \times \prod_{i\in J} M_i - \Delta(\tilde{X}_{2k+1}, \prod_{i\in J} M_i)$$

is spin and

$$\lambda(N) = -8X_1X_0^k\lambda(\prod_{i\in J} M_i).$$

Therefore we may assume that all $\alpha_{k,l,J}$ with $k > 0$ even and $l = 1$ vanish.

Next assume that $k > 0$ is even and $l = 0$. Then the dimension of $\prod_{i\in J} M_i$ with $\alpha_{k,l,J} \neq 0$ is congruent to $2 \mod 4$ and the action on this product is of even type. Therefore in this product there appears at least one factor M_{i_0} with i_0 odd and a second factor M_{i_1} with $i_1 > 0$. Let E be the principal S^1-bundle associated to the tautological line bundle over $\mathbb{C}P^k(\rho)$. Then the action on $\mathbb{C}P^k(\rho)$ lifts into E in such a way that it is trivial on the fibers of the fixed point component of codimension two in $\mathbb{C}P^k(\rho)$ and multiplication on the fiber over the isolated fixed point. Moreover, $N = E \times_{S^1} M_{i_0}$ is a Spin-manifold with semi-free S^1-manifold such that $\lambda(N) = X_0^k \lambda(M_{i_0})$ because $\dim M_{i_0} \equiv 2 \mod 4$. As in Construction 4.10, one sees that $N = E \times_{S^1} M_{i_0}$ admits an invariant metric of positive scalar curvature. Hence, after adding multiples of $\prod_{i \in J - \{i_0\}} M_i \times N$ to M, we may assume that all $\alpha_{k,l,J}$ with $k > 0$ even and $l = 0$ vanish.

Next let $k > 0$ be odd and $l = 1$. Then the dimension of $\prod_{i \in J} M_i$ with $\alpha_{k,l,J} \neq 0$ is divisible by four and the action on this product is of even type.

We will construct below two semi-free Spin S^1-manifolds N_1 and N_2 with even action from a semi-free Spin S^1-manifold M of dimension divisible by four with even action such that $\lambda(N_1) = 4X_1X_0(\lambda(M) - \bar{M})$ and $\lambda(N_2) = 2X_0^2(\lambda(M) - \bar{M})$, where $\bar{M}$ denotes the manifold M with trivial S^1-action. N_1 and N_2 admit invariant metrics of positive scalar curvature. Therefore after adding multiples of a manifold which is constructed by iterating these constructions we may assume that all $\alpha_{k,l,J}$ with $k > 0$, $l = 1$ and non-empty J vanish.

For the construction of N_1 consider the projectivization P of $\mathbb{C} \oplus \gamma \otimes \gamma$, where γ is the dual of the Hopf bundle over $\mathbb{C}P^1$. S^1 acts on P by multiplication on $\gamma \otimes \gamma$. Then P^{S^1} has two components and

$$\lambda(P) = 2X_1 - 2X_1.$$

Let γ' be the dual of the tautological bundle over P. Then the S^1-action lifts into $\gamma' \otimes \gamma'$ in such a way that the action over the two fixed point components in P are given by multiplication and multiplication with the inverse, respectively. This action induces a semi-free S^1-action on $P(\mathbb{C} \oplus \gamma' \otimes \gamma')$. The fixed point set of the S^1-action on $P(\mathbb{C} \oplus \gamma' \otimes \gamma')$ has four components and

$$\begin{aligned}\lambda(P(\mathbb{C} \oplus \gamma' \otimes \gamma')) &= 2X_1(X_0 - X_0) - 2X_1(-X_0 + X_0) \\ &= (2X_1X_0 + 2X_1X_0) - (2X_1X_0 + 2X_1X_0)\end{aligned}$$

Let E be the principal S^1-bundle associated to the dual of the tautological line bundle γ'' over $P(\mathbb{C} \oplus \gamma' \otimes \gamma')$. The S^1-action on $P(\mathbb{C} \oplus \gamma' \otimes \gamma')$ lifts into E in such a way that it is trivial over the two fixed point components corresponding to $-(2X_1X_0 + 2X_1X_0)$ and multiplication and multiplication with the inverse over the other two. Therefore it follows that for $N_1 = E \times_{S^1} M$, $\lambda(N_1) = 4X_1X_0(M - \bar{M})$. As in Construction 4.10 one sees that N_1 admits an invariant metric of positive scalar curvature. Because the action on M is even, N_1 is spin. The construction of N_2 is similar with P replaced by $\mathbb{C}P^1(\rho)$. We omit the details.

By the above constructions we may now assume that

$$\lambda(M) = \sum_{k=1}^{(n-1)/2} X_1 X_0^k \beta_k$$

with $\beta_k \in \Omega_*^{\mathrm{Spin}} \otimes \mathbb{Z}[\frac{1}{2}]$. But the $\mu(X_1 X_0^k)$ are part of a basis of $\Omega_*^{SO}(BU(1)) \otimes \mathbb{Z}[\frac{1}{2}]$. Therefore all the β_k must vanish and Theorem 4.11 is proved. $\square$

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S^1-equivariant bordism, invariant metrics of positive scalar curvature, and rigidity of elliptic genera

*Michael Wiemeler**

We construct geometric generators of the effective S^1-equivariant Spin- (and oriented) bordism groups with two inverted. We apply this construction to the question of which S^1-manifolds admit invariant metrics of positive scalar curvature.

As a further application of our results we give a new proof of the vanishing of the $\hat{A}$-genus of a spin manifold with non-trivial S^1-action originally proven by Atiyah and Hirzebruch. Moreover, based on our computations we can give a bordism-theoretic proof for the rigidity of elliptic genera originally proven by Taubes and Bott–Taubes.

1. Introduction

The problem of determining generators of S^1-equivariant bordism rings dates back to the 1970s. There first results were obtained by Uchida [Uch70], Ossa [Oss70], Kosniowski and Yahia [KY83] and Hattori and Taniguchi [HT72]. Most of these papers deal with oriented or unitary bordism. Moreover they construct additive generators.

Using these generators bordism-theoretic proofs of the Kosniowski formula and the Atiyah–Singer formula have been given [HT72], [KU71]. The Kosniowski formula expresses the T_y-genus of a unitary S^1-manifold in terms of fixed point data [Kos70]. The Atiyah–Singer formula expresses the signature of an oriented S^1-manifold in terms of the signatures of the fixed point components [AS68]. They have originally been proved using the Atiyah–Singer G-signature theorem.

More recently the problem of finding multiplicative generators for unitary bordism was studied by Sinha [Sin05].

Semi-free S^1-equivariant Spin-bordism has previously been considered by Borsari [Bor87]. Her motivation was a question of Witten, who asked if the equivariant indices of certain twisted Dirac operators are constants and suggested to approach this question via equivariant bordism theory [Wit85, pp. 258-259]. The indices of these twisted Dirac operators are coefficients in the Laurent expansion of the universal elliptic genus in the $\hat{A}$-cusp. Therefore a positive answer to Witten's question is implied by the rigidity of elliptic genera which was proven by Taubes [Tau89] and Bott–Taubes [BT89].

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However, contrary to what Witten suggested their proof was not based on equivariant bordism theory but instead used equivariant K-theory, the Lefschetz fixed point formula and some complex analysis. Later an alternative proof was given by Liu [Liu95] using modularity properties of the universal elliptic genus.

It seems that after their proofs appeared nobody carried out the bordism theoretic approach to the rigidity problem. We pick up this problem and prove the rigidity of elliptic genera via equivariant bordism theory, thus realizing Witten's original plan. Moreover, we give a bordism-theoretic proof for the vanishing of the $\hat{A}$-genus of a Spin-manifold which admits a non-trivial smooth S^1-action originally proved by Atiyah and Hirzebruch [AH70].

This classical result follows from general existence results for S^1-invariant metrics of positive scalar curvature (see Theorem 1.5 and Theorem 1.6). Up to a power of 2 these results are conclusive, thus finishing a line of thought begun in [BB83] and continued in [RW88], [Han08], [Wie15]. We remark that the Atiyah-Hirzebruch vanishing theorem mentioned before and the existence of S^1-invariant metrics of positive scalar curvature have not been considered as related subjects, so far.

These proofs are based on a construction of additive generators of the S^1-equivariant Spin- and oriented bordism groups. These generators are described in the following theorem. To state it we first have to fix some notations.

Let $G = SO$ or $G = \text{Spin}$. Denote by Ω_n^{G,S^1} the bordism group of n-dimensional manifolds with effective S^1-actions and G-structures on their tangent bundles. In the non-spin case we also assume that the closed strata, i.e. the submanifolds of H-fixed points M^H for any subgroup $H \subset S^1$, of our S^1-manifolds are orientable.

Moreover, denote by $\Omega_{\geq 4,n}^{G,S^1}$ similar groups of those S^1-manifolds which satisfy the above conditions and do not have fixed point components of codimension two. We also assume in this case that the bordisms between the manifolds do not have codimension-two fixed point components.

We also need the notion of a generalized Bott manifold. A generalized Bott manifold M is a manifold of the following type: There exists a sequence of fibrations

$$M = N_l \to N_{l-1} \to \dots N_1 \to N_0 = \{pt\},$$

such that N_0 is a point and each N_i is the projectivization of a Whitney sum of $n_i + 1$ complex line bundles over N_{i-1}. Then M has dimension $2n = 2\sum_{i=1}^{l} n_i$ and admits an effective action of an n-dimensional torus T which has a fixed point. Hence it is a so-called torus manifold (for details on the construction of this torus action see Section 2).

Theorem 1.1 *$\Omega_*^{G,S^1}[\frac{1}{2}]$ and $\Omega_{\geq 4,*}^{G,S^1}[\frac{1}{2}]$ are generated as modules over $\Omega_*^G[\frac{1}{2}]$ by manifolds of the following two types:*

1. *Semi-free S^1-manifolds, i.e. S^1-manifolds M, such that all orbits in M are free orbits or fixed points.*
2. *Generalized Bott manifolds M, equipped with a restricted S^1-action.*

The proof of this result is based on techniques first used by Kosniowski and Yahia [KY83] in combination with a result of Saihi [Sai01].

Using the above result, we can give a bordism-theoretic proof of the rigidity of elliptic genera. This gives the following theorem:

Theorem 1.2 ([BT89]) *Let $\varphi : \Omega_n^{Spin,S^1} \to H^{**}(BS^1, \mathbb{C}) = \mathbb{C}[[z]]$ be an equivariant elliptic genus. Then $\varphi(M)$ is constant, as a power series in z, for every effective S^1-manifold $[M] \in \Omega_n^{Spin,S^1}$.*

The idea of our proof of this theorem is as follows. By Theorem 1.1 one only has to prove the rigidity of elliptic genera for semi-free S^1-manifolds and generalized Bott manifolds. For semi-free S^1-manifolds this was done by Ochanine [Och88]. His proof can be modified in such a way that it gives the rigidity of T-equivariant elliptic genera of effective T-manifolds M such that all fixed point components have minimal codimension $2\dim T$. Here T denotes a torus. This minimality condition is satisfied for torus manifolds and therefore also for generalized Bott manifolds. So the theorem follows.

We also apply Theorem 1.1 to the question of which S^1-manifolds admit invariant metrics of positive scalar curvature. It is necessary for this application to consider the bordism groups $\Omega_{\geq 4,n}^{G,S^1}$ because the bordism principle which we will prove to attack this question only works for bordisms which do not have fixed point components of codimension two.

The existence question in the non-equivariant setting was finally answered by Gromov and Lawson [GL80] for high dimensional simply connected manifolds which do not admit spin structures and by Stolz [Sto92] for high dimensional simply connected manifolds which admit such a structure.

Their results are summarized by the following theorem.

Theorem 1.3 *Let M be a simply connected closed manifold of dimension at least five. Then the following holds:*

1. *If M does not admit a spin-structure, then M admits a metric of positive scalar curvature.*

2. *If M admits a spin-structure, then M admits a metric of positive scalar curvature if and only if $\alpha(M) = 0$.*

In the above theorem $\alpha(M)$ denotes the α-invariant of M. It is a KO-theoretic refinement of the $\hat{A}$-genus of M and an invariant of the spin-bordism type of M.

The proof of this theorem consists of two steps; one is geometric, the other is topological. The geometric step is to show that a manifold M which is constructed from another manifold N by surgery in codimension at least 3 admits a metric of positive scalar curvature if N admits such a metric. This is the so-called surgery principle. It has been shown independently by Gromov–Lawson [GL80] and Schoen–Yau [SY79].

From this principle it follows that a manifold of dimension at least five admits a metric of positive scalar curvature if and only if its class in a certain bordism group can be represented by a manifold with such a metric. This is called the bordism principle.

The final step in the proof of the above theorem is then to find all bordism classes which can be represented by manifolds which admit a metric of positive scalar curvature.

The answer to the existence question in the equivariant setting is less clear.

First of all, the question if there exists an invariant metric of positive scalar curvature on a G-manifold, G a compact Lie group acting effectively, has been answered positively by Lawson and Yau [LY74] for the case that the identity component of G is non-abelian. The proof of this result does not use bordism theory or surgery. It is based on the fact that a homogeneous G-space admits an invariant metric of positive scalar curvature, which is induced from an bi-invariant metric on G.

If the identity component of G is abelian, then the answer to this question is more complicated. In this case the existence question was first studied by Bérard Bergery [BB83].

He gave examples of simply connected manifolds with a non-trivial S^1-action which admit metrics of positive scalar curvature, but no S^1-invariant such metric. There are also examples of manifolds which admit S^1-actions, but no metric of positive scalar curvature. Such examples are given by certain homotopy spheres not bounding spin manifolds [Bre67], [Sch75], [Jos81].

Bérard Bergery also showed that the proofs of the surgery principle carry over to the equivariant setting for actions of any compact Lie group G. Based on this several authors have tried to adopt the proof of Theorem 1.3 to the equivariant situation.

At first a bordism principle has been proposed by Rosenberg and Weinberger [RW88] for finite cyclic groups G and actions without fixed point components of codimension two. The proof of this theorem is potentially problematic, because its proof needs more assumptions than those which are stated in the theorem (see the discussion following Corollary 16 in [Han08]). Based on Rosenberg's and Weinberger's theorem Farsi [Far92] studied Spin-manifolds of dimension less than eight with actions of cyclic groups of odd order.

Later Hanke [Han08] proved a bordism principle for actions of any compact Lie group G which also takes codimension-two singular strata into account. He used this result to prove the existence of invariant metrics of positive scalar curvature on certain non-spin S^1-manifolds which do not have fixed points and satisfy Condition C (see Definition 2.1).

In [Wie15] the author showed that every S^1-manifold with a fixed point component of codimension two admits an invariant metric of positive scalar curvature. In that paper existence results for invariant metrics of positive scalar curvature on semi-free S^1-manifolds without fixed point components of codimension two were also discussed.

Here we extend the results from that paper to certain non-semi-free S^1-manifolds. We prove the following existence results for metrics of positive scalar curvature on non-semi-free S^1-manifolds.

Theorem 1.4 *Let M be a connected effective S^1-manifold of dimension at least six such that $\pi_1(M_{max}) = 0$, M_{max} is not Spin and for all subgroups $H \subset S^1$, M^H is orientable. Then for some $k \geq 0$, the equivariant connected sum of 2^k copies of M admits an invariant metric of positive scalar curvature. Here M_{max} denotes the maximal stratum of M, i.e. the union of the principal orbits.*

Here a equivariant connected sum of two S^1-manifolds M_1, M_2 can be a fiber connected sum at principal orbits, a connected sum at fixed points or more generally the result of a zero-dimensional equivariant surgery on orbits $O_i \subset M_i$, $i = 1, 2$.

Now we turn to a similar result in the case that M is a spin manifold. For this we first note that on spin manifolds there are two types of actions. Those which lift to actions on the spin-structure and those which do not lift to the spin-structure. The actions of the first type are called actions of even type. Whereas the actions of the second kind are called actions of odd type.

Theorem 1.5 *Let M be a spin S^1-manifold with $\dim M \geq 6$, an effective S^1-action of odd type and $\pi_1(M_{max}) = 0$. Then there is a $k \in \mathbb{N}$ such that the equivariant connected sum of 2^k copies of M admits an invariant metric of positive scalar curvature.*

Theorem 1.6 *There is an equivariant bordism invariant $\hat{A}_{S^1}$ with values in $\mathbb{Z}[\frac{1}{2}]$, such that, for a spin S^1-manifold M with $\dim M \geq 6$, an effective S^1-action of even type and $\pi_1(M_{max}) = 0$, the following conditions are equivalent:*

1. *$\hat{A}_{S^1}(M) = 0$.*
2. *There is a $k \in \mathbb{N}$ such that the equivariant connected sum of 2^k copies of M admits an invariant metric of positive scalar curvature.*

For free S^1-manifolds M, $\hat{A}_{S^1}(M)$ is equal to the $\hat{A}$-genus of the orbit space of M. If the action on M is semi-free, then it coincides with a generalized $\hat{A}$-genus of the orbit space defined by Lott [Lot00]. In general we can identify $\hat{A}_{S^1}$ with the index of a Dirac-operator defined on a submanifold with boundary of $M_{\max}/S^1$.

$\hat{A}_{S^1}(M)$ can only be non-trivial if the dimension of M is $4k+1$. Moreover, the usual $\hat{A}$-genus of M is zero in these dimensions. It also vanishes, if M admits a metric of positive scalar curvature. Therefore Theorems 1.5 and 1.6 imply the following theorem which has originally been proved by Atiyah and Hirzebruch [AH70] using the Lefschetz fixed point formula and some complex analysis.

Theorem 1.7 *Let M be a Spin-manifold with a non-trivial S^1-action. Then $\hat{A}(M) = 0$.*

The proofs of the above results are based on a generalization (Theorem 5.9) of the bordism principle proved by Hanke [Han08] to S^1-manifolds with fixed points. When this is established the theorems follow from the fact that our generators of the second type of $\Omega^{G,S^1}_{\geq 4,*}[\frac{1}{2}]$ admit invariant metrics of positive scalar curvature and the existence results from [Wie15] for semi-free S^1-manifolds.

This paper is organized as follows. In Sections 2 and 3 we prove Theorem 1.1 for manifolds satisfying Condition C. Then in Section 4 we generalize these results to manifolds not satisfying Condition C. This completes the proof of Theorem 1.1.

Then we turn to the existence question for invariant metrics of positive scalar curvature on S^1-manifolds. In Section 5 we generalize the bordism principle of Hanke [Han08] to S^1-manifolds with fixed points. Then in Section 6 we show that, under mild assumptions on the isotropy groups of the codimension-two singular strata, normally symmetric metrics are dense in all invariant metrics on an S^1-manifold with respect to the C^2-topology. Here normally symmetric metrics are metrics which are invariant under certain extra S^1-symmetries which are defined on small neighborhoods of the codimension-two singular strata. In Section 7 we introduce our obstruction $\hat{A}_{S^1}$ to invariant metrics of positive scalar curvature on spin S^1-manifolds with actions of even type. Then in Section 8 we complete the proof of our existence results for metrics of positive scalar curvature on S^1-manifolds. Moreover, we give a new proof of the above mentioned result of Atiyah and Hirzebruch using our results.

In the last Section 9 we give a proof of the rigidity of elliptic genera based on our Theorem 1.1.

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2. Non-semi-free actions on non-spin manifolds

In this section we prove a version of Theorem 1.1 for the S^1-equivariant oriented bordism groups of manifolds which satisfy Condition C. At first we recall Condition C.

Definition 2.1 *Let T be a torus and M a compact T-manifold. We say that M satisfies Condition C if for each closed subgroup $H \subset T$, the T-equivariant normal bundle of the closed submanifold $M^H \subset M$ is equipped with the structure of a complex T-vector bundle such that the following compatibility condition holds: If $K \subset H \subset T$ are two closed subgroups, then the restriction of $N(M^K, M)$ to M^H is a direct summand of $N(M^H, M)$ as a complex T-vector bundle.*

Before we state and prove our main result of this section, we introduce some notations from [KY83].

Let M be a S^1-manifold satisfying Condition C and $x \in M$. Then the isotropy group S^1_x of x acts linearly on the tangent space of M at x. There is an isomorphism of S^1_x-representations $T_xM \cong \bar{V}_x \oplus V_x$, where $\bar{V}_x$ is a trivial S^1_x-representation and V_x is a unitary S^1_x-representation without trivial summands. The slice type of $x \in M$ is defined to be the pair $[S^1_x; V_x]$.

A S^1-slice-type is a pair $[H; W]$, where H is a closed subgroup of S^1 and W is a unitary H-representation without trivial summands. We call a slice-type $[H; W]$ effective (semi-free, resp.) if the S^1-action on $S^1 \times_H W$ is effective (semi-free, resp.). By a family $\mathcal{F}$ of slice-types we mean a collection of S^1-slice types such that if $[H; W] \in \mathcal{F}$ then for every $x \in S^1 \times_H W$, $[S^1_x, V_x] \in \mathcal{F}$.

A S^1-manifold M satisfying Condition C is of type $\mathcal{F}$, if for every $x \in M$, $[S^1_x, V_x] \in \mathcal{F}$. We denote by $\Omega_*^{C,S^1}[\mathcal{F}]$ the bordism groups of all oriented S^1-manifolds satisfying Condition C of type $\mathcal{F}$. We set $\Omega_*^{C,S^1} = \Omega_*^{C,S^1}[\text{All}]$, where All denotes the family of all slice-types. We denote by $\mathcal{AE}$ the family of all effective slice-types.

Let $\rho = [H; W]$ be a S^1-slice type. A complex S^1-vector bundle E over a manifold N is called of type ρ if the set of points in E having slice type ρ is precisely the zero section of E. Bordism of bundles of type ρ leads to a bundle bordism group $\Omega_*^{C,S^1}[\rho]$.

If $\mathcal{F} = \mathcal{F}' \cup \{\rho\}$ is a family of slice-types such that $\mathcal{F}'$ is a family of slice types, then $\Omega_*^{C,S^1}[\rho]$ is isomorphic to the relative equivariant bordism group $\Omega_*^{C,S^1}[\mathcal{F}, \mathcal{F}']$, which consists of those S^1-manifolds with boundary of type $\mathcal{F}$ whose boundary is of type $\mathcal{F}'$. Moreover, there is a long exact sequence of Ω_*^{SO}-modules

$$\dots \longrightarrow \Omega_n^{C,S^1}[\mathcal{F}'] \longrightarrow \Omega_n^{C,S^1}[\mathcal{F}] \xrightarrow{\nu} \Omega_n^{C,S^1}[\rho] \xrightarrow{\partial} \Omega_{n-1}^{C,S^1}[\mathcal{F}'] \longrightarrow \dots.$$

Here ν is the map which sends a S^1-manifold satisfying Condition C of type $\mathcal{F}$ to the normal bundle of the submanifold of points of type ρ. Moreover, ∂ assigns to a bundle of type ρ its sphere bundle.

This sequence provides an inductive method of calculating the bordism groups of S^1-manifolds satisfying Condition C of type $\mathcal{F}$.

Theorem 2.2 *Let $\mathcal{F} = \mathcal{AE}$ or $\mathcal{F} = \mathcal{AE} - \{[S^1; W]; \dim_{\mathbb{C}} W = 1\}$ be the family of all effective S^1-slice types with or without, respectively, the slice types of the form $[S^1; W]$ with W an unitary S^1-representation of dimension one. Then $\Omega_*^{C,S^1}[\mathcal{F}]$ is generated by semi-free S^1-manifolds and generalized Bott manifolds M with a restricted S^1-action which does not have fixed point components of codimension two.*

Before we prove this theorem we describe the torus action on the generalized Bott manifolds in more detail.

A generalized Bott manifold M is a manifold of the following type: There exists a sequence of fibrations

$$M = N_l \to N_{l-1} \to \dots N_1 \to N_0 = \{pt\},$$

such that N_0 is a point and each N_i is the projectivization a Whitney sum of $n_i + 1$ complex line bundles over N_{i-1}.

The torus action on these manifolds can be constructed inductively as follows. At first note that each N_{i-1} is simply connected. Therefore, if N_{i-1} has an effective action of an $(\sum_{j=1}^{i-1} n_j)$-dimensional torus T, then the natural map $H_T^2(N_{i-1}; \mathbb{Z}) \to H^2(N_{i-1}; \mathbb{Z})$ from equivariant to ordinary cohomology is surjective. Hence, by [HY76], the T-action lifts to an action on each of the $n_i + 1$ line bundles from which N_i is constructed. Together with the action of an $(n_i + 1)$-dimensional torus given by componentwise multiplication on each of these line bundles, this action induces an effective action of an $(\sum_{j=1}^{i} n_j)$-dimensional torus T' on N_i. Note that, by [HY76], the T'-action constructed in this way is unique up to automorphisms of T', i.e. does not depend on the actual choice of the lifts of the T-actions. So each N_i becomes a so-called torus manifold with this torus action, i.e. the dimension of the acting torus is half of the dimension of the manifold and there are fixed points.

Proof of Theorem 2.2. At first we define a sequence of families $\mathcal{F}_i$ of S^1-slice types such that

1. $\mathcal{F}_{i+1} = \mathcal{F}_i \cup \{\sigma_{i+1}\}$ for a slice type σ_{i+1},

2. $\mathcal{F} = \bigcup_{i=0}^{\infty} \mathcal{F}_i$,

3. $\mathcal{F}_0$ consists of semi-free S^1-slice types.

When this is done it is sufficient to prove that each $\Omega_*^{C,S^1}[\mathcal{F}_i]$ is generated by semi-free S^1-manifolds, generalized Bott manifolds and manifolds which bound in $\Omega_*^{C,S^1}[\mathcal{F}]$.

Before we do that we introduce some notations for the S^1-slice types. The irreducible non-trivial S^1-representations are denoted by

$$\dots, V_{-1}, V_1, V_2, \dots,$$

where V_i is $\mathbb{C}$ equipped with the S^1-action given by multiplication with s^i for $s \in S^1$. For $\mathbb{Z}_m \subset S^1$ we denote the irreducible non-trivial $\mathbb{Z}_m$-representations by $V_{-1}, \dots, V_{-m+1}$. Here V_i is $\mathbb{C}$ with $s \in \mathbb{Z}_m$ acting by multiplication with s^i.

The effective S^1-slice types are then of the forms

$$[S^1; V_{k(1)} \oplus V_{k(2)} \oplus \dots \oplus V_{k(n)}]$$

with $k(1) \geq k(2) \geq \dots \geq k(n)$, $n \geq 1$ and $\gcd\{k(1), \dots, k(n)\} = 1$ or

$$[\mathbb{Z}_m; V_{k(1)} \oplus V_{k(2)} \oplus \dots \oplus V_{k(n)}]$$

with $0 > k(1) \geq k(2) \geq \dots \geq k(n) > -m$, $n \geq 1$ if $m > 1$, and $\gcd\{k(1), \dots, k(n), m\} = 1$. Such a slice type is semi-free if it is of the form $[S^1; V_{k(1)} \oplus V_{k(2)} \dots \oplus V_{k(n)}]$ with $|k(i)| = 1$ for all i or $[\mathbb{Z}_1; 0]$.

The non-semi-free effective slice types fall into three different classes:

$$\begin{aligned}
\mathfrak{F} &= \{[H; W];\ H \text{ is finite}\} \\
\mathfrak{S}\mathfrak{F} &= \{[S^1; W];\ W = V_{k(1)} \oplus \dots \oplus V_{k(n)} \oplus V_{-m} \text{ with} \\
&\quad\ 0 > k(1) \geq k(2) \geq \dots \geq k(n) > -m\} \\
\mathfrak{T} &= \{\text{all other non-semi-free effective slice types}\}
\end{aligned}$$

For $\rho = [\mathbb{Z}_m; V_{k(1)} \oplus \dots \oplus V_{k(n)}] \in \mathfrak{F}$ we define $e(\rho) = [S^1; V_{k(1)} \oplus \dots \oplus V_{k(n)} \oplus V_{-m}] \in \mathfrak{S}\mathfrak{F}$.

We define an ordering of the non-semi-free slice types as follows similarly to the ordering in [KY83, Section 6]: Let

$$\begin{aligned}
\delta[S^1; V_{k(1)} \oplus \dots \oplus V_{k(n)}] &= \max\{|k(1)|, \dots, |k(n)|\} \\
d[S^1; V_{k(1)} \oplus \dots \oplus V_{k(n)}] &= n,
\end{aligned}$$

we order $\mathfrak{T} \cup \mathfrak{S}\mathfrak{F}$ at first by $\delta + d$, then by d and then lexicographically. This together with the following conditions gives an ordering on $\mathfrak{T} \cup \mathfrak{S}\mathfrak{F} \cup \mathfrak{F}$. If $\rho \in \mathfrak{F}$ and $\rho' \notin \mathfrak{F}$ then define $\rho < \rho'$ if $e(\rho) \leq \rho'$. If $\rho \in \mathfrak{F}$ and $\rho' \in \mathfrak{F}$ then let $\rho \leq \rho'$ if $e(\rho) \leq e(\rho')$.

Denote the elements of $\mathfrak{T} \cup \mathfrak{S}\mathfrak{F} \cup \mathfrak{F}$ by $\sigma_1, \sigma_2, \dots$ so that $\sigma_i < \sigma_j$ if $i < j$. Then one can check that

$$\begin{aligned}\mathcal{F}_0 &= \{\text{semi-free slice-types}\} - \{[S^1; V_i];\ |i| = 1\},\\ \mathcal{F}_i &= \mathcal{F}_0 \cup \{\sigma_1, \dots, \sigma_i\} \quad \text{for } i \geq 1\end{aligned}$$

are families of slice types. The difference between these families of slice types and the families of slice types $\mathcal{ST}_i$ appearing in [KY83] is that $\mathcal{F}_i = (\mathcal{ST}_{j(i)} \cap \mathcal{F}) \cup \mathcal{F}_0$ with $j(i) < j(i')$ if $i < i'$.

Now we prove by induction on i that $\Omega_*^{C,S^1}[\mathcal{F}_i]$ is generated by semi-free S^1-manifolds, generalized Bott manifolds and manifolds which bound in $\Omega_*^{C,S^1}[\mathcal{F}_{i+1}]$.

We may assume that $i \geq 1$. Then we have an exact sequence

$$\dots \longrightarrow \Omega_n^{C,S^1}[\mathcal{F}_{i-1}] \longrightarrow \Omega_n^{C,S^1}[\mathcal{F}_i] \xrightarrow{\nu_i} \Omega_n^{C,S^1}[\sigma_i] \xrightarrow{\partial_i} \Omega_{n-1}^{C,S^1}[\mathcal{F}_{i-1}] \longrightarrow .$$

At first assume that $\sigma_i \in \mathfrak{F}$. Then one can define a section q_i to ν_i as in [KY83, Section 6]. For $E \in \Omega_*^{C,S^1}[\sigma_i]$, $q_i(E)$ is represented by a sphere bundle associated to a unitary S^1-vector bundle of rank at least two over a manifold with trivial S^1-action.

Hence $\Omega_n^{C,S^1}[\mathcal{F}_i]$ is generated by manifolds of type $\mathcal{F}_{i-1}$ and the $q_i(E)$, $E \in \Omega_n^{C,S^1}[\sigma_i]$. Moreover, the disc bundle associated to E is bounded by $q_i(E)$ and is of type $\mathcal{F}_{i+1}$. Therefore the claim follows in this case from the induction hypothesis.

Next assume that $\sigma_i \in \mathfrak{S}\mathfrak{F}$. Then as in [KY83, Section 5], one sees that ∂_i induces an isomorphism $\Omega_n^{C,S^1}[\sigma_i] \to \operatorname{im} q_{i-1}$. Therefore we have an isomorphism $\Omega_*^{C,S^1}[\mathcal{F}_{i-2}] \to \Omega_*^{C,S^1}[\mathcal{F}_i]$. Hence the claim follows in this case from the induction hypothesis.

At the end assume that $\sigma_i \in \mathfrak{T}$. Then one defines a section q_i to ν_i as in [KY83, Section 7]. For $E \in \Omega_*^{C,S^1}[\sigma_i]$, $q_i(E)$ is represented by a projectivization $P(\tilde{E})$ of a unitary S^1-vector bundle $\tilde{E}$ of rank at least two over a manifold with trivial S^1-action or a manifold M_2 of the following form. First let M_1 be the projectivization of a unitary S^1-vector bundle $\tilde{E}_1$ over a manifold with trivial S^1-action. M_2 is then the projectivization of a unitary S^1-vector bundle E_2 of rank at least two over M_1.

Moreover, one sees from the definition of q_i in [KY83, Section 7] and the fact that σ_i is not semi-free that $q_i(E)$ does not have fixed point components of codimension two. Since $\Omega_*^{SO}(\prod_i BU(j_i))$ is generated as a module over Ω_*^{SO} by sums of line bundles over complex projective spaces, we may assume that the $q_i(E)$ are generalized Bott manifolds. Hence the claim follows in this case from the induction hypothesis. □

3. The spin case

In this section we prove the following theorem about the S^1-equivariant Spin-bordism groups of those manifolds which satisfy Condition C.

Theorem 3.1 *Let $\mathcal{F} = \mathcal{AE}$ or $\mathcal{F} = \mathcal{AE} - \{[S^1; W]; \dim_{\mathbb{C}} W = 1\}$ be the family of all effective S^1-slice types with or without, respectively, the slice types of the form $[S^1; W]$ with W an unitary S^1-representation of dimension one. Then $\Omega_*^{C,Spin,S^1}[\frac{1}{2}][\mathcal{F}]$ is generated by semi-free S^1-manifolds and generalized Bott manifolds with restricted S^1-action.*

For the proof of this theorem we will use the same families of slice types and the following exact sequences analogues to the ones used in the proof of Theorem 2.2.

$$\begin{tikzcd}
\dots \arrow[r] & \Omega_n^{C,Spin,S^1}[\mathcal{F}_{i-1}] \arrow[r] & \Omega_n^{C,Spin,S^1}[\mathcal{F}_i] \arrow[d] & & \\
& & \Omega_n^{C,Spin,S^1}[\sigma_i] \arrow[r] & \Omega_{n-1}^{C,Spin,S^1}[\mathcal{F}_{i-1}] \arrow[r] & \dots
\end{tikzcd}$$

Before we prove Theorem 3.1, we describe the groups $\Omega_n^{C,Spin,S^1}[\sigma_i]$. These groups have been computed by Saihi [Sai01]. She showed that

$$\Omega_n^{C,Spin,S^1}[H,V] \cong \Omega_{n-1-2\sum_{i=1}^k n_i}(B\Gamma_u \amalg B\Gamma_v, f),$$

where H is a finite group, Γ_u, Γ_v are connected two-fold covering groups of $\Gamma = SO \times S^1 \times_H \prod_i^k U(n_i)$ and $f = f_u \amalg f_v$ where $f_u : B\Gamma_u \to BSO, f_v : B\Gamma_v \to BSO$ are the fibrations induced by the projection $\Gamma \to SO$.

Lemma 3.2 *The two components of $B\Gamma_u \amalg B\Gamma_v$ are in one-to-one correspondence to the two spin-structures $\mathcal{U}, \mathcal{V}$ on $S^1 \times_H V$, in such a way that for $E \in \Omega_*(B\Gamma_u \amalg B\Gamma_v, f_u \amalg f_v)$ we have $E \in \Omega_*(B\Gamma_u, f_u)$ (, $E \in \Omega_*(B\Gamma_v, f_v)$, respectively) if and only if the restriction of the spin structure on E to an invariant tubular neighborhood of an orbit of type $[H,V]$ in E coincides with the spin structure $\mathcal{U}$ (, $\mathcal{V}$, respectively).*

Proof. The spin structures on $S^1 \times_H V \times \mathbb{R}^k$ are in one to one-to-one correspondence to the spin structures on $S^1 \times_H V$. Every spin structure on $S^1 \times_H V$ induces a homotopy class of lifts of the map $pt \to B\Gamma$ to $B\Gamma_u \amalg B\Gamma_v$. Moreover, every such lift induces a spin structure on $S^1 \times_H V$. Since the fiber of $B\Gamma_u \amalg B\Gamma_v \to B\Gamma$ has two components, there are exactly two homotopy classes of lifts. These lifts induce different spin structures on $S^1 \times_H V$ because they represent different elements in $\Omega_*(B\Gamma_u \amalg B\Gamma_v, f_u \amalg f_v)$. Since there are exactly two spin structures on $S^1 \times_H V$ the claim follows. □

After inverting two, we get the following isomorphism:

$$\begin{aligned}
\Omega_n^{C,Spin,S^1}[H,V][\frac{1}{2}] &\cong \Omega_{n-1-2\sum_{i=1}^k n_i}[\frac{1}{2}](B\Gamma \amalg B\Gamma, f) \\
&\cong \Omega^{Spin}_{n-1-2\sum_{i=1}^k n_i}[\frac{1}{2}](B(S^1 \times_H \prod_i^k U(n_i)) \amalg B(S^1 \times_H \prod_i^k U(n_i))) \\
&\cong \Omega^{Spin}_{n-1-2\sum_{i=1}^k n_i}[\frac{1}{2}](B(S^1/H \times \prod_i^k U(n_i)) \amalg B(S^1/H \times \prod_i^k U(n_i))).
\end{aligned}$$

Here the first isomorphism is induced by the two-fold coverings $\Gamma_u, \Gamma_v \to \Gamma$. Moreover, the third isomorphism is induced by the homomorphism

$$\begin{aligned} S^1 \times_H \prod_{i=1}^k U(n_i) \to S^1/H \times \prod_{i=1}^k U(n_i), \\ [z, u_1, \dots, u_k] \mapsto ([z], z^{\alpha_1} u_1, \dots, z^{\alpha_k} u_k). \end{aligned}$$

We also have

$$\begin{aligned} \Omega_n^{C,Spin,S^1}[S^1, V][\frac{1}{2}] &\cong \tilde{\Omega}_n^{Spin}[\frac{1}{2}](MU(n_1) \wedge \cdots \wedge MU(n_k)) \\ &\cong \tilde{\Omega}_n^{SO}[\frac{1}{2}](MU(n_1) \wedge \cdots \wedge MU(n_k)) \\ &\cong \Omega^{SO}_{n-2\sum_{i=1}^k n_i}[\frac{1}{2}](BU(n_1) \times \cdots \times BU(n_k)) \\ &\cong \Omega^{Spin}_{n-2\sum_{i=1}^k n_i}[\frac{1}{2}](BU(n_1) \times \cdots \times BU(n_k)). \end{aligned}$$

Here the first isomorphism has been shown by Saihi [Sai01], the second and fourth isomorphism are deduced from the fact that after inverting two Ω_*^{SO} and Ω_*^{Spin} become isomorphic.

Moreover, the third isomorphism is constructed as follows: Make a map $\phi : M \to MU(n_1) \wedge \cdots \wedge MU(n_k)$ transversal to the zero section of the classifying bundle over $BU(n_1) \times \cdots \times BU(n_k)$. Then restrict ϕ to the preimage of the zero section. This gives an element of $\Omega^{SO}_{*-2\sum_{i=1}^k n_i}(BU(n_1) \times \cdots \times BU(n_k))$.

We will need a basis of the $\Omega_*^{Spin}[\frac{1}{2}]$-module $\Omega_*^{Spin}[\frac{1}{2}](BU(n_1) \times \cdots \times BU(n_k))$. For odd $i > 0$ denote by X_i the tensor product of two copies of the tautological line bundle over $\mathbb{C}P^i$. For even $i > 0$ denote by X_i the tensor product of two copies of the tautological line bundle over $\mathbb{C}P(\gamma \otimes \gamma \oplus \mathbb{C})$. Here γ denotes the tautological bundle over $\mathbb{C}P^{i-1}$. Moreover for $i = 0$ denote by X_0 the trivial line bundle over a point. Then the

$$\prod_{i=1}^k \prod_{h=1}^{n_i} X_{j_{hi}} \qquad \text{with } 0 \leq j_{1i} \leq \cdots \leq j_{n_k i} \tag{1}$$

form a basis of $\Omega_*^{Spin}[\frac{1}{2}](BU(n_1) \times \cdots \times BU(n_k))$ as a $\Omega_*^{Spin}[\frac{1}{2}]$-module.

This can be seen as follows: First of all the X_i, $i \geq 0$, form a basis of $\Omega_*[\frac{1}{2}](BS^1)$ as a module over $\Omega_*[\frac{1}{2}]$ because the characteristic numbers $\langle c_1^i(X_i), [B_i] \rangle$ are powers of two [CF64, Theorem 18.1]. Moreover, for a torus $T = (S^1)^k$, $\Omega_*[\frac{1}{2}](BT)$ is isomorphic to $\bigotimes_{i=1}^k \Omega_*[\frac{1}{2}](BS^1)$. Now consider the Atiyah–Hirzebruch spectral sequences for

$$\Omega_*[\frac{1}{2}](BT)$$

and

$$\Omega_*[\frac{1}{2}](BU(n_1) \times \cdots \times BU(n_k)),$$

where T is a maximal torus of $U(n_1) \times \cdots \times U(n_k)$. They degenerate at the E^2-level. Hence, it follows that the products in (1) form a basis of $\Omega_*[\frac{1}{2}](BU(n_1) \times \cdots \times BU(n_k))$ from a comparison of the singular homologies of the two spaces (for details see [Koc96, Section 4.3]).

For the proof of Theorem 3.1 we will also need the following construction of twisted projective space bundles $\widetilde{\mathbb{C}TP}(E_1; E_2)$.

Construction 3.3 *Let $E_i \to B_i$, $i = 1, 2$, be unitary S^1-vector bundles such that the actions on the base spaces are trivial. Then E_i splits as a sum of unitary S^1-vector bundles*

$$E_i = \bigoplus_{j=1}^{k_i} E_i^{\beta_{ij}}$$

with $\beta_{i1}, \dots, \beta_{ik_i} \in \mathbb{Z}$ such that the action of $z \in S^1$ on each $E_i^{\beta_{ij}}$ is given by multiplication with $z^{\beta_{ij}}$. We call $E_i^{\beta_{ij}}$ the weight bundle of the weight β_{ij}.

Moreover, we write

$$E_i^+ = \bigoplus_{\beta_{ij} > 0} E_i^{\beta_{ij}}$$

and

$$E_i^- = \bigoplus_{\beta_{ij} < 0} E_i^{\beta_{ij}}.$$

Then we have $E_i = E_i^+ \oplus E_i^- \oplus E_i^0$.

Also let $\beta_{2\max} \in \{\beta_{21}, \dots, \beta_{2k_2}\}$ such that $|\beta_{2\max}| = \max_{1 \le j \le k_2} |\beta_{2j}|$.

In the following we assume that $\dim_{\mathbb{C}} E_2$ is even and $\dim_{\mathbb{C}} E_i^0 = 1$ for $i = 1, 2$. We also assume that one of the following three cases holds:

1. *$\dim_{\mathbb{C}} E_1$ is odd.*
2. *$\dim_{\mathbb{C}} E_1$ is even, $\beta_{2\max} = \pm 2$, $\dim_{\mathbb{C}} E_2^{\beta_{2\max}} = 1$, $|\beta_{2j}| = |\beta_{1j'}| = 1$ for $\beta_{2j} \neq \beta_{2\max}$ and all $j' \in \{1, \dots, k_1\}$.*
3. *$\dim_{\mathbb{C}} E_1$ is even, case 2 does not hold and $E_2^{\beta_{2\max}}$ splits off an S^1-invariant line bundle F.*

In each of these cases we define a free T^2-action on the product of sphere bundles $S(E_1) \times S(E_2)$.

In case 1 we define this action as follows

$$(s, t) \cdot ((e_1^+, e_1^-, e_1^0), (e_2^+, e_2^-, e_2^0)) = ((te_1^+, t^{-1}e_1^-, te_1^0), (tse_2^+, t^{-1}s^{-1}e_2^-, se_2^0)),$$

where $(s, t) \in S^1 \times S^1 = T^2$, $e_i^\pm \in E_i^\pm$ and $e_i^0 \in E_i^0$.

If, in case 2, $\beta_{2\max} < 0$, we define this action by

$$(s, t) \cdot ((e_1^+, e_1^-, e_1^0), (e_2^+, e_2^-, e_2^0, f)) = ((te_1^+, t^{-1}e_1^-, te_1^0), (tse_2^+, t^{-1}s^{-1}e_2^-, se_2^0, s^{-1}f)),$$

where $(s,t) \in S^1 \times S^1 = T^2$, $e_1^\pm \in E_1^\pm$, $e_2^+ \in E_2^+$, $e_2^- \in E_2^- \ominus E_2^{\beta_{2\max}}$, $e_i^0 \in E_i^0$ *and* $f \in E_2^{\beta_{2\max}}$.

If, in case 2, $\beta_{2\max} > 0$, *we define this action by*

$$(s,t)\cdot((e_1^+,e_1^-,e_1^0),(e_2^+,e_2^-,e_2^0,f)) = ((te_1^+,t^{-1}e_1^-,te_1^0),(tse_2^+,t^{-1}s^{-1}e_2^-,se_2^0,sf)),$$

where $(s,t) \in S^1 \times S^1 = T^2$, $e_1^\pm \in E_1^\pm$, $e_2^+ \in E_2^+ \ominus E_2^{\beta_{2\max}}$, $e_2^- \in E_2^-$, $e_i^0 \in E_i^0$ *and* $f \in E_2^{\beta_{2\max}}$.

If, in case 3, $\beta_{2\max} < 0$, *we define this action by*

$$(s,t)\cdot((e_1^+,e_1^-,e_1^0),(e_2^+,e_2^-,e_2^0,f)) = ((te_1^+,t^{-1}e_1^-,te_1^0),(tse_2^+,t^{-1}s^{-1}e_2^-,se_2^0,s^{-1}t^{-2}f)),$$

where $(s,t) \in S^1 \times S^1 = T^2$, $e_1^\pm \in E_1^\pm$, $e_2^+ \in E_2^+$, $e_2^- \in E_2^- \ominus F$, $e_i^0 \in E_i^0$ *and* $f \in F$.

If, in case 3, $\beta_{2\max} > 0$, *we define this action by*

$$(s,t)\cdot((e_1^+,e_1^-,e_1^0),(e_2^+,e_2^-,e_2^0,f)) = ((te_1^+,t^{-1}e_1^-,te_1^0),(tse_2^+,t^{-1}s^{-1}e_2^-,se_2^0,st^2f)),$$

where $(s,t) \in S^1 \times S^1 = T^2$, $e_1^\pm \in E_1^\pm$, $e_2^+ \in E_2^+ \ominus F$, $e_2^- \in E_2^-$, $e_i^0 \in E_i^0$ *and* $f \in F$.

We denote the orbit space of these actions by $\widetilde{\mathbb{C}TP}(E_1;E_2)$. *It is diffeomorphic to a* $\mathbb{C}P^{n_2}$*-bundle over a manifold* M, *where* M *is a* $\mathbb{C}P^{n_1}$*-bundle over* $B_1 \times B_2$. *Here we have* $n_i = \dim_{\mathbb{C}} E_i - 1$.

One can compute the cohomology of the total space of the $\mathbb{C}P^{n_1}$*-bundle* $\xi : M \to B_1 \times B_2$ *by using the Leray–Hirsch-Theorem as*

$$H^*(M;\mathbb{Z}_2) \cong H^*(B_1 \times B_2;\mathbb{Z}_2)[u]/(f(u)),$$

where u *is the* $\bmod 2$*-reduction of the first Chern-class of the tautological line bundle over* M *and* $f(u)$ *is a polynomial of degree* $2(n_1+1)$.

Moreover, its tangent bundle splits as a direct sum

$$\xi^*T(B_1 \times B_2) \oplus \eta,$$

where η *is the tangent bundle along the fibers of* ξ.

Note that η *is isomorphic to* $\gamma \otimes (E_1^+ \oplus E_1^0) \oplus \bar{\gamma} \otimes E_1^-$, *where* γ *denotes the tautological line bundle over* M.

From this fact it follows that the second Stiefel–Whitney class of M *is given by*

$$w_2(M) = w_2(B_1 \times B_2) + (n_1+1)u + w_2(E_1).$$

The same reasoning with M *replaced by* $\widetilde{\mathbb{C}TP}(E_1;E_2)$ *and* $B_1 \times B_2$ *replaced by* M *shows that the second Stiefel–Whitney class of* $\widetilde{\mathbb{C}TP}(E_1;E_2)$ *is given by*

$$w_2(\widetilde{\mathbb{C}TP}(E_1;E_2)) = w_2(B_1 \times B_2) + w_2(E_1) + w_2(E_2).$$

Therefore it follows that $\widetilde{\mathbb{C}TP}(E_1;E_2)$ *is a spin manifold if* B_1, B_2 *are spin-manifolds and* E_1, E_2 *are spin-vector bundles.*

The S^1-actions on E_1 and E_2 induce an S^1-action on $\widetilde{\mathbb{C}TP}(E_1; E_2)$. The submanifolds

$$N_{\beta_{1j},\beta_{2j'}} = (S(E_1^{\beta_{1j}}) \times S(E_2^{\beta_{2j'}}))/T^2 \subset \widetilde{\mathbb{C}TP}(E_1; E_2)^{S^1}$$

are fixed by this S^1-action on $\widetilde{\mathbb{C}TP}(E_1; E_2)$.

In the proof of Theorem 3.1 we want that the manifolds $\widetilde{\mathbb{C}TP}(E_1; E_2)$ are of type $\mathcal{F}_i$ for certain families $\mathcal{F}_i$ of slice types. For this it is important to know the weights of the S^1-representations on the normal bundles of $N_{\beta_{1j},\beta_{2j'}}$. In particular it is important how they differ from the weights at $N_{0,0}$. The weights of the S^1-action on the normal bundle of $N_{\beta_{1j},\beta_{2j'}}$ can be computed as listed in tables 1, 2, 3.

In the proof of Theorem 3.1 we will always have

$$\max_{1\leq j\leq k_1} |\beta_{1j}| < \max_{1\leq j\leq k_2} |\beta_{2j}| = |\beta_{2\max}|.$$

Assume that this holds and that $(\beta_{1j}, \beta_{2j'}) \neq (0, 0), (0, \pm\beta_{2\max})$, then we have in cases 1 and 3:

$$\max\{|\textit{weights at } N_{\beta_{1j},\beta_{2j'}}|\} < \max\{|\textit{weights at } N_{0,0}|\}.$$

In case 2 let $Y_{\beta_{1j},\beta_{2j'}}$ be the component of $\widetilde{\mathbb{C}TP}(E_1; E_2)^{S^1}$ containing $N_{\beta_{1j},\beta_{2j'}}$. Then we have

$$\operatorname{codim} Y_{\beta_{1j},\beta_{2j'}} < \dim E_1 + \dim E_2 - 2 = \operatorname{codim} Y_{0,0}$$

if $(\beta_{1j}, \beta_{2j'}) \neq (0, 0), (0, \pm 2), (\pm 1, \pm 2)$ or $(\beta_{1j}, \beta_{2j'}) \neq (0, 0), (0, \pm 2)$ and $\dim_{\mathbb{C}} E_1 \neq 2$.

If we have $(\beta_{1j}, \beta_{2j'}) = (\pm 1, \pm 2)$, $\dim_{\mathbb{C}} E_1 = 2$ and $\dim_{\mathbb{C}} E_2 \geq 4$, then the weight -2 appears in the weights at $N_{\beta_{1j},\beta_{2j'}}$ with multiplicity greater than one. Therefore the S^1-representation at $N_{\beta_{1j},\beta_{2j'}}$ does not coincide with the S^1-representation at $N_{0,0}$.

Note that in all three cases we have that the weights of the S^1-representation at $N_{0,\beta_{2\max}}$ are given by

$$\{-|\beta_{2\max}|, \beta_{11}, \dots, \beta_{1k}, \operatorname{sign} \beta_{2i}(|\beta_{2i}| - |\beta_{2\max}|);\ i = 1, \dots, k' \textit{ and } \beta_{2i} \neq 0\}.$$

Proof of Theorem 3.1. Now as in the non-spin case we construct sections to the maps

$$\nu_i : \Omega_n^{C,Spin,S^1}[\mathcal{F}_i][\frac{1}{2}] \to \operatorname{im} \nu_i \subset \Omega_n^{C,Spin,S^1}[\sigma_i][\frac{1}{2}].$$

At first assume that $\sigma_i = [H, V]$ with H finite and $V = \bigoplus_{i=1}^k V_{\alpha_i}$ with $0 > \alpha_1 \geq \cdots \geq \alpha_k > \alpha_{k+1} = -m = -\operatorname{ord} H$. Then, by Lemma 3.2, the two copies of $B\Gamma$ in $\Omega_n^{C,Spin,S^1}[\frac{1}{2}][\sigma_i]$ correspond to the two spin structures on

$$S^1 \times_H V.$$

$(\beta_{1j}, \beta_{2j'})$	weights at $N_{\beta_{1j},\beta_{2j'}}$
$(0,0)$	$\beta_{11},\dots,\beta_{1k},\beta_{21},\dots,\beta_{2k'}$
$(0,\gamma)$ with $\gamma \neq 0$	$-\lvert\gamma\rvert$, $\beta_{11},\dots,\beta_{1k}$, $\operatorname{sign}\beta_{2i}(\lvert\beta_{2i}\rvert-\lvert\gamma\rvert)$ for $i=1,\dots k'$ and $\beta_{2i}\neq 0$
$(\gamma,0)$ with $\gamma \neq 0$	$-\lvert\gamma\rvert$, $\operatorname{sign}\beta_{1i}(\lvert\beta_{1i}\rvert-\lvert\gamma\rvert)$ for $i=1,\dots k$ and $\beta_{1i}\neq 0$, $\operatorname{sign}\beta_{2i}(\lvert\beta_{2i}\rvert-\lvert\gamma\rvert)$ for $i=1,\dots k'$ and $\beta_{2i}\neq 0$
(γ_1,γ_2) with $\gamma_i \neq 0$	$-\lvert\gamma_1\rvert$, $\operatorname{sign}\beta_{1i}(\lvert\beta_{1i}\rvert-\lvert\gamma_1\rvert)$ for $i=1,\dots k$ and $\beta_{1i}\neq 0$, $\operatorname{sign}\beta_{2i}(\lvert\beta_{2i}\rvert-\lvert\gamma_2\rvert)$ for $i=1,\dots k'$ and $\beta_{2i}\neq 0$, $\lvert\gamma_1\rvert-\lvert\gamma_2\rvert$

Table 1: The weights of the S^1-action on $\widetilde{\mathbb{C}TP}$ in the first case.

$(\beta_{1j}, \beta_{2j'})$	weights at $N_{\beta_{1j},\beta_{2j'}}$
$(0,0)$	$\beta_{11},\dots,\beta_{1k},\beta_{21},\dots,\beta_{2k'}$
$(0,\pm1)$	$\beta_{11},\dots,\beta_{1k},\mp1$
$(0,\pm2)$	$\beta_{11},\dots,\beta_{1k},-2,\mp1$
$(\pm1,0)$, $(\pm1,\pm1)$	$-1,\pm2$
$(\pm1,\pm2)$	$-1,-2$

Table 2: The weights of the S^1-action on $\widetilde{\mathbb{C}TP}$ in the second case.

$(\beta_{1j}, \beta_{2j'})$	weights at $N_{\beta_{1j},\beta_{2j'}}$
$(0,0)$	$\beta_{11},\dots,\beta_{1k},\beta_{21},\dots,\beta_{2k'}$
$(0,\gamma)$ with $\gamma \neq 0$	$-\lvert\gamma\rvert$, $\beta_{11},\dots,\beta_{1k}$, $\operatorname{sign}\beta_{2i}(\lvert\beta_{2i}\rvert-\lvert\gamma\rvert)$ for $i=1,\dots k'$ and $\beta_{2i}\neq 0$
$(\gamma,0)$ with $\gamma \neq 0$	$-\lvert\gamma\rvert$, $\operatorname{sign}\beta_{1i}(\lvert\beta_{1i}\rvert-\lvert\gamma\rvert)$ for $i=1,\dots k$ and $\beta_{1i}\neq 0$, $\operatorname{sign}\beta_{2i}(\lvert\beta_{2i}\rvert-\lvert\gamma\rvert)$ for $i=1,\dots k'$ and $\beta_{2i}\neq 0,\alpha$, $\operatorname{sign}\alpha(\lvert\alpha\rvert-2\lvert\gamma\rvert)$
(γ_1,γ_2) with $\gamma_i \neq 0$, $\gamma_2 \neq \alpha$	$-\lvert\gamma_1\rvert$, $\operatorname{sign}\beta_{1i}(\lvert\beta_{1i}\rvert-\lvert\gamma_1\rvert)$ for $i=1,\dots k$ and $\beta_{1i}\neq 0$, $\operatorname{sign}\beta_{2i}(\lvert\beta_{2i}\rvert-\lvert\gamma_2\rvert)$ for $i=1,\dots k'$ and $\beta_{2i}\neq 0,\alpha$, $\lvert\gamma_1\rvert-\lvert\gamma_2\rvert$, $\operatorname{sign}\alpha(\lvert\alpha\rvert-\lvert\gamma_1\rvert-\lvert\gamma_2\rvert)$
(γ,α) with $\gamma \neq 0$	$-\lvert\gamma\rvert$, $\operatorname{sign}\beta_{1i}(\lvert\beta_{1i}\rvert-\lvert\gamma\rvert)$ for $i=1,\dots k$ and $\beta_{1i}\neq 0$, $\operatorname{sign}\beta_{2i}(\lvert\beta_{2i}\rvert-\lvert\alpha\rvert+\lvert\gamma\rvert)$ for $i=1,\dots k'$ and $\beta_{2i}\neq 0,\alpha$, $2\lvert\gamma\rvert-\lvert\alpha\rvert$

Table 3: The weights of the S^1-action on $\widetilde{\mathbb{C}TP}$ in the third case.

There are equivariant diffeomorphisms

$$S^1/H \times \bigoplus_{i=1}^{k} V_{\alpha_i} \to S^1 \times_H \bigoplus_{i=1}^{k} V_{\alpha_i}$$
$$([z], v_1, \dots, v_k) \mapsto [z, z^{-\alpha_1} v_1, \dots, z^{-\alpha_k} v_k]$$

and

$$S^1/H \times (V_{\alpha_1+m} \oplus \bigoplus_{i=2}^{k} V_{\alpha_i}) \to S^1 \times_H \bigoplus_{i=1}^{k} V_{\alpha_i}$$
$$([z], v_1, \dots, v_k) \mapsto [z, z^{-\alpha_1-m} v_1, \dots, z^{-\alpha_k} v_k],$$

where the action on the left hand spaces is given by the product action and on the right hand spaces the action is induced by left multiplication on the first factor.

Moreover, we can (non-equivariantly) identify

$$S^1/H \times \bigoplus_{i=1}^{k} V_{\alpha_i} \cong S^1 \times \bigoplus_{i=1}^{k} V_{\alpha_i}$$

and

$$S^1/H \times (V_{\alpha_i+m} \oplus \bigoplus_{i=2}^{k} V_{\alpha_i}) \cong S^1 \times (V_{\alpha_i+m} \oplus \bigoplus_{i=2}^{k} V_{\alpha_i}).$$

Composing these diffeomorphisms leads to the map

$$S^1 \times \mathbb{C}^k \cong S^1 \times \bigoplus_{i=1}^{k} V_{\alpha_i} \to S^1 \times (V_{\alpha_i+m} \oplus \bigoplus_{i=2}^{k} V_{\alpha_i}) \cong S^1 \times \mathbb{C}^k$$
$$(z, v_1, \dots, v_k) \mapsto (z, zv_1, \dots, v_k).$$

This map interchanges the two spin structures on $S^1 \times \mathbb{C}^k$. Therefore one of the two spin structures on $S^1 \times_H V$ equivariantly bounds $D^2 \times (\bigoplus_{i=1}^k V_{\alpha_i})$. The other equivariantly bounds $D^2 \times (V_{\alpha_i+m} \oplus \bigoplus_{i=2}^k V_{\alpha_i})$.

Let $E = \prod_{i=1}^{k+1} \prod_{h=1}^{n_i} X_{j_{hi}} \in \Omega_*^{Spin}[\frac{1}{2}](B\Gamma_{\text{first copy}})$. Then the sphere bundle associated to $\prod_{i=1}^{k+1} \prod_{h=1}^{n_i} X_{j_{hi},\alpha_i}$ gives the desired preimage. Here $X_{j,\alpha}$ denotes the bundle X_j equipped with the action of S^1 induced by multiplication with z^α for $z \in S^1$.

This can be seen as follows: The normal bundle of $S(E)^H$ in $S(E)$ bounds the normal bundle of $D(E)^H$ in $D(E)$. Therefore a tubular neighborhood of an orbit of type $[H, V]$ in $S(E)$, equipped with its natural spin structure, equivariantly bounds $D^2 \times V \times \mathbb{R}^k$. Hence, this spin structure is the one which corresponds to $B\Gamma_{\text{first copy}}$. In particular, $S(E)$ is the desired preimage of E.

Next assume that $E = \prod_{i=1}^{k+1} \prod_{h=1}^{n_i} X_{j_{hi}} \in \Omega_*[\frac{1}{2}](B\Gamma_{\text{second copy}})$, then, by an argument similar to the one from above, the sphere bundle associated to

$$(X_{j_{n_1 1},\alpha_1} \otimes \bar{X}_{j_{1(k+1)},m} \oplus X_{j_{1k+1},-m}) \times \prod_{h=1}^{n_1-1} X_{j_{h1},\alpha_1} \times \prod_{i=2}^{k} \prod_{h=1}^{n_i} X_{j_{hi},\alpha_i}$$

gives the desired preimage. Here $\bar{X}_{j_{1(k+1)}}$ is the dual vector bundle of $X_{j_{1(k+1)}}$. Moreover, $X_{j_{n_1 1}} \otimes \bar{X}_{j_{1(k+1)}} \oplus X_{j_{1k+1}}$ is the vector bundle over the base space $B_1 \times B_2$ of $X_{j_{n_1 1}} \times X_{j_{1k+1}}$ given by the Whitney sum of $pr_1^*(X_{j_{n_1 1}}) \otimes pr_2^*(\bar{X}_{j_{1(k+1)}})$ and $pr_2^*(X_{j_{1k+1}})$.

Next assume that $\sigma_i = [S^1, V]$ where $V = \bigoplus_{i=1}^k V_{\alpha_i}^{n_i}$ with $\alpha_i > \alpha_{i+1}$. For this case we will use Construction 3.3 of twisted projective space bundles $\widetilde{\mathbb{C}TP}(E_1;\ E_2)$.

Case 1: At first also assume, that all weights of V are negative and the minimal weight appears with multiplicity one. Then one can see that the map $\Omega_*^{C,Spin,S^1}[\frac{1}{2}][\sigma_i] \to \Omega_*^{C,Spin,S^1}[\frac{1}{2}][\mathcal{F}_{i-1}]$ has the same image as the first section constructed above for slice types with finite isotropy groups. Hence it is injective.

Case 2: Next assume that all weights except α_1 of V are negative, the minimal weight and α_1 appear with multiplicity one and $0 > \alpha_1 - m \geq \alpha_i$ for all $i > 1$, where $-m = \alpha_k$. Then one can see that the map $\Omega_*^{C,Spin,S^1}[\frac{1}{2}][\sigma_i] \to \Omega_*^{C,Spin,S^1}[\frac{1}{2}][\mathcal{F}_{i-1}]$ has the same image as the second section constructed above for slice types with finite isotropy groups.

Then

$$(X_{j_{11},\alpha_1-m} \otimes \bar{X}_{j_{1k},\alpha_k} \oplus X_{j_{1k},\alpha_k}) \times \prod_{i>1} \prod_{h=1}^{n_i} X_{j_{hi},\alpha_i}, \quad 0 \leq j_{1i} \leq j_{2i} \leq \cdots \leq j_{n_i i} \text{ for } i \geq 1$$

is a basis of $\Omega_*^{C,Spin}[\frac{1}{2}][\sigma_i]$ as an $\Omega_*^{Spin}[\frac{1}{2}]$-module. To see this, note that the map

$$\begin{gathered}\Omega_*[\frac{1}{2}](BU(1) \times BU(1) \times \prod_{i=2}^{k-1} BU(n_i)) \to \Omega_*[\frac{1}{2}](BU(1) \times BU(1) \times \prod_{i=2}^{k-1} BU(n_i))\\ X \oplus Y \oplus Z \;\mapsto\; X \otimes \bar{Y} \oplus Y \oplus Z\end{gathered}$$

is an isomorphism. Moreover, the elements given above are the images of the basis (1) under this isomorphism.

If $j_{11} \geq j_{n_2 2}$ with $\alpha_2 = \alpha_1 - m$ or $\alpha_1 - m > \alpha_2$, then the image of

$$X_{j_{11},\alpha_1-m} \otimes X_{j_{1k},\alpha_k} \oplus X_{j_{1k},\alpha_k} \times \prod_{k>i>1} \prod_{h=1}^{n_i} X_{j_{hi},\alpha_i}$$

under the map $\Omega_*^{C,Spin,S^1}[\frac{1}{2}][\sigma_i] \to \Omega_*^{C,Spin,S^1}[\frac{1}{2}][\mathcal{F}_{i-1}]$ is part of a basis of the image of the second section constructed for normal orbit types with finite isotropy groups. Therefore for these basis elements we do not have to define preimages under ν_i.

Therefore assume that $\alpha_2 = \alpha_1 - m$ and $j_{11} < j_{n_2 2}$. Then

$$\begin{aligned}&\widetilde{\mathbb{C}TP}(\mathbb{C} \times \prod_{h=1}^{n_2-1} X_{j_{h2},\alpha_2} \times \prod_{2<i<k} \prod_{h=1}^{n_i} X_{j_{hi},\alpha_i};\\ &\qquad\qquad (X_{j_{11},\alpha_1-m} \otimes \bar{X}_{j_{1k},\alpha_k} \oplus X_{j_{1k},\alpha_k}) \times X_{j_{2n_2},\alpha_2} \times \mathbb{C})\end{aligned}$$

is mapped by ν_i to

$$\begin{aligned}&\prod_{h=1}^{n_2-1} X_{j_{h2},\alpha_2} \times \prod_{2<i<k} \prod_{h=1}^{n_i} X_{j_{hi},\alpha_i} \times (X_{j_{11},\alpha_1-m} \otimes \bar{X}_{j_{1k},\alpha_k} \oplus X_{j_{1k},\alpha_k} \times X_{j_{2n_2},\alpha_2}\\ &\qquad\qquad - X_{j_{11},\alpha_1-m} \times X_{j_{2n_2},\alpha_2} \otimes \bar{X}_{j_{1k,\alpha_k}} \oplus X_{j_{1k},\alpha_k}).\end{aligned}$$

Here the two components of $\widetilde{\mathbb{C}TP}(E_1;E_2)^{S^1}$ which correspond to the two summands of the above sum are given by $\widetilde{\mathbb{C}TP}(\mathbb{C};\mathbb{C})$ and $\widetilde{\mathbb{C}TP}(\mathbb{C};X_{j_{1k},\alpha_k})$. The other fixed point components are of different slice types. Therefore we have found the generators in this case.

Case 3: Next assume that $\sigma_i = [S^1, V]$, such that all weights except α_1 of V are negative, the minimal weight and α_1 appear with multiplicity one and $\alpha_1 < -\alpha_k = m$, $\alpha_1 - m < \alpha_2$. Then

$$\widetilde{\mathbb{C}TP}(\mathbb{C} \times \prod_{h=1}^{n_2-1} X_{j_{h2},\alpha_2} \times \prod_{2<i<k} \prod_{h=1}^{n_i} X_{j_{hi},\alpha_i};\;\; X_{j_{11},\alpha_1} \times X_{j_{n_2 2},\alpha_2} \times X_{j_{1k},\alpha_k} \times \mathbb{C})$$

is mapped by ν_i to

$$\prod_{i=1}^{k} \prod_{h=1}^{n_i} X_{j_{hi},\alpha_i}.$$

Here the component of $\widetilde{\mathbb{C}TP}(E_1;E_2)^{S^1}$ which corresponds to this product is given by $\widetilde{\mathbb{C}TP}(\mathbb{C};\mathbb{C})^{S^1}$. The other components of the fixed point set are of different slice types. Therefore we have found the generators in this case.

Case 4: Next assume that $\sigma_i = [S^1, V]$, such that $\alpha_1 > 0$ appears with multiplicity at least two and the minimal weight appears with multiplicity one and is negative and $\alpha_1 < -\alpha_k = m$. Then

$$\widetilde{\mathbb{C}TP}(\mathbb{C} \times \prod_{h=3}^{n_1} X_{j_{h1},\alpha_1} \times \prod_{1<i<k} \prod_{h=1}^{n_i} X_{j_{hi},\alpha_i};\;\; X_{j_{11},\alpha_1} \times X_{j_{21},\alpha_1} \times X_{j_{1k},\alpha_k} \times \mathbb{C})$$

is mapped by ν_i to

$$\prod_{i=1}^{k} \prod_{h=1}^{n_i} X_{j_{hi},\alpha_i}$$

Therefore we have found the generators in this case.

Case 5: Next assume that $\sigma_i = [S^1, V]$, such that $\alpha_1 > \alpha_2 > 0$ and the minimal weight appears with multiplicity one and is negative and $\alpha_1 < -\alpha_k = m$. Then

$$\widetilde{\mathbb{C}TP}(\mathbb{C} \times \prod_{h=2}^{n_1} X_{j_{h1},\alpha_1} \times \prod_{h=2}^{n_2} X_{j_{h2},\alpha_2} \times \prod_{1<i<k} \prod_{h=1}^{n_i} X_{j_{hi},\alpha_i};\;\; X_{j_{11},\alpha_1} \times X_{j_{12},\alpha_2} \times X_{j_{1k},\alpha_k} \times \mathbb{C})$$

is mapped by ν_i to

$$\prod_{i=1}^{k} \prod_{h=1}^{n_i} X_{j_{hi},\alpha_i}$$

Therefore we have found the generators in this case.

Case 6: Next assume that $\alpha_1 = 2$ and $\alpha_2 = \pm 1$ are the only weights. Then

$$M = \mathbb{C}P(X_{j_{11}\alpha_1} \oplus \mathbb{C}) \times \mathbb{C}P(X_{j_{12},\alpha_2} \oplus \mathbb{C})$$

is a manifold which satisfies Condition C such that the weights at the four fixed point components are $(-2,-1),(-2,1),(2,-1),(2,1)$. If $\alpha_2=-1$, we conjugate the complex structure on the summand $N(M^{S^1},M^{\mathbb{Z}_2})$ of the normal bundle of the last fixed point component. In this way we get the desired generators in this case.

Case 7: In all other cases let $\beta\in\{\alpha_1,\alpha_k\}$ be a weight of maximal norm. Then there is a second weight $\delta\neq\beta,-\beta$. In these cases one of

$$\widetilde{\mathbb{C}TP}(\mathbb{C}\times\prod_{h=2}^{n_\delta}X_{j_{h\delta},\delta}\times\prod_{\alpha_i\neq\delta,-\beta,\beta}\prod_{h=1}^{n_i}X_{j_{hi},\alpha_i};\;X_{j_{1\delta},\delta}\times\prod_{h=1}^{n_\beta}X_{j_{h,\beta},\beta}\times\prod_{h=1}^{n_{-\beta}}X_{j_{h,-\beta},-\beta}\times\mathbb{C})$$

or

$$\widetilde{\mathbb{C}TP}(\mathbb{C}\times\prod_{\alpha_i\neq-\beta,\beta}\prod_{h=1}^{n_i}X_{j_{hi},\alpha_i};\;\prod_{h=1}^{n_\beta}X_{j_{h,\beta},\beta}\times\prod_{h=1}^{n_{-\beta}}X_{j_{h,-\beta},-\beta}\times\mathbb{C})$$

is spin and mapped by ν_i to

$$\prod_{i=1}^{k}\prod_{h=1}^{n_i}X_{j_{hi},\alpha_i}.$$

Therefore we have found the generators in this case.

Hence we have found generators of the image of ν_i in all cases. This completes the proofs of the theorem. □

4. S^1-manifolds not satisfying Condition C

In this section we prove Theorem 1.1. In view of Theorems 2.2, 3.1 it suffices to prove the following lemma.

Lemma 4.1 *For $G=SO$ or $G=Spin$ and $\mathcal{F}=\mathcal{AE}$ or $\mathcal{F}=\mathcal{AE}-\{[S^1,W];\dim_{\mathbb{C}}W=1\}$, the natural map*

$$\Omega_n^{C,G,S^1}[\frac{1}{2}][\mathcal{F}]\to\Omega_n^{G,S^1}[\frac{1}{2}][\mathcal{F}]$$

is surjective. Here $\Omega_n^{G,S^1}[\mathcal{F}]$ denotes the bordism group of n-dimensional G-manifolds with effective S^1-action of type $\mathcal{F}$ such that, for all subgroups $H\subset S^1$, M^H is orientable.

To prove this lemma, we use an induction as in Section 3. To do so, we first recall that the real slice types of S^1, which appear in an orientable S^1-manifold, are given by

$$[\mathbb{Z}_m;\prod_{-m<\alpha\leq-\frac{m}{2}}V_\alpha^{j_\alpha}]\qquad\qquad[S^1;\prod_{\alpha<0}V_\alpha^{j_\alpha}]$$

Therefore for each real slice type ρ there is a preferred complex slice type σ such that after forgetting the complex structure we have $\rho=\sigma$. Note that this slice type is the first complex slice type representing ρ with respect to the ordering introduced in Section 2. We call two complex slice types equivalent if they represent the same real slice type.

The ordering of preferred complex slice types induces an ordering of the real slice types. Moreover, this leads to a sequence $\mathcal{G}_i$ of families of slice types such that

$$\begin{aligned}
\mathcal{G}_0 &= \{\text{semi-free slice types}\} - \{[S^1, V_{-1}]\} \\
\mathcal{G}_{i+1} &= \mathcal{G}_i \amalg \{\rho_i\} \text{ for a slice type } \rho_i \\
\bigcup_{i=0}^{\infty} \mathcal{G}_i &= \{\text{effective slice types without codimension two fixed point sets}\}
\end{aligned}$$

We will show by induction on i that every $[M] \in \Omega_*^{G,S^1}[\frac{1}{2}][\mathcal{G}_i]$ can be represented by an effective S^1-manifold which satisfies Condition C.

We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
\Omega_*^{C,G,S^1}[\frac{1}{2}][\mathcal{F}_{h(i)-1}] \arrow[r] \arrow[d] & \Omega_*^{C,G,S^1}[\frac{1}{2}][\mathcal{F}_{h(i)}] \arrow[r, "\nu_i^C"] \arrow[d] & \Omega_*^{C,G,S^1}[\frac{1}{2}][\sigma_{h(i)}] \arrow[r, "\partial_{h(i)}^C"] \arrow[d] & \Omega_*^{C,G,S^1}[\frac{1}{2}][\mathcal{F}_{h(i)-1}] \arrow[d] \\
\Omega_*^{G,S^1}[\frac{1}{2}][\mathcal{G}_{i-1}] \arrow[r] & \Omega_*^{G,S^1}[\frac{1}{2}][\mathcal{G}_i] \arrow[r, "\nu_i"] & \Omega_*^{G,S^1}[\frac{1}{2}][\rho_i] \arrow[r, "\partial_i"] & \Omega_*^{G,S^1}[\frac{1}{2}][\mathcal{G}_{i-1}]
\end{tikzcd}$$

Here $\sigma_{h(i)}$ denotes the preferred complex slice type representing ρ_i.

A diagram chase shows that we are done with the induction step if we can show that the composition of maps

$$\Omega_*^{C,G,S^1}[\frac{1}{2}][\mathcal{F}_i] \to \Omega_*^{G,S^1}[\frac{1}{2}][\mathcal{G}_i] \to \ker \partial_i$$

is surjective.

The following isomorphisms can be constructed in a similar way as in the case of the groups $\Omega_*^{C,G,S^1}[\frac{1}{2}][\sigma]$.

$$\begin{aligned}
\Omega_n^{SO,S^1}[\mathbb{Z}_m, \prod_{\alpha=-m}^{-\frac{m}{2}} V_\alpha^{j_\alpha}][\frac{1}{2}] &\cong \Omega^{SO}_{n-1-2\sum_{\alpha=-m}^{-m/2} j_\alpha}[\frac{1}{2}](B(S^1 \times_{\mathbb{Z}_m} SO(2j_{-m/2})) \\
&\quad \times B(\prod_{\alpha=-m}^{-(m+1)/2} U(j_\alpha))) \\
\Omega_n^{Spin,S^1}[\mathbb{Z}_m, \prod_{\alpha=-m}^{-\frac{m}{2}} V_\alpha^{j_\alpha}][\frac{1}{2}] &\cong \Omega^{Spin}_{n-1-2\sum_{\alpha=-m}^{-m/2} j_\alpha}[\frac{1}{2}](B(S^1 \times_{\mathbb{Z}_m} SO(2j_{-m/2})) \\
&\quad \times B(\prod_{\alpha=-m}^{-(m+1)/2} U(j_\alpha))) \text{ if } j_{-m/2} > 0
\end{aligned}$$

$$\Omega_n^{Spin,S^1}[\mathbb{Z}_m, \prod_{\alpha=-m}^{-\frac{m+1}{2}} V_\alpha^{j_\alpha}][\frac{1}{2}] \cong \Omega^{Spin}_{n-1-2\sum_{\alpha=-m}^{-m/2} j_\alpha}[\frac{1}{2}](B(S^1/\mathbb{Z}_m) \times \prod_{\alpha=-m}^{-(m+1)/2} BU(j_\alpha)$$

$$\amalg\, B(S^1/\mathbb{Z}_m) \times \prod_{\alpha=-m}^{-(m+1)/2} BU(j_\alpha))$$

$$\cong \Omega^{Spin}_{n-1-2\sum_{\alpha=-m}^{-m/2} j_\alpha}[\frac{1}{2}](B\Gamma_{\text{first copy}} \amalg B\Gamma_{\text{second copy}})$$

$$\Omega_n^{G,S^1}[S^1, \prod_{\alpha<0} V_\alpha^{j_\alpha}][\frac{1}{2}] \cong \Omega^G_{n-2\sum_\alpha j_\alpha}[\frac{1}{2}](\prod_{\alpha<0} BU(j_\alpha)),$$

For the construction of these isomorphisms in the spin case see [Sai01].

In particular, it follows that the maps $\Omega_n^{C,G,S^1}[\sigma_{h(i)}][\frac{1}{2}] \to \Omega_n^{G,S^1}[\rho_i][\frac{1}{2}]$ are surjective.

Indeed, if $\rho_i \neq [\mathbb{Z}_m, \prod_{\alpha=-m}^{-\frac{m}{2}} V_\alpha^{j_\alpha}]$ with $j_{-\frac{m}{2}} > 0$, then these maps are isomorphisms.

In the case that $\rho_i = [\mathbb{Z}_m, \prod_{\alpha=-m}^{-\frac{m}{2}} V_\alpha^{j_\alpha}]$ with $j_{-\frac{m}{2}} > 0$, we consider the map

$$f : BT \to B\tilde{G},$$

where T is a maximal torus of $\tilde{G} = (S^1/\mathbb{Z}_{\frac{m}{2}}) \times SO(2j_{\frac{m}{2}}) \times \prod_{\alpha\neq\frac{m}{2}} U(j_\alpha)$. This map induces an surjective map in singular homology with coefficients in $\mathbb{Z}[\frac{1}{2}]$ because $\tilde{G}$ does not have odd torsion.

Since $H_*(BT;\mathbb{Z}[\frac{1}{2}])$ and $H_*(B\tilde{G};\mathbb{Z}[\frac{1}{2}])$ are concentrated in even degrees, the Atiyah–Hirzebruch spectral sequence for the bordism groups (with two inverted) for these spaces degenerates at the E^2-level. Hence it follows that f induces a surjective map on bordism groups.

Because this map factors through $B\tilde{G}_C$, where $G_C = (S^1/\mathbb{Z}_{\frac{m}{2}}) \times U(j_{\frac{m}{2}}) \times \prod_{\alpha\neq\frac{m}{2}} U(j_\alpha)$ and G and G_C are two-fold covering groups of

$$(S^1 \times_{\mathbb{Z}_m} SO(2j_{\frac{m}{2}})) \times \prod_{\alpha\neq\frac{m}{2}} U(j_\alpha) \qquad \text{and} \qquad (S^1 \times_{\mathbb{Z}_m} U(j_{\frac{m}{2}})) \times \prod_{\alpha\neq\frac{m}{2}} U(j_\alpha),$$

respectively, it follows from the above isomorphism that $\Omega_n^{C,G,S^1}[\sigma_{h(i)}][\frac{1}{2}] \to \Omega_n^{G,S^1}[\rho_i][\frac{1}{2}]$ is surjective.

We have a section $q_{h(i)} : \operatorname{im} \nu^C_{h(i)} \to \Omega^{C,G,S^1}[\mathcal{F}_{h(i)}][\frac{1}{2}]$. Hence, it suffices to consider those real slice types for which $\nu^C_{h(i)}$ is not surjective. These are of the form

$$[S^1; V_{\alpha_1}^{j_1} \times \cdots \times V_{\alpha_{k-1}}^{j_{k-1}} \times V_{-m}]$$

with $0 > \alpha_1 > \alpha_2 > \cdots > \alpha_k = -m$.

Case 1: At first assume that $\alpha_1 < -\frac{m}{2}$. Then $\Omega^{C,G,S^1}[\sigma_{h(i-1)}][\frac{1}{2}]$ and $\Omega^{G,S^1}[\rho_{i-1}][\frac{1}{2}]$ are isomorphic. Moreover, $\partial^C_{h(i)}$ is a injection into the image of $q_{h(i)-1}$. In the non-spin case we have $\operatorname{im} q_{h(i-1)} = \operatorname{im} \partial^C_{h(i)}$. In the spin case we have $q_{h(i-1)}(B\Gamma_{\text{first copy}}) = \operatorname{im} \partial^C_{h(i)}$.

Since $\Omega^{C,G,S^1}[\sigma_{h(i)}][\frac{1}{2}]$ and $\Omega^{G,S^1}[\rho_i][\frac{1}{2}]$ are isomorphic, it follows that ∂_i is also injective. Therefore in this case the maps $\nu^C_{h(i)}$ and ν_i are the zero maps. Therefore in this case the statement follows from the induction hypothesis.

Case 2: Next assume that $\alpha_1 > -\frac{m}{2}$ and $\alpha_{i_0} > -\frac{m}{2} > \alpha_{i_0+1}$ for some i_0. Then $\sigma'_{h(i)}$ is equivalent to $\tilde{\sigma} = [S^1; V^{j_1}_{-\alpha_1} \times \cdots \times V^{j_{i_0}}_{-\alpha_{i_0}} \times V^{j_{i_0+1}}_{\alpha_{i_0+1}} \times \cdots \times V_{-m}]$.

In the non-spin-case the map $\nu^C_{\tilde{\sigma}}$ is surjective for this slice type. Moreover, the section $q_{\tilde{\sigma}}$ to $\nu^C_{\tilde{\sigma}}$ has the following property: If $X \in \Omega^{SO,S^1}[\frac{1}{2}][\tilde{\sigma}]$, then there is no point in $q_{\tilde{\sigma}}(X)$ with slice type σ' such that $\sigma_{h(i)} \leq \sigma' < \tilde{\sigma}$. Therefore in the family of slice types $\mathcal{F}_{h(i)}$ we can replace $\sigma_{h(i)}$ by $\tilde{\sigma}$. The map $q_{\tilde{\sigma}}$ is then still a section to the map $\nu^C_{\tilde{\sigma}}$ with image in $\Omega^{C,SO,S^1}_*[\mathcal{F}_{h(i)}]$.

In the spin case we have to look at three cases:

- $\alpha_2 > -\frac{m}{2}$
- α_1 is the only weight with $\alpha_1 > -\frac{m}{2}$ and appears with multiplicity greater than one.
- α_1 is the only weight with $\alpha_1 > -\frac{m}{2}$ and appears with multiplicity one.

As seen in Cases 4 and 5 of the proof of Theorem 3.1 in the first two cases the map $\nu^C_{\tilde{\sigma}}$ is surjective. Moreover, one can check that the section $q_{\tilde{\sigma}}$ has the same property as in the non-spin case. Therefore one can argue as in the non-spin case to get the conclusion in these two cases.

In the last case we can replace $\sigma_{h(i)}$ by $\tilde{\sigma}$ by the same reason as in the first two cases. Therefore we have the commutative diagram

$$\begin{tikzcd}
\Omega^{C,G,S^1}_*[\frac{1}{2}][\tilde{\sigma}] \arrow[r] \arrow[d] & \Omega^{C,G,S^1}_*[\frac{1}{2}][\mathcal{F}_{i-1}] \arrow[r] \arrow[d] & \Omega^{C,G,S^1}_*[\frac{1}{2}][\sigma_{i-1}] \arrow[d] \\
\Omega^{C,G,S^1}_*[\frac{1}{2}][\rho_i] \arrow[r] & \Omega^{C,G,S^1}_*[\frac{1}{2}][\mathcal{G}_{i-1}] \arrow[r] & \Omega^{C,G,S^1}_*[\frac{1}{2}][\rho_{i-1}]
\end{tikzcd}$$

where the vertical maps on the left and right are isomorphisms. Moreover, in Case 2 of the proof of Theorem 3.1 it was shown that the restriction of the upper right map to the image of the upper left map is injective. Therefore it follows that the kernel of the lower left map is contained in the image of the kernel of the upper left map under the isomorphism on the left.

From this it follows by diagram chasing that the map $\Omega^{C,G,S^1}_*[\mathcal{F}_{h(i)}] \to \Omega^{G,S^1}_*[\mathcal{G}_i]$ is surjective.

Case 3: Next assume $\alpha_1 > -\frac{m}{2}$ and $\alpha_{i_0} = -\frac{m}{2}$. Then σ_i is equivalent to $\tilde{\sigma} = [S^1; V^{j_1}_{-\alpha_1} \times V^{j_{i_0}}_{-\alpha_{i_0}} \times \prod_{i \neq 1, i_0} V^{j_i}_{\alpha_i}]$. As shown in Case 5 of the proof of Theorem 3.1, for this slice type $\nu^C_{\tilde{\sigma}}$ is surjective in the spin case. In the non-spin case this map is surjective because $-\alpha_1 > 0$. Moreover, the section $q_{\tilde{\sigma}}$ to $\nu^C_{\tilde{\sigma}}$ has the following property: If $X \in \Omega^{G,S^1}[\frac{1}{2}][\tilde{\sigma}]$, then there is no point in $q_{\tilde{\sigma}}(X)$ with slice type σ' such that $\sigma_{h(i)} \leq \sigma' < \tilde{\sigma}$. Therefore we can argue as in the previous cases to get the conclusion.

Case 4: At last assume that $\alpha_1 = -\frac{m}{2}$. Then we have a commutative diagram:

$$\begin{tikzcd}
\Omega^{G,S^1}[\frac{1}{2}][\sigma_i] \arrow[r, "\cong"] \arrow[ddd, "\nu_{i-1}\circ\partial_i"] & \Omega_*[\frac{1}{2}](BS^1/\mathbb{Z}_m \times U(j_1)) \arrow[d] & \Omega_*[\frac{1}{2}](BS^1/\mathbb{Z}_{\frac{m}{2}} \times U(j_1)) \arrow[l, "\cong"'] \arrow[d] \\
 & \Omega_*[\frac{1}{2}](BS^1 \times_{\mathbb{Z}_m} U(j_1)) \arrow[d] & \Omega_*[\frac{1}{2}](BS^1 \times_{\mathbb{Z}_{\frac{m}{2}}} U(j_1)) \arrow[l, "\cong"'] \arrow[d] \\
 & \Omega_*[\frac{1}{2}](BS^1 \times_{\mathbb{Z}_m} SO(2j_1)) \arrow[d] & \Omega_*[\frac{1}{2}](BS^1 \times_{\mathbb{Z}_{\frac{m}{2}}} SO(2j_1)) \arrow[l, "\cong"'] \arrow[d] \\
\Omega^{G,S^1}[\frac{1}{2}][\sigma_{i-1}] \arrow[r, "\cong"] & \Omega_*[\frac{1}{2}](BS^1/\mathbb{Z}_{\frac{m}{2}} \times_{\mathbb{Z}_2} SO(2j_1)) & \Omega_*[\frac{1}{2}](BS^1/\mathbb{Z}_{\frac{m}{2}} \times SO(2j_1)) \arrow[l, "\cong"']
\end{tikzcd}$$

Here we have omitted the $BU(j_i)$ factors with $i > 1$ in the middle and right hand column.

Moreover, the left hand horizontal maps are the isomorphisms from the previous page. The right hand horizontal maps are induced by two-fold coverings of the respective groups.

The right hand vertical maps are as follows:

The first map maps a pair (X, Y) where X is a line bundle over some manifold M and Y is an j_1-dimensional complex vector bundle over M to $(X, X \otimes Y)$.

The second map is the map which forgets the complex structure on Y. The kernel of this map can be described as follows.

There are natural actions of the Weyl-groups $W(U(j_1))$ and $W(SO(2j_1))$ on $\Omega_*(BT^{j_1})$, where $S^1 \times \cdots \times S^1 = T^{j_1} \subset U(j_1) \subset SO(2j_1)$ is a maximal torus. Note that $W(SO(2j_1))$ can be identified with a semi-direct product $\mathbb{Z}_2^{j_1-1} \rtimes S_{j_1}$. Moreover $W(U(j_1))$ can be identified with the permutation subgroup S_{j_1}. An element of $\mathbb{Z}_2^{j_1-1}$ acts on $H^2(BT) = H^2(BS^1) \oplus \cdots \oplus H^2(BS^1)$ by multiplication with -1 on an even number of summands. An element of S_{j_1} acts by permuting the summands. It follows from an inspection of the relevant Atiyah–Hirzebruch spectral sequences that the kernels of the natural surjective maps

$$\Omega_*[\frac{1}{2}](BS^1 \times_{\mathbb{Z}_{\frac{m}{2}}} T^{j_1}) \to \Omega_*[\frac{1}{2}](BS^1 \times_{\mathbb{Z}_{\frac{m}{2}}} U(j_1))$$

and

$$\Omega_*[\frac{1}{2}](BS^1 \times_{\mathbb{Z}_{\frac{m}{2}}} T^{j_1}) \to \Omega_*[\frac{1}{2}](BS^1 \times_{\mathbb{Z}_{\frac{m}{2}}} SO(2j_1))$$

are generated by elements of the form $(X, Z - \gamma Z)$ with $\gamma \in W(U(j_1))$ or $\gamma \in W(SO(2j_1))$, respectively. Therefore the kernel of the map

$$\Omega_*[\frac{1}{2}](BS^1 \times_{\mathbb{Z}_{\frac{m}{2}}} U(j_1)) \to \Omega_*[\frac{1}{2}](BS^1 \times_{\mathbb{Z}_{\frac{m}{2}}} SO(2j_1))$$

is generated by elements of the form $(X, Z - \gamma Z)$ with $\gamma \in W(SO(2j_1))/W(U(j_1)) = \mathbb{Z}_2^{j_1-1}$.

The third map is induced by the isomorphism $S^1 \times_{\mathbb{Z}_{\frac{m}{2}}} SO(2j_1) \to S^1/\mathbb{Z}_{\frac{m}{2}} \times SO(2j_1)$. This isomorphism exists since $\mathbb{Z}_{\frac{m}{2}}$ acts trivially on $SO(2j_1)$.

Since the right hand horizontal maps are induced by two-fold coverings, it follows that, if $j_1 > 1$, then the kernel of the composition of the middle vertical maps is generated by bundles of the form

$$(X, Z - (Z_1 \oplus \bar{Z}_2 \otimes \bar{X})),$$

where X and Z are complex vector bundles over the same base manifold with dimension one and j_1 respectively, γ is an element of $W(SO(2j_1))/W(U(j_1))$. Moreover, we have $Z = Z_1 \oplus Z_2$ with $\gamma Z_1 = Z_1$ and $\gamma Z_2 = \bar{Z}_2$ and $\bar{X}$, $\bar{Z}_2$ denote the conjugated bundles of X and Z_2, respectively.

If $j_1 = 1$ then this map is injective. Therefore in the following we will assume that $j_1 > 1$. In this case a bundle as above is the image under ν_i of

$$\widetilde{\mathbb{C}TP}(Y \oplus Z_1 \oplus \mathbb{C};\ X \oplus Z_2 \oplus \mathbb{C}),$$

where $Z = Z_1 \oplus Z_2$, X is a complex line bundle and Y is a complex vector bundle induced by the projection $BS^1 \times BU(j_1) \times \prod_{i>1} BU(j_i) \to \prod_{i>1} BU(j_i)$. Note that since $W(SO(2j_1))$ acts on $H^2(BT)$ in the way described above, Z_2 is always even-dimensional. Therefore the above manifold is spin if the involved bundles X, Y, Z_1, Z_2 are spin bundles and the base space is a spin manifold. This is the case for our generators of $\Omega_*^{\mathrm{Spin},S^1}[\sigma_i][\frac{1}{2}]$ considered in Section 3. Therefore the lemma is proved.

5. Resolving singularities

Now we turn to the construction of invariant metrics of positive scalar curvature on S^1-manifolds. In this and the next two sections we prepare the proof of Theorems 1.4, 1.5 and 1.6 which will be carried out in Section 8.

In this section we discuss a general construction for invariant metrics of positive scalar curvature on certain S^1-manifolds. These S^1-manifolds are not semi-free and have S^1-fixed points. The construction is a generalization of a construction from [Han08], where it was done for fixed point free S^1-manifolds.

For this construction we need the following technical definition.

Definition 5.1 (cf. [Han08, Definition 21]) *Let M be a manifold (possibly with boundary) with an action of a torus T and an T-invariant Riemannian metric g.*

1. *Assume $T = S^1$ and $M^{S^1} = \emptyset$. We say that g is* scaled *if the vector field generated by the S^1-action is of constant length.*

2. *We call g* normally symmetric in codimension two *if the following holds: Let $H \subset T$ be a closed subgroup and F a component of M^H of codimension two in M. Then there is an T-invariant tubular neighborhood $N_F \subset M$ of F together with an isometric S^1-action σ_F on N_F which commutes with the T-action and has fixed point set F.*

The next lemma is a generalization of Lemma 23 of [Han08] to S^1-manifolds with fixed points.

Lemma 5.2 *Let M be a compact S^1-manifold,* $\dim M \geq 3$*, such that* $\operatorname{codim} M^{S^1} \geq 4$*, and V a small tubular neighborhood of M^{S^1}. If M admits an invariant metric g of positive scalar curvature, then there is another metric $\tilde{g}$ of positive scalar curvature on M such that $\tilde{g}$ is scaled on $M - V$. If g is normally symmetric in codimension two on $M - V$, then the same can be assumed for $\tilde{g}$.*

Proof. Let $V' \subset V$ be a slightly smaller tubular neighborhood of M^{S^1} in M. By an equivariant version of Theorem 2' of [Gaj87] , there is a metric g_1 of positive scalar curvature on $M - V'$, such that in a collar of $SM^{S^1} = \partial V'$, g_1 is of the form $h_1 + ds^2$, where h_1 is the restriction of a metric of the form $g|_{N(M^{S^1},M)} +_H h$ to the normal sphere bundle of M^{S^1} of sufficiently small radius δ, where h is the metric induced by g on M^{S^1} and H is the normal connection. Here ds^2 denotes the standard metric on $[0, 1]$. Moreover, g_1 coincides with g on the complement of V.

We have $N(M^{S^1}, M) = \bigoplus_{i>0} E_i$, where E_i is a complex S^1-bundle such that each $z \in S^1$ acts by multiplication with z^i on E_i. Since g is S^1-invariant, the E_i are orthogonal to each other with respect to $g|_{N(M^{S^1},M)}$. Using O'Neill's formula for the fibration $S_\delta \hookrightarrow SM^{S^1} \to M^{S^1}$, one sees that there is a $\delta' > 0$ such that h_1 is isotopic via S^1-invariant metrics of positive scalar curvature to the restriction h_2 of $\delta'(\sum_{i>0} \frac{1}{i^2} g|_{E_i}) +_H h$ to SM^{S^1}. (The isotopy is given by convex combination of the metrics $\sum_{i>0} \frac{1}{i^2} g|_{E_i} +_H h$ and $g|_{N(M^{S^1},M)} +_H h$ and rescaling the fibers.) Note that h_2 is scaled.

Therefore by Lemma 3 of [GL80], there is a invariant metric g_2 of positive scalar curvature on $M - V'$ such that g_2 restricted to a collar of $\partial V'$ is of the form $h_2 + ds^2$ and g_2 restricted to $M - V$ is equal to g.

Now let $X : M - V' \to T(M - V')$ be the vector field generated by the S^1-action. Because there are no fixed points in $M - V'$, X is nowhere zero. Denote by $\mathcal{V}$ the one-dimensional subbundle of $T(M - V')$ generated by X and $\mathcal{H}$ its orthogonal complement with respect to g_2. For $p \in M - V'$, define $f(p) = \|X(p)\|_{g_2}$. Note that f is constant in a small neighborhood of $\partial V'$.

Next we describe some local Riemannian submersions which are useful to show that our scaled metrics have positive scalar curvature. Let $S^1 \times_H D(W) \subset M - V'$ be a tube around an orbit in $M - V'$. We pull back the metric g_2 via the covering $S^1 \times D(W) \to S^1 \times_H D(W)$. This yields a metric which is invariant under the free circle action on the first factor. Let g_2' be the induced quotient metric on $D(W)$. Then the argument from the proof of Theorem C of [BB83], shows that the metric

$$f^{\frac{2}{\dim M - 2}} \cdot g_2'$$

on $D(W)$ has positive scalar curvature.

Now let dt^2 be the metric on $\mathcal{V}$ for which X has constant length one. By O'Neill's formula applied in the above local fibration, the scalar curvature of the metric

$$g_{\epsilon,3} = (\epsilon^2 \cdot dt^2) + (f^{\frac{2}{\dim M - 2}} \cdot g_2|_{\mathcal{H}})$$

on $M - V'$ is given by

$$\mathrm{scal}_{g_{\epsilon,3}} = \mathrm{scal}_{f^{\frac{2}{\dim M-2}} g_2'} - \epsilon^2 \|A\|_{g_{1,3}}, \tag{2}$$

where A is the A-tensor for the connection induced by $g_{1,3}$ in the fibration $S^1 \times D(W) \to D(W)$. Since M is compact it follows that there is an $\epsilon_0 > 0$ such that for all $\epsilon_0 > \epsilon > 0$ the metric $g_{\epsilon,3}$ has positive scalar curvature.

Moreover, since the restriction of g_2 to a collar of $\partial V'$ was a product metric and f is constant in this neighborhood $g_{\epsilon,3}$ is also a product metric on this collar.

Next we show that $g_{\epsilon,3}|_{\partial V}$ and h_2 are isotopic via invariant metrics of positive scalar curvature. We have

$$\begin{aligned} g_{\epsilon,3}|_{\partial V'} &= (\epsilon^2 \cdot dt^2) + (f^{\frac{2}{\dim M-2}} \cdot g_2|_{\mathcal{H}\cap T\partial V'}) \\ h_2 &= f^{-\frac{2}{\dim M-2}}((f^2 f^{\frac{2}{\dim M-2}} \cdot dt^2) + (f^{\frac{2}{\dim M-2}} \cdot g_2|_{\mathcal{H}\cap T\partial V'})), \end{aligned}$$

where $f > 0$ is constant.

In other words, h_2 is equal to the metric $g_{f \cdot f^{\frac{1}{\dim M-2}},3}$ up to scaling. Hence, it follows from formula (2) that the metrics h_2 and $g_{\epsilon,3}|_{\partial V}$ are isotopic via invariant metrics of positive scalar curvature because both metrics have positive scalar curvature. One only has to increase or decrease the parameter ϵ and then rescale the metric. Since h_2 is isotopic to h_1 it follows from Lemma 3 of [GL80], that there is an invariant metric of positive scalar curvature on M whose restriction to $M - V'$ is $g_{\epsilon,3}$.

The remark about the normally symmetric metrics of positive scalar curvature can be seen as follows. Because the local S^1-actions σ_F commute with the global S^1-action, they respect the decomposition $T(M - V) = \mathcal{H} \oplus \mathcal{V}$. Therefore the new metric is invariant under these actions. This completes the proof. □

Remark 5.3 *Note that every $g_{\epsilon,3}$, $0 < \epsilon < \epsilon_0$, can be extended to an invariant metric of positive scalar curvature on M and that the restrictions of all these metrics to $\mathcal{H}$ are the same. This shows that the metric $g_{\epsilon,3}$ can be scaled down on $\mathcal{V}$ without effecting the restriction of the metric to $\mathcal{H}$ and the fact that the $g_{\epsilon,3}$ can be extended to a metric of positive scalar curvature on M.*

Next we describe a resolution of singularities for singular strata of codimension two from [Han08]. Let M be a S^1-manifold of dimension $n \geq 3$ and

$$\phi : S^1 \times_H (S^{n-3} \times D(W)) \hookrightarrow M$$

be an S^1-equivariant embedding where H is a finite subgroup of S^1 and W is an one-dimensional unitary effective H-representation. Here H acts trivially on S^{n-3}.

Since $S(W)/H$ can be identified with S^1, the S^1-principal bundle

$$S^1 \hookrightarrow S^1 \times_H (S^{n-3} \times S(W)) \to S^{n-3} \times S(W)/H$$

is trivial. Choose a trivialization

$$\chi : S^1 \times_H (S^{n-3} \times S(W)) \to S^1 \times S^{n-3} \times S(W)/H$$

and consider $S(W)/H$ as the boundary of D^2. Then we can glue the free S^1-manifold $S^1 \times S^{n-3} \times D^2$ to $M - \operatorname{im}\phi$ to get a new S^1-manifold M'. We say that M' is obtained from M by resolving the singular stratum $\phi(S^1 \times_H (S^{n-3} \times \{0\}))$.

Now we can state the following generalization of Theorem 25 of [Han08].

Theorem 5.4 *Let M be a closed S^1-manifold of dimension $n \geq 3$ such that* $\operatorname{codim} M^{S^1} \geq 4$ *and $H \subset S^1$ a finite subgroup. Let V be an invariant tubular neighborhood of M^{S^1}. Moreover, let*

$$\phi : S^1 \times_H (S^{n-3} \times D(W)) \hookrightarrow M - V$$

be an S^1-equivariant embedding where W is a unitary effective H-representation of dimension one. Let M' be obtained from M by resolving the singular stratum $\phi(S^1 \times_H (S^{n-3} \times \{0\})) \subset M$. If M admits an invariant metric of positive scalar curvature which is normally symmetric in codimension 2 outside of V, then also M' admits such a metric.

Since the proof of Theorem 25 of [Han08] is mainly a local argument in a neighborhood of the singular stratum which is resolved with some down scaling at the end, its proof is also valid in the case where M has fixed points components of codimension at least four, by Lemma 5.2 and Remark 5.3. Therefore we have the above theorem.

The next step is the following generalization of Lemma 24 of [Han08]:

Lemma 5.5 *Let Z be a compact orientable S^1-bordism between S^1-manifolds X and Y such that for all subgroups $H \subset S^1$ all components of codimension two of Z^H are orientable. Assume that X carries an invariant metric of positive scalar curvature which is normally symmetric in codimension 2. If Z admits a decomposition into special S^1-handles of codimension at least 3, then Y carries an invariant metric of positive scalar curvature which is normally symmetric in codimension 2.*

Proof. First recall that a special S^1-handle is an S^1-handle of the form

$$S^1 \times_H (D^d \times D(W)),$$

where H is a subgroup of S^1, W is an orthogonal H-representation, $D(W)$ is the unit disc in W and H acts trivially on the d-dimensional disc D^d. Here the codimension of the handle is given by $\dim W$.

Therefore as in the proof of Lemma 24 in [Han08] we may assume that Y may be constructed from X by equivariant surgery on $S = S^1 \times_H (S^{d-1} \times D(W))$.

First assume that $d = 0$. Then we have $S = \emptyset$. Hence the surgery on S produces a new component $S^1 \times_H (D^0 \times S(W))$ of Y. Since Z is orientable, there is a homomorphism $\phi : H \to SO(n)$ which corresponds to the H-representation W. As a subgroup of S^1, H has a dense cyclic subgroup. Therefore $\phi(H)$ is contained in a maximal torus of $SO(n)$.

Hence W is isomorphic to $\mathbb{R}^k \oplus \bigoplus_i W_i$, where $\mathbb{R}^k$ denotes the trivial H-representation and the W_i are complex one-dimensional H-representations.

If $K \subset S^1$ is a subgroup such that $S^1 \times_H (D^0 \times S(W))^K \neq \emptyset$, then K is contained in H. If S^K has codimension two in S, then there is exactly one W_{i_0} such that K acts non-trivially on W_{i_0}. Therefore the S^1-action on $S^1 \times_H (D^0 \times S(W))$, which is induced by complex multiplication on W_{i_0} commutes with the original S^1-action and leaves all connection metrics induced from the round metric on $S(W)$ invariant. Since there are such connection metrics with positive scalar curvature on $S^1 \times_H (D^0 \times S(W))$ the theorem follows in this case.

Now assume that $d \geq 1$. Then S is non-empty. The proof proceeds as in Hanke's paper. This is done as follows. As in Hanke's paper we may assume that there is a $K \subset H$ and a component $F \subset X^K$ of codimension 2 with

$$S^1 \times_H (S^{d-1} \times \{0\}) \subset F.$$

Moreover, the extra symmetry σ induces an orthogonal action σ of S^1 on the third factor of $S^1 \times_H (S^{d-1} \times D(W))$. This action extends to an orthogonal action on the third factor of the handle $S^1 \times_H (S^{d-1} \times D(W))$.

Indeed, if $d > 1$, then S^{d-1} is connected and hence the action extends.

If $d = 1$, then $S^{d-1} = \{\pm 1\}$ has two components. And in principle the $H \times S^1$-representations $\sigma_\pm$ on $\{\pm 1\} \times W$ might have different isomorphism types. If this happens the S^1-action on $S^1 \times_H (S^{d-1} \times D(W)$ cannot be extended to an action on the handle $S^1 \times_H (D^d \times D(W))$.

Therefore we have to rule out this case. This is done as follows. By assumption $F' = (F \times I) \cup S^1 \times_H ([-1, +1] \times D(W)^K) \subset Z^K$ is orientable. Therefore the structure group of $N(F', Z)$ is $SO(2) = U(1)$ and the S^1-action σ extends to an action which is defined on a neighborhood of F'.

Since the actions $\sigma_\pm$ are restrictions of this action their isomorphism types coincide. Therefore the actions $\sigma_\pm$ extend to an orthogonal action on $S^1 \times_H ([-1, 1] \times D(W))$ with fixed point set $S^1 \times_H ([-1, 1] \times D(W)^K)$.

Now one can construct an invariant metric of positive scalar curvature which is normally symmetric in codimension 2 on Y as in the proof of Lemma 24 of [Han08]. $\square$

Remark 5.6 *It follows from an argument of Edmonds [Edm81] that the condition on the codimension-two singular strata of Z in the above lemma is always satisfied if Z is spin (see also section 10 of Bott-Taubes [BT89]).*

Let $p : B \to BO$ be a fibration. A B-structure on a manifold M is a lift $\hat{\nu} : M \to B$ of the classifying map $\nu \to BO$ of the stable normal bundle of M. We denote by Ω^B_{n,S^1} the equivariant bordism group of S^1-manifolds with B-structures on maximal strata, i.e. an element of Ω^B_{n,S^1} is represented by a pair $(M, \hat{\nu})$, where M is an n-dimensional S^1-manifold and $\hat{\nu} : M_{\max} \to B$ is a B-structure. Moreover, such a pair represents zero if there is an $n+1$-dimensional S^1-manifold with boundary W with a B-structure $f : W_{\max} \to B$ on its maximal stratum such that $\partial W = M$ and $f|_{M_{\max}} = \hat{\nu}$.

Lemma 5.7 *Let M be a connected S^1-manifold of dimension $n \geq 6$ with a B-structure $\hat{\nu} : M_{max} \to B$ on its maximal stratum such that $\hat{\nu}$ is a two-equivalence, i.e. $\hat{\nu}$ induces an isomorphism on π_1 and a surjection on π_2. Let W be a connected equivariant B-bordism between M and another S^1-manifold N. Then there is a B-bordism W' between M and N such that $M_{max} \hookrightarrow W'_{max}$ is a two-equivalence and there is a diffeomorphism $V \to V'$, where V and V' are open neighborhoods of $\partial W \cup W_{sing}$ and $\partial W' \cup W'_{sing}$, respectively.*

Proof. We first show that surgery on $W_{\max}$ can be used to construct a B-bordism W_1 such that the inclusion $M_{\max} \hookrightarrow W_{1,\max}$ induces an isomorphism on fundamental groups. We have the following commutative diagram with exact rows.

$$\begin{tikzcd}
\pi_1(S^1) \arrow[r] \arrow[d, "\mathrm{Id}"] \arrow[ddd, bend left, "\mathrm{Id}"] & \pi_1(M_{\max}) \arrow[r] \arrow[d] \arrow[ddd, bend left, "\mathrm{Id}"] & \pi_1(M_{\max}/S^1) \arrow[r] \arrow[d] \arrow[ddd, bend left, "\mathrm{Id}"] & 0 \\
\pi_1(S^1) \arrow[r] \arrow[dd, "\mathrm{Id}"] & \pi_1(W_{\max}) \arrow[r] \arrow[d, "f_*"] & \pi_1(W_{\max}/S^1) \arrow[r] \arrow[dd, dashed, "\phi"] & 0 \\
 & \pi_1(B) \arrow[d, "\hat{\nu}_*^{-1}"] & & \\
\pi_1(S^1) \arrow[r] & \pi_1(M_{\max}) \arrow[r] & \pi_1(M_{\max}/S^1) \arrow[r] & 0
\end{tikzcd}$$

Here the dashed map ϕ is induced by the commutativity of the diagram and the universal property of the quotient group. It follows from an easy diagram chase that $\ker\phi \cong \ker\hat{\nu}_*^{-1} \circ f_*$.

Let $c : S^1 \to W_{\max}/S^1$ be an embedding that represents an element of $\ker\phi$ and does not meet the boundary. Then there is an equivariant embedding $c' : S^1 \times S^1 \to W_{\max}$, where S^1 acts by rotation on the first factor, such that $c = \pi \circ c' \circ \iota_2$, where $\iota_2 : S^1 \to S^1 \times S^1$ is the inclusion of the second factor and $\pi : W_{\max} \to W_{\max}/S^1$ is the orbit map. Since $c \in \ker\phi$, there is a map $c'' : S^1 \to S^1$ such that $c' \circ (\iota_2 * (\iota_1 \circ c''))$ is contained in the kernel of f_*. Therefore by precomposing c' with

$$(s,t) \mapsto (sc''(t), t)$$

we may assume that $c' \circ \iota_2 \in \ker\hat{\nu}_*^{-1} \circ f_*$. Therefore we can do equivariant surgery on c' to construct a new B-bordism W' such that $\pi_1(W'_{\max}/S^1) = \pi_1(W_{\max}/S^1)/\langle c\rangle$, where $\langle c\rangle$ denotes the normal subgroup of $\pi_1(W_{\max}/S^1)$ which is generated by c.

Since $\pi_1(W_{\max}/S^1)$ and $\pi_1(M_{\max}/S^1)$ are finitely presentable, it follows that $\ker\phi$ is finitely generated as a normal subgroup of $\pi_1(W_{\max}/S^1)$. Therefore after a finite number of iterations of the above surgery step we may achieve that $\ker\hat{\nu}_*^{-1} \circ f_* = 0$. This is equivalent to the fact that $\pi_1(M_{\max}) \to \pi_1(W_{\max})$ is an isomorphism.

The next step is to make the map $\pi_2(M_{\max}) \to \pi_2(W_{\max})$ surjective. We have the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \pi_2(M_{\max}) \arrow[r] \arrow[d] & \pi_2(M_{\max}/S^1) \arrow[r] \arrow[d] & \pi_1(S^1) \arrow[r] \arrow[d, "\cong"] & \pi_1(M_{\max}) \arrow[d, "\cong"] \\
0 \arrow[r] & \pi_2(W_{\max}) \arrow[r] & \pi_2(W_{\max}/S^1) \arrow[r] & \pi_1(S^1) \arrow[r] & \pi_1(W_{\max})
\end{tikzcd}$$

It follows from an application of the snake lemma that there is an isomorphism

$$\pi_2(W_{\max})/\pi_2(M_{\max}) \to \pi_2(W_{\max}/S^1)/\pi_2(M_{\max}/S^1).$$

Let $c : S^2 \to W_{\max}$ be a representative of a class in $\pi_2(W_{\max})/\pi_2(M_{\max})$ such that $\pi \circ c$ is an embedding. Then there is an equivariant embedding $c' : S^1 \times S^2 \hookrightarrow W_{\max}$ such that $c' \circ \iota_2 = c$. Since $\hat{\nu}_* : \pi_2(M_{\max}) \to \pi_2(B)$ is surjective we may assume that $f_*c = 0$. Therefore we can do equivariant surgery on c' to construct a new B-bordism W' such that

$$\pi_2(W'_{\max}/S^1) = \pi_2(W_{\max}/S^1)/\langle \pi_* c\rangle,$$

where $\langle \pi_* c\rangle$ denotes the $\mathbb{Z}[\pi_1(W_{\max}/S^1)]$-submodule of $\pi_2(W_{\max}/S^1)$ which is generated by $\pi_* c$.

Since $\pi_2(W_{\max})/S^1$ is a finitely generated $\mathbb{Z}[\pi_1(W_{\max}/S^1)]$-module we get after a finite number of repetitions of this step a bordism W' for which

$$\pi_2(W'_{\max})/\pi_2(M_{\max}) = 0.$$

This completes the proof. □

Remark 5.8 *If in the situation of the above lemma M_{max} is spin, then $p : B = B\pi_1(M_{max}) \times BSpin \to BO$ and $\hat{\nu} = f \times s : M_{max} \to B$, where p is the composition of the projection on the second factor with the natural fibration $BSpin \to BO$, f is the classifying map of the universal covering of M_{max} and s is a Spin-structure on M_{max}, satisfies the assumptions on B.*

If M_{max} is orientable, not spin with universal covering not spin, then $p : B = B\pi_1(M_{max}) \times BSO \to BO$ and $\hat{\nu} = f \times s : M_{max} \to B$, where p is the composition of the projection on the second factor with the natural fibration $BSO \to BO$, f is the classifying map of the universal covering of M_{max} and s is a orientation on M_{max}, satisfies the assumptions on B.

If M_{max} is orientable, not spin with universal covering spin, then it follows that $w_2(M_{max}) = f^*(\beta)$ for some $\beta \in H^2(B\pi_1(M_{max}; \mathbb{Z}/2\mathbb{Z}))$. Let $Y(\pi_1(M_{max}), \beta)$ be the pullback of $\beta : B\pi_1(M_{max}) \to B\mathbb{Z}/2\mathbb{Z}$ and $w_2 : BSO \to B\mathbb{Z}/2\mathbb{Z}$. Then $f \times s : M_{max} \to B\pi_1(M_{max}) \times BSO$ as in the previous case lifts to a map $\hat{\nu} : M_{max} \to Y(\pi_1(M_{max}), \beta)$. If we let $p : B = Y(\pi_1(M_{max}), \beta) \to BO$ be the composition of the natural fibrations $Y(\pi_1(M_{max}), \beta) \to BSO$ and $BSO \to BO$, then $\hat{\nu}$ and B satisfy the assumptions from the above lemma.

For more details see [RS94].

Now we can prove the following generalization of Theorem 34 of [Han08].

Theorem 5.9 *Let Z be a compact connected oriented S^1-bordism between closed S^1-manifolds X and Y. Assume that for all subgroups $H \subset S^1$ all components of codimension two of Z^H are orientable and that the following holds*

1. $\dim Z/S^1 \geq 6$,
2. $\operatorname{codim} Z^{S^1} \geq 4$,
3. *There is a B-structure on Z_{max}, whose restriction to Y_{max} induces a two-equivalence $Y_{max} \to B$.*

Then, if X admits an S^1-invariant metric of positive scalar curvature which is normally symmetric in codimension 2, then Y admits an S^1-invariant metric of positive scalar curvature which is normally symmetric in codimension 2 outside a tubular neighborhood of Y^{S^1}.

Proof. Let $\dim Z = n+1$. The first step in the proof is to replace the bordism Z by a B-bordism Z' between X and Y' such that

1. All codimension two strata in Z' meet Y'.
2. Y can be constructed from Y' by resolving singular strata.

By (2) and Theorem 5.4 it is sufficient to construct a metric of positive scalar curvature which is normally symmetric in codimension 2 on Y'. The construction of Z' is as follows. Let F be a codimension 2 singular stratum in Z which does not meet Y and $\Omega \subset F$ an orbit. By the slice theorem there is a S^1-invariant tubular neighborhood N of Ω in Z which is S^1-equivariantly diffeomorphic to

$$S^1 \times_H (D^{n-2} \times D(W)).$$

Then we have $\partial N = S^1 \times_H (D^{n-2} \times S(W) \cup S^{n-3} \times D(W))$. Let $B \subset \partial N$ be a tubular neighborhood of an orbit in $S^1 \times_H (D^{n-2} \times S(W))$. Then B is equivariantly diffeomorphic to $S^1 \times D^{n-1}$. Since $Z_{\max}/S^1$ is connected, there is an embedding

$$\Psi : S^1 \times D^{n-1} \times [0,1] \hookrightarrow Z_{\max},$$

such that

$$\begin{aligned} &S^1 \times D^{n-1} \times \{0\} \subset Y_{\max} \\ &S^1 \times D^{n-1} \times \{1\} = B \\ &S^1 \times D^{n-1} \times]0,1[\subset Z - (Y \cup N). \end{aligned}$$

We set $Z' = Z - (N \cup \operatorname{im} \Psi)$.

As in the proof of Theorem 34 of [Han08] one sees that (1) and (2) hold. To be more precise we have an equivariant diffeomorphism

$$Y \cong (Y' - \phi'(S^1 \times_H (S^{n-3} \times D(W)))) \cup \phi(S^1 \times S^{n-3} \times D^2),$$

where $\phi' : S^1 \times_H (S^{n-3} \times D(W)) \to Y'$ and $\phi : S^1 \times S^{n-3} \times D^2 \to Y$ are equivariant embeddings, such that $\operatorname{im} \phi$ is contained in an equivariant coordinate chart of Y.

Since $\pi_2(B)$ is finitely generated as a $\mathbb{Z}[\pi_1(B)]$-module we can assume that $\operatorname{im} \Psi$ avoids a finite set of embedded two-spheres which are mapped by the B-structure ν to the generators of $\pi_2(B)$. Hence, we may assume that

$$\nu'_* : \pi_2(Y'_{\max}) \to \pi_2(B)$$

is still surjective.

But there might be a non-trivial linking sphere $S^1 \subset Y'_{\max}/S^1$ of $\Sigma/S^1 \subset Y'/S^1$ where $\Sigma \subset Y'$ is the singular stratum $\phi'(S^1 \times_H (S^{n-3} \times \{0\})$. This problem can be dealt with as in Hanke's paper by attaching a 2-handle to Z which can be canceled by a 3-handle. Therefore the same argument as in Hanke's paper leads to an isomorphism $\pi_1(Y') \to \pi_1(B)$.

Now it follows from Lemma 5.7 and Theorem 15 of [Han08] (with the refinement of Lemma 5.5), that Y' admits an invariant metric of positive scalar curvature which is normally symmetric in codimension 2. Therefore it follows from Theorem 5.4 that Y admits such a metric. □

Remark 5.10 *Given the other conditions on Z from the above Theorem, the condition* $\operatorname{codim} Z^{S^1} \geq 4$ *cannot be relaxed. This can be seen as follows. Let Y be a free simply connected S^1-manifold, whose orbit space does not admit a metric of positive scalar curvature. Then Y is necessarily spin and the S^1-action is of even type. Let Z be the trace of an equivariant surgery on an orbit in M, as in Lemma 3.1 of [Wie15]. Then Z is a semi-free S^1-manifold, not spin and has a codimension two fixed point component which meets the boundary component X which is not equal to Y. By Theorem 2.4 of [Wie15], X admits an invariant metric of positive scalar curvature. But Z_{max} is homotopy equivalent to Y and therefore admits a Spin-structure. Therefore, by Remark 5.8, all assumptions of Theorem 5.9 except the one about the codimension of the fixed point set are satisfied.*

We have the following corollaries to Theorem 5.9:

Corollary 5.11 *Let M be a Spin S^1-manifold of dimension at least six with simply connected maximal stratum and without fixed point components of codimension two. Then M admits an invariant metric of positive scalar curvature which is normally symmetric in codimension 2 if and only if M is equivariantly spin-bordant to a manifold M' which admits such a metric and has no fixed point components of codimension two, such that the bordism Z between M and M' does not have fixed point components of codimension two.*

Proof. Let Z be an equivariant Spin cobordism between M and M' as above.

Then, by Remark 5.6, all codimension two strata of Z are orientable. Therefore the corollary follows from Theorem 5.9. $\square$

Corollary 5.12 *Let M be a S^1-manifold of dimension at least six with simply connected non-spin maximal stratum and without fixed point components of codimension two such that all singular strata of M are orientable. Then M admits an invariant metric of positive scalar curvature which is normally symmetric in codimension 2 if and only if there is an equivariant bordism whose singular strata are orientable between M and a manifold M' which admits such a metric and has no fixed point components of codimension two.*

Proof. Let Z be a equivariant bordism between M and M'. Then there might be codimension two fixed point components in Z. But by assumption they do not meet the boundary. Therefore we can cut them out of Z. This construction leads to a new bordism Z' between M and $M' \amalg M_1 \amalg \cdots \amalg M_k$, where the S^1 acts freely on the M_i. After attaching handles of codimension at least 3 to Z', we may assume that the M_i/S^1 are simply connected and not spin. Since the M_i/S^1 are not spin, it follows from Theorem C of [BB83] that there is an invariant metric of positive scalar curvature on each M_i. Hence, the corollary follows from Theorem 5.9. $\square$

Note that if the T-manifold M satisfies condition C, then for all closed subgroups $H \subset T$ the fixed point set M^H is orientable.

Lemma 5.13 *Let M be a S^1_0-manifold, where $S^1_0 = S^1$, which satisfies condition C such that there is a fixed point component F of codimension two. Then there is an invariant metric of positive scalar curvature on M which is normally symmetric in codimension two.*

Proof. This follows from an inspection of the proof of Theorem 2.4 of [Wie15] and Lemma 24 of [Han08]. We use the same notation as in the proof of Theorem 2.4 of [Wie15].

Since M satisfies condition C, this also holds for the $S^1_0 \times S^1_1$-manifold $Z \times D^2$. Hence it follows from Lemma 24[1] of [Han08] , that

$$\partial(Z \times D^2) = SF \times D^2 \cup Z \times S^1,$$

admits a $S^1_0 \times S^1_1$-invariant metric of positive scalar curvature which is normally symmetric in codimension two.

For $H \subsetneq S^1_0$, we have $p^{-1}(M^H) = (\partial(Z \times D^2))^H$, where

$$p : \partial(Z \times D^2) \to M = \partial(Z \times D^2)/\operatorname{diag}(S^1_0 \times S^1_1)$$

[1]The proof of this lemma also holds for T-manifolds where T is a torus, instead of S^1-manifolds.

is the orbit map of the $\mathrm{diag}(S^1_0 \times S^1_1)$-action. Hence, it follows from the construction in the proof of Theorem 2.2 of [Wie15] that M admits a S^1_0-invariant metric of positive scalar curvature which is normally symmetric in codimension two. $\square$

Using the above lemma we can prove the following theorem:

Theorem 5.14 *Let M be a connected S^1-manifold satisfying condition C, such that*

$$\pi_1(M_{max}) = 0$$

and M_{max} is not Spin. Moreover, let $J \subset \Omega^*_{C,S^1}$ be the ideal generated by connected manifolds with non-trivial S^1-actions. If $\dim M \geq 6$ and $[M] \in J^2$, then M admits an S^1-invariant metric of positive scalar curvature.

Proof. By Theorem 2.4 of [Wie15] we may assume that $\operatorname{codim} M^{S^1} \geq 4$. Let M_i, N_i be connected manifolds with non-trivial S^1-action satisfying Condition C, such that

$$[M] = \sum_i [M_i \times N_i].$$

Since $\Omega^1_{C,S^1} = 0$, we may assume that $\dim M_i, \dim N_i \geq 2$ for all i. Hence, by Lemma 3.1 of [Wie15] we may assume that all M_i and N_i have S^1-fixed point components of codimension two. Therefore by Lemma 5.13, we may assume that $M_i \times N_i$ admits an S^1-invariant metric of positive scalar curvature which is normally symmetric in codimension two. Hence the theorem follows from Corollary 5.12. $\square$

6. Normally symmetric metrics are generic

In this section we prove that under mild conditions on the isotropy groups of the singular strata of codimension two in an S^1-manifold M, any invariant metric g on M can be deformed to a metric which is normally symmetric in codimension two.

The main result of this section is as follows:

Theorem 6.1 *Let M be an orientable effective S^1-manifold. Moreover, let g be an invariant metric on M.*

If there are no codimension two singular strata with isotropy group $\mathbb{Z}_2$, then there is an invariant metric g' on M which is C^2-close to g and normally symmetric in codimension two.

Proof. Let $N \subset M$ be a codimension-two open singular stratum of M. Let $U = S^1 \times_H W \times \mathbb{R}^{n-3}$ be a neighborhood of an orbit in N. Here W can be assumed to be the standard one-dimensional complex representation of $H \subset S^1$ because the S^1-action on M is effective.

We pull back g to a metric $\tilde{g}$ on $\tilde{U} = S^1 \times W \times \mathbb{R}^{n-3}$. This metric is $S^1 \times H$-invariant. Let

$$h : T\tilde{U} \otimes T\tilde{U} \to \mathbb{R}$$

be the Taylor expansion of $\tilde{g}$ in directions tangent to W up to terms of degree two. Then h might be thought of as an invariant function on

$$(S^1 \times W \times \mathbb{R}^{n-3}) \times (W \times \mathbb{R}^{n-2}) \times (W \times \mathbb{R}^{n-2}),$$

which is linear in the copies of $W \times \mathbb{R}^{n-2}$ and a polynomial of degree two in the first copy of W. Therefore h can be identified with a map $\mathbb{R}^{n-3} \to ((S^0W^* \oplus S^1W^* \oplus S^2W^*) \otimes_{\mathbb{R}} (W^* \oplus_{\mathbb{R}} \mathbb{R}^{n-2}) \otimes_{\mathbb{R}} (W^* \oplus \mathbb{R}^{n-2}))^H$, where S^iW^* denotes the i-th symmetric product of W^*. There are $a_i \in \mathbb{N}$, such that

$$\begin{aligned}
&\Big((S^0W^* \oplus S^1W^* \oplus S^2W^*) \otimes_{\mathbb{R}} ((W^* \oplus \mathbb{R}^{n-2})) \otimes_{\mathbb{R}} (W^* \oplus \mathbb{R}^{n-2})\Big)^H \otimes \mathbb{C} \\
&\subset \Big((\mathbb{C} \oplus (W^* \oplus W) \oplus (W^* \oplus W) \otimes_{\mathbb{C}} (W^* \oplus W)) \otimes_{\mathbb{C}} ((W^* \oplus W \oplus (n-2)\mathbb{C}) \\
&\quad \otimes_{\mathbb{C}} (W^* \oplus W \oplus (n-2)\mathbb{C})\Big)^H \\
&= \Big(W^{*\otimes 4} \oplus W^{\otimes 4} \oplus a_3(W^{*\otimes 3} \oplus W^{\otimes 3}) \oplus a_2(W^{*\otimes 2} \oplus W^{\otimes 2}) \\
&\quad \oplus a_1(W^* \oplus W) \oplus a_0\mathbb{C}\Big)^H
\end{aligned}$$

Moreover, for H of order greater than 4 and $b \leq 4$, we have

$$(W^{\otimes b})^H = (W^{\otimes b})^{S^1} \qquad\qquad (W^{*\otimes b})^H = (W^{*\otimes b})^{S^1}.$$

Hence, it follows that h is invariant under the rotational action of S^1 on W if the order of H is greater than 4.

Now we can deform $\tilde{g}$ so that it coincides with h in a neighborhood of $S^1 \times \{0\} \subset \tilde{U}$. This metric induces a metric on U which is invariant under the rotational action of S^1 on W.

Since N is orientable the rotational action on W extends to an action on a neighborhood of N in M with fixed point set N. Therefore we can glue the metrics on different neighborhoods of orbits in N. This implies the claim if there are no singular strata of codimension two with isotropy group $\mathbb{Z}_k$, $k \leq 4$.

Now assume that that there is a singular stratum of codimension two with isotropy group $\mathbb{Z}_3$. Then we have to show that the projection $\bar{h}$ of h to $a_3(W^{*\otimes 3} \oplus W^{\otimes 3})$ is trivial. This projection is of the following form

$$\bar{h} = \alpha_1(u)zdzdz + \beta_1(u)\bar{z}d\bar{z}d\bar{z} + \sum_j(\alpha_{2j}(u)z^2dzdu_j + \beta_{2j}(u)\bar{z}^2d\bar{z}du_j).$$

Here z denotes the complex coordinates in W and u denotes the coordinates in $S^1 \times \mathbb{R}^{n-3}$.

In polar coordinates $z = re^{i\varphi}$ the above expression is equal to

$$\begin{aligned}\bar{h} &= r(\alpha_1(u)e^{i3\varphi} + \beta_1(u)e^{-i3\varphi})drdr + r^2 i(\alpha_1(u)e^{3i\varphi} - \beta_1(u)e^{-3i\varphi})drd\varphi \\ &+ \sum_j r^2(\alpha_{2j}(u)e^{3i\varphi} + \beta_{2j}(u)e^{-3i\varphi})drdu_j \\ &- r^3(\alpha_1(u)e^{3i\varphi} + \beta_1(u)e^{-3i\varphi})d\varphi d\varphi + \sum_j ir^3(\alpha_{2j}(u)e^{3i\varphi} - \beta_{2j}(u)e^{-3i\varphi})d\varphi du_j.\end{aligned}$$

But by the generalized Gauss Lemma [Gra04, Section 2.4] we may assume that g is of the form

$$drdr + h'(u, r, \varphi),$$

where $h'(u, r, \varphi)$ is a metric on $S^1 \times \mathbb{R}^{n-3} \times S^1_r$. Here S^1_r denotes the circle of radius r in W.

Hence, we may assume that $\alpha_1 = \beta_1 = \alpha_{2j} = \beta_{2j} = 0$. Therefore the metric can be deformed as in the first case.

The case of singular strata with isotropy group $\mathbb{Z}_4$ is similar and left to the reader. □

If M is spin and the S^1-action on M is of even type then there are no components of $M^{\mathbb{Z}_2}$ of codimension two in M. Therefore we get the following corollary to the above theorem.

Corollary 6.2 *Let M be a spin S^1-manifold with an effective action of even type. Then M admits an invariant metric of positive scalar curvature if and only if it admits an invariant metric of positive scalar curvature which is normally symmetric in codimension two.*

7. An obstruction to invariant metrics of positive scalar curvature

Before we prove existence results for invariant metrics of positive scalar curvature on Spin-S^1-manifolds, we introduce an obstruction to the existence of such metrics. Throughout this section we only deal with Spin-S^1-manifolds with actions of even type.

Assume that M is such a manifold and that there is no codimension two stratum in M and M admits a metric of positive scalar curvature. Let N be a tubular neighborhood of a minimal stratum M^H. Then, since $\operatorname{codim} M^H \geq 4$, there is an invariant metric of positive scalar curvature on $M_1 = M - N$ which is scaled and a connection metric on the boundary of M_1, whose restriction to the fibers of $\partial N \to M^H$ is given by a metric which is constructed from the round metric on S^k by a certain deformation [GL80]. Moreover, this metric on M_1 is a product metric near the boundary.

We continue this construction in the same manner with M replaced by M_1. Since there are only finitely many orbit types in M after a finite number of steps we will reach some M_k with a free S^1-action, such that

- M_k has a metric of positive scalar curvature,
- the restriction of this metric to a neighborhood of the boundary of M_k is a product metric,
- the restriction of this metric to the boundary has positive scalar curvature,
- the restriction of this metric to an open stratum of ∂M_k is a connection metric with fibers isometric to open subsets in deformed spheres.

By [BB83, Theorem C], there is a metric of positive scalar curvature on M_k/S^1. Therefore the index of the Dirac-operator on M_k/S^1 vanishes.

If the invariant metric on M does not have positive scalar curvature, we still can construct M_k and the metric on the boundary of M_k is still the same as in the above construction. We still have an Dirac-operator on M_k/S^1. Its index is an invariant of M_k/S^1 together with the metric on its boundary. We define $\hat{A}_{S^1}(M)$ to be the index of this Dirac-operator. One easily sees that $\hat{A}_{S^1}(M)$ is an invariant of the equivariant spin bordism type of M.

Now assume that M has strata of codimension two. Then by using the above construction we get a manifold M_k which has only minimal strata of codimension two. Moreover, there is a metric of positive scalar curvature on the boundary. This metric on the boundary can be assumed to be normally symmetric in codimension two by the discussion in section 6.

Now we can apply the following desingularization process to get a manifold M_{k+1} with boundary and without codimension two singular strata. It is similar to the desingularization process in [Han08, Section 4].

Let N be a codimension two singular stratum in M_k. Then a neighborhood of N in M_k is diffeomorphic to a fiber bundle E with fiber $S^1 \times_H D(W)$ and structure group $S^1 \times_H SO(W) = S^1 \times_H S^1_1$ over N/S^1. Here W is a one-dimensional unitary representation of H which depends on N.

The boundary of this neighborhood ∂E is given by the principal $S^1 \times_H S^1_1$-bundle P associated to E.

Now let $S^1_2 \subset S^1 \times_H S^1_1$ be a circle subgroup with $S^1 \cap S^1_2 = \{1\}$. Denote by E' the $S^1 \times D^2$-bundle $P \times_{S^1_2} D^2$ where S^1_2 acts by rotation on D^2. Then E' has the same boundary as E and we define

$$M_{k+1} = (M_k - E) \cup_P E'.$$

In this way we can construct a free S^1-manifold with boundary M_{k+1}.

Note here that M_{k+1} itself might depend on the choice of S^1_2. But its orbit space M_{k+1}/S^1 does not depend on this choice.

Since the constructions in Section 4 of [Han08] are mainly local arguments, they also hold in our desingularization process. This means that, if M_k admits a metric of positive scalar curvature which is normally symmetric in codimension two, then M_{k+1} also admits such a metric.

In any case we have a metric on M_{k+1} whose restriction to the boundary has positive scalar curvature. Therefore as in the first case we can define $\hat{A}_{S^1}(M)$ as the index of the Dirac-operator on M_{k+1}/S^1. It vanishes if there is an invariant metric of positive scalar curvature on M which is normally symmetric in codimension two. Moreover, one can see that $\hat{A}_{S^1}(M)$ is an invariant of the equivariant spin bordism type of M.

For semi-free S^1-manifolds M, $\hat{A}_{S^1}(M)$ coincides with the index obstruction to metrics of positive scalar curvature defined by Lott [Lot00].

8. Invariant metrics of positive scalar curvature and a result of Atiyah and Hirzebruch

Now we can complete the proofs of our existence results for invariant metrics of positive scalar curvature. These are as follows:

Theorem 8.1 *Let M be a connected effective S^1-manifold of dimension at least six which satisfies Condition C such that $\pi_1(M_{max}) = 0$ and M_{max} is not Spin. Then the equivariant connected sum of two copies of M admits an invariant metric of positive scalar curvature.*

Proof. By Theorem 2.4 of [Wie15], we may assume that there is no codimension two fixed point component in M. Therefore, by Corollary 5.12, it is sufficient to show that $2M$ is equivariantly bordant to a manifold M' which admits an invariant metric of positive scalar curvature which is normally symmetric in codimension two such that $\operatorname{codim} M'^{S^1} \geq 4$. By Theorem 2.2, there is an equivariant bordism Z which satisfies Condition C between M and $M' = M_1 \amalg M_2$, where M_1 is a semi-free S^1-manifold and M_2 is a S^1-manifold which admits an invariant metric of positive scalar curvature which is normally symmetric in codimension two and $\operatorname{codim} M'^{S^1} \geq 4$. After attaching S^1-handles to Z we may assume that all components of M_1 are simply connected and not Spin.

Hence, the Theorem follows from Theorem 4.7 of [Wie15]. $\square$

Now we turn to the proof of a similar result for Spin-manifolds.

Theorem 8.2 *Let M be a spin S^1-manifold with $\dim M \geq 6$, an effective S^1-action of odd type and $\pi_1(M_{max}) = 0$. Then there is a $k \in \mathbb{N}$ such that the equivariant connected sum of 2^k copies of M admits an invariant metric of positive scalar curvature which is normally symmetric in codimension two.*

Theorem 8.3 *For a spin S^1-manifold with $\dim M \geq 6$, an effective S^1-action of even type and $\pi_1(M_{max}) = 0$, we have $\hat{A}_{S^1}(M) = 0$ if and only if there is a $k \in \mathbb{N}$ such that the equivariant connected sum of 2^k copies of M admits an invariant metric of positive scalar curvature which is normally symmetric in codimension two.*

Proof. By Theorem 3.1 and Lemma 4.1, the connected sum of 2^l copies of M is equivariantly bordant to a union $M_1 \amalg M_2$, where M_1 is a semi-free simply connected S^1-manifold and M_2 is a S^1-manifold which admits an invariant metric of positive scalar curvature which is normally symmetric in codimension two.

If the S^1-action on M is of even type, we have $\hat{A}_{S^1}(M) = 2^{-l}\hat{A}_{S^1}(M_1/S^1)$. Now the theorems follow from Theorems 4.7 and 4.11 of [Wie15]. □

As an application of our results we give a new proof of the following result of Atiyah and Hirzebruch [AH70].

Theorem 8.4 ([AH70]) *Let M be a spin manifold with a non-trivial action of S^1. Then $\hat{A}(M)$ vanishes.*

Proof. We may assume that $\dim M = 4k$ and that the S^1-action is effective. Then by Theorem 3.1 and Lemma 4.1, $2^l M$ is equivariantly spin bordant to a union $M_1 \amalg M_2$, where M_1 is simply connected and semi-free and M_2 admits an invariant metric of positive scalar curvature. By Theorems 4.7 and 4.11 of [Wie15], the obstruction $\hat{A}_{S^1}(M_1/S^1)$ that $2^{l'}M_1$ admits an invariant metric of positive scalar curvature vanishes by dimension reasons. Hence it follows that $2^{l+l'}\hat{A}(M) = \hat{A}(2^{l'}M_1) + \hat{A}(2^{l'}M_2) = 0$. This implies $\hat{A}(M) = 0$. □

9. Rigidity of elliptic genera

In this section we give a proof of the rigidity of elliptic genera. At first we recall the definition of an equivariant genus. We follow [Och88] for this definition.

A Λ-genus is a ring homomorphism $\varphi : \Omega_*^{SO} \to \Lambda$ where Λ is a $\mathbb{C}$-algebra. For such a homomorphism one denotes by

$$g(u) = \sum_{i \geq 0} \frac{\varphi[\mathbb{C}P^{2i}]}{2i+1} u^{2i+1} \in \Lambda[[u]]$$

the logarithm of φ and by $\Phi \in H^{**}(BSO;\Lambda)$ the total Hirzebruch class associated to φ. Φ is uniquely determined by the property that for the canonical line bundle γ over BS^1, $\Phi(L)$ is given by $\frac{u}{g^{-1}(u)}$.

Then for every oriented manifold M one has

$$\varphi[M] = \langle \Phi(TM), [M] \rangle.$$

For a compact Lie group G, the G-equivariant genus φ_G associated to φ is defined as

$$\varphi_G[M] = p_* \Phi(TM_G) \in H^{**}(BG;\Lambda).$$

Here M is a G-manifold, TM_G is the Borel construction of the tangent bundle of M. It is a vector bundle over the Borel construction M_G of M. Moreover, p_* denotes the integration over the fiber in the fibration $M \to M_G \to BG$.

It follows from this definition that if H is a closed subgroup of G, then we have

$$\varphi_H[M] = f^*\varphi_G[M],$$

where $f : BH \to BG$ is the map induced by the inclusion $H \hookrightarrow G$.

Lemma 9.1 *Let $G = S^1$ and M be an oriented G-manifold. Then the equivariant genus $\varphi_G[M]$ depends only on the G-equivariant bordism type of M.*

Proof. It is sufficient to show that if $M = \partial W$ is an equivariant boundary, then $\varphi_G[M] = 0$. Since the homology of $BS^1 = \mathbb{C}P^\infty$ is concentrated in even degrees and generated by the fundamental classes of the natural inclusions $\iota_n : \mathbb{C}P^n \hookrightarrow \mathbb{C}P^\infty$, $n \geq 0$, it is sufficient to show that

$$0 = \langle p_*\Phi(TM_G), [\mathbb{C}P^n]\rangle,$$

for all n. Now we have

$$\begin{aligned}\langle p_*\Phi(TM_G), [\mathbb{C}P^n]\rangle &= \langle p_*\Phi(TM_G|_{\mathbb{C}P^n}), [\mathbb{C}P^n]\rangle \\ &= \langle \Phi(TM_G|_{\mathbb{C}P^n}), [M_G|_{\mathbb{C}P^n}]\rangle = 0.\end{aligned}$$

Here the first two equations follow from the properties of p_*. Moreover, the last equality follows because $M_G|_{\mathbb{C}P^n}$ bounds $W_G|_{\mathbb{C}P^n}$. This proves the lemma. □

A Λ-genus φ is called elliptic if there are $\delta, \epsilon \in \Lambda$ such that its logarithm is given by

$$g(u) = \int_0^u \frac{dz}{\sqrt{1 - 2\delta z^2 + \epsilon z^4}}.$$

We call an equivariant genus φ_{S^1} of an S^1-manifold M rigid, if $\varphi_{S^1}[M] \in H^{**}(BS^1; \Lambda) = \Lambda[[u]]$ is constant in u. The following has been proved by Ochanine [Och88].

Theorem 9.2 *The elliptic genus of a semi-free Spin-S^1-manifold is rigid.*

In view of the above lemma and Theorem 1.1 it suffices to show the following lemma to prove the rigidity of elliptic genera (Theorem 1.2). In an effective T-manifold the codimension of the fixed point set is at least $2 \dim T$. The next lemma states that the T-equivariant elliptic genus of an effective T-manifold is constant if the codimension of all components of the fixed point set is minimal.

Lemma 9.3 *Let M be an effective Spin-T^n-manifold, such that all fixed point components have codimension $2n$. Then the T^n-equivariant elliptic genera of M are rigid.*

Proof. It suffices to consider the equivariant elliptic genus $\varphi_{T^n}(M)$ defined in Section 2.1 of Ochanine's paper.

This is defined as follows: For a lattice $W \subset \mathbb{C}$ and a non-trivial homomorphism $r : W \to \mathbb{Z}_2$ there exists a unique meromorphic function x on $\mathbb{C}$ such that

1. x is odd,

2. the poles of x are exactly the points in W; they are all simple and the residues of x in $w \in W$ is given by $(-1)^{r(w)}$,

3. for all $w \in W$ we have
$$x(u+w) = (-1)^{r(w)}x(u).$$

From this one defines a genus φ such that $g(u)^{-1}$ is the Taylor expansion of $1/x$ in the point $u = 0$.

With this definition, $\varphi_{T^n}[M]$ can be identified with a meromorphic function on $\mathbb{C}^n$.

Let $F \subset M$ be a fixed point component and $\lambda_{1,F}, \dots, \lambda_{n,F} : \mathbb{Z}^n \to \mathbb{Z}$ the weights of the T^n-action on the normal bundle to F. Since the T^n-action is effective and $\operatorname{codim} F = 2n$, it follows that
$$(\lambda_{1,F}, \dots, \lambda_{n,F}) : \mathbb{Z}^n \to \mathbb{Z}^n$$
is an isomorphism. In particular each $\lambda_{i,F}$ is surjective.

As in the proof of Proposition 7 in Ochanine's paper [Och88] one sees that the genus $\varphi_{T^n}(M)$ is a polynomial in $x \circ \lambda_{i,F,\mathbb{C}}$ and $y \circ \lambda_{i,F,\mathbb{C}}$, $i = 1, \dots, n$. Here $F \subset M^{T^n}$ is a component of M^{T^n}, and $\lambda_{i,F,\mathbb{C}}$ is the linear extension of $\lambda_{i,F}$ to $\mathbb{C}^n$. Moreover, y is the derivative of x.

In particular, the poles of $\varphi_{T^n}(M)$ lie in the union of the following hyperplanes:
$$\ker \lambda_{i,F,\mathbb{C}} + z$$
with $z \in W^n$. Since the singular set of a meromorphic function on $\mathbb{C}^n$ is empty or an analytic set of codimension one, we may assume that the singular set of φ_{T^n} has a non-empty open intersection with one of the hyperplanes above.

Since M is spin, the mod two reduction of $\sum_{i=1}^n \lambda_{i,F}$ does not depend on the component F. It follows from the defining equation (3) for x, that a non-empty open set in the hyperplane $\ker \lambda_{i,F,\mathbb{C}}$ is singular for φ_{T^n}.

But the restriction of φ_{T^n} to this hyperplane equals $\varphi_{T^{n-1}}$ where T^{n-1} is the codimension one subtorus of T^n which is defined by $\lambda_{i,F}$. Since $\varphi_{T^{n-1}}$ is a meromorphic function on $\mathbb{C}^{n-1}$, it follows that the intersection of the singular set of φ_{T^n} with $\ker \lambda_{i,F}$ can only be open if it is empty. Therefore φ_{T^n} does not have singular points. This implies that it is constant. $\square$

At the end we want to compare our proof of the rigidity of elliptic genera with the proof of Bott–Taubes. The difference between our proof and the proof of Bott–Taubes is that they prove that the equivariant universal elliptic genus of an Spin-S^1-manifold M equals some twisted elliptic genus of some auxiliary manifold M' by using the Lefschetz fixed point formula. Using this fact they can show that the equivariant universal elliptic genus of M does not have poles. Therefore it may be identified with a bounded holomorphic function. Hence, it is constant.

We use the fact that we only have to prove the theorem for our generators of the S^1-equivariant spin bordism ring. For semi-free S^1-manifolds this has been done by

Ochanine [Och88], by using localization in equivariant cohomology and some elementary complex analysis. The proof in the semi-free case is simpler than in the non-semi-free case because one sees directly from the fixed point formula that the elliptic genus does not have poles. Therefore one does not need the auxiliary manifolds in this case.

So, we only have to show that an S^1-equivariant elliptic genus of a generalized Bott manifold, which is spin, is constant. We do this by showing that the T-equivariant elliptic genera of a generalized Bott manifold M^{2n} are constant. The main new technical observation in the proof of this fact is that Ochanine's proof carries over to the situation where the codimension of the T-fixed point set is minimal.

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Part 2

Non-negative sectional curvature

Torus manifolds and non-negative curvature

Michael Wiemeler

A torus manifold M is a $2n$-dimensional orientable manifold with an effective action of an n-dimensional torus such that $M^T \neq \emptyset$. In this paper we discuss the classification of torus manifolds which admit an invariant metric of non-negative curvature. If M is a simply connected torus manifold which admits such a metric, then M is diffeomorphic to a quotient of a free linear torus action on a product of spheres. We also classify rationally elliptic torus manifolds M with $H^{\text{odd}}(M;\mathbb{Z}) = 0$ up homeomorphism.

1. Introduction

The study of non-negatively curved manifolds has a long history in geometry. In this note we discuss the classification of these manifolds in the context of torus manifolds. A torus manifold M is a $2n$-dimensional closed orientable manifold with an effective action of an n-dimensional torus T such that $M^T \neq \emptyset$.

Recently Spindeler [Spi14] proved the Bott-conjecture for simply connected torus manifolds. This conjecture implies that a non-negatively curved manifold is rationally elliptic. Our first main result deals with rationally elliptic torus manifolds:

Theorem 1.1 (Theorem 4.1) *Let M be a simply connected rationally elliptic torus manifold with $H^{odd}(M;\mathbb{Z}) = 0$. Then M is homeomorphic to a quotient of a free linear torus action on a product of spheres.*

Our second main theorem is as follows.

Theorem 1.2 (Theorem 6.1) *Let M be a simply connected non-negatively curved torus manifold. Then M is equivariantly diffeomorphic to a quotient of a free linear torus action on a product of spheres.*

In the situation of the theorem the torus action on the quotient N/T' of a product of spheres $N = \prod_{i<r} S^{2n_i} \times \prod_{i\geq r} S^{2n_i+1}$ by a free linear action of a torus T' is defined as follows. Let T be a maximal torus of $\prod_{i<r} SO(2n_i+1) \times \prod_{i\geq r} SO(2n_i+2)$. Then there is a natural linear action of T on N. Moreover, T' can be identified with a subtorus of T. Therefore T/T' acts on N/T'. If the dimension of T' is equal to the number of odd-dimensional factors in the product N, then N/T' together with the action of T/T' is a torus manifold.

We also show that the fundamental group of a non-simply connected torus manifold of dimension $2n$ with an invariant metric of non-negative curvature is isomorphic to $\mathbb{Z}_2^k$

with $k \leq n-1$. In particular, every such manifold is finitely covered by a manifold as in Theorem 1.2.

Theorems 1.1 and 1.2 are already known in dimension four. As is well known a simply connected rationally elliptic four-manifold is homeomorphic to S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, $\mathbb{C}P^2 \# \mathbb{C}P^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Furthermore, a simply connected non-negatively curved four-dimensional torus manifold is diffeomorphic to one of the manifolds in the above list (see [GGS11], [SY94], [Kle90], [GW13]). Moreover, by [DJ91] or [GGK14], the T^2-actions on these spaces are always equivalent to a torus action as described above.

We should note here that it has been shown by Grove and Searle [GS94] that a simply connected torus manifold which admits an invariant metric of positive curvature is diffeomorphic to a sphere or a complex projective space. Moreover, it has been shown by Gurvich in his thesis [Gur08] that the orbit space of a rationally elliptic quasitoric manifold is face-preserving homeomorphic to a product of simplices. This condition on the orbit space is satisfied if and only if the quasitoric manifold is a quotient of a free torus action on a product of odd-dimensional spheres.

In dimension six Theorem 1.1 follows from these results of Gurvich, the classification of simply connected six dimensional torus manifolds with $H^3(M;\mathbb{Z}) = 0$ given by Kuroki [Kur13] and a characterization of the cohomology rings of simply connected six-dimensional rationally elliptic manifolds given by Herrmann [Her14].

In [GS94] $(2n+1)$-dimensional positively curved manifolds with isometric actions of an $(n+1)$-dimensional manifolds were also classified. We expect that a result similar to Theorem 1.2 holds for isometric actions of $n+1)$-dimensional tori on non-negatively curved $2n+1$-dimensional manifolds with one-dimensional orbits. We will discuss the details of this in a subsequent paper.

We apply Theorem 1.1 to rigidity problems in toric topology. As a consequence we get the following theorem:

Theorem 1.3 (Corollary 5.3) *Let M be a simply connected torus manifold with $H^*(M;\mathbb{Z}) \cong H^*(\prod_i \mathbb{C}P^{n_i};\mathbb{Z})$. Then M is homeomorphic to $\prod_i \mathbb{C}P^{n_i}$.*

This theorem is a stronger version of a result of Petrie [Pet73] related to his conjecture on circle actions on homotopy complex projective spaces. This conjecture states that if $f: M \to \mathbb{C}P^n$ is a homotopy equivalence and S^1 acts non-trivially on the manifold M, then $f^*(p(\mathbb{C}P^n)) = p(M)$, where $p(M)$ denotes the total Pontrjagin class of M. Petrie showed that his conjecture holds if there is an action of an n-dimensional torus on M. Moreover, the conjecture has been shown by Dessai and Wilking [DW04] for the case that there is an effective action of a torus of dimension greater than $\frac{n+1}{4}$ on M. For more results related to this conjecture see the references in [DW04].

The proof of Theorem 1.1 consists of two steps. In a first step we show that the homeomorphism type of a simply connected torus manifold M, whose cohomology with integer coefficients vanishes in odd degrees, depends only on the isomorphism type of the face poset of M/T and the characteristic function of M. Then all possible face posets of M/T under the condition that M is rationally elliptic are determined. By the first step all such manifolds are homeomorphic to quotients of the moment angle complex associated to these posets. As it turns out these moment angle complexes are products of spheres.

The proof of Theorem 1.2 is based on the computations of the face posets from above. With Spindeler's results from [Spi14] one can see that all faces of M/T are diffeomorphic after smoothing the corners to standard discs. When this is established, generalizations of results from [Wie13] imply the theorem.

This paper is organized as follows. In Section 2 we discuss results of Masuda and Panov about torus manifolds with vanishing odd degree cohomology. In Section 3 we introduce a construction which simplifies the torus action on a torus manifold. In Sections 4, 5 and 6 we prove Theorems 1.1, 1.3 and 1.2, respectively. In the last Section 7 we discuss non-simply connected non-negatively curved torus manifolds.

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2. Preliminaries

Before we prove our results, we review some results of Masuda and Panov [MP06] about torus manifolds with vanishing odd degree cohomology.

They have shown that a smooth torus manifold M with $H^{\text{odd}}(M;\mathbb{Z}) = 0$ is locally standard. This means that each point in M has an invariant neighborhood which is weakly equivariantly homeomorphic to an open invariant subset of the standard T^n-representation on $\mathbb{C}^n$. Moreover, the orbit space is a nice manifold with corners such that all faces of M/T are acyclic [MP06, Theorem 9.3]. Here a manifold with corners is called nice if each of its codimension-k faces is contained in exactly k codimension-one faces. A codimension-one face of M/T is also called a facet of M/T. Moreover, following Masuda and Panov, we count M/T itself as a codimension-zero face of M/T.

The faces of M/T do not have to be contractible. But we show in Section 3 that the action on M can be changed in such a way that all faces become contractible without changing the face-poset of M/T. This new action might be non-smooth (see Remark 3.5). But it always admits a canonical model over a topological nice manifold with corners as described below.

For a facet F of $Q = M/T$ denote by $\lambda(F)$ the isotropy group of a generic point in $\pi^{-1}(F)$, where $\pi : M \to M/T$ is the orbit map. Then $\lambda(F)$ is a circle subgroup of T. Let

$$M_Q(\lambda) = Q \times T/\sim,$$

where two points $(x_i, t_i) \in Q \times T$, $i = 1, 2$, are identified if and only if $x_1 = x_2$ and $t_1 t_2^{-1}$ is contained in the subtorus of T which is generated by the $\lambda(F)$ with $x_1 \in F$. There is a T-action on $M_Q(\lambda)$, induced by multiplication on the second factor in $Q \times T$. Then, by [MP06, Lemma 4.5], there is an equivariant homeomorphism

$$M_Q(\lambda) \to M.$$

For every map $\lambda : \{\text{facets of } Q\} \to \{\text{one-dimensional subtori of } T\}$ such that T is isomorphic to $\lambda(F_1) \times \cdots \times \lambda(F_n)$, whenever the intersection of $F_1 \cap \cdots \cap F_n$ is non-empty, the model $M_Q(\lambda)$ is a manifold.

The canonical model is equivariantly homeomorphic to a quotient of a free torus action on the moment angle complex Z_Q associated to Q. Here Z_Q is defined as follows:

$$Z_Q = Q \times T_Q / \sim .$$

Here T_Q is the torus $S^1_1 \times \cdots \times S^1_k$, where k is the number of facets of Q. The equivalence relation $\sim$ is defined as follows. Two points $(q_i, t_i) \in Q \times T_Q$ are identified if $q_1 = q_2$ and $t_1 t_2^{-1} \in \prod_{i \in S(q_1)} S^1_i$, where $S(q_1)$ is the set of those facets of Q which contain q_1.

The torus which acts freely on Z_Q with quotient $M_Q(\lambda)$ is given by the kernel of a homomorphism $\psi : T_Q \to T$, such that the restriction of ψ to S^1_i induces an isomorphism $S^1_i \to \lambda(F_i)$.

Example 2.1 *If $Q = \Delta^n$ is an n-dimensional simplex, then T_Q is an $(n+1)$-dimensional torus. Moreover, Z_Q is equivariantly homeomorphic to $S^{2n+1} \subset \mathbb{C}^{n+1}$ with the standard linear torus action.*

Example 2.2 *If $Q = \Sigma^n$ is the orbit space of the standard linear torus action on S^{2n}, then T_Q is n-dimensional. Moreover, Z_Q is equivariantly homeomorphic to $S^{2n} \subset \mathbb{C}^n \oplus \mathbb{R}$ with the standard linear torus action.*

Example 2.3 *Let Q_1 and Q_2 be two nice manifolds with corners. If $Q = Q_1 \times Q_2$, then $T_Q \cong T_{Q_1} \times T_{Q_2}$ and Z_Q is equivariantly homeomorphic to $Z_{Q_1} \times Z_{Q_2}$.*

Now assume that M is a torus manifold with $H^{\text{odd}}(M; \mathbb{Q}) = 0$. This condition is always satisfied if M is rationally elliptic because $\chi(M) = \chi(M^T) > 0$. Then the torus action on M might not be locally standard and M/T might not be a manifold with corners. But M/T still has a face-structure induced by its stratification by connected orbit types. It is defined as in [CS74]. A k-dimensional face is a component C of $M^{T^{n-k}}/T$ such that the identity component of the isotropy group of a generic point in C is equal to T^{n-k}, where T^{n-k} is a subtorus of codimension k in T. The faces of M/T defined in this way have the following properties:

- It follows from localization in equivariant cohomology that the cohomology of every $M^{T^{n-k}}$ is concentrated in even degrees. Therefore every component of $M^{T^{n-k}}$ contains a T-fixed point. This is equivalent to saying that each face of M/T contains at least one vertex, i.e. a face of dimension zero.
- By an investigation of the local weights of the action, one sees that each face of M/T of codimension k is contained in exactly k faces of codimension 1.
- The vertex-edge-graph of each face is connected. (see [CS74, Proposition 2.5])

3. Simplifying torus actions

In this section we describe an operation on locally standard torus manifolds M which simplifies the torus action on M. For this construction we need the following two lemmas.

Lemma 3.1 *Let M be a topological n-manifold with $H^*(M;\mathbb{Z}) \cong H^*(S^n;\mathbb{Z})$. Then there is a contractible compact $(n+1)$-manifold X such that $\partial X = M$. Moreover, X is unique up to homeomorphism relative M. In particular, every homeomorphism of M extends to a homeomorphism of X.*

Proof. For $n \leq 2$, this follows from the classification of manifolds of dimension n. If $n = 3$ then this follows from the proof of Corollary 9.3C and Corollary 11.1C of [FQ90]. For $n \geq 4$ this follows from the proof of [FQ90, Corollary 11.1]. $\square$

Lemma 3.2 *Let Q_1, Q_2 be two nice manifolds with corners of the same dimension such that all faces of Q_i, $i = 1,2$, are contractible. If there is an isomorphism of their face-posets $\phi\colon \mathcal{P}(Q_1) \to \mathcal{P}(Q_2)$, then there is a face-preserving homeomorphism $f\colon Q_1 \to Q_2$, such that, for each face F of Q_1, $f(F) = \phi(F)$.*

Proof. We construct f by induction on the n-skeleton of Q_1. There is no problem to define f on the 0-skeleton. Therefore assume that f is already defined on the $(n-1)$-skeleton.

Let F be a n-dimensional face of Q_1. Then f restricts to an homeomorphism $\partial F \to \partial\phi(F)$. Because F and $\phi(F)$ are contractible manifolds with boundary ∂F, f extends to a homeomorphism $F \to \phi(F)$. This completes the proof. $\square$

Now let Q be a nice manifold with corners and F a face of Q of positive codimension which is a homology disc. Let X be a homology disc with $\partial X = \partial F$. Then $X \cup_{\partial F} F$ is a homology sphere and therefore bounds a contractible manifold Y. We equip Y with a face structure such that the facets of Y are given by F and X and the lower dimensional faces coincide with the faces of F in $\partial F = F \cap X$. With this face-structure X and F become nice manifolds with corners.

Let $k = \dim Q - \dim F$. Then define

$$S_{X,F} = Y \times \Delta^{k-1} \cup_{X \times \Delta^{k-1}} X \times \Delta^k.$$

Then $F' = F \times \Delta^{k-1}$ is a facet of $S_{X,F}$ and we define

$$\alpha_{X,F}(Q) = Q - (F \times \Delta^k) \cup_{F'} S_{X,F}.$$

Then $\alpha_{X,F}(Q)$ is naturally a nice manifolds with corners.

Lemma 3.3 *Let Q, F, X as above. If $dimF \geq 3$ and all faces of Q of dimension greater than* $\dim F$ *are contractible, then $Q' = \alpha_{F,X}(\alpha_{X,F}(Q))$ and Q are face-preserving homeomorphic.*

Proof. It is clear from the construction above that $\mathcal{P}(Q')$ and $\mathcal{P}(Q)$ are isomorphic. Moreover, the $\dim F$-skeleta of Q and Q' are face-preserving homeomorphic. Therefore, by using Lemma 3.1, the statement follows by an induction as in the proof of Lemma 3.2. $\square$

If M is a locally standard torus manifold over Q, then we can construct a torus manifold M' with orbit space $\alpha_{X,F}(Q)$ as follows. Let T^k act on $Y' = \partial(Y \times D^{2k})$ (without the face-structure) by the standard action on the second factor. Choose an isomorphism $T \cong T^k \times T^{n-k}$ which maps $\lambda(F)$ to the first factor T^k. If $\dot{F}$ denotes F with a small collar of its boundary removed, then a small neighborhood of $\pi^{-1}(\dot{F})$ is equivariantly homeomorphic to $\dot{F} \times T^{n-k} \times D^{2k}$.

We define

$$\beta_{X,F}(M) = M - (\pi^{-1}(\dot{F}) \times D^{2k}) \cup_{\partial(\dot{F} \times D^{2k} \times T^{n-k})} (Y' - \dot{F} \times D^{2k}) \times T^{n-k}.$$

Since Y' is the simply connected boundary of a contractible manifold, it is homeomorphic to a sphere. Moreover, if F is contractible, then $\dot{F} \times D^{2k}$ is contractible with simply connected boundary. Hence, it follows from Schoenflies' Theorem that $\dot{F} \times D^{2k}$ is a disc and that we may assume that $F \times D^{2k}$ is embedded in Y' as the upper hemisphere. Therefore it follows that $\beta_{X,F}(M)$ is homeomorphic to M if F is contractible.

Theorem 3.4 *Let M be a simply connected torus manifold with $H^{odd}(M;\mathbb{Z}) = 0$. Then M is determined by $(\mathcal{P}(M/T), \lambda)$ up to homeomorphism.*

Proof. The first step is to simplify the action on M in such a way that all faces F of M/T become contractible. Then the statement will follow from Lemma 3.2 and [MP06, Lemma 4.5].

At first assume that $\dim M \leq 6$. Then because $\pi_1(M) = 0$ all faces of M/T are contractible. Therefore the theorem follows in this case.

Next assume that $\dim M \geq 8$. Because M is simply connected, M/T is contractible. We simplify the torus action on M by using the operations $\beta_{X,F}$ applied by a downwards induction on the dimension of F beginning with the codimension one faces. At first assume that $\dim F \geq 3$ and that all faces of dimension greater than $\dim F$ are already contractible. Let X be a contractible manifold with boundary ∂F. Then $\beta_{F,X}(\beta_{X,F}(M))$ is homeomorphic to $\beta_{X,F}(M)$. But by Lemma 3.3, $\alpha_{F,X}(\alpha_{X,F}(M/T))$ is face-preserving homeomorphic to M/T. Therefore it follows that $\beta_{F,X}(\beta_{X,F}(M))$ is homeomorphic to M. Hence, $\beta_{X,F}(M)$ is also homeomorphic to M.

If $\dim F \leq 2$, it follows from the classification of two- and one-dimensional manifolds that there is nothing to do. $\square$

Remark 3.5 *Since not every three-dimensional homology sphere bounds a contractible smooth manifold, the torus action on $\beta_{X,F}(M)$ might be non-smooth. Therefore with our methods we cannot prove that the diffeomorphism type of M is determined by $(\mathcal{P}(M/T), \lambda)$.*

Using results of Masuda and Panov [MP06] and the author's methods from [Wie12] together with Theorem 3.4, one can prove the following partial generalization of Theorem 2.2 of [Wie12].

Theorem 3.6 *Let M and M' be simply connected torus manifolds of dimension $2n$ with $H^*(M;\mathbb{Z})$ and $H^*(M';\mathbb{Z})$ generated in degree 2. Let m, m' be the numbers of characteristic submanifolds of M and M', respectively. Assume that $m \leq m'$. Furthermore, let $u_1, \dots, u_m \in H^2(M)$ be the Poincaré-duals of the characteristic submanifolds of M and $u'_1, \dots, u'_{m'} \in H^2(M')$ the Poincaré-duals of the characteristic submanifolds of M'. If there is a ring isomorphism $f : H^*(M) \to H^*(M')$ and a permutation $\sigma : \{1, \dots, m'\} \to \{1, \dots, m'\}$ with $f(u_i) = \pm u'_{\sigma(i)}$, for $i = 1, \dots, m$, then M and M' are homeomorphic.*

Here a characteristic submanifold of a torus manifold is a codimension-two-submanifold which is fixed by some circle-subgroup of the torus and contains a T-fixed point. Each characteristic submanifold is the preimage of some facet of M/T under the orbit map.

In [Pet73] Petrie proved that if an n-dimensional torus acts on a homotopy complex projective space M of real dimension $2n$, then the Pontrjagin classes of M are standard. In fact a much stronger statement holds.

Corollary 3.7 *Let M be a torus manifold which is homotopy equivalent to $\mathbb{C}P^n$. Then M is homeomorphic to $\mathbb{C}P^n$.*

Proof. By Corollary 7.8 of [MP06], the cohomology ring of M can be computed from the face-poset of the orbit space. In particular all the Poincaré duals of the characteristic submanifolds of M are generators of $H^2(M;\mathbb{Z})$. Now it follows from Theorem 3.6 that M is homeomorphic to $\mathbb{C}P^n$. □

4. Rationally elliptic torus manifolds

In this section we prove the following theorem.

Theorem 4.1 *Let M be a simply connected rationally elliptic torus manifold with $H^{odd}(M;\mathbb{Z}) = 0$. Then M is homeomorphic to a quotient of a free linear torus action on a product of spheres.*

Since the proof of this theorem is very long we give a short outline of its proof.

Sketch of proof. In a first step (Lemma 4.2) we will show that each two-dimensional face of the orbit space of M contains at most four vertices. Then we will show in Proposition 4.5 that a nice manifold with corners Q, whose two-dimensional faces contain at most four vertices and all of whose faces are acyclic, is combinatorially equivalent to a product $\prod_{i<r} \Sigma^{n_i} \times \prod_{i \geq r} \Delta^{n_i}$. Here Σ^n is the orbit space of the linear T^n-action on S^{2n} and Δ^n is the n-dimensional simplex.

When this is achieved Theorem 4.1 will follow from Theorem 3.4 and the structure results described in Section 2.

The proof of Proposition 4.5 is by induction on the dimension of Q. The case $\dim Q = 2$ is obvious. Therefore we may assume that $n = \dim Q > 2$.

We consider a facet F of Q. By the induction hypothesis we know that there is a combinatorial equivalence

$$F \cong \prod_{i<r} \Sigma^{n_i} \times \prod_{i \geq r} \Delta^{n_i}. \tag{1}$$

The facets G_k of Q which meet F intersect F in a disjoint union of facets of F. Since the facets of F are all of the form

$$\tilde{F} \times \prod_{i_0 \neq i < r} \Sigma^{n_i} \times \prod_{i_0 \neq i \geq r} \Delta^{n_i},$$

where $\tilde{F}$ is a facet of the i_0-th factor in the product (1).

Hence, it follows that each G_k "belongs" to a factor $\Gamma_{j(k)}$ of the product (1). There are seven cases of how the G_k which belong to the same factor can intersect. We call four of these cases exceptional (these are the cases 1a, 1b, 2a and 3a in the list in the proof of Proposition 4.5) and three of them generic (these are the cases 1c, 2b, 3b).

Depending on which cases occur we determine the combinatorial type of Q in Lemmas 4.6, 4.7, 4.8 and 4.9. With these Lemmas we complete the proof of Proposition 4.5.

The proofs of the above lemmas are again subdivided into several sublemmas. In Sublemmas 4.10, 4.11, 4.12 and 4.13 we determine the combinatorial type of those facets of Q which belong to a factor Γ_{j_0} where one of the exceptional cases occurs.

Then in Sublemma 4.14 we use this information and Lemma 4.4 to show that F has at most one factor where one of the exceptional cases can appear. In Lemma 4.4 we show that a certain poset is not the poset of a nice manifold with corners with only acyclic faces.

Assuming that one of the exceptional cases appears at the factor Γ_{j_0} of F the combinatorial types of the G_k which do not belong to Γ_{j_0} are then determined in Sublemma 4.15. With the information gained in this sublemma together with the results from Sublemmas 4.10, 4.11, 4.12 and 4.13 we then can determine the combinatorial type of Q for the case that at one factor of F one of the exceptional cases occurs. This completes the proofs of Lemmas 4.6, 4.7 and 4.8.

So we are left with the case where no exceptional case appears at the factors of F. For this case we determine in Sublemma 4.16 the combinatorial types of the G_k. Moreover, in Sublemma 4.17 we show that there is exactly one facet H of Q which does not meet F. We determine the combinatorial type of H in the same sublemma. With the information on the combinatorial types of all facets of Q we then can determine the combinatorial type of Q. This completes the proof of Lemma 4.9. □

For the proof of Theorem 4.1 we need the following lemmas.

Lemma 4.2 *Let M be a torus manifold with $H^{odd}(M;\mathbb{Q}) = 0$. Assume that M admits an invariant metric of non-negative sectional curvature or M is rationally elliptic. Then each two-dimensional face of M/T contains at most four vertices.*

Proof. At first note that, by localization in equivariant cohomology, the odd-degree cohomology of all fixed point components of all subtori of T vanishes.

A two-dimensional face F is the image of a fixed point component M_1 of a codimension-two subtorus of T under the orbit map. Therefore it follows from the classification of four-dimensional T^2-manifolds given in [OR70] and [OR74] that the orbit space is homeomorphic to a two-dimensional disk. If M admits an invariant metric of non-negative sectional curvature, then the same holds for M_1. Therefore it follows from the argument in the proof of [GGS11, Lemma 4.1] that there are at most four vertices in F.

Now assume that M is rationally elliptic. Then, by [AP93, Corollary 3.3.11], the minimal model $\mathcal{M}(M_1)$ of M_1 is elliptic. The number of vertices in F is equal to the number of fixed points in M_1. Since $\chi(M_1^T) = \chi(M_1)$, it is also equal to the Euler-characteristic of M_1. By [FHT01, Theorem 32.6] and [FHT01, Theorem 32.10], we have

$$4 \geq 2 \dim \Pi_\psi^2(M_1) = 2b_2(M_1).$$

Here $\Pi_\psi^2(M_1)$ denotes the second pseudo-dual rational homotopy group of M_1. Therefore, $\chi(M_1) \leq 4$ and there are at most four vertices in F. □

Remark 4.3 *If M is a rationally elliptic torus manifold, then we always have $H^{odd}(M;\mathbb{Q}) = 0$ since $\chi(M) = \chi(M^T) > 0$. Therefore $H^{odd}(M;\mathbb{Z}) = 0$ if and only if $H^*(M;\mathbb{Z})$ is torsion-free.*

Lemma 4.4 *For $n > 2$, there is no n-dimensional nice manifold with corners whose faces are all acyclic, such that each facet is combinatorially equivalent to an $(n-1)$-dimensional cube and the intersection of any two facets has two components.*

Proof. Assume that there is such a manifold Q with corners. Then the boundary of Q is a homology sphere. Moreover by applying the construction α from Section 3 to Q we may assume that all faces of Q of codimension at least one are contractible.

Let $Q' = [-1,1]^n/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $[-1,1]^n$ by multiplication with -1 on each factor. Then the boundary of Q' is a real projective space of dimension $n-1$.

Moreover, for each facet F_i of Q and each facet F_i' of Q' there is an isomorphism of face posets $\mathcal{P}(F_i) \to \mathcal{P}(F_i')$ such that $F_i \cap F_j$ is mapped to $F_i' \cap F_j'$.

Since there are automorphisms of $\mathcal{P}(F_i)$ which interchange the two components of $F_i \cap F_j$ and leave the other facets of F_i unchanged, we can glue these isomorphisms together to get an isomorphism of face posets

$$\mathcal{P}(Q) \to \mathcal{P}(Q').$$

Since the faces of Q and Q' of codimension at least one are contractible we obtain a homeomorphism $\partial Q \to \partial Q'$. This is a contradiction because ∂Q is a homology sphere and $\partial Q'$ a projective space. $\square$

Proposition 4.5 *Let Q be a nice manifold with only acyclic faces such that each two-dimensional face of Q has at most four vertices. Then $\mathcal{P}(Q)$ is isomorphic to the face poset of a product $\prod_i \Sigma^{n_i} \times \prod_i \Delta^{n_i}$. Here Σ^m is the orbit space of the linear T^m-action on S^{2m} and Δ^m is an m-dimensional simplex.*

Proof. We prove this proposition by induction on the dimension of Q. If $\dim Q = 2$, there is nothing to show.

Therefore let us assume that $\dim Q > 2$ and that all facets of Q are combinatorially equivalent to a product of Σ^{n_i}'s and Δ^{n_i}'s.

Let F be a facet of Q, such that F is combinatorially equivalent to $\prod_i \Gamma_i$, where $\Gamma_i = \Sigma^{n_i}$ for $i < r$ and $\Gamma_i = \Delta^{n_i}$ for $i \geq r$. We fix this facet F for the rest of this section.

In the following we will denote $\tilde{\Gamma}_i = \Sigma^{n_i - 1}$ and $\bar{\Gamma}_i = \Sigma^{n_i+1}$ if $i < r$ or $\tilde{\Gamma}_i = \Delta^{n_i-1}$ and $\bar{\Gamma}_i = \Delta^{n_i+1}$ if $i \geq r$.

Each facet of Q which meets F intersects F in a union of facets of F. Since $\mathcal{P}(F) \cong \mathcal{P}(\prod_i \Gamma_i)$, the facets of F are of the form

$$F_j \times \prod_{i \neq j} \Gamma_i,$$

where F_j is a facet of Γ_j. Therefore each facet G_k of Q which meets F "belongs" to a factor $\Gamma_{j(k)}$ of F, i.e.

$$F \cap G_k \cong \prod_{i \neq j(k)} \Gamma_i \times \tilde{F}_k,$$

where $\tilde{F}_k$ is a union of facets of the $j(k)$-th factor $\Gamma_{j(k)}$ in F.

If $\dim \Gamma_j = 1$, then Γ_j is combinatorially equivalent to an interval. Hence, it has two facets which do not intersect. Therefore in this case there are at most two facets of Q which belong to Γ_j. If there is exactly one such facet G_k, then the intersection $F \cap G_k$ has two components. Otherwise the intersections $F \cap G_k$ are connected.

If $\Gamma_j = \Sigma^{n_j}$ with $n_j > 1$, then Γ_j has exactly n_j facets. These facets have pairwise non-trivial intersections. Therefore $F \cap G_k$ is connected if $j(k) = j$. And there are exactly n_j facets of Q which belong to Γ_j.

If $\Gamma_j = \Delta^{n_j}$ with $n_j > 1$, then Γ_j has exactly $n_j + 1$ facets. These facets have pairwise non-trivial intersections. Hence, $F \cap G_k$ is connected if $j(k) = j$ and there are exactly $n_j + 1$ facets of Q which belong to Γ_j.

Therefore there are the following cases:

1. $\dim \Gamma_j = 1$ and one of the following statements holds:
   a) There is exactly one facet G_k which belongs to Γ_j.
   b) There are exactly two facets G_{k_1}, G_{k_2} which belong to Γ_j and $G_{k_1} \cap G_{k_2} \neq \emptyset$.
   c) There are exactly two facets G_{k_1}, G_{k_2} which belong to Γ_j and $G_{k_1} \cap G_{k_2} = \emptyset$.

2. $\Gamma_j \cong \Sigma^{n_j}$ with $n_j > 1$ and one of the following statements holds:
   a) There are exactly n_j facets $G_{k_1}, \dots, G_{k_{n_j}}$ which belong to Γ_j and the union of those components of $\bigcap_{i=1}^{n_j} G_{k_i}$ which meet F is connected.
   b) There are exactly n_j facets $G_{k_1}, \dots, G_{k_{n_j}}$ which belong to Γ_j and the union of those components of $\bigcap_{i=1}^{n_j} G_{k_i}$ which meet F is not connected.
3. $\Gamma_j \cong \Delta^{n_j}$ with $n_j > 1$ and one of the following statements holds:
   a) There are exactly $n_j + 1$ facets $G_{k_1}, \dots, G_{k_{n_j+1}}$ which belong to Γ_j and $\bigcap_{i=1}^{n_j+1} G_{k_i} \neq \emptyset$.
   b) There are exactly $n_j + 1$ facets $G_{k_1}, \dots, G_{k_{n_j+1}}$ which belong to Γ_j and $\bigcap_{i=1}^{n_j+1} G_{k_i} = \emptyset$.

The proof of the proposition will be completed by the following lemmas.

Lemma 4.6 *If, in the above situation, there is a j_0 such that*

- $\Gamma_{j_0} \cong \Sigma^{n_{j_0}}$ *with* $n_{j_0} > 1$ *and the union of those components of* $\bigcap_{k;\, j(k)=j_0} G_k$ *which meet F is connected, or*
- $\Gamma_{j_0} \cong \Delta^{n_{j_0}}$ *with* $n_{j_0} > 1$ *and* $\bigcap_{k;\, j(k)=j_0} G_k \neq \emptyset$,

i.e., one of the cases 2a and 3a appears at Γ_{j_0}, then there is an isomorphism of face posets $\mathcal{P}(Q) \to \mathcal{P}(\bar{\Gamma}_{j_0} \times \prod_{i \neq j_0} \Gamma_i)$ which sends each G_k to a facet belonging to the $j(k)$-th factor and F to a facet belonging to $\bar{\Gamma}_{j_0}$.

Lemma 4.7 *If, in the above situation, there is a j_0 such that $\dim \Gamma_{j_0} = 1$ and there is exactly one facet G_{k_0} which belongs to Γ_{j_0}, i.e., the case 1a appears at Γ_{j_0}, then there is an isomorphism of face posets $\mathcal{P}(Q) \to \mathcal{P}(\Sigma^2 \times \prod_{i \neq j_0} \Gamma_i)$, which sends each G_k to a facet belonging to the $j(k)$-th factor and F to a facet belonging to Σ^2.*

Lemma 4.8 *If, in the above situation, there is a j_0 such that $\dim \Gamma_{j_0} = 1$ and there are exactly two facets G_{k_0} and $G_{k_0'}$ which belong to Γ_{j_0} and $G_{k_0} \cap G_{k_0'} \neq \emptyset$, i.e., the case 1b appears at Γ_{j_0}, then there is an isomorphism of face posets $\mathcal{P}(Q) \to \mathcal{P}(\Delta^2 \times \prod_{i \neq j_0} \Gamma_i)$ which sends each G_k to a facet belonging to the $j(k)$-th factor and F to a facet belonging to Δ^2.*

In particular, if one of the cases 1a, 1b, 2a and 3a appears at Γ_{j_0}, then at the other Γ_j only the cases 1c, 2b and 3b can appear.

Lemma 4.9 *If, in the above situation, at all factors Γ_j only the cases 1c, 2b and 3b appear, then there is an isomorphism of face posets $\mathcal{P}(Q) \to \mathcal{P}(F \times [0,1])$ which sends each G_k to $(G_k \cap F) \times [0,1]$ and F to $F \times \{0\}$.*

$\square$

Now we prove by induction on the dimension of Q the lemmas from above. For $\dim Q = 2$, these lemmas are obvious. Therefore we may assume that $n = \dim Q > 2$ and that all the lemmas are proved in dimensions less than n.

Sublemma 4.10 *Assume that the case 2a appears at the factor Γ_{j_0} of F. Let G_{k_0} be a facet of Q which belongs to Γ_{j_0}. Then the following holds:*

1. *The facets of G_{k_0} are given by the components of the intersections $G_k \cap G_{k_0}$ and $F \cap G_{k_0}$.*
2. *$G_k \cap G_{k_0}$ is connected if and only if $F \cap G_k$ is connected.*
3. *There is a combinatorial equivalence*

$$\mathcal{P}(G_{k_0}) \to \mathcal{P}(\prod_i \Gamma_i),$$

such that $F \cap G_{k_0}$ corresponds to a facet of the j_0-th factor and the components of $G_k \cap G_{k_0}$ correspond to facets of the $j(k)$-th factor.

Proof. We consider the inclusion of $F \cap G_{k_0} \hookrightarrow G_{k_0}$, where G_{k_0} is a facet of Q which belongs to Γ_{j_0}. Then $F \cap G_{k_0}$ is a facet of G_{k_0} and there is a combinatorial equivalence

$$F \cap G_{k_0} \cong \Sigma^{n_{j_0}-1} \times \prod_{i \neq j_0} \Gamma_i$$

such that each component of $G_{k_0} \cap G_k \cap F$ corresponds to a facet of the $j(k)$-th factor. Moreover, the facets of G_{k_0} which meet $F \cap G_{k_0}$ are given by those components of the $G_k \cap G_{k_0}$ which meet F. Since the case 2a appears at the factor Γ_{j_0} of F it follows that there is only one component of $\bigcap_{k;\, j(k)=j_0} G_k = \bigcap_{k;\, j(k)=j_0} (G_k \cap G_{k_0})$ which meets F. Therefore one of the cases 2a or 1a appears at the factor $\Sigma^{n_{j_0}-1}$ of $F \cap G_{k_0}$.

Hence, it follows from the induction hypothesis that G_{k_0} is combinatorially equivalent to $\prod_i \Gamma_i$ in such a way that the component of $G_{k_0} \cap G_k$ which meets F is mapped to a facet which belongs to the $j(k)$-th factor and $F \cap G_{k_0}$ is mapped to a facet which belongs to the j_0-th factor.

Since all facets of $\prod_i \Gamma_i$ meet the facet which corresponds to $F \cap G_{k_0}$ it follows that each facet of G_{k_0} is a component of some intersection $G_k \cap G_{k_0}$.

Moreover, $G_k \cap G_{k_0}$ is connected for all k with $n_{j(k)} > 1$ because for these k the facets of the factor $\Gamma_{j(k)}$ of G_{k_0} have pairwise non-trivial intersections. If $n_{j(k)} = 1$ and $j(k) \neq j_0$, then $G_k \cap G_{k_0}$ is disconnected if and only if $G_k \cap G_{k_0} \cap F$ is disconnected because $F \cap G_{k_0}$ and the components of $G_k \cap G_{k_0}$ are facets of different factors of $G_{k_0} \cong \prod_i \Gamma_i$. Since G_k and G_{k_0} belong to different factors of F, it follows that this last statement is true if and only if $G_k \cap F$ is disconnected. $\square$

Sublemma 4.11 *Assume that the case 3a appears at the factor Γ_{j_0} of F. Let G_{k_0} be a facet of Q which belongs to Γ_{j_0}. Then the following holds:*

1. *The facets of G_{k_0} are given by the components of the intersections $G_k \cap G_{k_0}$ and $F \cap G_{k_0}$.*

2. *$G_k \cap G_{k_0}$ is connected if and only if $F \cap G_k$ is connected.*

3. *There is a combinatorial equivalence*

$$\mathcal{P}(G_{k_0}) \to \mathcal{P}(\prod_i \Gamma_i),$$

such that $F \cap G_{k_0}$ corresponds to a facet of the j_0-th factor and the components of $G_k \cap G_{k_0}$ correspond to facets of the $j(k)$-th factor.

Proof. We consider the inclusions of $F \cap G_{k_0} \hookrightarrow G_{k_0}$ where G_{k_0} is a facet which belongs to Γ_{j_0}. Then $F \cap G_{k_0}$ is a facet of G_{k_0} and there is a combinatorial equivalence

$$F \cap G_{k_0} \cong \Delta^{n_{j_0}-1} \times \prod_{i \neq j_0} \Gamma_i$$

such that each component of $G_{k_0} \cap G_k \cap F$ corresponds to a facet of the $j(k)$-th factor.

The facets of G_{k_0} which meet $F \cap G_{k_0}$ are given by the components C_{ki} of $G_k \cap G_{k_0}$ which meet F. The induction hypothesis then implies that G_{k_0} is combinatorially equivalent to one of the following spaces

1. $\Delta^{n_{j_0}} \times \prod_{i \neq j_0} \Gamma_i$ or

2. $\bar{\Gamma}_{j_1} \times \tilde{\Gamma}_{j_0} \times \prod_{i \neq j_1, j_0} \Gamma_i$ or

3. $[0,1] \times \tilde{\Gamma}_{j_0} \times \prod_{i \neq j_0} \Gamma_i$

in such a way that the C_{ki} with $j(k) = j_0$ correspond to facets of the j_0-th factor. In particular, $G_k \cap G_{k_0} = C_{k1}$ is connected for all k with $j(k) = j_0$.

Indeed, if $n_{j_0} > 2$, then all facets of $\tilde{\Gamma}_{j_0}$ and Γ_{j_0} have pairwise non-trivial intersections. Hence, $G_k \cap G_{k_0}$ has only one component in this case.

If $n_{j_0} = 2$, then $\tilde{\Gamma}_{j_0}$ has two facets. Moreover, the intersection of G_{k_0} with another facet of Q which belongs to the factor Γ_{j_0} of F is a non-empty union of facets of $\tilde{\Gamma}_{j_0}$. Since besides G_{k_0} there are two other facets of Q which belong to Γ_{j_0}, these intersections must be connected.

Since $\bigcap_{k;\, j(k)=j_0} (G_k \cap G_{k_0}) = \bigcap_{k;\, j(k)=j_0} G_k \neq \emptyset$, it follows from the induction hypothesis that we are in case 1. In this case an isomorphism of $\mathcal{P}(G_{k_0}) \to \mathcal{P}(\prod_i \Gamma_i)$ is induced by

$$F \mapsto H_{k_0} \qquad\qquad C_{ki} \mapsto H_{ki},$$

where H_{k_0} is a facet of the j_0-th factor in the product and the other H_{ki} are facets of the $j(k)$-th factor in the product.

Since all facets of $\prod_i \Gamma_i$ meet the facet which corresponds to $F \cap G_{k_0}$, it follows that all facets of G_{k_0} are components of intersections $G_k \cap G_{k_0}$.

Moreover, $G_k \cap G_{k_0} = C_{k1}$ is connected for all k with $n_{j(k)} > 1$ because all facets of these $\Gamma_{j(k)}$ have pairwise non-trivial intersection. If $n_{j(k)} = 1$ and $j(k) \neq j_0$, then $G_k \cap G_{k_0}$ is disconnected if and only if $G_k \cap F$ is disconnected. This last statement can be seen as in the proof of Sublemma 4.10. □

Sublemma 4.12 *Assume that the case 1a appears at the factor Γ_{j_0} of F. Let G_{k_0} be the facet of Q which belongs to Γ_{j_0}. Then the following holds:*

1. *The facets of G_{k_0} are given by the components of the intersections $G_k \cap G_{k_0}$ and $F \cap G_{k_0}$.*

2. *$G_k \cap G_{k_0}$ is connected if and only if $F \cap G_k$ is connected.*

3. *There is a combinatorial equivalence*

$$\mathcal{P}(G_{k_0}) \to \mathcal{P}(\prod_i \Gamma_i),$$

such that $F \cap G_{k_0}$ corresponds to a facet of the j_0-th factor and the components of $G_k \cap G_{k_0}$ correspond to facets of the $j(k)$-th factor.

Proof. At first we describe the combinatorial type of G_{k_0} where G_{k_0} is the facet of Q which belongs to Γ_{j_0}.

We consider the inclusion of a component C of $F \cap G_{k_0}$ in G_{k_0}. Then C is a facet of G_{k_0} and there is a combinatorial equivalence

$$C \cong \prod_{i \neq j_0} \Gamma_i$$

such that each component of $G_{k_0} \cap G_k \cap C$ corresponds to a facet of the $j(k)$-th factor.

Then the facets of G_{k_0} which meet C are given by the components C_{ki} of $G_k \cap G_{k_0}$ which meet C. It follows from the induction hypothesis that G_{k_0} is combinatorially equivalent to one of the following spaces:

1. $[0,1] \times \prod_{i \neq j_0} \Gamma_i$ or

2. $\Sigma^{n_{j_1}+1} \times \prod_{i \neq j_0, j_1} \Gamma_i$ or

3. $\Delta^{n_{j_1}+1} \times \prod_{i \neq j_0, j_1} \Gamma_i$,

such that each C_{ki} corresponds to a facet of the $j(k)$-th factor and C corresponds to a facet of the first factor. By the condition 1a, there is a facet of G_{k_0} which does not meet C. This facet is the other component of the intersection $F \cap G_{k_0}$. Hence, it follows that we are in case 1.

In this case an isomorphism of $\mathcal{P}(G_{k_0}) \to \mathcal{P}([0,1] \times \prod_{i \neq j_0} \Gamma_i)$ is induced by

$$C \mapsto H_{k_2} \qquad\qquad C' \mapsto H'_{k_2} \qquad\qquad C_{ki} \mapsto H_{ki} \text{ for } j(k) \neq j_0,$$

where H_{ki} is a facet of the $j(k)$-th factor in the product, H_{k_2} and H'_{k_2} are the facets of the j_0-th factor $[0,1]$ of the product and C' is the other component of $F \cap G_{k_0}$.

Since all facets of $\prod_i \Gamma_i$ except the facet corresponding to C' meet the facet corresponding to C, it follows that all facets of G_{k_0} are components of intersections $G_k \cap G_{k_0}$.

As in the proof of Sublemma 4.10 one sees, moreover, that $G_k \cap G_{k_0} = C_k$ is connected for all k with $n_{j(k)} > 1$. If $n_{j(k)} = 1$ and $j(k) \neq j_0$, then $G_k \cap G_{k_0}$ is disconnected if and only if $G_k \cap F$ is disconnected. □

Sublemma 4.13 *Assume that the case 1b appears at the factor Γ_{j_0} of F. Let G_{k_0} be a facet of Q which belongs to Γ_{j_0}. Then the following holds:*

1. *The facets of G_{k_0} are given by the components of the intersections $G_k \cap G_{k_0}$ and $F \cap G_{k_0}$.*
2. *$G_k \cap G_{k_0}$ is connected if and only if $F \cap G_k$ is connected.*
3. *There is a combinatorial equivalence*

$$\mathcal{P}(G_{k_0}) \to \mathcal{P}(\prod_i \Gamma_i),$$

such that $F \cap G_{k_0}$ corresponds to a facet of the j_0-th factor and the components of $G_k \cap G_{k_0}$ correspond to facets of the $j(k)$-th factor.

Proof. Let G_{k_0} and $G_{k_0'}$ be the two facets of Q, which belong to Γ_{j_0}.

Then the intersection $G_{k_0} \cap G_{k_0'}$ is non-empty. Therefore G_{k_0} has a facet which is not equal to $F \cap G_{k_0}$ or a component of $G_k \cap G_{k_0}$ with $j(k) \neq j_0$. Therefore, as in the proof of Sublemma 4.12, one sees that G_{k_1} is combinatorially equivalent to $[0,1] \times \prod_{i \neq j_0} \Gamma_i$.

The other statements of the Sublemma can be seen as in the proof of Sublemma 4.10. □

Next we show that, if there is a factor Γ_{j_0} of F where one of the cases 1a, 1b, 2a or 3a occurs, then at the other factors of F only the cases 1c, 2b or 3b can occur.

Sublemma 4.14 *There is at most one factor Γ_{j_0} of F, where one of the cases 1a, 1b, 2a or 3a appears.*

Proof. Assume that one of the cases 2a and 3a occurs at the factor Γ_{j_0} of F and one of the cases 1a, 1b, 2a and 3a appears at another factor Γ_{j_1} of F. Then we consider the intersection $G_{k_0} \cap G_{k_1}$ with $j(k_0) = j_0$ and $j(k_1) = j_1$. It follows from the description of the combinatorial type of G_{k_0} given in the Sublemmas 4.10 and 4.11, that there is an isomorphism of posets

$$\mathcal{P}(C) \to \mathcal{P}(\Gamma_{j_0} \times \tilde{\Gamma}_{j_1} \times \prod_{i \neq j_0, j_1} \Gamma_i),$$

where C is a component of $G_{k_1} \cap G_{k_0}$ such that $F \cap G_{k_0} \cap G_{k_1}$ and the $G_k \cap G_{k_0} \cap G_{k_1}$, $j(k) = j_0$, correspond to the facets belonging to the factor Γ_{j_0}. Hence, it follows that the intersection of $G_{k_0} \cap G_{k_1}$ with $\bigcap_{k;\, j(k)=j_0} G_k$ is non-empty (or connected) if case 3a (or 2a, respectively) appears at the factor Γ_{j_0} of F.

From the description of the combinatorial type of G_{k_1} given in Sublemmas 4.10, 4.11, 4.12 and 4.13 it follows that there is an isomorphism of posets

$$\mathcal{P}(G_{k_0} \cap G_{k_1}) \to \mathcal{P}(\tilde{\Gamma}_{j_0} \times \Gamma_{j_1} \times \prod_{i \neq j_0, j_1} \Gamma_i),$$

such that the $G_k \cap G_{k_0} \cap G_{k_1}$, $j(k) = j_0$, correspond to the facets belonging to the factor $\tilde{\Gamma}_{j_0}$. Hence, it follows that the intersection of $G_{k_0} \cap G_{k_1}$ with $\bigcap_{k;\, j(k)=j_0} G_{kj_0}$ is empty (or non-connected) if case 3a (or 2a, respectively) appears at the factor $\tilde{\Gamma}_{j_0}$ of F. Therefore we have a contradiction.

Next assume that at one factor Γ_{j_0} of F the case 1a appears and at another factor Γ_{j_1} the case 1b appears. Then it follows from the description of the combinatorial type of G_{k_0} given in Sublemma 4.12 that the intersection $G_{k_0} \cap G_{k_1}$ is connected. Here G_{k_0} is the facet belonging to the factor Γ_{j_0} and G_{k_1} is a facet belonging to the factor Γ_{j_1}. But the description of the combinatorial type of G_{k_1} given in Sublemma 4.13 implies that this intersection is disconnected.

Next assume that the case 1b occurs at two factors Γ_{j_0} and Γ_{j_1}. Let G_{k_0} and $G_{k_0'}$ be the facets belonging to Γ_{j_0}. Moreover, let G_{k_1} be a facet of Q belonging to Γ_{j_1}. Then it follows from the description of the combinatorial type of G_{k_0} given in Sublemma 4.13 that the intersection $G_{k_0} \cap G_{k_0'} \cap G_{k_1}$ is non-empty. But from the description of the combinatorial type of G_{k_1} it follows that this intersection is empty.

Last we show that the case 1a occurs at at most one factor of F. Assume that the case 1a appears at the factors Γ_j with $j < s$ and not at the factors Γ_j with $j \geq s$. Then for each $j \geq s$ choose n_j facets $G_{k_{j1}}, \dots, G_{k_{jn_j}}$ of Q belonging to that factor. Consider a component K of $\bigcap_{j \geq s} \bigcap_{i=1}^{n_j} G_{k_{ji}}$ which meets F. Since at none of the factors Γ_j, $j \geq s$, the cases 1a or 2a appear, the intersection $F \cap K$ is connected. Therefore also the intersections $G_k \cap K$, $j(k) < s$ are connected by Sublemma 4.12.

By the description of the combinatorial type of the G_k given in Sublemma 4.12 the intersections $G_k \cap K$ are combinatorially equivalent to cubes and have pairwise non-connected intersection. Therefore it follows from Lemma 4.4, that K has dimension two. Hence it follows that s is two, i.e. there is only one factor of F where the case 1a appears.

Therefore there is at most one factor Γ_{j_0} of F where one of the cases 1a, 1b, 2a and 3a appears. $\square$

Sublemma 4.15 *Assume that one of the cases 1a, 1b, 2a or 3a appears at the factor Γ_{j_0} of F. Let G_{k_1} be a facet of Q which meets F and belongs to Γ_{j_1} with $j_1 \neq j_0$. Then the following holds:*

1. *The facets of G_{k_1} are given by the components of the intersections $G_k \cap G_{k_1}$.*
2. *$G_{k_1} \cap G_k$ is disconnected if and only if $\Gamma_{j_1} = \Sigma^2$ and $j(k) = j(k_1) = j_1$.*
3. *There is a combinatorial equivalence*

$$G_{k_1} \cong \bar{\Gamma}_{j_0} \times \tilde{\Gamma}_{j_1} \times \prod_{i \neq j_0, j_1} \Gamma_i,$$

such that $F \cap G_{k_1}$ corresponds to a facet of the j_0-th factor and the $G_k \cap G_{k_1}$ correspond to disjoint unions of facets of the $j(k)$-th factor.

Proof. We consider the inclusion $F \cap G_{k_1} \hookrightarrow G_{k_1}$. $F \cap G_{k_1}$ is a facet of G_{k_1} and there is a combinatorial equivalence

$$G_{k_1} \cap F \cong \tilde{\Gamma}_{j_1} \times \prod_{i \neq j_1} \Gamma_i$$

such that each component of $G_k \cap G_{k_1} \cap F$ corresponds to a facet of the $j(k)$-th factor. Moreover, the facets of G_{k_1} which meet $G_{k_1} \cap F$ are given by the components C_{ki} of $G_k \cap G_{k_1}$ which meet $G_{k_1} \cap F$.

If $j(k) = j_0$, then $G_k \cap F$ and $F \cap G_{k_1}$ are facets of different factors of F. Since $G_k \cap G_{k_0}$ and $G_{k_0} \cap G_{k_1}$ are facets of different factors of $G_{k_0} \cong \prod_i \Gamma_i$ and because $F \cap G_{k_0}$ is a facet of the j_0-th factor of G_{k_0}, it follows from Sublemma 4.11 that $\bigcap_{k;\, j(k)=j_0} (G_k \cap G_{k_1}) \neq \emptyset$ if the case 3a appears at the factor Γ_{j_0} of F.

Furthermore, by the same argument $\bigcap_{k;\, j(k)=j_0} (G_k \cap G_{k_1}) \neq \emptyset$ is connected if the case 2a appears at the factor Γ_{j_0} of F. Here one uses Sublemma 4.10.

If the case 1a appears at the factor Γ_{j_0} one can argue as follows. It follows from the description of the combinatorial type of G_{k_0}, $j(k_0) = j_0$, given in Sublemma 4.12 that $G_{k_0} \cap G_{k_1}$ is connected. Moreover, it follows from the same description that $G_{k_0} \cap G_{k_1} \cap F$ has two components.

Therefore the case 1a also appears at the factor Γ_{j_0} of $G_{k_1} \cap F$.

If the case 1b appears at the factor Γ_{j_0}, then it follows from the combinatorial description of the G_k with $j(k) = j_0$ given in Sublemma 4.13 that for these k the intersection of G_k with G_{k_1} is connected. Moreover, the intersection $\bigcap_{k;\, j(k)=j_0} G_k \cap G_{k_1}$ is non-empty. Therefore the case 1b appears at the factor Γ_{j_0} of $G_{k_1} \cap F$.

Hence, the case which appears for the factor Γ_{j_0} of F also appears at the factor Γ_{j_0} of $G_{k_1} \cap F$. Therefore from the induction hypothesis we get an isomorphism of posets

$$\mathcal{P}(G_{k_1}) \to \mathcal{P}(\bar{\Gamma}_{j_0} \times \tilde{\Gamma}_{j_1} \times \prod_{i \neq j_0, j_1} \Sigma^{n_i} \times \prod_{i \neq j_0 j_1} \Delta^{n_i}),$$

such that $G_{k_1} \cap F$ is mapped to a facet of the j_0-th factor and C_{ki} to a facet of the $j(k)$-th factor. Here C_{ki} is a compoenent of $G_k \cap G_{k_1}$.

Since all pairs of facets of $\bar{\Gamma}_{j_0}$ have non-trivial intersection, all facets of G_{k_1} meet $G_{k_1} \cap F$. Moreover, it follows that $G_k \cap G_{k_1}$ is connected if $j(k) \neq j_1$ or $\Gamma_{j_1} \neq \Sigma^2$. Otherwise this intersection has two components.

Indeed, if $j(k_2) \neq j_1, j_0$, then it follows from the description of the combinatorial type of F that $F \cap G_{k_1} \cap G_{k_2} = (F \cap G_{k_1}) \cap (F \cap G_{k_2})$ is connected. Because G_{k_2} and F belong to different factors of G_{k_1} it follows that $G_{k_2} \cap G_{k_1}$ is connected.

Next assume that $j(k_2) = j_1$ and $\dim \Gamma_{j_1} \geq 3$. Then all pairs of facets of $\tilde{\Gamma}_{j_1}$ have non-trivial intersections. Therefore $G_{k_2} \cap G_{k_1}$ is connected in this case.

Assume now that $\Gamma_{j_1} = \Delta^2$. Then besides G_{k_1} there are two other facets of Q which belong to Γ_{j_1}. These two facets have non-trivial intersections with G_{k_1}. Moreover, the components of these intersections are facets of the factor $\tilde{\Gamma}_{j_1}$ of G_{k_1}. Since $\tilde{\Gamma}_{j_1}$ has two facets, the intersections $G_{k_2} \cap G_{k_1}$ with $j(k_2) = j_1$ are connected.

Next assume that $\Gamma_{j_1} = \Sigma^2$. Then besides G_{k_1} there is exactly one other facet G_{k_2} of Q which belongs to Γ_{j_1}. Moreover, $F \cap G_{k_2} \cap G_{k_1}$ has two components. Since $F \cap G_{k_1}$ and the components of $G_{k_2} \cap G_{k_1}$ are facets of different factors of G_{k_1}. It follows that $G_{k_2} \cap G_{k_1}$ has two components.

At last assume that $\dim \Gamma_{j_1} = 1$. Then since $F \cap G_{k_1}$ is connected. There is another facet G_{k_2} of Q which belongs to Γ_{j_1}. Since $F \cap G_{k_1} \cap G_{k_2}$ is empty and all facets of G_{k_1} meet $F \cap G_{k_1}$ it follows that $G_{k_1} \cap G_{k_2}$ is empty. □

Now we can prove the Lemmas 4.6, 4.7, and 4.8.

Proof of Lemmas 4.6, 4.7 and 4.8. For $j \neq j_0$, let $\tilde{n}_j = n_j$ and $\tilde{n}_{j_0} = n_{j_0} + 1$. Moreover, let $G_0 = F$ and $j(0) = j_0$. Let $P = \prod_{i<r} \Sigma^{\tilde{n}_i} \times \prod_{i \geq r} \Delta^{\tilde{n}_i}$. Denote by H_k the facets of P. We have shown in Sublemmas 4.10, 4.11, 4.12, 4.13 and 4.15 that there are isomorphisms of posets

$$\mathcal{P}(G_k) \to \mathcal{P}(H_k)$$

such that $(G_k \cap G_{k'}) \mapsto (H_k \cap H_{k'})$ where H_k and $H_{k'}$ are facets of the $j(k)$-th and $j(k')$-th factor of P, respectively.

If $\bigcap_{k \in K} H_k \neq \emptyset$, then this intersection has 2^m components, where m is the number of j_1's with $j_1 < r$ and $K \supset I_{j_1} = \{k;\ j(k) = j_1\}$. If $K = I_{j_1}$ as above, then $\bigcap_{k \in K} H_k$ has two components C_1 and C_2. Moreover, there is an automorphism of $\mathcal{P}(P)$, which interchanges C_1 and C_2 and fixes all faces of P not contained in $C_1 \cup C_2$. Therefore, after composing some of the isomorphisms $\mathcal{P}(G_k) \to \mathcal{P}(H_k)$ with these automorphisms if necessary, we can extend these isomorphisms to an isomorphism

$$\mathcal{P}(Q) \to \mathcal{P}(P),$$

with $G_k \mapsto H_k$. This completes the proof of the lemmas. $\square$

For the proof of Lemma 4.9 we need two more sublemmas.

Sublemma 4.16 *Assume that we are in the situation of Lemma 4.9. Let G_{k_0} be a facet of Q belonging to the factor Γ_{j_0} of F. Then there is a combinatorial equivalence $\mathcal{P}(G_{k_0}) \to \mathcal{P}((F \cap G_{k_0}) \times [0,1])$ which sends each $G_k \cap G_{k_0}$ to $(F \cap G_{k_0} \cap G_k) \times [0,1]$ and $F \cap G_{k_0}$ to $(F \cap G_{k_0}) \times \{0\}$.*

Proof. We consider the inclusion $F \cap G_{k_0} \hookrightarrow G_{k_0}$. Then $F \cap G_{k_0}$ is a facet of G_{k_0} and there is a combinatorial equivalence

$$F \cap G_{k_0} \cong \tilde{\Gamma}_{j_0} \times \prod_{i \neq j_0} \Gamma_i$$

such that each component of $G_{k_0} \cap G_k \cap F$ corresponds to a facet of the $j(k)$-th factor. The facets of G_{k_0} which meet $F \cap G_{k_0}$ are given by the components of $G_{k_0} \cap G_k$ which meet F. We show that if one of the cases 1c, 2b and 3b appears at the factor Γ_j of F then the same holds for the factor Γ_j of $F \cap G_{k_0}$. If one of the cases 1c and 3b appears this is clear because

$$\bigcap_{k;\, j(k)=j} (G_k \cap G_{k_0}) \subset \bigcap_{k;\, j(k)=j} G_k = \emptyset.$$

Therefore assume that case 2b occurs at Γ_j.

At first assume that $j(k) = j_0$ and $n_j = 2$. Then there is only one G_k which belongs to Γ_{j_0} and is not equal to G_{k_0}. The intersection of G_k and G_{k_0} has two components which meet F because $G_k \cap G_{k_0} \cap F$ has two components and the union of those components of $G_k \cap G_{k_0}$ which meet F is disconnected. Therefore we have that case 1c appears at the factor $\tilde{\Gamma}_{j_0}$ of $F \cap G_{k_0}$.

Next assume that $j(k) = j \neq j_0$ or $n_{j_0} > 2$. Then there is only one component B_k of $G_{k_0} \cap G_k$ which meets F because $G_{k_0} \cap G_k \cap F$ is connected. Clearly $\bigcap_{k;\, j(k)=j} B_k$ is contained in $\bigcap_{k;\, j(k)=j} G_k \cap G_{k_0}$. By dimension reasons $\bigcap_{k;\, j(k)=j} B_k$ is a union of components of $\bigcap_{k;\, j(k)=j} G_k \cap G_{k_0}$. In fact, the union of those components of $\bigcap_{k;\, j(k)=j} B_k$ which meet F is equal to the union of those components of $\bigcap_{k;\, j(k)=j} G_k \cap G_{k_0}$ which meet F.

Since every component of $\bigcap_{k;\, j(k)=j} G_k \cap F$ contains exactly one component of $\bigcap_{k;\, j(k)=j} G_k \cap G_{k_0} \cap F$, each component of $\bigcap_{k;\, j(k)=j} G_k$ which meets F contains a component of $\bigcap_{k;\, j(k)=j} B_k$ which meets F. Because the union of those components of $\bigcap_{k;\, j(k)=j} G_k$ which meet F is disconnected, the same holds for the union of those components of $\bigcap_{k;\, j(k)=j} B_k$ which meet F. Hence, the case 2b appears at the factor Γ_j of $F \cap G_{k_0}$.

Therefore it follows from the induction hypotheses that there is an isomorphism of posets $\mathcal{P}(G_{k_0}) \to \mathcal{P}((F \cap G_{k_0}) \times [0,1])$, such that F is mapped to $(F \cap G_{k_0}) \times \{0\}$ and the component C of $G_{k_0} \cap G_k$ which meets F is mapped to $(F \cap C) \times [0,1]$. In particular, all components of $G_{k_0} \cap G_k$ meet F. $\square$

Sublemma 4.17 *In the situation of Lemma 4.9, there is exactly one facet H of Q which does not meet F. Moreover, the following holds:*

1. *Under the isomorphism constructed in the previous sublemma, $G_{k_0} \cap H$ corresponds to $(F \cap G_{k_0}) \times \{1\}$.*
2. *$\mathcal{P}(H) \cong \mathcal{P}(F)$.*

Proof. Let H_{k_0} be the facet of Q which intersects G_{k_0} in the facet of G_{k_0} which corresponds to $(F \cap G_{k_0}) \times \{1\}$. If q is a vertex of G_{k_0} corresponding to $(p, 1)$, where p is a vertex of $F \cap G_{k_0}$, then H_{k_0} is the facet of Q which is perpendicular to the edge of G_{k_0} which corresponds to $\{p\} \times [0, 1]$. We claim that all H_{k_0} are the same.

If $j_1 \neq j_2$, then the intersection of $G_{k_1} \cap F$ and $G_{k_2} \cap F$, where $j(k_1) = j_1$ and $j(k_2) = j_2$, is non-empty. If $j_1 = j_2$ and $\dim G_{k_1} \cap F = \dim G_{k_2} \cap F > 0$, there is a $G_{k'}$ such that $G_{k'} \cap G_{k_i} \cap F \neq \emptyset$ for $i = 1, 2$. Hence, in these cases $H_{k_1} = H_{k_2}$ because there is a vertex in $G_{k_1} \cap G_{k_2} \cap F$. Since $\dim Q > 2$, F cannot be an interval. Hence, it follows that all H_k are equal, so that we can drop the indices. Since the vertex-edge-graph of H is connected, every face of Q contains at least one vertex and each vertex is contained in exactly $n - 1$ facets of H, it follows that the facets of H are given by the $G_k \cap H$.

Indeed, if there is another facet of H, then it contains a vertex v of H. Since the vertex-edge-graph of H is connected, we may assume that v is connected by an edge to a vertex $v' \in G_k \cap H$. It follows from the description of the combinatorial type of G_k that v' is contained in $n - 1$ facets of H of the form $G_{k'} \cap H$. Therefore each edge which meets v' is contained in a facet of the form $G_{k'} \cap H$. Hence, $v \in G_{k'}$. Therefore it follows from the description of the combinatorial type of $G_{k'}$ that all facets of H which contain v are of the form $G_{k''} \cap H$. This is a contradiction to the assumption that v is contained in a facet which is not of this form.

Therefore there is an isomorphism of posets $\phi : \mathcal{P}(F) \to \mathcal{P}(H)$, such that $\phi(C_K \cap F) = C_K \cap H$. Here C_K is a component of the intersection $\bigcap_{k \in K} G_k$. Since the vertex-edge-graph of Q is connected, every face of Q contains at least one vertex and each vertex is contained in exactly n facets, F, H, G_k is a complete list of facets of Q. □

Proof of Lemma 4.9. It follows from Sublemmas 4.16 and 4.17 that the face posets of Q and $F \times [0, 1]$ are isomorphic. An isomorphism is given by

$$C_K \mapsto (C_K \cap F) \times [0, 1] \qquad (F \cap C_K) \mapsto (F \cap C_K) \times \{0\}$$
$$(H \cap C_K) \mapsto (F \cap C_K) \times \{1\}.$$

Here C_K is a component of the intersection $\bigcap_{k \in K} G_k$. Therefore the sublemma is proved. □

Proof of Theorem 4.1. It follows from Lemmas 4.2 and 4.5 that M/T is combinatorially equivalent to $P = \prod_{i<r} \Sigma^{n_i} \times \prod_{i \geq r} \Delta^{n_i}$. Therefore, by Theorem 3.4, M is homeomorphic to a torus manifold M' over P. The manifold M' can be constructed as the model $M_P(\lambda)$, where λ is the characteristic map of M. Now M' is the quotient

of a free torus action on the moment angle complex Z_P associated to P. But Z_P is equivariantly homeomorphic to a product of spheres with linear torus action. Therefore the theorem is proved. $\square$

5. Applications to rigidity problems in toric topology

A torus manifold M is called quasitoric if it is locally standard and M/T is face-preserving homeomorphic to a simple convex polytope. In toric topology there are two notions of rigidity one for simple polytopes and one for quasitoric manifolds. These are:

Definition 5.1 *Let M be a quasitoric manifold over the polytope P.*

- *M is called rigid if any other quasitoric manifold N with $H^*(N;\mathbb{Z}) \cong H^*(M;\mathbb{Z})$ is homeomorphic to M.*
- *P is called rigid if any other simple polytope Q, such that it exists a quasitoric manifold N over Q and a quasitoric manifold M' over P with $H^*(N;\mathbb{Z}) \cong H^*(M';\mathbb{Z})$, is combinatorially equivalent to P.*

It has been shown by Choi, Panov and Suh [CPS10] that a product of simplices is a rigid polytope. As a consequence of Theorem 4.1 we have the following partial generalization of their result.

Theorem 5.2 *Let M_1 and M_2 be two simply connected torus manifolds with $H^*(M_1;\mathbb{Q}) \cong H^*(M_2;\mathbb{Q})$ and $H^{odd}(M_i;\mathbb{Z}) = 0$. Assume that $\mathcal{P}(M_1/T)$ is isomorphic to $\mathcal{P}(\prod_i \Sigma^{n_i} \times \prod_i \Delta^{n_i})$. Then the face posets of the orbit spaces of M_1 and M_2 are isomorphic.*

Proof. By Theorem 3.4, M_1 is homeomorphic to a quotient of a free linear torus action on a product of spheres. Since the cohomology of such a quotient is intrinsically formal, M_1 and M_2 are rationally homotopy equivalent and rationally elliptic.

Therefore both M_1 and M_2 are homeomorphic to quotients of free torus actions on products of spheres S_i, $i = 1, 2$, where the dimension of the acting torus T_i is equal to the number of odd dimensional spheres in the product. Moreover, each factor in these products has at least dimension 3. And each factor in S_i corresponds to a factor of the face-poset of M_i/T which is combinatorially equivalent to $\prod_j \Sigma^{n_{ji}} \times \prod_j \Delta^{n_{ji}}$. Therefore we have

$$\dim \pi_2(M_i) \otimes \mathbb{Q} = \dim T_i \qquad \dim \pi_2(S_i) \otimes \mathbb{Q} = 0$$
$$\dim \pi_j(M_i) \otimes \mathbb{Q} = \dim \pi_j(S_i) \otimes \mathbb{Q}$$

for $i = 1, 2$ and $j > 2$. Since two products of spheres have the same rational homotopy groups if and only if they have the same number of factors of each dimension, it follows that the face posets of M_1 and M_2 are isomorphic. $\square$

It is known that $\prod_i \mathbb{C}P^{n_i}$, is rigid among quasitoric manifolds (see [CS12] and the references therein). The next corollary shows that $\prod_i \mathbb{C}P^{n_i}$ is rigid among simply connected torus manifolds.

Corollary 5.3 *Let M be a simply connected torus manifold with $H^*(M;\mathbb{Z}) \cong H^*(\prod_i \mathbb{C}P^{n_i};\mathbb{Z})$. Then M is homeomorphic to $\prod_i \mathbb{C}P^{n_i}$.*

Proof. By Theorem 5.2, we know that $\mathcal{P}(M/T)$ is isomorphic to $\mathcal{P}(\prod_i \Delta^{n_i})$. Denote by λ the characteristic function of M. Then from a canonical model we can construct a quasitoric manifold M_1 over $\prod_i \Delta^{n_i}$ with characteristic function λ. By Theorem 3.4, M and M_1 are homeomorphic. Moreover, by Corollary 1.3 of [CS12], M_1 is homeomorphic to $\prod_i \mathbb{C}P^{n_i}$. Therefore the corollary follows. $\square$

6. Non-negatively curved torus manifolds

In this section we prove the following:

Theorem 6.1 *Let M be a simply connected non-negatively curved torus manifold. Then M is equivariantly diffeomorphic to a quotient of a free linear torus action on a product of spheres.*

For the proof of this theorem we need the following result of Spindeler.

Theorem 6.2 ([Spi14, Theorem 3.28 and Lemma 3.30]) *Let M be a closed non-negatively curved fixed point homogeneous Riemannian manifold. Then for every maximal fixed point component F there exists a smooth invariant submanifold $N \subset M$ such that M decomposes as the union of the normal disc bundles of N and F:*

$$M \cong D(F) \cup_E D(N). \tag{2}$$

Here $E = \partial D(F) \cong \partial D(N)$. Further N is invariant under the group $U = \{f \in \mathrm{Iso}(M);\ f(F) = F\}$. Moreover, the decomposition (2) is U-equivariant with respect to the natural action of U on $D(F)$, $D(N)$ and M.

Here a Riemannian G-manifold is called fixed point homogeneous if there is a component F of M^G, such that, for every $x \in F$, G acts transitively on the normal sphere $S(N_x(F, M))$. Such a component F is called maximal component of M^G.

The above mentioned natural U-actions on the normal disc bundles are given by the restrictions of the natural actions on the normal bundles given by differentiating the original action on M.

Now let M be a torus manifold and $F \subset M$ a characteristic submanifold. Then M is naturally a fixed point homogeneous manifold with respect to the $\lambda(F)$-action on M. Moreover, the torus T is contained in the group U from the above theorem. In this situation we have the following lemma.

Lemma 6.3 *Let M be a simply connected torus manifold with an invariant metric of non-negative curvature. Then M is locally standard and M/T and all its faces are diffeomorphic (after-smoothing the corners) to standard discs D^k. Moreover, $H^{odd}(M;\mathbb{Z}) = 0$.*

Proof. We prove this lemma by induction on the dimension of M. If $2n = \dim M \leq 2$, then this is obvious. Therefore assume that $\dim M \geq 4$ and that the lemma is proved in all dimensions less than $\dim M$.

By Theorem 6.2, we have a decomposition

$$M = D(N) \cup_E D(F),$$

where F is a characteristic submanifold of M and E the S^1-bundle associated to the normal bundle of F. Spindeler proved that $\operatorname{codim} N \geq 2$ and $\pi_1(F) = 0$ if M is simply connected [Spi14, Lemma 3.29 and Theorem 3.35] (see also the proof of Lemma 7.1 below).

Since F is totally geodesic in M, it admits an invariant metric of non-negative curvature.

It follows from the exact homotopy sequence for the fibration $\pi_F : E \to F$ that $\pi_1(E)$ is cyclic and generated by the inclusion of a fiber of π_F.

The circle subgroup $\lambda(F)$ of T, which fixes F, acts freely on E by multiplication on the fibers of π_F. It follows from the exact homotopy sequence for the fibration $\pi_N : E \to N$, that $\pi_1(N)$ is generated by the curve

$$\gamma_0 : S^1 = \lambda(F) \to N, \qquad z \mapsto zx_0,$$

where $x_0 \in N$ is any base point of N.

Let $x \in F$ be a T-fixed point. Then, since the T-action on M is effective, up to an automorphism of T, the T-representation on the tangent space at x is given by the standard representation on $\mathbb{C}^n$. Therefore T decomposes as $T \cong (S^1)^n$, where each S^1-factor acts non-trivially on exactly one factor of $T_xM \cong \mathbb{C}^n$. It acts on this factor by complex multiplication. Since $\lambda(F)$ acts trivially on $T_xF \subset T_xM$, $\lambda(F)$ is equal to one of these S^1-factors.

Let T' be the product of the other factors. Then the fiber of π_F over x is a T-orbit of type T/T'.

Then there are two cases:

1. $\dim \pi_N(\pi_F^{-1}(x)) = 0$
2. $\dim \pi_N(\pi_F^{-1}(x)) = 1$

In the first case $\pi_N(\pi_F^{-1}(x))$ is a T-fixed point $\bar{x}_1$ in N. Because N is T-invariant, it follows from an investigation of the T-representation $T_{\bar{x}_1}M$ that N is a fixed point component of some subtorus T'' of T with $2\dim T'' = \operatorname{codim} N$. Therefore N is a torus manifold. Since N is totally geodesic in M it follows that the induced metric on N has non-negative curvature. Moreover N is simply connected since γ_0 is constant for $x_0 = \bar{x}_1$. Hence, it follows from the induction hypothesis that N is locally standard and

N/T is diffeomorphic after smoothing the corners to a standard disc. Hence it follows that the T-actions on $D(N)$ and $D(F)$ are locally standard and

$$D(N)/T \cong N/T \times \Delta^k \cong D^n,$$
$$D(F)/T \cong F/T \times I \cong D^n.$$

Since $E/T \cong F/T$ is also diffeomorphic to a disc, it follows that M/T is diffeomorphic to a standard disc. In particular, $\partial M/T$ is connected.

Note that by the above arguments all characteristic submanifolds of M are simply connected and admit an invariant metric of non-negative curvature. Therefore from the induction hypothesis, we know that if a facet $\tilde{F}$ of M/T contains a vertex, then all faces contained in $\tilde{F}$ are diffeomorphic after smoothing the corners to standard discs. In particular each such face contains a vertex.

Since $\partial M/T$ is connected, it follows that every facet $\tilde{F}$ of M/T contains a vertex. Because each proper face of M/T is contained in a facet, it follows that all faces of M/T are diffeomorphic to standard discs. By [MP06, Theorem 2], we have $H^{\text{odd}}(M;\mathbb{Z}) = 0$. Hence, the lemma follows in this case.

In the second case $\pi_N(\pi_F^{-1}(x)))$ is a one-dimensional orbit. Moreover, $\pi_F^{-1}(x)$ is an orbit of type T/T'. Since the T-action on M is effective and T' is a subtorus of T of codimension one, it follows from dimension reasons and the slice theorem that there is an invariant neighborhood of $\pi_F^{-1}(x)$ which is equivariantly diffeomorphic to

$$\lambda(F) \times \mathbb{C}^{n-1} \times \mathbb{R}, \tag{3}$$

where $\mathbb{C}^{n-1}$ is a faithful T'-representation and $\mathbb{R}$ is a trivial representation. Since E has an invariant collar in $D(F)$ and $D(N)$, the $\mathbb{R}$-factor is normal to E.

Since π_N is a equivariant, $\pi_N(\pi_F^{-1}(x)))$ is an orbit of type $T/(H_0 \times T')$, where H_0 is a finite subgroup of $\lambda(F)$.

By an argument similar to the argument given above for $\pi_F^{-1}(x)$, $\pi_N(\pi_F^{-1}(x)))$ has an invariant neighborhood in M which is diffeomorphic to

$$\lambda(F) \times_{H_0} \mathbb{C}^{n-1} \times \mathbb{R}, \tag{4}$$

where T' acts effectively on $\mathbb{C}^{n-1}$ and the H_0-action on $\mathbb{C}^{n-1} \times \mathbb{R}$ commutes with the T'-action. Moreover, the factor $\mathbb{R}$ is normal to N because the $\mathbb{R}$-factor in (3) is normal to E and π_N is an equivariant submersion.

The restriction of the tangent bundle of N to the orbit $\pi_N(\pi_F^{-1}(x)) \cong \lambda(F)/H_0$ is an invariant subbundle of the restriction of the tangent bundle of M to this orbit. The latter is isomorphic to $\lambda(F) \times_{H_0} \mathbb{C}^{n-1} \times \mathbb{R}$.

Because T' has dimension $n-1$ and acts effectively on $\mathbb{C}^{n-1}$, the invariant subvector bundles of this bundle are all of the form

$$\lambda(F) \times_{H_0} \mathbb{C}^k \times \mathbb{R}^l,$$

with $0 \leq k \leq n-1$ and $l = 0, 1$. Since the $\mathbb{R}$-factor is normal to N and M has even dimension, it follows that N has odd dimension.

Claim: $\lambda(F)$ acts freely on N.

Assume that there is an $H \subset \lambda(F)$, $H \neq \{1\}$, such that H has a fixed point $x_2 \in N$. We may assume that H has order equal to a prime p. Then H acts freely on the fiber of π_N over x_2. This fiber is diffeomorphic to S^{2k}. Since $2 = \chi(S^{2k}) \equiv \chi((S^{2k})^H) \mod p$, it follows that $p = 2$. In this case the restriction of E to the orbit $\lambda(F)x_2$ is a non-orientable sphere bundle. Hence N is not orientable. Therefore $\pi_1(N)$ has even order.

Let

$$\gamma_1 : [0, \frac{1}{2}] \to N, \qquad\qquad y \mapsto \exp(i2\pi y)x_2.$$

Then γ_0 is homotopic to $2\gamma_1$. Since $\pi_1(N)$ is cyclic and generated by γ_0, it follows that

$$[\gamma_0] = 2[\gamma_1] = 2k[\gamma_0],$$

for some $k \in \mathbb{Z}$. Hence, $0 = (2k-1)[\gamma_0]$, which implies that $\pi_1(N)$ is of odd order. This gives a contradiction. Therefore $\lambda(F)$ acts freely on N. In particular H_0 is trivial.

Now it follows from (4), that $N/\lambda(F)$ is a torus manifold with $\pi_1(N/\lambda(F)) = 0$. Hence, N is orientable, because the stable tangent bundle of N is isomorphic to the pullback of the stable tangent bundle of $N/\lambda(F)$. Moreover, by (4), N is a codimension-one submanifold of a fixed point component N' of a subtorus $T'' \subset T$ with $2\dim T'' = \operatorname{codim} N'$. The normal bundle of N' in M splits as a sum of complex line bundles. Therefore N' is orientable and the normal bundle of N in N' is trivial. Hence, the structure group of the normal bundle of N in M (and also of $E \to N$) is given by T''.

Let T''' be a complimentary subtorus of T to T'' with $T''' \supset \lambda(F)$. The T-action on E can be described as follows. T'' acts linearly on the sphere S^{2k} with $2k = \operatorname{codim} N - 1$. Let P be the principal T''-bundle associated to $E \to N$. Then we have

$$E \cong P \times_{T''} S^{2k}.$$

The T'''-action on N lifts to an action on P. Together with the T''-action on S^{2k} this action induces the T-action on E.

Let $H \subset T'''/\lambda(F)$ be the isotropy group of some point $y \in N/\lambda(F)$. Then H acts on the fiber of $E/\lambda(F)$ over y via a homomorphism $\phi : H \to T''$. This ϕ depends only on the component of $(N/\lambda(F))^H$ which contains y. Since $H' = \operatorname{graph} \phi^{-1} \subset T'' \times T'''/\lambda(F)$ acts trivially on the fiber of $E/\lambda(F) \to N/\lambda(F)$ over y, it follows that

$$\operatorname{codim}(N/\lambda(F))^H = \operatorname{codim}(E/\lambda(F))^{H'}$$

and that H' is the isotropy group of generic points in the fiber over y. Since $E/\lambda(F)$ is equivariantly diffeomorphic to F and F is locally standard by the induction hypothesis, it follows that H is a torus and $2\dim H = \operatorname{codim}(N/\lambda(F))^H$. Therefore $N/\lambda(F)$ is locally standard. Hence it follows that M is locally standard in a neighborhood of N. Since M is also locally standard in a neighborhood of F, it follows that M is locally standard everywhere.

Now we have the following sequence of diffeomorphisms

$$\begin{aligned} D^{n-1} \cong F/T \cong E/T = (E/T'')/T''' &= (P \times_{T''} (\Sigma^k))/T''' \\ &= N/T''' \times \Sigma^k \cong N/T''' \times D^k. \end{aligned}$$

Hence there is a diffeomorphism $D^n \cong N/T''' \times D^{k+1} \cong N/T''' \times \Sigma^{k+1} \cong D(N)/T$. Now the statement follows as in the first case. □

For the proof of Theorem 6.1 we need some more preparation.

Lemma 6.4 *Let Q be a nice manifold with corners such that all faces of Q are diffeomorphic (after smoothing the corners) to standard discs. Then the diffeomorphism type of Q is uniquely determined by $\mathcal{P}(Q)$.*

Proof. This follows directly from results of Davis [Dav14, Theorem 4.2]. □

In analogy to line shellings for polytopes we define shellings for nice manifolds with corners.

Definition 6.5 *Let Q be a nice manifold with corners such that all faces of Q are contractible. An ordering $F_1, \dots, F_s$ of the facets of Q is called a shelling if*

1. *F_1 has a shelling.*
2. *For $1 < j \leq s$, $F_j \cap \bigcup_{i=1}^{j-1} F_i$ is the beginning of a shelling of F_j, i.e.*

$$F_j \cap \bigcup_{i=1}^{j-1} F_i = G_1 \cup \dots \cup G_r$$

   for some shelling $G_1, \dots, G_r, \dots, G_t$ of F_j.

3. *If $j < s$, then $\bigcup_{i=1}^{j} F_i$ is contractible.*

Q is called shellable if it has a shelling.

Example 6.6 *Δ^n and Σ^n are shellable and any ordering of their facets is a shelling. This follows by induction on the dimension n because the intersection of any facet F_j with a facet F_i with $i < j$ is a facet of F_j.*

Lemma 6.7 *Let Q_1 and Q_2 be two nice manifolds with corners such that all faces of Q_1 and Q_2 are contractible. If $F_1, \dots, F_s$ and $G_1, \dots, G_r$ are shellings of Q_1 and Q_2, respectively, then*

$$F_1 \times Q_2, \dots, F_{s-1} \times Q_2, Q_1 \times G_1, \dots, Q_1 \times G_r, F_s \times Q_2$$

is a shelling of $Q_1 \times Q_2$.

Proof. We prove this lemma by induction on the dimension of $Q_1 \times Q_2$. For $\dim Q_1 \times Q_2 = 0$ there is nothing to show. Therefore assume that $\dim Q_1 \times Q_2 > 0$ and the lemma is proved for all products $\tilde{Q}_1 \times \tilde{Q}_2$ of dimension less than $\dim Q_1 \times Q_2$.

1. It follows from the induction hypothesis that $F_1 \times Q_2$ has a shelling.

2. For $j \leq s-1$, we have

$$(F_j \times Q_2) \cap \bigcup_{i=1}^{j-1} (F_i \times Q_2) = (F_j \cap \bigcup_{i=1}^{j-1} F_i) \times Q_2. \tag{5}$$

   Because $\bigcup_{i=1}^{j} F_i$ is contractible, $F_j \cap \bigcup_{i=1}^{j-1} F_i \neq \partial F_j$ is the beginning of a shelling of F_j. By the induction hypothesis it follows that (5) is the beginning of a shelling of $F_j \times Q_2$.

   For $1 \leq j \leq r$ we have

$$(Q_1 \times G_j) \cap \left(\bigcup_{i=1}^{s-1} (F_i \times Q_2) \cup \bigcup_{i=1}^{j-1} (Q_1 \times G_i) \right) = \bigcup_{i=1}^{s-1} F_i \times G_j \cup \left(Q_1 \times (\bigcup_{i=1}^{j-1} G_j \cap G_i) \right). \tag{6}$$

   Therefore it follows from the induction hypothesis that (6) is the beginning of a shelling of $Q_1 \times G_j$.

   The intersection of $F_s \times Q_2$ with

$$\bigcup_{i=1}^{s-1} (F_i \times Q_2) \cup \bigcup_{i=1}^{r} (Q_1 \times G_i)$$

   is the whole boundary of $F_s \times Q_2$. Since by assumption there are shellings for F_s and Q_2, it follows from the induction hypothesis that it is a beginning of a shelling for $F_s \times Q_2$.

3. The verification that $\bigcup_{i=1}^{j} F_i \times Q_2 \cup \bigcup_{i=1}^{j'} Q_1 \times G_i$ is contractible for $j \leq s-1$ and $j' \leq r$ is left to the reader.

□

It follows from the above lemma that $\prod_i \Sigma^{n_i} \times \prod_i \Delta^{n_i}$ is shellable. Now we can prove the following lemma in the same way as Theorem 5.6 in [Wie13].

Lemma 6.8 *Let M be a locally standard torus manifold over a shellable nice manifold with corners Q. Then M is determined up to equivariant diffeomorphism by the characteristic function λ_M.*

Now we can prove Theorem 6.1.

Proof of Theorem 6.1. As in the proof of Theorem 4.1, it follows that $(\mathcal{P}(M/T), \lambda)$ is isomorphic to $(\mathcal{P}(M'/T), \lambda')$, where M' is the quotient of a free linear torus action on a product of spheres. This quotient admits an invariant metric of non-negative curvature and is simply connected. Therefore, by Lemmas 6.3, 6.4 and 6.8, M and M' are equivariantly diffeomorphic. $\square$

7. Non-simply connected non-negatively curved torus manifolds

Now we discuss some results for non-simply connected non-negatively curved torus manifolds.

Lemma 7.1 *Let M be a $2n$-dimensional torus manifold with an invariant metric of non-negative sectional curvature. Then we have $|\pi_1(M)| = 2^k$ for some $0 \leq k \leq n-1$.*

Proof. We prove this lemma by induction on the dimension $2n$ of M. For $n = 1$ the only $2n$-dimensional torus manifold is S^2. Therefore the lemma is true in this case.

Now assume that the lemma is true for all torus manifolds of dimension less than $2n$. Let M be a torus manifold of dimension $2n$ with a metric of non-negative curvature and F a characteristic submanifold of M. Then by Theorem 6.2 we have a decomposition of M as a union of two disk bundles:

$$M = D(N) \cup_E D(F).$$

Moreover, F is a torus manifold of dimension $2(n-1)$ which admits an invariant metric of non-negative curvature. Therefore, by the induction hypothesis, we have $|\pi_1(F)| = 2^k$ with $0 \leq k \leq n-2$.

At first assume that $\operatorname{codim} N \geq 3$. Then it follows from the exact homotopy sequence for the fibration $E \to N$ that $\pi_1(E) \to \pi_1(N)$ is an isomorphism. Hence, it follows from Seifert–van Kampen's theorem that $\pi_1(M) = \pi_1(F)$. So the claim follows in this case.

Next assume that $\operatorname{codim} N = 2$. Let $x \in F$ be a T-fixed point. Then with an argument similar to that in the proof of Lemma 6.3, one sees that $\pi_N(\pi_F^{-1}(x)) = \{y\}$ is a single point. Here $\pi_N : E \to N$ and $\pi_F : E \to F$ denote the bundle projections.

Therefore $y \in N$ is a T-fixed point. Hence, N is a characteristic submanifold of M. Denote by $\lambda(N) \subset T$ the circle subgroup of T which fixes N. Then we have an exact sequence

$$\pi_1(\lambda(N)) \to \pi_1(E) \to \pi_1(N) \to 1.$$

Here the first map is induced by the inclusion of an $\lambda(N)$-orbit. Now it follows from Seifert–van Kampen's theorem, that $\pi_1(M) = \pi_1(F)/\langle \pi_1(\lambda(N)) \rangle$. Here $\langle \pi_1(\lambda(N)) \rangle$ denotes the normal subgroup of $\pi_1(F)$ which is generated by the image of the map $\pi_1(\lambda(N)) \to \pi_1(F)$ induced by the inclusion of a $\lambda(N)$-orbit. Since there are T-fixed points in F, the $\lambda(N)$-orbits in F are null-homotopic. Therefore it follows that $\pi_1(M) = \pi_1(F)$. Hence the claim follows in this case.

Now assume that $\operatorname{codim} N = 1$. Then the map $E \to N$ is a two-fold covering. Therefore we have an exact sequence

$$1 \to \pi_1(E) \to \pi_1(N) \to \mathbb{Z}_2 \to 1. \tag{7}$$

In particular, $\pi_1(E)$ is a normal subgroup of $\pi_1(N)$.

Since $\operatorname{codim} F = 2$, we get the following exact sequence from the exact homotopy sequence for the fibration $E \to F$

$$\pi_1(\lambda(F)) \to \pi_1(E) \to \pi_1(F) \to 1.$$

Therefore it follows from Seifert–van Kampen's theorem that

$$\pi_1(M) = \pi_1(N)/\langle \pi_1(\lambda(F)) \rangle.$$

Here $\langle \pi_1(\lambda(F)) \rangle$ denotes the normal subgroup of $\pi_1(N)$ which is generated by the image of the inclusion $\pi_1(\lambda(F)) \to \pi_1(E) \to \pi_1(N)$.

Since $\pi_1(E) \subset \pi_1(N)$ is normal, we have $\langle \pi_1(\lambda(F)) \rangle \subset \pi_1(E)$. Therefore from (7) we get the following exact sequence

$$1 \to \pi_1(E)/\langle \pi_1(\lambda(F)) \rangle \to \pi_1(M) \to \mathbb{Z}_2 \to 1.$$

Since there is a surjection $\pi_1(F) = \pi_1(E)/\pi_1(\lambda(F)) \to \pi_1(E)/\langle \pi_1(\lambda(F)) \rangle$, the claim now follows. $\square$

As a corollary to Lemma 7.1 we get:

Corollary 7.2 *Let M be a $2n$-dimensional torus manifold which admits an invariant metric of non-negative sectional curvature. Then the universal covering $\tilde{M}$ of M is a simply connected torus manifold which admits an invariant metric of non-negative curvature. Moreover, the action of the torus on $\tilde{M}$ commutes with the action of the deck transformation group.*

Proof. By Lemma 7.1, $\tilde{M}$ is a closed manifold. Since there are T-fixed points in M, the principal orbits of the T-action on M are null-homotopic in M. Hence it follows that the T-action lifts to an action on $\tilde{M}$.

This action on $\tilde{M}$ has a fixed point and normalizes the deck transformation group G. Since T is connected and G discrete it follows that the T- and G-actions on $\tilde{M}$ commute.

The metric on M lifts to an metric on $\tilde{M}$ which clearly has non-negative sectional curvature and is invariant under the lifted torus action. Hence the claim follows. $\square$

Now we can determine the isomorphism type of the fundamental group of a non-simply connected non-negatively curved torus manifold.

Theorem 7.3 *Let M be a non-negatively curved torus manifold of dimension $2n$. Then there is a $0 \leq k \leq n-1$, such that $\pi_1(M) = \mathbb{Z}_2^k$.*

Proof. By Corollary 7.2, the universal covering $\tilde{M}$ of M is a torus manifold. Moreover, the action of $G = \pi_1(M)$ on $\tilde{M}$ commutes with the action of the torus T on $\tilde{M}$.

Therefore it induces a G-action on $\mathcal{P}(\tilde{M}/T)$. Moreover, for any $g \in G$ and all faces F of $\tilde{M}/T$, we have

$$\lambda(gF) = \lambda(F).$$

Hence, the intersection of gF and F is empty if $gF \neq F$.

Since $\mathcal{P}(\tilde{M}/T) \cong \mathcal{P}(\prod_{i<r} \Sigma^{n_i} \times \prod_{i \geq r} \Delta^{n_i})$ and all facets of Σ^{n_i} and Δ^{n_i} have non-trivial intersection if $n_i \geq 2$, it follows that $gF = F$ for all facets of $\tilde{M}/T$ belonging to a factor of dimension at least two. Moreover, the facets which belong to the other factors are mapped to facets that belong to the same factor.

Since G acts freely on $\tilde{M}^T$, the G-action on $\mathcal{P}(\tilde{M}/T)$ is effective. Therefore G might be identified with a subgroup of $\operatorname{Aut}(\mathcal{P}(\tilde{M}/T))$. This subgroup is contained in the subgroup H of $\operatorname{Aut}(\mathcal{P}(\tilde{M}/T))$ which contains all automorphisms which leave all facets belonging to factors of dimension greater or equal to two invariant and maps facets belonging to a factor of dimension one to facets belonging to the same factor.

We will show that H is isomorphic to $\mathbb{Z}_2^{l+r-1}$ where l is the number of $i \geq r$ with $n_i = 1$. Using Lemma 7.1 one sees that this implies the theorem.

We define a homomorphism $\psi : H \to \mathbb{Z}_2^{l+r-1}$ as follows.

At first assume that the factor Σ^{n_i} has dimension at least two. Then we set $\psi(h)_i = 0$ if and only if h leaves all components of $\bigcap_j F_j$ invariant, where the intersection is taken over all facets F_j of $\tilde{M}/T$ belonging to the factor Σ^{n_i}. Note that this intersection has two components.

Now assume that the factor Γ_i has dimension one. Then we set $\psi(h)_i = 0$ if and only if h leaves the two facets belonging to Γ_i invariant.

The homomorphism ψ has an obvious inverse $\phi : \mathbb{Z}_2^{l+r-1} \to H$. It is defined as follows. For $a \in \mathbb{Z}_2^{l+r-1}$, $\phi(a)$ leaves all facets of $\tilde{M}/T$ which do not belong to a factor of dimension two invariant.

Now assume that the factor Σ^{n_i} has dimension at least two. Then $\phi(a)$ interchanges the two components of $\bigcap_j F_j$, where the intersection is taken over all facets belonging to Σ^{n_i}, if and only if $a_i \neq 0$. Otherwise it leaves these components invariant.

Now assume that the factor Γ_i has dimension one. Then $\phi(a)$ interchanges the two facets belonging to Γ_i if and only if $a_i \neq 0$. Otherwise it leaves these facets invariant. It is easy to check that $\phi(a)$ defined as above extends to an automorphism of $\mathcal{P}(\tilde{M}/T)$.

So we see that ψ is an isomorphism and the theorem is proved. □

We give an example to show that the bound on the order of the fundamental group given in the above theorem is sharp.

Example 7.4 *Let $\tilde{M} = \prod_{i=1}^{n} S^2$ with the torus action induced by rotating each factor. The product metric of the standard metrics on each factor is invariant under the action of the torus and has non-negative curvature. On each factor there is an isometric involution ι_1 given by the antipodal map.*

We define an action of $\mathbb{Z}_2^{n-1}$ on $\tilde{M}$ as follows. Let $e_1, \dots, e_{n-1}$ a generating set of $\mathbb{Z}_2^{n-1}$. Then each e_i acts on the i-th factor and $(i+1)$-st factor of $\tilde{M}$ by ι_1 and trivially *on the other factors. This defines a free orientation preserving action of $\mathbb{Z}_2^{n-1}$ which commutes with the torus action.*

Therefore $\tilde{M}/\mathbb{Z}_2^{n-1}$ is a torus manifold with an invariant metric of non-negative sectional curvature and fundamental group $\mathbb{Z}_2^{n-1}$.

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Positively curved GKM-manifolds

Oliver Goertsches and Michael Wiemeler

Let T be a torus of dimension $\geq k$ and M a T-manifold. M is a GKM$_k$-manifold if the action is equivariantly formal, has only isolated fixed points, and any k weights of the isotropy representation in the fixed points are linearly independent.

In this paper we compute the cohomology rings with real and integer coefficients of GKM$_3$- and GKM$_4$-manifolds which admit invariant metrics of positive sectional curvature.

1. Introduction

The classification of positively curved manifolds is a longstanding problem in Riemannian geometry. So far only few examples of such manifolds are known. In dimension greater than 24 all such examples are diffeomorphic to compact rank one symmetric spaces (CROSSs). All examples admit non-trivial S^1-actions and it is conjectured that all positively curved manifolds admit a non-trivial S^1-action. Based on this conjecture several authors (see for example [GS94], [Wil03], [FR05], [AK13] and others) have considered the above classification problem under the extra assumption that there is a isometric circle or torus action. Usually a lower bound on the dimension of the acting torus is assumed which grows with the dimension of the manifold.

In this paper we study positively curved manifolds M which admit an isometric torus action of GKM-type [GKM98]. These actions have only finitely many fixed points (hence, M is necessarily even-dimensional) and the union of all one-dimensional orbits is two-dimensional. These assumptions on the strata of the action are stronger than what is usually assumed, but our methods work for three- and four-dimensional tori, independent of the dimension of M. In the appendix we observe that all known examples of positively curved manifolds in even dimensions admit isometric GKM actions.

Our main result states that under a slightly more restrictive assumption on the strata we can determine the real cohomology ring of M:

Theorem 1.1 *Let M be a compact connected positively curved orientable Riemannian manifold satisfying $H^{\mathrm{odd}}(M;\mathbb{R}) = 0$.*

1. *Assume that M admits an isometric torus action of type GKM$_4$, i.e., an action with finitely many fixed points such that at each fixed point, any four weights of the isotropy representation are linearly independent. Then M has the real cohomology ring of S^{2n} or $\mathbb{C}P^n$.*

2. Assume that M admits an isometric torus action of type GKM_3, i.e., an action with finitely many fixed points such that at each fixed point, any three weights of the isotropy representation are linearly independent. Then M has the real cohomology ring of a compact rank one symmetric space.

In both cases, the real Pontryagin classes of M are standard, i.e. there exists an isomorphism of rings $f : H^*(M;\mathbb{R}) \to H^*(K;\mathbb{R})$ which preserves Pontryagin classes, where K is a compact rank one symmetric space.

As a corollary we get the following:

Corollary 1.2 *Let M be a compact connected positively curved orientable Riemannian manifold with $H^{\mathrm{odd}}(M;\mathbb{R}) = 0$ which admits an isometric torus action of type GKM_3 and an invariant almost complex structure. Then M has the real cohomology ring of a complex projective space.*

Under even stronger conditions on the weights of the isotropy representations (see Section 6) we can also prove versions of the above theorem for cohomology with integer coefficients. By combining these versions with results from rational homotopy theory we can conclude that up to diffeomorphism there are only finitely many GKM manifolds which admit invariant metrics of positive sectional curvature and satisfy these stronger conditions, see Remark 6.5.

We also have a version of Theorem 1.1 for non-orientable manifolds (see Corollary 7.2).

This paper is organized as follows. In Section 2 we review the basics of GKM theory and give the precise GKM condition. In Section 3 we give an outline of the proofs of the main results. In Section 4 we describe the GKM graph of natural torus actions on CROSSs. In Section 5 we prove our main results. In Sections 6 and 7 we discuss extensions of our main results to cohomology with integer coefficients and to non-orientable manifolds. In the appendix we describe the GKM graphs of natural torus actions on the known nonsymmetric examples of positively curved manifolds.

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2. GKM theory

Throughout this paper, we consider an action of a compact torus T on a compact differentiable manifold M, which we will always assume to be connected. Then its equivariant cohomology is defined as

$$H_T^*(M;R) = H^*(M \times_T ET;R),$$

where R is the coefficient ring and $ET \to BT$ the classifying bundle of T. For the moment we will consider only the real numbers as coefficient ring and thus omit the coefficients from the notation, but we will also consider the case of the integers in Section 6 below. The equivariant cohomology $H^*_T(M)$ has, via the projection $M \times_T ET \to BT$, the natural structure of an $H^*(BT) = S(\mathfrak{t}^*)$-algebra.

We say that the T-action is *equivariantly formal* if $H^*_T(M)$ is a free $H^*(BT)$-module. For such actions, the ordinary (de Rham) cohomology ring of M can be computed from the equivariant cohomology algebra because of the following well-known statement [AP93, Theorem 3.10.4 and Corollary 4.2.3]: If the T-action on M is equivariantly formal, then the natural map $H^*_T(M) \to H^*(M)$ is surjective and induces a ring isomorphism

$$H^*_T(M)/(H^{>0}(BT)) \cong H^*(M).$$

For torus actions with only finitely many fixed points this condition is also equivalent to the fact that $H^{\mathrm{odd}}(M) = 0$. In this case it follows that the number of fixed points of the torus action, which is equal to the Euler characteristic of M, is given by the total dimension of $H^*(M)$.

The Borel localization theorem [AP93, Corollary 3.1.8] implies that the canonical restriction map

$$H^*_T(M) \longrightarrow H^*_T(M^T)$$

has as kernel the $H^*(BT)$-torsion submodule of $H^*_T(M)$; hence, for equivariantly formal actions, this map is injective, and one can try to compute $H^*_T(M)$ by understanding its image in $H^*_T(M^T)$.

The Chang-Skjelbred Lemma [CS74, Lemma 2.3] describes this image in terms of the *one-skeleton*

$$M_1 = \{p \in M \mid \dim T \cdot p \leq 1\}.$$

of the action: if the T-action is equivariantly formal, then the sequence

$$0 \longrightarrow H^*_T(M) \longrightarrow H^*_T(M^T) \longrightarrow H^*_T(M_1, M^T)$$

is exact, where the last arrow is the boundary operator of the long exact sequence of the pair (M_1, M^T). Thus, the image of $H^*_T(M) \to H^*_T(M^T)$ is the same as the image of the restriction map $H^*_T(M_1) \to H^*_T(M^T)$.

GKM theory, named after Goresky, Kottwitz and MacPherson [GKM98], now poses additional conditions on the action in order to simplify the structure of the one-skeleton of the action. We assume first of all that the action has only finitely many fixed points. Then at each fixed point $p \in M$ the isotropy representation decomposes into its weight spaces

$$T_pM = \bigoplus_{i=1}^{n} V_i,$$

corresponding to weights $\alpha_i : \mathfrak{t} \to \mathbb{R}$ which are well-defined up to multiplication by -1.

Definition 2.1 *We say that the action is GKM$_k$, $k \geq 2$, if it is equivariantly formal, has only finitely many fixed points, and at each fixed point p any k weights of the isotropy representation are linearly independent.*

For $k = 2$ one obtains the usual GKM conditions. If M is a GKM$_k$-manifold with respect to some action of a torus T and $T' \subset T$ is a subtorus of codimension $< k$, then the condition on the weights implies that every component of its fixed point set $M^{T'}$ is of dimension at most $2k$. In particular, for T' of codimension one, each component is either a point or a two-dimensional T-manifold with fixed points. Hence, it can only be either S^2 or $\mathbb{R}P^2$, with $\mathbb{R}P^2$ occurring only if M is non-orientable.

Consider first the case that M is orientable, i.e., that the one-skeleton M_1 is a union of two-spheres. Then the GKM graph Γ_M of the action is by definition the graph with one vertex for each fixed point, and one edge connecting two vertices for each two-sphere in M_1 containing the corresponding fixed points. We label the edge with the isotropy weight of the associated two-sphere.

In the non-orientable case, the graph encodes the possible presence of $\mathbb{R}P^2$'s in the one-skeleton via edges that connect a vertex corresponding to the unique fixed point contained in the $\mathbb{R}P^2$ with an auxiliary vertex representing the exceptional orbit; in [GM14] these vertices are drawn as stars. However, in our situation of a torus action these edges have no impact on the equivariant cohomology at all, so we could as well leave them out from the graph without losing any cohomological information.

Summarizing, one obtains the following description of the equivariant cohomology algebra of an action of type GKM ([GKM98, Theorem 7.2]; the non-orientable case is an easy generalization, see [GM14, Section 3]):

Theorem 2.2 *Consider an action of a torus T on a compact manifold M of type GKM$_2$, with fixed points $p_1, \ldots, p_n$. Then, via the natural restriction map*

$$H_T^*(M) \longrightarrow H_T^*(M^T) = \bigoplus_{i=1}^n S(\mathfrak{t}^*),$$

the equivariant cohomology algebra $H_T^*(M)$ is isomorphic to the set of tuples (f_i), with the property that if the vertices i and j in the associated GKM graph are joined by an edge with label $\alpha \in \mathfrak{t}^*$, then $f_i|_{\ker \alpha} = f_j|_{\ker \alpha}$.

If M is of class GKM$_3$, the *two-skeleton* $M_2 = \{p \in M \mid \dim T \cdot p \leq 2\}$ of M is a union of four-dimensional T-invariant submanifolds. The T-action induces on each of these submanifolds an effective action of a two-dimensional torus. We call a subgraph of Γ_M which corresponds to the intersection of M_1 with one of these four-dimensional manifolds a two-dimensional face of Γ_M. It is easy to see that for each pair (e_1, e_2) of edges of Γ_M emanating from the same vertex v there is exactly one two-dimensional face of Γ_M which contains e_1 and e_2.

For a $2m$-dimensional GKM manifold M with fixed points $p_1, \ldots, p_n$ the restriction of the total equivariant (real) Pontryagin class $p^T(M)$ of M to M^T is given by

$$\sum_{i=1}^n \prod_{j=1}^m (1 + \alpha_{ij}^2) \in \bigoplus_{i=1}^n S(\mathfrak{t}^*), \tag{1}$$

where the α_{ij} are the weights of the T-representation $T_{p_i}M$ at $p_i \in M^T$ (see for example [Kaw91, Lemma 6.10] and [MS74, Corollary 15.5]). Since the restriction $H_T^*(M) \to$

$H_T^*(M^T)$ is injective, it follows that the equivariant Pontryagin classes of M are determined by the GKM graph of M.

3. Strategy of the proof

Let us indicate here the strategy of the proof of Theorem 1.1 and Corollary 1.2. In Section 4 we will determine all possible GKM graphs of torus actions on compact rank one symmetric spaces. Then in Section 5 we will show that under the given assumptions the GKM graph necessarily coincides with one of these, say of an action on a CROSS N. Therefore by the GKM-description of equivariant cohomology we have an isomorphism of $H^*(BT)$-algebras $H_T^*(M) \cong H_T^*(N)$. Since $H^*(M) \cong H_T^*(M)/(H^{>0}(BT))$ and similarly for N, it follows that $H^*(M) \cong H^*(N)$. The remark about the Pontryagin classes follows because $p^T(M)$ is mapped to $p(M)$ by the natural map $H_T^*(M) \to H^*(M)$ and $p^T(M)$ is determined by the GKM graph of M by (1).

4. Torus actions on compact rank one symmetric spaces

Let M be an even-dimensional compact simply-connected symmetric space of rank one, i.e., either S^{2n}, $\mathbb{C}P^n$, $\mathbb{H}P^n$ or $\mathbb{O}P^2$. In the following we describe certain GKM actions on these manifolds and determine their GKM graphs, including their labeling with weights.

4.1. The spheres

Let $\alpha_i : T \to S^1$, $i = 1, \ldots, n$ be characters of the torus T. Denote by V_{α_i} the T-representation

$$T \times \mathbb{C} \to \mathbb{C} \qquad\qquad (t, z) \mapsto \alpha_i(t)z.$$

Here and in the rest of Section 4 we will denote the weight of this representation, i.e., the linear form $d\alpha_i \in \mathfrak{t}^*$, again by α_i.

Assume that for any $j_1, j_2 \in \{1, \ldots, n\}$, $j_1 \neq j_2$, α_{j_1} and α_{j_2} are linearly independent. Then the restriction of the T-action to the unit sphere of $V_{\alpha_1} \oplus \cdots \oplus V_{\alpha_n} \oplus \mathbb{R}$ defines a GKM$_2$ action of T on S^{2n}. This action has 2 fixed points v_1, v_2 corresponding to the two points in the intersection of the sphere with the $\mathbb{R}$-summand. These two vertices are joined in the GKM graph of S^{2n} by exactly n edges. Moreover, the weights of these edges are given by the $\pm\alpha_i$.

Thus, any labeling of $\Gamma_{S^{2n}}$ with pairwise linearly independent weights is realized by a GKM action on S^{2n}.

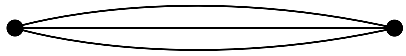

Figure 1: GKM graph of S^6

4.2. The complex projective spaces

Let $\alpha_i : T \to S^1$, $i = 0, \dots, n$, be characters of the torus T. Denote by V_{α_i} the T-representation

$$T \times \mathbb{C} \to \mathbb{C} \qquad (t, z) \mapsto \alpha_i(t)z.$$

Assume that, if $n > 1$, for any pairwise distinct $i, j_1, j_2 \in \{0, \dots, n\}$, $\alpha_{j_1} - \alpha_i$ and $\alpha_{j_2} - \alpha_i$ are linearly independent. If $n = 1$ assume that $\alpha_1 \neq \alpha_0$. Then the projectivization of $V_{\alpha_0} \oplus \cdots \oplus V_{\alpha_n}$ defines a GKM action of T on $\mathbb{C}P^n$. This action has $n + 1$ fixed points $v_0, \dots, v_n$ corresponding to the weight spaces of the above T-representation. The GKM graph of this action is a complete graph on these fixed points. Moreover, the weight of the edge from v_i to v_j is given by $\pm(\alpha_i - \alpha_j)$.

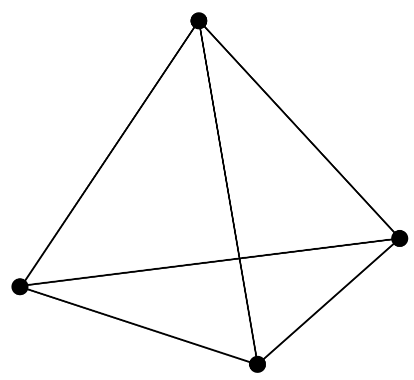

Figure 2: GKM graph of $\mathbb{C}P^3$

In particular, in this type of example an arbitrary set of non-zero pairwise linearly independent weights $\gamma_1, \dots, \gamma_n$, such that, if for any pairwise distinct $i, j_1, j_2 \in \{1, \dots, n\}$, $\gamma_{j_1} - \gamma_i$ and $\gamma_{j_2} - \gamma_i$ are linearly independent, can occur as labels of the edges adjacent to v_0.

4.3. The hyperbolic projective spaces

Let $\alpha_i : T \to S^1$, $i = 0, \dots, n$, be characters of the torus T. Denote by V_{α_i} the quaternionic T-representation

$$T \times \mathbb{H} \to \mathbb{H} \qquad (t, q) \mapsto \alpha_i(t)q.$$

Assume that, if $n > 1$, for any pairwise distinct $i, j_1, j_2 \in \{0, \dots, n\}$, $\alpha_{j_1} \pm \alpha_i$ and $\alpha_{j_2} \pm \alpha_i$ are linearly independent. If $n = 1$ assume that $\alpha_0 + \alpha_1$ and $\alpha_0 - \alpha_1$ are linearly independent. Because the action of T on $V_{\alpha_0} \oplus \cdots \oplus V_{\alpha_n} = \mathbb{H}^{n+1}$ commutes with the action of $\mathbb{H}^*$ given by multiplication from the right, we get a GKM action of T on $\mathbb{H}P^n = (\mathbb{H}^{n+1} \setminus \{0\})/\mathbb{H}^*$. This action has $n + 1$ fixed points $v_0, \dots, v_n$ corresponding to the weight spaces of the above T-representation. In the GKM graph of this action any two fixed points are joined by exactly two edges. Moreover, the weights of the edges from v_i to v_j are given by $\pm(\alpha_i - \alpha_j)$ and $\pm(\alpha_i + \alpha_j)$.

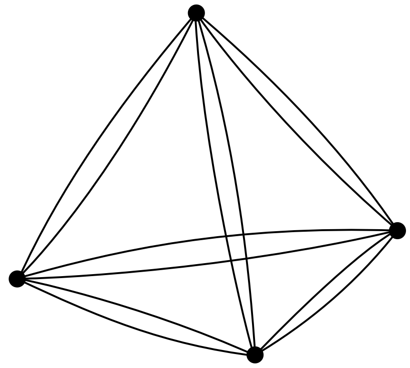

Figure 3: GKM graph of $\mathbb{H}P^3$

Note that in this case, the set of weights at the edges adjacent to, say, v_0, is not arbitrary. Indeed, the weights corresponding to the two edges connecting v_0 with v_i add up (after choosing the right signs) to $2\alpha_0$, independent of i.

4.4. The Cayley plane

The Cayley plane $\mathbb{O}P^2$ can be defined as the homogeneous space $F_4/\operatorname{Spin}(9)$. This is a homogeneous space of the form G/K satisfying $\operatorname{rank} G = \operatorname{rank} K$. For a homogeneous space G/K of this type, let $T \subset K$ be a maximal torus and consider the T-action on G/K by left multiplication. It is known that this action is equivariantly formal and is of type GKM$_2$ [GHZ06]. The GKM graph of an action of this type can be described completely in terms of the root systems Δ_G and Δ_K of G and K, see [GHZ06, Theorem 2.4]: first of all, the set of fixed points of the action is given by the quotient of Weyl groups W_G/W_K. At the origin eK, the tangent space of G/K is K-equivariantly isomorphic to the quotient $\mathfrak{g}/\mathfrak{k}$, which implies that the weights of the isotropy representation at eK are given by those roots of G which are not roots of K. Moreover, two vertices corresponding to elements wW_K and $w'W_K$ of W_G/W_K are on a common edge if and only if $wW_K = \sigma_\alpha w' W_K$ for some $\alpha \in \Delta_G \setminus \Delta_K$, where σ_α denotes the reflection at α.

Remark 4.1 *Of course, also the other compact rank one symmetric spaces are homogeneous spaces of this type, and we could have used the results of [GHZ06] to describe the GKM graphs of the torus actions on these spaces as well.*

Because $\dim H^*(\mathbb{O}P^2) = 3$, the action of a maximal torus $T \subset \operatorname{Spin}(9)$ has exactly three fixed points, which correspond to three vertices in the associated GKM graph. Since $\dim \mathbb{O}P^2 = 16$, from each vertex emerge eight edges. Because of the W_{F_4}-action on the graph which is transitive on the vertices, it follows that any two vertices are connected by four edges.

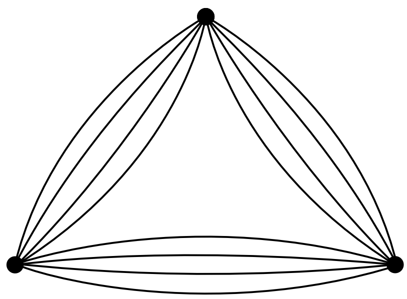

Figure 4: GKM graph of $\mathbb{O}P^2$

The root system of F_4 is described explicitly in [Kna02, Proposition 2.87]: Consider $\mathbb{R}^4$, equipped with the standard inner product and standard basis $e_1, \ldots, e_4$,

$$\Delta_{F_4} = \begin{cases} \pm e_i & \\ \pm e_i \pm e_j & \text{for } i \neq j \\ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4). & \end{cases}$$

The root system of Spin(9) is contained in Δ_{F_4} as the subroot system

$$\Delta_{\mathrm{Spin}(9)} = \begin{cases} \pm e_i & \\ \pm e_i \pm e_j & \text{for } i \neq j, \end{cases}$$

see [Ada96, p. 55].

Now we can determine the labeling of the GKM graph of $\mathbb{O}P^2$ from the fact that the labeling of the graph is invariant under the action of W_{F_4} on the graph.

More precisely, if α and β are weights in $\Delta_{F_4} \setminus \Delta_{\mathrm{Spin}(9)}$, then the reflection $\sigma_\alpha \beta$ of β at α is $\pm\beta$ if and only if the number of minus signs in α and β coincide modulo two, and if the number of signs is not congruent modulo two, then $\sigma_\alpha \beta$ is one of the roots $\pm e_i$. (See also the proof of Proposition 2.87 in [Kna02] for a description of the action of the Weyl group of F_4 on Δ_{F_4}.) This implies that if v_0, v_1, v_2 are the vertices of $\Gamma_{\mathbb{O}P^2}$, then the weights at the edges between two of them, say v_0 and v_1, are given (up to sign) by

$$\frac{1}{2}(-e_1 - e_i + \sum_{j \neq i,1} e_j) \quad (i = 2, 3, 4) \quad \text{and} \quad \frac{1}{2}\sum_{j=1}^{4} e_j,$$

the weights at the edges between v_0 and v_2 are given (up to sign) by

$$\frac{1}{2}(-e_i + \sum_{j \neq i} e_j) \qquad (i = 1, \ldots, 4),$$

and the weights at the edges between v_1 and v_2 are given by $\pm e_1, \ldots, \pm e_4$.

Remark 4.2 *The examples given in this section show that every simply connected compact symmetric space of rank one admits a torus action of type GKM$_3$. Note that in order to check whether an action of a torus $T \subset K$ on a homogeneous space G/K with* $\operatorname{rank} G = \operatorname{rank} K$ *is GKM$_3$, we only need to check the* 3*-independence of the weights in one of the fixed points: the* 3*-independence in the other fixed points then follows from the $W(G)$-action on the GKM graph which is transitive on the vertices.*

Moreover, one sees that there are actions on these spaces for which all weights appearing in the GKM graph of the action are primitive vectors in the weight lattice of T. To see this just take the α_i in the construction of the actions on S^{2n}, $\mathbb{C}P^n$ and $\mathbb{H}P^n$ to be primitive vectors and pairwise linear independent. For the action on $\mathbb{O}P^2$ one sees that the weights are primitive vectors in the weight lattice of a maximal torus of F_4, because this lattice is generated by the roots of F_4.

5. The GKM graphs of positively curved manifolds

In this section we determine the GKM graphs of positively curved GKM$_3$- and GKM$_4$-manifolds. The key observation is the following lemma.

Lemma 5.1 *Let M be an orientable GKM$_3$-manifold with an invariant metric of positive sectional curvature. Then all two dimensional faces of the GKM graph Γ_M of M have two or three vertices.*

Proof. A two-dimensional face of Γ_M is the GKM graph of a four-dimensional invariant submanifold N of M on which a two-dimensional torus acts effectively. N is a fixed point component of the action of some subtorus $T' \subset T$. Therefore it is totally geodesic in M and the induced metric has positive sectional curvature. The classification results of four-dimensional T^2-manifolds with positive sectional curvature given in [GS94] imply that N is diffeomorphic to S^4 or $\mathbb{C}P^2$. As the Euler characteristic of these manifolds is at most 3, the claim follows. $\square$

Let M be a positively curved GKM$_3$-manifold and $N \subset M$ a four-dimensional invariant submanifold corresponding to a two-dimensional face of Γ_M which is a triangle. Then N is equivariantly diffeomorphic to $\mathbb{C}P^2$ equipped with one of the actions described in Section 4.2. [OR70, Theorem 4.3], [Mel82, Theorem 2]. Let α and β denote the weights at two of the three edges in Γ_N. Note that these weights are determined up to sign. Then the weight at the third edge of $\Gamma_N = \Gamma_{\mathbb{C}P^2}$ is given by $\pm\alpha \pm \beta$.

Lemma 5.2 *An orientable GKM$_4$-manifold with an invariant metric of positive sectional curvature has the same GKM graph as a torus action on S^{2n} or $\mathbb{C}P^n$.*

Proof. For three distinct vertices v_1, v_2, v_3 in Γ_M we denote by K_{ij} the set of edges between v_i and v_j. Then each pair $(e_1, e_2) \in K_{12} \times K_{13}$ spans a two-dimensional face of Γ_M. By Lemma 5.1, the vertices of this face are exactly v_1, v_2 and v_3. We denote the unique edge contained in this face which connects v_2 and v_3 by $\phi(e_1, e_2)$; in this way we obtain a map $\phi : K_{12} \times K_{13} \to K_{23}$.

If α_1, α_2 are the weights at e_1 and e_2, respectively, then the weight at $\phi(e_1, e_2)$ is given by $\pm\alpha_1 \pm \alpha_2$. Therefore if $(e'_1, e'_2) \in K_{12} \times K_{13}$ is another pair with $\phi(e'_1, e'_2) = \phi(e_1, e_2)$, then we get the relation:

$$\pm\alpha_1 \pm \alpha_2 = \pm\alpha'_1 \pm \alpha'_2.$$

This contradicts the 4-independence of the weights at v_1. We have thus shown that ϕ is injective, i.e., $\#K_{12} \cdot \#K_{13} \leq \#K_{23}$. But this relation holds also for any permutation of v_1, v_2 and v_3, hence if one $\#K_{ij} > 1$, the other two K_{ij} must be empty. It follows that no edge of Γ_M is contained at the same time in a two-dimensional face with three vertices and another two-dimensional face with two vertices.

Now the GKM-graph of a GKM$_3$-manifold, all of whose two-dimensional faces are biangles, is $\Gamma_{S^{2n}}$, the graph with two vertices and n edges. The GKM-graph of a GKM$_3$-manifold, all of whose two-dimensional faces are triangles, is the complete graph $\Gamma_{\mathbb{C}P^n}$ on $n + 1$ vertices.

As argued in Subsection 4.1, any labeling of $\Gamma_{S^{2n}}$ with pairwise linearly independent weights is realized by a torus action on S^{2n}. For $\Gamma_{\mathbb{C}P^n}$, the statement that the weights are those of a torus action on $\mathbb{C}P^n$ will be shown in Lemma 5.3 below (for later use already for GKM$_3$-actions). □

Lemma 5.3 *Consider a GKM$_3$-manifold M with the GKM graph $\Gamma_{\mathbb{C}P^n}$. Then the induced labeling of the GKM graph is the same as that of a torus action on $\mathbb{C}P^n$.*

Proof. Let $v_0, \ldots, v_n$ be the vertices of the GKM graph, and γ_{ij} the weight of the edge between v_i and v_j (which is determined up to sign). By the considerations in Subsection 4.2 we only have to show that we can choose the signs in such a way that $\gamma_{ij} = \gamma_{0i} - \gamma_{0j}$. We can assume without loss of generality that

$$\gamma_{1i} = \gamma_{01} - \gamma_{0i}$$

for all i, and that

$$\gamma_{ij} = \gamma_{0i} \pm \gamma_{0j}.$$

This implies

$$\gamma_{ij} = \gamma_{0i} \pm \gamma_{0j} = \gamma_{01} - \gamma_{1i} \pm (\gamma_{01} - \gamma_{1j}),$$

but on the other hand γ_{ij} is also a linear combination of γ_{1i} and γ_{1j}. Hence, the 3-independence shows that $\gamma_{ij} = \gamma_{0i} - \gamma_{0j}$. □

As explained in Section 3, the first part of Theorem 1.1 follows from Lemma 5.2. The second part will follow from the following sequence of lemmas.

Lemma 5.4 *Let M be an orientable GKM$_3$-manifold with an invariant metric of positive sectional curvature. Then Γ_M is equal to a graph which is constructed from a simplex by replacing each edge by k edges, $k \in \mathbb{N}$.*

Remark 5.5 *Note that all the GKM graphs listed in Section 4 arise in this way. The graph of S^{2n}, for example, is obtained from the 1-simplex.*

Proof. Assume that Γ_M has at least three vertices, i.e., is not a single point and not $\Gamma_{S^{2n}}$. Then the statement of the lemma follows as soon as we can show that for every choice of vertices v_1, v_2, v_3 in a two-dimensional face of Γ_M we have $\#K_{12} = \#K_{13} = \#K_{23}$.

But for that it is sufficient to show that $\#K_{12} \leq \#K_{23}$, by the symmetry of the statement. For $f \in K_{13}$, consider the map $\psi : K_{12} \to K_{23}$, $e_2 \mapsto \phi(e_2, f)$, where $\phi : K_{12} \times K_{13} \to K_{23}$ is the map defined in Lemma 5.2, sending two edges to the unique third edge in the two-dimensional face spanned by them. Now if $\psi(e_1) = \psi(e_2)$, then $(\psi(e_1), f)$ and $(\psi(e_2), f)$ span the same triangle in Γ_M. Therefore we must have $e_1 = e_2$ in this case, i.e., ψ is injective, which shows that $\#K_{12} \leq \#K_{23}$. $\square$

The following lemma implies Corollary 1.2.

Lemma 5.6 *A GKM$_3$-manifold with an invariant metric of positive sectional curvature and an invariant almost complex structure has the same GKM graph as $\mathbb{C}P^n$.*

Proof. If there is an almost complex structure on a GKM-manifold N, one can modify the definition of the GKM graph so that it contains information about the complex structure (see for example [GZ00, Section 1]). In this case the weights of the T-representations T_xN, $x \in N^T$, have a preferred sign. We label the oriented edges emanating from $x \in \Gamma_N$ by the weights of the representation T_xM. For an oriented edge e denote by $\alpha(e)$ the weight at e. Then we have $\alpha(\bar{e}) = -\alpha(e)$, where $\bar{e}$ denotes e equipped with the inverse orientation.

Now assume that N is four-dimensional and let v_1, v_2 be two vertices in Γ_N joined by an edge e, denote by e_1 and e_2 the other oriented edges emanating from v_1 and v_2, then we must have $\alpha(e_1) = \alpha(e_2) \mod \alpha(e)$.

From these two properties we get a contradiction if we assume that Γ_N has only two vertices and two edges.

Therefore there is no biangle in Γ_M. Now, by Lemma 5.4, the claim follows. $\square$

Remark 5.7 *As pointed out by one of the referees, one can also prove the above lemma as follows. If M admits an invariant almost complex structure, then all components of $M^{T'}$ for any subtorus $T' \subset T$ admit almost complex structures. A two-dimensional face F of Γ_M is the GKM graph of such a fixed point component N. F is a biangle if and only if N is diffeomorphic to S^4. But S^4 does not admit any almost complex structure. Hence, there are no biangles in Γ_M. Therefore the lemma follows as in the above proof.*

Lemma 5.8 *Let M be an orientable GKM$_3$-manifold with an invariant metric of positive sectional curvature. Then Γ_M is equal to $\Gamma_{S^{2n}}$ or a graph which is constructed from a simplex by replacing each edge by k edges, $k = 1, 2, 4$.*

Proof. Assume that Γ_M is not $\Gamma_{S^{2n}}$. Then there is a triangle v_1, v_2, v_3 in Γ_M. Denote by α_i the weights of the edges between v_1 and v_2, by β_i the weights of the edges between v_1 and v_3, by γ_i the weights of the edges between v_2 and v_3.

Since the weights are only determined up to sign we may assume that

$$\gamma_i = \alpha_1 - \beta_i = \epsilon_{ij}\alpha_j + \delta_{ij}\beta_{\sigma_j(i)} \tag{2}$$

with $\epsilon_{ij}, \delta_{ij} \in \{\pm 1\}$ and permutations σ_j, where $\sigma_1 = \mathrm{Id}$.

Sublemma 5.9 *For $i \neq j$ the permutations $\sigma_j^{-1} \circ \sigma_i$ are all of order two. Moreover, they do not have fixed points; in other words, for each l, the map $j \mapsto \sigma_j(l)$ is a permutation as well.*

Proof of Sublemma 5.9. Fix $i \neq j$; we consider the permutation $\sigma_j^{-1} \circ \sigma_i$ and want to show that it is of order 2 without fixed points.

To show that it has no fixed points assume $\sigma_i(l) = \sigma_j(l)$ for some l. Then (2) applied twice gives

$$\epsilon_{li}\alpha_i + \delta_{li}\beta_{\sigma_i(l)} = \gamma_l = \epsilon_{lj}\alpha_j + \delta_{lj}\beta_{\sigma_j(l)},$$

a contradiction to the 3-independence. Therefore there are no fixed points.

Fix a number A, and let $B = \sigma_j^{-1} \circ \sigma_i(A)$, i.e. $\sigma_i(A) = \sigma_j(B)$. We have

$$\gamma_A = \epsilon_{Ai}\alpha_i + \delta_{Ai}\beta_{\sigma_i(A)} = \epsilon_{Aj}\alpha_j + \delta_{Aj}\beta_{\sigma_j(A)} \tag{3}$$

and

$$\gamma_B = \epsilon_{Bi}\alpha_i + \delta_{Bi}\beta_{\sigma_i(B)} = \epsilon_{Bj}\alpha_j + \delta_{Bj}\beta_{\sigma_j(B)}. \tag{4}$$

Then the term $\epsilon_{Bi}\gamma_A - \epsilon_{Ai}\gamma_B$ is equal to the following two expressions:

$$\begin{aligned} \epsilon_{Bi}\gamma_A - \epsilon_{Ai}\gamma_B &= \epsilon_{Bi}\delta_{Ai}\beta_{\sigma_i(A)} - \epsilon_{Ai}\delta_{Bi}\beta_{\sigma_i(B)} \\ &= (\epsilon_{Bi}\epsilon_{Aj} - \epsilon_{Ai}\epsilon_{Bj})\alpha_j + \epsilon_{Bi}\delta_{Aj}\beta_{\sigma_j(A)} - \epsilon_{Ai}\delta_{Bj}\beta_{\sigma_j(B)}. \end{aligned} \tag{5}$$

Assume that $\epsilon_{Bi}\epsilon_{Aj} = -\epsilon_{Ai}\epsilon_{Bj}$. Then $\epsilon_{Bi}\gamma_A + \epsilon_{Ai}\gamma_B$ is equal to the following:

$$\epsilon_{Bi}\gamma_A + \epsilon_{Ai}\gamma_B = \epsilon_{Bi}\delta_{Aj}\beta_{\sigma_j(A)} + \epsilon_{Ai}\delta_{Bj}\beta_{\sigma_j(B)}. \tag{6}$$

By adding and subtracting equations (6) and (5) we get

$$2\epsilon_{Bi}\gamma_A = \epsilon_{Bi}\delta_{Aj}\beta_{\sigma_j(A)} + \epsilon_{Ai}\delta_{Bj}\beta_{\sigma_j(B)} + \epsilon_{Bi}\delta_{Ai}\beta_{\sigma_i(A)} - \epsilon_{Ai}\delta_{Bi}\beta_{\sigma_i(B)} \tag{7}$$

and

$$2\epsilon_{Ai}\gamma_B = \epsilon_{Bi}\delta_{Aj}\beta_{\sigma_j(A)} + \epsilon_{Ai}\delta_{Bj}\beta_{\sigma_j(B)} - \epsilon_{Bi}\delta_{Ai}\beta_{\sigma_i(A)} + \epsilon_{Ai}\delta_{Bi}\beta_{\sigma_i(B)}. \tag{8}$$

Since $\beta_{\sigma_i(A)} = \beta_{\sigma_j(B)}$, γ_A or γ_B is a linear combination of $\beta_{\sigma_j(A)}$ and $\beta_{\sigma_i(B)}$. This gives a contradiction to the 3-independence of the weights. Hence our assumption is wrong; and we have $\epsilon_{Bi}\epsilon_{Aj} = \epsilon_{Ai}\epsilon_{Bj}$. Therefore it follows from equation (5) and the 3-independence that $\sigma_j(A) = \sigma_i(B)$. This shows that $\sigma_j^{-1} \circ \sigma_i$ has order two. □

In particular, the sublemma shows that all the $\sigma_j = \sigma_1^{-1} \circ \sigma_j$, for $j \neq 1$, are of order two. Again by the sublemma, they also commute because

$$\sigma_j \circ \sigma_i = \sigma_j^{-1} \circ \sigma_i = (\sigma_j^{-1} \circ \sigma_i)^{-1} = \sigma_i^{-1} \circ \sigma_j = \sigma_i \circ \sigma_j.$$

Let G be the subgroup of the permutation group generated by the σ_j. Then we have an epimorphism $\mathbb{Z}_2^{k-1} \to G$. Since the $\sigma_j^{-1} \circ \sigma_i$ do not have fixed points, G acts transitively on $\{1, \ldots, k\}$. Therefore we have $\{1, \ldots, k\} \cong \mathbb{Z}_2^{k-1}/H$ for a subgroup H of $\mathbb{Z}_2^{k-1}$. In particular k is a power of two.

Let $i \in \{1, \ldots, k\}$.

Sublemma 5.10 *For $j > 1$ we have $\delta_{ij} = 1$ and $\epsilon_{\sigma_j(i)j} = \epsilon_{ij}$, i.e.,*

$$\gamma_i = \epsilon_{ij}\alpha_j + \beta_{\sigma_j(i)} \qquad\qquad \gamma_{\sigma_j(i)} = \epsilon_{ij}\alpha_j + \beta_i.$$

Proof of Sublemma 5.10. From the above relation (2) it follows that

$$\begin{aligned}
\alpha_1 &= \gamma_i + \beta_i = \epsilon_{ij}\alpha_j + \delta_{ij}\beta_{\sigma_j(i)} + \beta_i \\
\alpha_1 &= \gamma_{\sigma_j(i)} + \beta_{\sigma_j(i)} = \epsilon_{\sigma_j(i)j}\alpha_j + \delta_{\sigma_j(i)j}\beta_i + \beta_{\sigma_j(i)}.
\end{aligned}$$

Now the three-independence of the weights implies that we have $\delta_{ij} = 1$ and $\epsilon_{\sigma_j(i)j} = \epsilon_{ij}$. □

Sublemma 5.11 *For $j > 2$ we have $\epsilon_{\sigma_j(i)2} = -\epsilon_{i2}$.*

Proof of Sublemma 5.11. From Sublemma 5.10 it follows that

$$\begin{aligned}
\epsilon_{i2}\alpha_2 &= \gamma_{\sigma_2(i)} - \beta_i = \epsilon_{\sigma_2(i)j}\alpha_j + \beta_{\sigma_j \circ \sigma_2(i)} - \beta_i \\
\epsilon_{\sigma_j(i)2}\alpha_2 &= \gamma_{\sigma_j(i)} - \beta_{\sigma_j \circ \sigma_2(i)} = \epsilon_{ij}\alpha_j + \beta_i - \beta_{\sigma_j \circ \sigma_2(i)}.
\end{aligned}$$

Now the three-independence of the weights implies that $\epsilon_{i2} = -\epsilon_{\sigma_j(i)2}$. □

Because we know that k is a power of two, if $k > 2$, then $k \geq 4$, so we may choose $j > j' > 2$. Then, by applying Sublemma 5.11 twice, we have

$$\epsilon_{\sigma_j(\sigma_{j'}(i))2} = -\epsilon_{\sigma_{j'}(i)2} = \epsilon_{i2}.$$

Since, by Sublemma 5.9, there is a j'' such that $\sigma_{j''}(i) = \sigma_j \circ \sigma_{j'}(i)$ and $\sigma_j \circ \sigma_{j'}$ does not have fixed points, it follows, again by Sublemma 5.11, that $j'' = 2$. Because this holds for each i, it follows that $\sigma_2 = \sigma_j \circ \sigma_{j'}$. Hence it follows that $k \leq 4$. This proves the lemma. □

Lemma 5.12 *If we have $k = 4$ in the situation of Lemma 5.8, then Γ_M is a triangle with each edge replaced by four edges.*

Proof. Assume that Γ_M is a higher dimensional simplex. Then there are four vertices $v_0, \dots, v_3$ in Γ_M. Denote by α_i the weights at the edges between v_0 and v_1, by β_i the weights at the edges between v_0 and v_2, by γ_i the weights at the edges between v_0 and v_3.

By Sublemma 5.10 we may assume that the weights at the edges between v_1 and v_2 are given by

$$\alpha_1 - \beta_i = \pm\alpha_2 + \beta_{\sigma(i)} \tag{9}$$

and that the weights at the edges between v_1 and v_3 are given by

$$\alpha_1 - \gamma_i = \epsilon_i \alpha_2 + \gamma_{\sigma'(i)}, \tag{10}$$

for some permutations σ, σ' and $\epsilon_i \in \{\pm 1\}$.

At first we show that we may assume that the weights at the eges between v_2 and v_3 are given by $\beta_1 - \gamma_i$. Since these weights are determined only up to sign we may assume that there are $c_j \in \{\pm 1\}$ such that these weights are given by

$$\beta_1 + c_j \gamma_j.$$

Hence, for each $j = 1, \dots, 4$ there are $a, b \in \{\pm 1\}$ and $i \in \{1, \dots, 4\}$ such that

$$\alpha_1 - \beta_1 = a(\alpha_1 - \gamma_i) + b(\beta_1 + c_j \gamma_j). \tag{11}$$

At first assume that $i \neq j$. By 3-independence at v_0 we have $a = -1$ and $b = 1$. Then we have

$$(\alpha_1 - \gamma_j) - (\beta_1 - \gamma_j) = -(\alpha_1 - \gamma_i) + (\beta_1 + c_j \gamma_j).$$

We get a contradiction to 3-independence at v_3 if $c_j = -1$. Hence, we must have $c_j = 1$. By Sublemma 5.10 there is a $k \in \{1, \dots, 4\}$ such that

$$\alpha_1 - \gamma_i = \pm\alpha_k + \gamma_j.$$

Therefore we get

$$2\alpha_1 - 2\beta_1 = \gamma_i + \gamma_j = \alpha_1 \mp \alpha_k.$$

This is a contradiction to the 3-independence at v_0.

Therefore we have $i = j$. In this case it follows from 3-independence at v_0 and Equation (11) that $a = 1$ and $b = c_j = -1$. Therefore, by Sublemma 5.10, the weights at the edges between v_2 and v_3 are given by

$$\beta_1 - \gamma_i = \delta_i \beta_2 + \gamma_{\sigma''(i)}, \tag{12}$$

where σ'' is a permutation and $\delta_i \in \{\pm 1\}$. Note that by Sublemma 5.11 not all δ_i are equal.

We may assume that $\sigma(1) = 2$. Then we have, using Equations (9), (11), (10) and (12), that

$$\begin{aligned} \pm\alpha_2 + \beta_2 = \alpha_1 - \beta_1 &= (\alpha_1 - \gamma_i) - (\beta_1 - \gamma_i) \\ &= (\epsilon_i \alpha_2 + \gamma_{\sigma'(i)}) - (\delta_i \beta_2 + \gamma_{\sigma''(i)}). \end{aligned}$$

Let m be the order of $\sigma'^{-1}\sigma''$. Then we have:

$$m(\pm\alpha_2+\beta_2) = \sum_{l=0}^{m-1}((\epsilon_{(\sigma'^{-1}\sigma'')^l(i)}\alpha_2 + \gamma_{\sigma'(\sigma'^{-1}\sigma'')^l(i)}) - (\delta_{(\sigma'^{-1}\sigma'')^l(i)}\beta_2 + \gamma_{\sigma''(\sigma'^{-1}\sigma'')^l(i)})) = \sum_{l=0}^{m-1}(\epsilon_{(\sigma'^{-1}\sigma'')^l(i)}\alpha_2 - \delta_{(\sigma'^{-1}\sigma'')^l(i)}\beta_2).$$

Therefore it follows that $\delta_i = -1$ for all i, a contradiction to the fact that not all δ_i are equal. □

We have shown that the GKM graph of an isometric GKM$_3$-action on a positively curved manifold is either $\Gamma_{S^{2n}}$, $\Gamma_{\mathbb{C}P^n}$, $\Gamma_{\mathbb{H}P^n}$ or $\Gamma_{\mathbb{O}P^2}$. To complete the proof of Theorem 1.1 we need to show that the induced labeling of the GKM graph is one of those described in Section 4. For $\Gamma_{S^{2n}}$ there is nothing to show, and for $\Gamma_{\mathbb{C}P^n}$ we already proved this in Lemma 5.3. Let us consider now $\Gamma_{\mathbb{H}P^n}$.

Lemma 5.13 *Consider an orientable GKM$_3$-manifold M with GKM graph $\Gamma_{\mathbb{H}P^n}$. Then the induced labeling of the GKM graph is the same as that of a torus action on $\mathbb{H}P^n$.*

Proof. Let $v_0, v_1, \ldots, v_m$ be the vertices of Γ_M, and denote by $\gamma_{ij}, \gamma'_{ij}$ the weights at the edges between v_i and v_j. These weights are well-defined up to sign, and up to permutation of γ_{ij} and γ'_{ij}. We can choose $\gamma_{0k}, \gamma_{1k}, \gamma'_{1k}$ and γ'_{01} in such a way that for all $j>1$ we have (with Sublemma 5.10 in mind)

$$\gamma_{1j} = \gamma_{0j} - \gamma_{01} = \epsilon_j\gamma'_{0j} + \gamma'_{01}$$
$$\gamma'_{1j} = \gamma_{0j} - \gamma'_{01} = \epsilon_j\gamma'_{0j} + \gamma_{01}$$

for some signs ϵ_j. Then we fix the sign of γ'_{0j}, $j>1$, such that $\epsilon_j = -1$. It follows that $\gamma_{01} + \gamma'_{01} = \gamma_{0j} + \gamma'_{0j}$ for all $j>1$, hence

$$\gamma_{0i} + \gamma'_{0i} = \gamma_{0j} + \gamma'_{0j} \tag{13}$$

for all $i,j>1$. We define

$$\alpha_0 = \frac{1}{2}(\gamma_{01} + \gamma'_{01}), \qquad \alpha_j = \frac{1}{2}(\gamma_{0j} - \gamma'_{0j}) \quad (j>1)$$

so that

$$\gamma_{0j} = \alpha_0 + \alpha_j \quad \text{and} \quad \gamma'_{0j} = \alpha_0 - \alpha_j.$$

We have to show that we can choose γ_{ij} and γ'_{ij} $(i,j>1)$ in such a way that $\gamma_{ij} = \alpha_i + \alpha_j$ and $\gamma'_{ij} = \alpha_i - \alpha_j$.

For that we choose them such that

$$\gamma_{ij} = \gamma_{0j} + \eta_{ij}\gamma'_{0i} = \pm\gamma'_{0j} \pm \gamma_{0i} \tag{14}$$

$$\gamma'_{ij} = -\gamma_{0j} + \eta'_{ij}\gamma_{0i} = \pm\gamma'_{0j} \pm \gamma'_{0i} \tag{15}$$

for some signs η_{ij}, η'_{ij}. Subtracting (13) from (14) would give a contradiction to 3-independence if $\eta_{ij} = 1$; hence $\eta_{ij} = -1$. Similarly, adding (13) to (15) shows $\eta'_{ij} = 1$. It thus follows (using (13) and (14))

$$\alpha_i + \alpha_j = \frac{1}{2}(\gamma_{0i} - \gamma'_{0i} + \gamma_{0j} - \gamma'_{0j}) = \gamma_{0j} - \gamma'_{0i} = \gamma_{ij}$$

and

$$\alpha_i - \alpha_j = \frac{1}{2}(\gamma_{0i} - \gamma'_{0i} - \gamma_{0j} + \gamma'_{0j}) = -\gamma_{0j} + \gamma_{0i} = \gamma'_{ij}$$

as desired. □

And finally we consider $\Gamma_{\mathbb{O}P^2}$.

Lemma 5.14 *Consider an orientable GKM$_3$-action with GKM graph $\Gamma_{\mathbb{O}P^2}$. Then the induced labeling of the GKM graph is the same as that of a torus action on $\mathbb{O}P^2$.*

Proof. Let v_0, v_1, v_2 be the vertices of Γ_M. Denote by α_i the weights at the edges between v_0 and v_1, by β_i the weights at the edges between v_1 and v_2, and by γ_i the weights at the edges between v_2 and v_0.

Then by Sublemmas 5.9, 5.10 and 5.11, we may assume that the following relations hold:

$$\gamma_1 = \alpha_1 - \beta_1 = \alpha_2 + \beta_2 \tag{16}$$

$$\gamma_2 = \alpha_1 - \beta_2 = \alpha_2 + \beta_1 \tag{17}$$

$$\gamma_3 = \alpha_1 - \beta_3 = -\alpha_2 + \beta_4 \tag{18}$$

$$\gamma_4 = \alpha_1 - \beta_4 = -\alpha_2 + \beta_3 \tag{19}$$

By adding equations (17) and (18) we get that

$$\alpha_1 = \frac{1}{2}(\beta_1 + \beta_2 + \beta_3 + \beta_4).$$

Hence, the γ_i are of the form $\frac{1}{2}(-\beta_i + \sum_{j\neq i}\beta_j)$. By Sublemma 5.10 we may assume that

$$\gamma_1 = \alpha_1 - \beta_1 = \alpha_2 + \beta_2 = \alpha_3 + \beta_3 = \alpha_4 + \beta_4.$$

Thus, for $i > 1$,

$$\alpha_i = \frac{1}{2}(-\beta_1 - \beta_i + \sum_{j\neq i,1}\beta_j).$$

This proves the lemma. □

6. Integer coefficients

In this section we prove a version of our main theorem for integer coefficients. To do so we have to generalize some of the results from Section 2 to integer coefficients.

The two ingredients we need for our theorem to hold are that the ordinary cohomology is encoded in the equivariant cohomology, i.e., that

$$H_T^*(M;\mathbb{Z}) \longrightarrow H^*(M;\mathbb{Z}) \tag{20}$$

is surjective, and that the equivariant cohomology algebra is encoded in the combinatorics of the one-skeleton M_1, i.e., that a Chang-Skjelbred Lemma holds. This can be seen to be true only under additional assumptions on the isotropy groups of the action, see [FP11, Corollary 2.2]: if for all $p \notin M_1$ the isotropy group T_p is contained in a proper subtorus of T, and $H_T^*(M;\mathbb{Z})$ is a free module over $H^*(BT)$, then

$$0 \longrightarrow H_T^*(M;\mathbb{Z}) \longrightarrow H_T^*(M^T;\mathbb{Z}) \longrightarrow H_T^*(M_1, M^T;\mathbb{Z})$$

is exact.

Because the fixed point set is always finite and M is orientable in our situation, freeness of $H_T^*(M;\mathbb{Z})$ is equivalent to $H^{\mathrm{odd}}(M;\mathbb{Z}) = 0$ by [MP06, Lemma 2.1]. Moreover, under the assumption that $H^{\mathrm{odd}}(M;\mathbb{Z}) = 0$, (20) is surjective: this follows either from the proof of [MP06, Lemma 2.1] or via the fact that in this situation the Leray spectral sequence collapses.

Moreover, since $H_T^*(M^T;\mathbb{Z}) \to H_T^*(M^T;\mathbb{R})$ is injective the formula (1) also holds for the equivariant (integer) Pontryagin classes of M.

Denote by $\mathbb{Z}_\mathfrak{t}^* \subset \mathfrak{t}^*$ the weight lattice of the torus T. Then we call two weights $\alpha, \beta \in \mathbb{Z}_\mathfrak{t}^*$ coprime if there are primitive elements of $\mathbb{Z}_\mathfrak{t}^*$, α', β', and $a, b \in \mathbb{Z}$ such that $\alpha = a\alpha'$ and $\beta = b\beta'$ and a and b are coprime. Note that, if $\alpha \neq 0$, a and α' are uniquely determined up to sign by α.

Lemma 6.1 *If, for an orientable GKM manifold with vanishing odd-degree integer cohomology, at each fixed point any two weights are coprime, then the Chang-Skjelbred Lemma holds for integer coefficients.*

Proof. Let p be a prime and denote by G the maximal $\mathbb{Z}/p\mathbb{Z}$-torus in T. By [FP11, Theorem 2.1], it is sufficient to show that M^G is contained in M_1.

At first note that by an iterated application of [Bre72, Theorem VII.2.2, p. 376], we have that $\dim H^{\mathrm{odd}}(M^G;\mathbb{Z}/p\mathbb{Z}) \leq \dim H^{\mathrm{odd}}(M;\mathbb{Z}/p\mathbb{Z}) = 0$. Therefore every component of M^G is a T-invariant submanifold with non-trivial Euler characteristic. As the Euler characteristic always equals the Euler characteristic of the fixed point set, it follows that in all these components there is a T-fixed point x.

Now consider the T-representation T_xM. Then $T_x(M^G) = (T_xM)^G$ is an invariant subrepresentation and therefore a direct sum of weight spaces V_α. Let $q \in \mathbb{Z}$ and $\alpha' \in \mathbb{Z}_\mathfrak{t}^*$ a primitive element such that $\alpha = q\alpha'$. Then V_α is fixed by G if and only if p divides q. Since by assumption the weights at x are coprime, it follows that T_xM^G contains at most one weight space. Thus, the component of M^G which contains x is contained in a two-dimensional sphere fixed by a corank-one torus T' of T. Hence M^G is contained in M_1. □

Remark 6.2 *As pointed out to us by one of the referees, Lemma 6.1 was conjectured for Hamiltonian torus actions of GKM type on symplectic manifolds by Tolman and Weitsman [TW99]. This conjecture has been shown in [Sch01] by Schmid under the assumption that all weights of the GKM graph are primitive vectors in $\mathbb{Z}^*_{\mathfrak{t}}$.*

Theorem 6.3 *Let M be a positively curved orientable manifold with $H^{\mathrm{odd}}(M;\mathbb{Z}) = 0$ which admits an isometric torus action with finitely many fixed points such that*

- *At each fixed point any three weights of the isotropy representation are linearly independent and*
- *At each fixed point any two weights are coprime.*

Then M has the integer cohomology ring of a CROSS. Moreover, the total Pontryagin class of M is standard, i.e. there is a CROSS K and an isomorphism of rings $f : H^*(M;\mathbb{Z}) \to H^*(K;\mathbb{Z})$ such that $f(p(M)) = p(K)$.

Proof. By Lemma 6.1, we have a GKM description of the equivariant cohomology of M with integer coefficients. Therefore the statement follows as in the proof of the second part of Theorem 1.1. □

Remark 6.4 *If M is simply-connected and has the integer cohomology of $\mathbb{C}P^n$, then M is already homotopy equivalent to $\mathbb{C}P^n$: choose a map $f : M \to K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$ such that the pullback of a generator of $H^2(\mathbb{C}P^\infty;\mathbb{Z})$ generates $H^2(M;\mathbb{Z})$. This map can be deformed to a map which takes values in the $2n$-skeleton of $\mathbb{C}P^\infty$, which is $\mathbb{C}P^n$, and this deformed map is then the desired homotopy equivalence.*

Remark 6.5 *By [AP93, Corollary 2.7.9], a simply connected manifold which has the same rational cohomology as a compact rank one symmetric space is formal in the sense of rational homotopy theory. Therefore it follows from [KT91, Theorem 2.2] which is a generalization of [Sul77, Theorem 12.5] that up to diffeomorphism there are only finitely many simply connected integer cohomology $\mathbb{K}P^n$'s, $\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O}$, with standard Pontryagin classes of dimension greater than four. Thus there are only finitely many diffeomorphism types of simply connected GKM$_3$-manifolds as in Theorem 6.3.*

7. Non-orientable GKM-manifolds

In this section we prove an extension of Theorem 1.1 to non-orientable GKM$_3$-manifolds.

Lemma 7.1 *If M^{2n} is a non-orientable GKM$_3$-manifold with an invariant metric of positive curvature, then the GKM graph of M coincides with the GKM graph of a linear torus action on $\mathbb{R}P^{2n}$, i.e., it has only a single vertex.*

Proof. Denote by $\tilde{M}$ the orientable double cover of M. Then the torus action on M lifts to a torus action on $\tilde{M}$. With this lifted action $\tilde{M}$ is a GKM$_3$-manifold: Indeed, it is obvious that the torus action on $\tilde{M}$ has isolated fixed points. Moreover, the isotropy representation at a fixed point $x \in \tilde{M}$ is isomorphic to the isotropy representation at $p(x)$ where $p : \tilde{M} \to M$ is the covering map. Therefore the 3-independence of the weights of $T_x\tilde{M}$ follows.

Now we show that $H^{\text{odd}}(\tilde{M};\mathbb{R}) = 0$ which is equivalent to equivariant formality of the torus action on $\tilde{M}$. Since the torus action on M is equivariantly formal and has only finitely many fixed points, we have $H^{\text{odd}}(M;\mathbb{R}) = 0$. M is the quotient of a free $\mathbb{Z}_2$-action on $\tilde{M}$. Therefore, by [Bre72, Chapter III], $H^*(M;\mathbb{R})$ is isomorphic to $H^*(\tilde{M};\mathbb{R})^{\mathbb{Z}_2}$. Hence, it follows that $\mathbb{Z}_2$ acts on $H^{\text{odd}}(\tilde{M};\mathbb{R})$ and $H^{2n}(\tilde{M};\mathbb{R})$ by multiplication with -1. Now assume that $H^{\text{odd}}(\tilde{M};\mathbb{R}) \neq 0$. Then, by Poincaré duality, there are $\alpha_1, \alpha_2 \in H^{\text{odd}}(\tilde{M};\mathbb{R})$ such that $0 \neq \alpha_1\alpha_2 \in H^{2n}(\tilde{M};\mathbb{R})$. But this is a contradiction to the description of the $\mathbb{Z}_2$-action given above. Therefore we must have $H^{\text{odd}}(\tilde{M};\mathbb{R}) = 0$.

The one-skeleton of the action on $\tilde{M}$ is a double covering of the one-skeleton of the action on M. Therefore Γ_M is a quotient of a $\mathbb{Z}_2$-action on $\Gamma_{\tilde{M}}$ which is free on the vertices. By the results in Section 5, $\Gamma_{\tilde{M}}$ is one of the graphs described in Section 4. It is easy to see that if $\Gamma_{\tilde{M}}$ is not $\Gamma_{S^{2n}}$, then every vertex of Γ_M is contained in an edge which contains two vertices and an edge which contains only one vertex. But this is impossible. Indeed, the only non-orientable T^2-manifold in dimension four, which admits an invariant metric of positive sectional curvature is $\mathbb{R}P^4$ [GS94]. Therefore there is no two-dimensional face of Γ_M which contains an edge which connects two vertices and an edge which contains only a single vertex. Hence, $\Gamma_{\tilde{M}}$ must be isomorphic to $\Gamma_{S^{2n}}$. In this case there is only a single vertex in Γ_M. This implies that $\Gamma_M = \Gamma_{\mathbb{R}P^{2n}}$.

Therefore the lemma is proved. □

From the above lemma we get immediately the following corollary.

Corollary 7.2 *Let M be a compact positively curved non-orientable Riemannian manifold with $H^{\text{odd}}(M;\mathbb{R}) = 0$ which admits an isometric torus action of type GKM$_3$. Then $H^*(M;\mathbb{R}) = H^0(M;\mathbb{R}) = \mathbb{R}$.*

A. GKM actions on the nonsymmetric examples

Apart from the compact rank one symmetric spaces, the only known examples of even-dimensional positively curved manifolds are the homogeneous spaces $\mathrm{SU}(3)/T^2$, $\mathrm{Sp}(3)/\mathrm{Sp}(1)^3$ and $F_4/\mathrm{Spin}(8)$ [Wal72], and the biquotient $\mathrm{SU}(3)//T^2$ [Esc84]. Because these examples do not have the rational cohomology of a compact rank one symmetric space, Theorem 1.1 implies that they do not admit an isometric action of type GKM$_3$, but we will see in this section that all of them admit an isometric action of type GKM_2, and we will determine their GKM graphs.

The homogeneous examples admit a GKM_2-action by the general results of [GHZ06]: for any homogeneous space of the form G/H, where G and H are Lie groups of equal rank, the action of a maximal torus in H (or G) is of type GKM_2. Let us first determine the GKM graphs of the three homogeneous examples above.

A.1. $\mathrm{SU}(3)/T^2$

This example was considered in [GHZ06], Section 5.3, where it was shown that the GKM graph of the T^2-isotropy action is the bipartite graph $K_{3,3}$. Let us provide a slightly

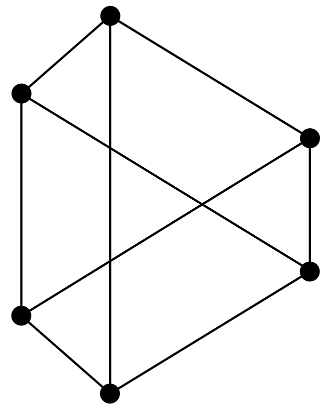

Figure 5: GKM graph of $\mathrm{SU}(3)/T^2$ and $\mathrm{SU}(3)//T^2$

different argument which will turn out to be generalizable to the other two homogeneous examples. The homogeneous space $\mathrm{SU}(3)/T^2$ admits a homogeneous fibration

$$S^2 \longrightarrow \mathrm{SU}(3)/T^2 \longrightarrow \mathbb{C}P^2,$$

which implies that the GKM graph in question projects onto the GKM graph of $\mathbb{C}P^2$ (a triangle), with fibers the GKM graph of S^2 (a line). Now by [GHZ06], Theorem 2.4, the vertices of the graph are in one-to-one correspondence with the elements of the Weyl group $W(\mathrm{SU}(3))$ of $\mathrm{SU}(3)$, which is the symmetric group S_3. Moreover, by the same theorem, two vertices are not joined by an edge if the corresponding elements w, w' in the Weyl group satisfy the condition that $w^{-1}w'$ is not of order two. Thus, two vertices are not joined by an edge if they correspond to elements of $W(\mathrm{SU}(3)) = S_3$ whose orders are congruent modulo 2. This leaves $K_{3,3}$ as the only possibility for the graph.

A.2. $\mathrm{Sp}(3)/\mathrm{Sp}(1)^3$

Consider the GKM_2-action on $\mathrm{Sp}(3)/\mathrm{Sp}(1)^3$ of a maximal torus $T^3 \subset \mathrm{Sp}(1)^3$. This homogeneous space admits a homogeneous fibration

$$S^4 \longrightarrow \mathrm{Sp}(3)/\mathrm{Sp}(1)^3 \longrightarrow \mathbb{H}P^2,$$

so the graph projects to a triangle with each edge doubled, with biangles as fibers. The vertices are in one-to-one correspondence to elements of the Weyl group quotient $W(\mathrm{Sp}(3))/W(\mathrm{Sp}(1)^3)$, and because the Weyl group of $\mathrm{Sp}(1)^3$, $\mathbb{Z}_2^3$, is normal in the Weyl group $\mathbb{Z}_2^3 \rtimes S_3$ of $\mathrm{Sp}(3)$, with quotient S_3, similar considerations as above hold true. We conclude that the GKM graph is the bipartite graph $K_{3,3}$, with each edge doubled.

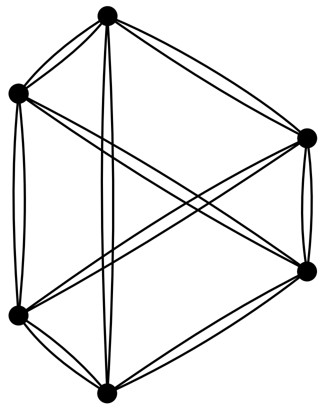

Figure 6: GKM graph of $\mathrm{Sp}(3)/\operatorname{Sp}(1)^3$

A.3. $F_4/\operatorname{Spin}(8)$

Also this example admits a homogeneous fibration:

$$S^8 \longrightarrow F_4/\operatorname{Spin}(8) \longrightarrow \mathbb{O}P^2$$

which implies that the GKM graph projects onto a triangle with each edge replaced by four edges, with fibers the graph with two vertices and four edges. Again, the Weyl group of $\mathrm{Spin}(8)$ is normal in the Weyl group of F_4, with quotient S_3, see [Ada96], Theorem 14.2. The GKM graph is therefore $K_{3,3}$, with each edge replaced by four edges.

A.4. $\mathrm{SU}(3)//T^2$

Finally, we consider the biquotient $\mathrm{SU}(3)//T^2$. It is the quotient of $\mathrm{SU}(3)$ by the free T^2-action given by

$$(t, w) \cdot g = \operatorname{diag}(t, w, tw) \cdot g \cdot \operatorname{diag}(1, 1, t^{-2}w^{-2}).$$

We denote the projection $\mathrm{SU}(3) \to \mathrm{SU}(3)//T^2$ by $g \mapsto [g]$. This manifold admits an action of a two-dimensional torus by

$$(a, b) \cdot [g] = [\operatorname{diag}(a, a, a^{-2}) \cdot g \cdot \operatorname{diag}(b, b^{-1}, 1)]$$

The six fixed points of the action are given by the elements $[g]$, where $g \in \mathrm{SU}(3)$ is a matrix that has (maybe after multiplying with -1) as column vectors a permutation

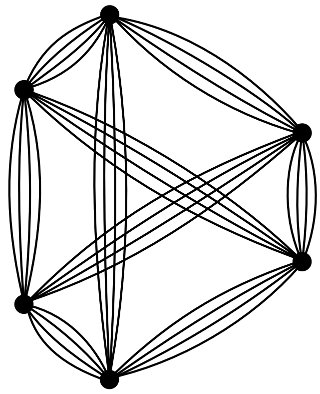

Figure 7: GKM graph of $F_4/\operatorname{Spin}(8)$

of the standard basis vectors e_1, e_2, e_3. This action is of type GKM_2; two fixed points $[g_1], [g_2]$, with g_i as above, are connected by a two-dimensional isotropy submanifold if and only if g_1 and g_2 have one identical column. For example, $\left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right]$ and $\left[-\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right]$ are joined by the projection of the T^2-invariant submanifold $\mathrm{S}(\mathrm{U}(1)\times \mathrm{U}(2)) = \{\begin{pmatrix}a&0&0\\0&b&c\\0&d&e\end{pmatrix} \in \mathrm{SU}(3)\}$ of $\mathrm{SU}(3)$ to $\mathrm{SU}(3)/\!/T^2$.

It follows that the GKM graph of $\mathrm{SU}(3)/\!/T^2$ is again the bipartite graph $K_{3,3}$: the vertices corresponding to the elements $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$, $\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$ and $\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$ are pairwise not connected, and also not the vertices corresponding to $-\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$, $-\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$ and $-\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$.

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Part 3

Moduli spaces

Moduli spaces of invariant metrics of positive scalar curvature on quasitoric manifolds

*Michael Wiemeler**

We show that the higher homotopy groups of the moduli space of torus-invariant positive scalar curvature metrics on certain quasitoric manifolds are non-trivial.

1. Introduction

In recent years it became fashionable to study the homotopy groups of the space of Riemannian metrics of positive scalar curvature on a given closed, connected manifold and its moduli space, see for example the papers [BHSW10], [HSS14], [BERW14], [Wal14], [Wal13], [Wal11], [Wra16] and the book [TW15]. As far as the moduli space is concerned these results are usually only for the so-called observer moduli space of positive scalar curvature metrics, not for the naive moduli space.

The definition of the naive and the observer moduli space are as follows. The diffeomorphism group of a manifold M acts by pullback on the space of metrics of positive scalar curvature on M. The naive moduli space of metrics of positive scalar curvature on M is the orbit space of this action.

The observer moduli space of metrics is the orbit space of the action of a certain subgroup of the diffeomorphism group, the so-called observer diffeomorphism group. It consists out of those diffeomorphisms φ, which fix some point $x_0 \in M$ and whose differential $D_{x_0}\varphi : T_{x_0}M \to T_{x_0}M$ at x_0 is the identity.

This group does not contain any compact Lie subgroup and therefore acts freely on the space of metrics on M. Hence, the observer moduli space can be treated from a homotopy theoretic view point more easily than the naive moduli space.

In this paper we deal with the equivariant version of the above problem: We assume that there is a torus T acting effectively on the manifold and that all our metrics are invariant under this torus action. To be more specific we study invariant metrics on so-called torus manifolds and quasitoric manifolds.

A torus manifold is a $2n$-dimensional manifold with a smooth effective action of an n-dimensional torus such that there are torus fixed points in the manifold. Such a manifold is called locally standard if it is locally weakly equivariantly diffeomorphic to the standard representation of $T = (S^1)^n$ on $\mathbb{C}^n$. If M is locally standard, then the orbit space of the T-action on M is naturally a manifold with corners. M is called quasitoric if it is locally standard and M/T is diffeomorphic to a simple convex polytope.

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In this paper we use the following notations: Let M be a compact manifold. For a compact connected Lie subgroup G of $\mathrm{Diff}(M)$ we denote by

- $\mathcal{R}(M,G)$ the space of G-invariant metrics on M
- $\mathcal{R}^+(M,G)$ the space of G-invariant metrics of positive scalar curvature on M.
- $D(M,G) = N_{\mathrm{Diff}(M)}(G)/G$ the normalizer of G in $\mathrm{Diff}(M)$ modulo G.
- $\mathcal{M}(M,G) = \mathcal{R}(M,G)/D(M,G)$
- $\mathcal{M}^+(M,G) = \mathcal{R}^+(M,G)/D(M,G)$.

With this notation our main result is as follows:

Theorem 1.1 (Theorem 4.2) *There are quasitoric manifolds M of dimension $2n$ such that for $0 < k < \frac{n}{6} - 7$, n odd and $k \equiv 0 \mod 4$, $\pi_k(\mathcal{M}^+) \otimes \mathbb{Q}$ is non-trivial, where $\mathcal{M}^+$ is some component of $\mathcal{M}^+(M;T^n)$.*

Note that $\mathcal{M}^+(M;T^n)$ is the analogue of the naive moduli space of metrics of positive scalar curvature in the equivariant situation and not the analogue of the observer moduli space for which so far most results have been proven.

Moreover, we think that the above theorem is the first step to understand the topology of the full naive moduli space of metrics of positive scalar curvature on quasitoric manifolds. This is because this moduli space is stratified by the rank of the isometry groups of metrics on M. The above theorem is a non-triviality result for the homotopy type of a minimal stratum of $\mathcal{M}^+(M;\{\mathrm{Id}\})$. If one also has non-triviality results for all higher strata, one might expect non-triviality results for the full moduli space.

2. The action of $D(M,T)$ on $\mathcal{R}(M,T)$ for M a torus manifold

In this section we describe the action of $D(M,T)$ on $\mathcal{R}(M,T)$ where M is a torus manifold. We give sufficient criteria for the rational homotopy groups of $\mathcal{M}(M,T)$ to be isomorphic to the rational homotopy groups of the classifying space of $D(M,T)$.

Lemma 2.1 *Let M be a closed manifold. If T is a maximal torus in $\mathrm{Diff}(M)$, then the isotropy groups of the natural $D(M,T)$-action on $\mathcal{R}(M,T)$ are finite.*

Proof. The isotropy group of the $D(M,T)$-action of an element $g \in \mathcal{R}(M,T)$ is the normalizer W of the torus T in the isometry group K of g modulo T. Since M is compact K is a compact Lie group. Moreover, because T is a maximal torus of K, W is the Weyl group of K which is a finite group. Therefore the statement follows. □

For each torus manifold M there is a natural stratification of the orbit space M/T by the identity components of the isotropy groups of the orbits. That is, the open strata of M/T are given by the connected components of

$$(M/T)_H = \{Tx \in M/T;\ (T_x)^0 = H\}$$

for connected closed subgroups H of T. We call the closure of an open stratum a closed stratum. The closed strata are naturally ordered by inclusion. We denote by $\mathcal{P}$ the poset of closed strata of M/T.

There is a natural map

$$\lambda : \mathcal{P} \to \{\text{closed connected subgroups of } T\}$$

such that $\lambda((M/T)_H) = H$. We call $(\mathcal{P}, \lambda)$ the characteristic pair of M.

An automorphism of $(\mathcal{P}, \lambda)$ is a pair (f, g) such that f is an automorphism of the poset $\mathcal{P}$ and g is an automorphism of the torus T so that $\lambda(f(x)) = g(\lambda(x))$ for all $x \in \mathcal{P}$. The automorphisms of $(\mathcal{P}, \lambda)$ naturally form a group $\operatorname{Aut}(\mathcal{P}, \lambda)$.

There is a natural action of $D(M, T)$ on M/T which preserves the above stratification. Therefore $D(M, T)$ acts by automorphisms on the characteristic pair $(\mathcal{P}, \lambda)$.

Lemma 2.2 *Let M be a torus manifold. Then there is a finite index subgroup G of $D(M, T)$ which acts freely on $\mathcal{R}(M, T)$. To be more precise, G is the kernel of the natural homomorphism $D(M, T) \to \operatorname{Aut}(\mathcal{P}, \lambda) \subset \operatorname{Aut}(\mathcal{P}) \times \operatorname{Aut}(T)$, where $(\mathcal{P}, \lambda)$ is the characteristic pair associated to M.*

Proof. Denote by $(\mathcal{P}, \lambda)$ the labeled face poset of M/T.

At first we show that $\operatorname{Aut}(\mathcal{P}, \lambda)$ is a finite group. To see this note that $\operatorname{Aut}(\mathcal{P})$ is finite because $\mathcal{P}$ is finite. Moreover, the natural map $\operatorname{Aut}(\mathcal{P}, \lambda) \to \operatorname{Aut}(\mathcal{P})$ has finite kernel, because T is generated by the image of λ (see [Wie14, Section 2] for details).

Let G be the kernel of the natural map $D(M, T) \to \operatorname{Aut}(\mathcal{P}, \lambda)$. Then G has finite index since $\operatorname{Aut}(\mathcal{P}, \lambda)$ is a finite group.

Let $T \subset H \subset \tilde{G}$ be a compact Lie group which fixes some metric $g \in \mathcal{R}(M)$, where $\tilde{G}$ is the preimage of G in $N_{\mathrm{Diff}(M)}(T)$. Then each element of H fixes every $x \in M^T$ and commutes with T. Hence, the differential of the H-action on T_xM gives an injective homomorphism $H \to O(2n)$. Since T is identified with a maximal torus of $O(2n)$ under this map, it follows that the centralizer of T in $O(2n)$ is T itself. Hence it follows that $H = T$. Therefore $G = \tilde{G}/T$ acts freely on $\mathcal{R}(M, T)$. □

Definition 2.3 *A $2n$-dimensional quasitoric manifold M is a locally standard torus manifold, such that M/T is diffeomorphic to a simple convex n-dimensional polytope P. We denote by $\pi : M \to P$ the orbit map.*

Lemma 2.4 *For M a quasitoric manifold, the group of T-equivariant diffeomorphisms of M is naturally isomorphic to $C^\infty(M/T, T) \rtimes \operatorname{Diff}(M/T, \lambda)$ as topological groups. Here $\operatorname{Diff}(M/T, \lambda)$ denotes the group of those diffeomorphisms of M/T, which leave λ invariant.*

In particular the group G of the previous lemma is homotopy equivalent to the group of all diffeomorphisms of M/T, which leave all faces of M/T invariant.

Proof. First we show that the kernel of the natural map $\varphi : \mathrm{Diff}_T(M) \to \mathrm{Diff}(M/T, \lambda)$ is isomorphic to $C^\infty(M/T, T)$. Since T is abelian, there is a natural map from $C^\infty(M/T, T)$ to the kernel of φ given by $f \mapsto F$, where $F(x) = f(Tx)x$ for $x \in M$.

We show that this map is a homeomorphism. To do so let $F \in \ker \varphi$. Then F leaves all T-invariant subsets of M invariant. Since M is quasitoric, there is a covering of M by open invariant subsets $U_1, \dots, U_k$ which are weakly equivariantly diffeomorphic to $\mathbb{C}^n$ with the standard T-action.

The restriction of F to $U_j \cong \mathbb{C}^n$ is of the form

$$F(z_1, \dots, z_n) = (z_1 f_1(z_1, \dots, z_n), \dots, z_n f_n(z_1, \dots, z_n)),$$

where $(z_1, \dots, z_n) \in \mathbb{C}^n$ and $f_k(z_1, \dots, z_n) \in S^1$ for $k = 1, \dots, n$ depends only on $(|z_1|^2, \dots, |z_n|^2)$.

We have to show that f_k is smooth for all k.

Smoothness in points with $z_k \neq 0$ follows from the smoothness of F. We show that f_k is also smooth in points with $z_k = 0$.

Since F is smooth, we have for $(z_1, \dots, z_n) \in \mathbb{C}^n$,

$$z_k f_k(z_1, \dots, z_n) = F_k(z_1, \dots, z_n) = \int_0^1 (D_{z_k} F_k(z_1, \dots, z_{k-1}, z_k t, z_{k+1}, \dots, z_n))(z_k)\, dt,$$

where

$$(D_{z_k} F_k(z_1, \dots, z_n))(z) = \left(\frac{\partial F_k}{\partial x_k}(z_1, \dots, z_n), \frac{\partial F_k}{\partial y_k}(z_1, \dots, z_n) \right) (x, y)^t$$

with $z_l = x_l + i y_l$ for $l = 1, \dots, n$ and $z = x + iy$, $x_l, x, y_l, y \in \mathbb{R}$.

Since F is T-equivariant, it follows that

$$\begin{aligned} z_k f_k(z_1, \dots, z_n) &= \int_0^1 (D_{z_k} F_k(z_1, \dots, z_{k-1}, z_k t, z_{k+1}, \dots, z_n))(z_k)\, dt \\ &= \int_0^1 z_k (D_{z_k} F_k(z_1, \dots, z_{k-1}, \frac{|z_k|}{z_k} z_k t, z_{k+1}, \dots, z_n))(1)\, dt \\ &= z_k \int_0^1 (D_{z_k} F_k(z_1, \dots, z_{k-1}, |z_k| t, z_{k+1}, \dots, z_n))(1)\, dt. \end{aligned}$$

Since F is T-equivariant, it follows that $t \mapsto (D_{z_k} F_k(z_1, \dots, z_{k-1}, t, z_{k+1}, \dots, z_n))(1)$, $t \in \mathbb{R}$, is an even function. Therefore the integrand in the last integral depends smoothly on $(z_1, \dots, z_n)$ and f_k is smooth everywhere. Because f_k is T-invariant, it induces a smooth map on the orbit space.

Hence it is sufficient to show that there is a section to φ.

There is a canonical model $M = ((M/T) \times T)/\sim$, where $(x, t) \sim (x', t')$ if and only if $x = x'$ and $t' t^{-1} \in \lambda(x)$. Therefore every diffeomorphism f of M/T, which leave λ-invariant, lifts to a homeomorphism of M given by $f \times \mathrm{Id}$. One can show (see [GP13, Lemma 2.3]), that this homeomorphism is actually a diffeomorphism. Therefore we have a section of φ and the first statement follows.

The second statement follows because $H = C^\infty(M/T,T)/T$ is contractible because M/T is contractible. $\square$

Lemma 2.5 *If in the situation of Lemma 2.2 M is quasitoric and the natural homomorphism* $\mathrm{Aut}(\mathcal{P},\lambda) \to \mathrm{Aut}(\mathcal{P})$ *is trivial, then* $\pi_k(\mathcal{M}(M,T)) \otimes \mathbb{Q} \cong \pi_k(BD(M,T)) \otimes \mathbb{Q}$ *for* $k > 1$*.*

Proof. Since G acts freely and properly on $\mathcal{R}(M,T)$, it follows from Ebin's slice theorem [Ebi70] (see also [Bou75]) that $\mathcal{R}(M,T) \to \mathcal{R}(M,T)/G$ is a locally trivial fiber bundle. Because $\mathcal{R}(M,T)$ is contractible, $\mathcal{R}(M,T)/G$ is weakly homotopy equivalent to BG.

Let H be as in the proof of the previous lemma. Then H is contractible. Hence it follows that $\mathcal{R}(M,T)$ and $\mathcal{R}(M,T)/H$ are weakly homotopy equivalent.

It follows from Ebin's slice theorem that all H-orbits in $\mathcal{R}(M,T)$ are closed. Since there is a $D(M,T)$-invariant metric on $\mathcal{R}(M,T)$, it follows that $\mathcal{R}(M,T)/H$ is metriziable. Hence, $\mathcal{R}(M,T)/H$ is paracompact and completely regular.

The $D(M,T)$-invariant metric on $\mathcal{R}(M,T)$ can be constructed as follows. Ebin constructs in his paper a sequence of Hilbert manifolds $\mathcal{R}^s$, $s \in \mathbb{N}$, such that $\mathcal{R}(M,\{\mathrm{Id}\}) = \bigcap_{s\in\mathbb{N}} \mathcal{R}^s$. On each $\mathcal{R}^s$ he constructs a $\mathrm{Diff}(M)$-invariant Riemannian structure. This structure induces a $\mathrm{Diff}(M)$-invariant metric d^s on $\mathcal{R}^s$. The restrictions of all these metrics d^s to $\mathcal{R}(M,\{\mathrm{Id}\})$ together induce the C^∞-topology on $\mathcal{R}(M,\{\mathrm{Id}\})$. Therefore the metric

$$d(x,y) = \sum_{s\in\mathbb{N}} \min\{d^s(x,y), 2^{-s}\}$$

is $\mathrm{Diff}(M)$-invariant and induces the C^∞-topology on $\mathcal{R}(M,\{\mathrm{Id}\})$.

Since $\mathrm{Aut}(\mathcal{P},\lambda) \to \mathrm{Aut}(\mathcal{P})$ is trivial, there is a splitting $\psi : \mathrm{Aut}(\mathcal{P},\lambda) \to D(M,T)$. Here an element $\tau = (g,f) \in \mathrm{Aut}(\mathcal{P},\lambda)$ acts on $M = ((M/T) \times T)/\sim$ as identity on the first factor and by $f \in \mathrm{Aut}(T)$ on the second. To see that this is a diffeomorphism of M, we note that there are invariant charts $U \subset M$ which are weakly equivariantly diffeomorphic to $\mathbb{C}^n$ such that $U \cap ((M/T) \times (\mathbb{Z}_2)^n))/\sim$ is mapped to $\mathbb{R}^n \subset \mathbb{C}^n$. For a construction of such charts see [GP13, Section 2]. The action of τ in this chart is given by complex conjugation on some of the factors of $\mathbb{C}^n$.

Note that H and G are normalized by $\operatorname{im}\psi$. Moreover, $\operatorname{im}\psi$ commutes with G/H in $D(M,T)/H$.

Since $H_1 = \langle T, \operatorname{im}\psi\rangle$ is a compact Lie subgroup of $\mathrm{Diff}(M)$, there is an H_1-invariant metric on M.

Therefore it follows from [Bre72, Chapter II.6] that $\mathcal{R}(M,T)/H/\operatorname{im}\psi$ is simply connected. Moreover, by [Bre72, Theorem III.7.2], one sees that $\mathcal{R}(M,T)/H/\operatorname{im}\psi$ is rationally acyclic.

Hence, by the Whitehead theorem, all rational homotopy groups of $\mathcal{R}(M,T)/H/\operatorname{im}\psi$ vanish.

By Lemma 2.2, we know that the identity components of G and $D(M,T)$ are the same. Therefore the higher homotopy groups of BG and $BD(M,T)$ are naturally isomorphic.

Therefore, by Lemma 2.4 and Ebin's slice theorem, it now suffices to show that G/H acts freely on $\mathcal{R}(M,T)/H/\operatorname{im}\psi$.

Let $g \in \mathcal{R}(M,T)$, $h_1 \in G$, $h_2 \in H$ such that $h_1 g = \tau h_2 g$ with $\tau \in \operatorname{im}\psi$. Then we have

$$\tau^{-1}h_1 g = \tau^{-1}\tau h_2 g = h_2 g.$$

Since the isotropy group of g in $D(M,T)$ is finite, it follows that $\tau^{-1}h_1$ has finite order in $D(M,T)/H$.

In particular, h_1 induces a diffeomorphism of finite order m on M/T which leaves all faces of M/T invariant because τ induces the identity on this space. Since the principal isotropy group of a $\mathbb{Z}_m$-action on a manifold with boundary is equal to the principal isotropy group of the restricted action on the boundary, it follows by induction on the dimension of the faces of M/T that the diffeomorphism induced by h_1 on M/T is trivial. This means that h_1 is contained in H and the lemma is proved. □

Example 2.6 *We give an example of quasitoric manifolds satisfying the assumptions of the previous lemma.*

Let M_0 be the projectivization of a sum of $n-1$ complex line bundles $E_0, \dots, E_{n-2}$ over $\mathbb{C}P^1$, such that $c_1(E_0) = 0$ and the first Chern classes of the other bundles are non-trivial, not equal to one and pairwise distinct. Then M_0 is a generalized Bott manifold and in particular a quasitoric manifold over $I \times \Delta^{n-2}$, where I is the interval and Δ^{n-2} denotes an $n-2$-dimensional simplex.

Let $M_1 = \mathbb{C}P^1 \times M_0$ and M_2 the blow up of M_1 at a single point. The orbit space of M_1 is $I \times I \times \Delta^{n-2}$. The orbit space of M_2 is the orbit space of M_1 with a vertex cut off.

The combinatorial types of the facets of M_2/T are given as in table 1 below. Since the combinatorial types of facets in the lines in this table are pairwise distinct, it follows that the lines in the table are invariant under the action of $\operatorname{Aut}(\mathcal{P},\lambda)$. Therefore the facets in the first two lines are fixed by the action of this group. The facets in lines 3 and 4 are fixed, because in each of these lines there appears one facet F with $\lambda(F) = \{(z,1,\dots,1) \in T^n;\ z \in S^1\}$ but the values of λ on the other facets are distinct.

Finally the facets $F_1, \dots, F_{n-2}$ in the last line are fixed, by all $(f,g) \in \operatorname{Aut}(\mathcal{P},\lambda)$ because g must permute the subgroups $\lambda(F_1), \dots, \lambda(F_{n-2})$, which are the coordinate subgroups in $\{(1,1)\} \times (S^1)^{n-2}$, and must also fix the subgroups $\lambda(F')$ with F' from line 3.

Note that depending on the choices of the bundles $E_0, \dots, E_{n-2}$, M_2 can be spin or non-spin.

3. The homotopy groups of $D(M,T)$ for M a quasitoric manifold

In this section we show that, for quasitoric manifolds of dimension $2n$, n odd, the rational homotopy groups of $D(M,T)$ are non-trivial in certain degrees.

	combinatorial type	$(\alpha_1, \dots, \alpha_n)$
1	Δ^{n-1}	$(1, \dots, 1)$
1	$I \times I \times \Delta^{n-3}$	$(0, 0, 1, \dots, 1)$
2	$I \times \Delta^{n-2}$	$(1, 0, 0, \dots, 0)$ $(0, 1, k_1, \dots, k_{n-2})$ with k_i pairwise distinct and non-zero
2	$I \times \Delta^{n-2}$ with vertex cut off	$(1, 0, 0, \dots, 0)$ $(0, 1, 0, \dots, 0)$
$n-2$	$I \times I \times \Delta^{n-3}$ with vertex cut off	$(0, 0, 0, \dots, 0, 1, 0, \dots, 0)$

Table 1: The combinatorial types of the facets of M_2/T. In the first column the numbers of facets of these type are given. In the last column the values of $\lambda(F) = \{(z^{\alpha_1}, \dots, z^{\alpha_n}) \in T^n;\ z \in S^1\}$ are given.

Now let M be a quasitoric manifold with orbit polytope P.

Let $D^n \hookrightarrow P$ be an embedding into the interior of P such that $K = P - D^n$ is a collar of P. Then we have a decomposition

$$M = (D^n \times T^n) \cup \pi^{-1}(K) = (D^n \times T^n) \cup N.$$

From this decomposition we get a homomorphism $\psi : \mathrm{Diff}(D^n, \partial D^n) \to D(M, T^n)$ by letting a diffeomorphism of D^n act on M in the natural way on D^n and by the identity on T^n and N.

Note that the natural map $\mathrm{Diff}(D^n, \partial D^n) \to \mathrm{Diff}(P)$ factors through ψ.

Lemma 3.1 *For $0 < k < \frac{n}{6} - 8$, n odd and $k \equiv -1 \mod 4$. The natural map*

$$\pi_k(\mathrm{Diff}(D^n, \partial D^n)) \otimes \mathbb{Q} \to \pi_k(\mathrm{Diff}(P)) \otimes \mathbb{Q}$$

is injective and non-trivial. In particular ψ induces an injective non-trivial homomorphism on these homotopy groups.

Proof. We have exact sequences

$$1 \to \mathrm{Diff}(D^n, \partial D^n) \to \widetilde{\mathrm{Diff}}(P) \to \mathrm{Diff}(K),$$

where $\widetilde{\mathrm{Diff}}(P)$ is the group of diffeomorphisms of P which preserve K, and

$$1 \to \mathrm{Diff}(K, \partial D^n) \to \mathrm{Diff}(K) \to \mathrm{Diff}(\partial D^n).$$

Note that $\widetilde{\mathrm{Diff}}(P)$ is weakly homotopy equivalent to $\mathrm{Diff}(P)$, by the uniqueness of collars of P up to isotopy. Moreover, the images of the right-hand maps in the above sequences have finite index.

In the first sequence this is because the group of those diffeomorphisms of a sphere which extend to diffeomorphisms of the disc has finite index in all diffeomorphisms of the sphere.

For the second sequence one can argue as follows to see that $\mathrm{Diff}(K) \to \mathrm{Diff}(\partial D^n)$ is surjective. Since K is homeomorphic to $\partial D^n \times I$ and every diffeomorphism φ of ∂D^n is isotopic to a diffeomorphism of ∂D^n which is the identity on some big embedded disc $D^{n-1} \subset \partial D^n$, φ can be extend to a homeomorphism ϕ of K which satisfies:

There is a facet F of P such that:

- ϕ is a diffeomorphism on $\partial D^n \times [0, \frac{1}{2}] \cup \mathring{F} \times [0, 1]$.
- ϕ is the identity on a neighborhood of $(\partial P - \mathring{F}) \times [\frac{1}{2}, 1]$.

Hence ϕ is a diffeomorphism of K.

Therefore we get exact sequences of rational homotopy groups

$$\pi_{k+1}(\mathrm{Diff}(P)) \otimes \mathbb{Q} \to \pi_{k+1}(\mathrm{Diff}(K)) \otimes \mathbb{Q} \to \pi_k(\mathrm{Diff}(D^n, \partial D^n) \otimes \mathbb{Q} \to \pi_k(\mathrm{Diff}(P)) \otimes \mathbb{Q}$$

and

$$\pi_{k+1}(\mathrm{Diff}(K, \partial D^n)) \otimes \mathbb{Q} \to \pi_{k+1}(\mathrm{Diff}(K)) \otimes \mathbb{Q} \to \pi_{k+1}(\mathrm{Diff}(\partial D^n)) \otimes \mathbb{Q}.$$

By Farrell and Hsiang [FH78], we have $\pi_{k+1}(\mathrm{Diff}(\partial D^n)) \otimes \mathbb{Q} = 0$.

Moreover every family of diffeomorphisms of K which lies in the image of $\pi_{k+1}(K, \partial D^n)$ extends to a family of diffeomorphisms of P, by defining the extension to be the identity on D^n.

Therefore the map $\pi_{k+1}(\mathrm{Diff}(K)) \otimes \mathbb{Q} \to \pi_k(\mathrm{Diff}(D^n, \partial D^n) \otimes \mathbb{Q}$ is the zero map and the claim follows from Farrell and Hsiang [FH78]. $\square$

4. $\pi_k(\mathcal{M}^+)$ is non-trivial

In this section we show that $\pi_k(\mathcal{M}^+(M, T))$ is non-trivial for manifolds as in Example 2.6.

To do so, we need the following theorem which is an equivariant version of Theorem 2.13 of [Wal11].

Theorem 4.1 *Let G be a compact Lie group. Let X be a smooth compact G-manifold of dimension n and B a compact space. Let $B = \{g_b \in \mathcal{R}^+(X, G) : b \in B\}$ be a continuous family of invariant metrics of positive scalar curvature. Moreover, let $\iota : G \times_H (S(V) \times D_1(W)) \to X$ be an equivariant embedding, with $H \subset G$ compact, V, W orthogonal H-representations with $\dim G - \dim H + \dim V + \dim W = n + 1$ and $\dim W > 2$.*

Finally let $g_{G/H}$ be any G-invariant metric on G/H and g_V be any H-invariant metric on $S(V)$.

Then, for some $1 > \delta > 0$, there is a continuous map

$$\begin{aligned} B &\to \mathcal{R}^+(X, G) \\ b &\mapsto g^b_{std} \end{aligned}$$

satisfying

1. *Each metric g^b_{std} makes the map $G \times_H (S(V) \times D_\delta(W)) \to (G/H, g_{G/H})$ into a Riemannian submersion. Each fiber of this map is isometric to $(S(V) \times D_\delta(W), g_V + g_{tor})$, where g_{tor} denotes a torpedo metric on $D_\delta(W)$. Moreover g^b_{std} is the original metric outside a slightly bigger neighborhood of $G \times_H (S(V) \times \{0\})$.*

2. *The the original map $B \to \mathcal{R}^+(X, G)$ is homotopic to the new map.*

The proof of this theorem is a direct generalization of the proof of Theorem 2.13 of [Wal11] using the methods of the proof of Theorem 2 in [Han08]. Therefore we leave it to the reader.

Let E be the total space of a Hatcher disc bundle [Goe01] over S^k and fiber D^n with structure group $\mathrm{Diff}(D^n, \partial D^n)$, that is a disc bundle over S^k such that its classifying map $S^k \to B\,\mathrm{Diff}(D^n, \partial D^n)$ represents a non-trivial element in $\pi_k(B\,\mathrm{Diff}(D^n, \partial D^n))$.

Moreover, let

$$F = (E \times T^n) \cup (S^k \times N)$$

with N as in the previous section. Let $M_1 \subset N$ be a characteristic submanifold and denote by $\tilde{M_1}$ a small equivariant tubular neighborhood of M_1. Then F is a bundle over S^k with fiber the quasitoric manifold M and structure group $\mathrm{Diff}(D^n, \partial D^n)$. Note that F has a natural fiberwise T^n-action.

By Theorem 2.9 of [BHSW10], we have a metric on E with fiberwise positive scalar curvature. Together with an invariant metric of non-negative scalar curvature on $T^n \times D^2$ we get from this metric an invariant metric g on $\partial(E \times T^n \times D^2)$ with fiberwise positive scalar curvature, i.e. the restriction to any fiber of the bundle has positive scalar curvature.

On $(N - \tilde{M_1}) \times D^2$, there is an equivariant Morse function without critical orbits of coindex less than three. The global minimum of this Morse function is attained on $(\partial N) \times D^2$. Using this Morse function and Theorem 4.1 we get a fiberwise invariant metric of positive scalar curvature on $\partial((F - (S^k \times \tilde{M_1})) \times D^2)$.

Indeed, using the Morse function, we get an equivariant handle decomposition of $(N - \tilde{M_1}) \times D^2$, without handles of codimension less than three. Moreover, the restriction of the bundle $\partial(E \times T^n \times D^2) \to S^k$ to $(\partial E) \times T^n \times D^2$ is trivialized by assumption. So the restriction of g to the fibers of this bundle gives a compact family of invariant metrics of positive scalar curvature on $(\partial N) \times D^2$. Therefore, by Theorem 4.1, we can do equivariant surgery on $(\partial N) \times D^2$ to get a fiberwise invariant metric of positive scalar curvature on $\partial((F - (S^k \times \tilde{M_1})) \times D^2)$.

Note that Berard Bergery's result [BB83] on the existence of a metric of positive scalar curvature on the orbit space of a free torus action, generalizes directly to a family version. This is because Berard Bergery shows that if g is an invariant metric of positive scalar curvature on a free S^1-manifold M, then $f^{2/\dim M - 2} \cdot g^*$ has positive scalar curvature, where g^* is the quotient metric of g and f is the length of the S^1-orbits in M. This construction clearly generalizes to families of metrics. Moreover the metrics on the orbit space will be invariant under under every Lie group action which is induced on M/S^1 from an action on M which commutes with S^1 and leaves the metrics on M invariant (see [Wie16, Theorem 2.2] for the case of a single metric).

Moreover, note that F is the orbit space of the free action of the diagonal in $\lambda(M_1) \times S^1$ on $\partial((F - (S^k \times \tilde{M_1})) \times D^2)$, where S^1 is the circle group which acts by rotation on D^2.

Hence, with the remarks from above one gets an invariant metric of fiberwise positive scalar curvature on F in the same way as in the case of a single metric (see [Wie16, Proof of Theorem 2.4] for details).

This metric defines an element γ in $\pi_k(\mathcal{M}^+(M,T)) \otimes \mathbb{Q}$. The image of γ in

$$\pi_k(\mathcal{M}(M,T)) \otimes \mathbb{Q} \cong \pi_k(BD(M,T)) \otimes \mathbb{Q}$$

is represented by the classifying map for our Hatcher bundle E.

Therefore it follows from the lemmas in the previous two sections, that γ is non-trivial if M is as in example 2.6 because the classifying map of a Hatcher bundle represents a non-trivial element in the homotopy groups of $B\operatorname{Diff}(D^n, \partial D^n)$.

Therefore we have proved the following theorem:

Theorem 4.2 *Let M be a quasitoric manifold of dimension $2n$ as in example 2.6. Then for $0 < k < \frac{n}{6} - 7$, n odd and $k \equiv 0 \mod 4$, $\pi_k(\mathcal{M}^+) \otimes \mathbb{Q}$ is non-trivial, where $\mathcal{M}^+$ is some component of $\mathcal{M}^+(M; T^n)$.*

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On moduli spaces of positive scalar curvature metrics on certain manifolds

*Michael Wiemeler**

We show that the homotopy groups of moduli spaces of positive scalar curvature metrics on certain simply connected spin manifolds are non-trivial.

1. Introduction

In recent years a lot of work was devoted on the study of spaces $\mathcal{R}^+(M)$ of Riemannian metrics of positive scalar curvature on a given spin manifold M. Here we only want to mention the works [BERW14] and [HSS14].

The first work relates the study of the homotopy type of these spaces to certain infinite loop spaces. It is shown in this paper that infinitely many homotopy groups of these spaces are non-trivial.

In the second work it was shown that in degrees d much smaller than the dimension n of the manifold M there are classes in the homotopy groups $\pi_d(\mathcal{R}^+(M))$, which descend to non-trivial classes in the homotopy groups of the observer moduli space of positive scalar curvature metrics on M, if M fulfills some technical condition.

In this note we combine the results and methods of the above mentioned papers to prove the following theorem.

Theorem 1.1 *Let M be a closed spin manifold of dimension $n \geq 6$ which admits a metric of positive scalar curvature. If $k = 4s - n - 1 > 2n$, $s \in \mathbb{Z}$ and $g_0 \in \mathcal{R}^+(M)$, then*

$$\pi_k(\mathcal{R}^+(M), g_0) \otimes \mathbb{Q} \to \pi_k(\mathcal{M}^+_{observer}(M), g_0) \otimes \mathbb{Q}$$

has non-trivial image.

If moreover, there is no non-trivial orientation-preserving action of a finite group on M, then

$$\pi_k(\mathcal{R}^+(M), g_0) \otimes \mathbb{Q} \to \pi_k(\mathcal{M}^+_0(M), g_0) \otimes \mathbb{Q}$$

has non-trivial image.

Here $\mathcal{M}^+_0(M) = \mathcal{R}^+(M)/\operatorname{Diff}_0(M)$ denotes the moduli space of metrics of positive scalar curvature with respect to the group $\operatorname{Diff}_0(M)$ of orientation-preserving diffeomorphisms of M.

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Examples of manifolds on which no finite group acts non-trivially and orientation-preserving have been given by Puppe [Pup95]. These examples are simply connected spin manifolds of dimension six. Therefore they admit metrics of positive scalar curvature.

Moreover, we show the following result:

Theorem 1.2 *Let $n > 1$. Then in the image of the map*

$$\pi_1(\mathcal{R}^+(S^{4n-2})) \to \pi_1(\mathcal{M}_0^+(S^{4n-2}))$$

there are elements of infinite order.

As far as we know, these are the first examples of elements in the fundamental group of the space of positive scalar curvature metrics which decent to elements of infinite order in the fundamental group of $\mathcal{M}_0^+(S^{4n-2})$. Note that, by [Wal14], $\pi_1(\mathcal{R}^+(S^{4n-2}))$ is abelian. The space $\mathcal{R}^+(S^2)$ is known to be contractible [RS01]. Hence, the above theorem is false for $n = 1$.

This note is structured as follows. In the next Section 2 we proof Theorem 1.1. In Section 3 we recall some transversality results in the context of infinite dimensional manifolds. Then in Section 4 we recall Ebin's slice theorem and some of its consequences. In the Section 5 we give the proof of Theorem 1.2. Finally in Section 6 we show that our methods are not fine enough to deal with the moduli space of positive scalar curvature metrics with respect to the full diffeomorphism group.

2. The proof of Theorem 1.1

Our proof of Theorem 1.1 is based on the following result:

Theorem 2.1 ([BERW14, Theorem A]) *Let M be a closed spin manifold of dimension $n \geq 6$. If $k = 4s - n - 1 \geq 0$ and $g_0 \in \mathcal{R}^+(M)$, then the map*

$$A_k \otimes \mathbb{Q} : \pi_k(\mathcal{R}^+(M), g_0) \otimes \mathbb{Q} \to \mathbb{Q}$$

is surjective.

Here A_k denotes the secondary index invariant for metrics of positive scalar curvature metrics. There are two definitions for this invariant one is by Hitchin [Hit74], the other is by Gromov and Lawson [GL83].

As shown in [Ebe14], these two definitions lead to the same invariant. This is important for our work because we use results of [BERW14], where mainly the first definition is used, and of [HSS14], where the definition of Gromov and Lawson is used.

We also need the following definition from [HSS14]:

Definition 2.2 *Let M be an oriented closed smooth manifold. We call M a $\hat{A}$-multiplicative fiber in degree k if for every oriented fiber bundle $M \to E \to S^{k+1}$ we have $\hat{A}(E) = 0$.*

For $\hat{A}$-multiplicative manifolds the following is known:

Theorem 2.3 *Let M be a closed spin manifold and $k \geq 2$. If M is a $\hat{A}$-multiplicative fiber in degree k, then the following holds:*

1. *The map $A_k \otimes \mathbb{Q}$ from above factors through $\pi_k(\mathcal{M}^+_{observer}(M), g_0) \otimes \mathbb{Q}$.*
2. *If there is no non-trivial smooth orientation-preserving action of a finite group on M then $A_k \otimes \mathbb{Q}$ factors through $\pi_k(\mathcal{M}^+_0(M), g_0) \otimes \mathbb{Q}$.*

Proof. The first statement is proved in [HSS14, Section 2]. The proof of the second statement is similar. Since there is no non-trivial orientation-preserving smooth action of a finite group on M the isometry group of any Riemannian metric on M contains at most one orientation-reversing involution and the identity. Therefore the group $\mathrm{Diff}_0(M)$ of orientation-preserving diffeomorphisms acts freely on $\mathcal{R}^+(M)$.

In particular, there is an exact sequence

$$\pi_k(\mathrm{Diff}_0(M)) \otimes \mathbb{Q} \to \pi_k(\mathcal{R}^+(M)) \otimes \mathbb{Q} \to \pi_k(\mathcal{M}^+_0) \otimes \mathbb{Q}$$

Since M is an $\hat{A}$-multiplicative fiber in degree k it follows from the arguments in [HSS14, Section 2] that $A_k \otimes \mathbb{Q}$ vanishes on the image of $\pi_k(\mathrm{Diff}_0(M)) \otimes \mathbb{Q}$. Therefore the theorem follows. □

Our Theorem 1.1 now follows from the above two theorems and the following lemma.

Lemma 2.4 *Let M be a closed oriented smooth manifold of dimension $n \geq 1$. If $k > 2n$, then M is a $\hat{A}$-multiplicative fiber in degree k.*

Proof. Let $M \to E \to S^{k+1}$ be a smooth oriented fiber bundle. The tangent bundle of E is isomorphic to $p^*(TS^{k+1}) \oplus V$ where V is the bundle along the fiber. Since TS^{k+1} is stably trivial, it follows that TE and V are stably isomorphic. Therefore the Pontrjagin classes of E are concentrated in degrees smaller or equal to $2n$. Moreover, since $n < k$, it follows from an inspection of the Serre spectral sequence for the fibration that $H^j(E;\mathbb{Q}) = 0$ for $n < j < k+1$. Because of $2n < k$ the Pontrjagin classes of E are concentrated in degrees smaller or equal to n. Moreover it follows that all products of these classes of degree greater than n must vanish. In particular all the Pontrjagin numbers of E vanish and the theorem is proved. □

3. Transversality

In this section we start to collect the necessary material for the proof of Theorem 1.2. We need the following transversality result for maps into infinite dimensional manifolds.

Lemma 3.1 *Let F be a topological vector space and $p : F \to \mathbb{R}^{n+1}$ a continuous linear surjective map and $U \subset F$ an open neighborhood of zero.*

If $f : D^n \to F$ is a continuous map and $K_1, \ldots, K_m \subset D^n$ are compact such that there are open $U_1, \ldots, U_m \subset U$ with $f(K_i) \subset U_i$, then there is a continuous map $f' : D^n \to U$, such that

1. $f'(D^n_{1/2}) \subset U - \{0\}$,

2. there is a homotopy h_t from f to f', such that $h_t(K_i) \subset U_i$ for all t and i and $h_t|_{\partial D^n} = f|_{\partial D^n}$.

Proof. The existence of p guarantees that F is isomorphic as a topological vector space to $\mathbb{R}^{n+1} \times F'$ for some closed subvector space $F' \subset F$. Therefore the Lemma follows from the finite dimensional case $F = \mathbb{R}^{n+1}$. This case follows from transversality considerations. $\square$

Remark 3.2 *If there is a continuous scalar product on an infinite dimensional topological vector space, then there is always a map p as above. One can take the orthogonal projection onto some $n+1$-dimensional subvector space.*

4. Consequences of Ebin's slice theorem

Ebin [Ebi70] showed the following theorem:

Theorem 4.1 *Let $\mathcal{R}$ be the space of Riemannian metrics on a closed manifold M, $g \in \mathcal{R}$.*

Then there is a neighborhood N of g, which is homeomorphic to an open subset of $V \times E$, where V is a neighborhood of $\mathrm{Id} \in \mathrm{Diff}(M)/\mathrm{Iso}(g)$ and E is an infinite dimensional $\mathrm{Iso}(g)$-representation.

Moreover he gives a description of the $\mathrm{Diff}(M)$-action on $V \times E$.

From the proof of this theorem it follows that E is isomorphic as a topological vector space to $E^{\mathrm{Iso}(g)} \times F$, where F is a $\mathrm{Iso}(g)$-invariant closed subspace of E. Moreover, there is a continuous scalar product on E so that the remark of the previous section applies to E and F. Note that Bourguignon [Bou75, Proposition III.20] has shown that F is infinite dimensional if $\mathrm{Iso}(g)$ is non-trivial.

Furthermore, it follows that the minimal stratum in N is given by $V \times E^{\mathrm{Iso}(g)}$. In particular, maps from finite dimensional manifolds to N can be made transverse to the minimal stratum.

For $g' \in N$, $\mathrm{Iso}(g')$ is conjugated in $\mathrm{Diff}(M)$ to a subgroup of $\mathrm{Iso}(g)$.

If $\gamma : I \to N/\mathrm{Diff}(M) = E/\mathrm{Iso}(g)$ is a path, then this path can be lifted to a path in N (see [Bre72, Chapter II.6]).

Denote by $\mathcal{M}$ the moduli space of all Riemannian metrics on the closed manifold M and by $\mathcal{M}_{m,k}$ the moduli space of Riemannian metrics on M with $\dim \mathrm{Iso}(g) < m$ or $(\dim \mathrm{Iso}(g) = m$ and $|\mathrm{Iso}(g)/\mathrm{Iso}(g)^0| \leq k)$.

Then we have:

1. $\mathcal{M}$ is locally homeomorphic to orbit spaces of infinite dimensional representations of compact Lie groups. We call the charts which correspond to these local models "Ebin charts", denoted by E/G.

2. $\mathcal{M}_{m,k-1} \subset \mathcal{M}_{m,k} \subset \mathcal{M}$ is open.

Since S^n and D^{n+1} are compact for all $n \in \mathbb{N}$ the following statements follow from the second point above.

3. $\pi_n(\mathcal{M}_{m,0}) = \lim_k \pi_n(\mathcal{M}_{m-1,k})$
4. $\pi_n(\mathcal{M}) = \lim_m \pi_n(\mathcal{M}_{m,0})$
5. $\ker(\pi_n(\mathcal{M}_{m,k}) \to \pi_n(\mathcal{M}_{m+1,0})) = \lim_j \ker(\pi_n(\mathcal{M}_{m,k}) \to \pi_n(\mathcal{M}_{m,j}))$
6. $\ker(\pi_n(\mathcal{M}_{m,0}) \to \pi_n(\mathcal{M})) = \lim_j \ker(\pi_n(\mathcal{M}_{m,0}) \to \pi_n(\mathcal{M}_{j,0}))$
7. similar statements hold for homology groups.

5. The proof of Theorem 1.2

In the following $\mathcal{R}$, $\mathcal{M}$, etc. can be defined as in the previous section, or as $\mathrm{Diff}(M)$-invariant open subspaces.

Lemma 5.1 *The map $\pi_1(\mathcal{M}_{m,k-1}) \to \pi_1(\mathcal{M}_{m,k})$ is surjective.*

Proof. Let $\gamma : I \to \mathcal{M}_{m,k}$ with $\gamma(0) = \gamma(1) \in \mathcal{M}_{m,k-1}$.

Then there are finitely many intervals $[a_j, b_j] \subset]0,1[$, so that $\gamma([a_j, b_j])$ is contained in an Ebin chart and $\gamma^{-1}(\mathcal{M}_{m,k} - \mathcal{M}_{m,k-1}) \subset \bigcup_j [a_j, b_j]$.

The path $\gamma|_{[a_j,b_j]}$ can be lifted to a path in an Ebin slice E. There it can be made transversal to the minimal stratum. Since the complement of the minimal stratum in an Ebin chart is connected, γ is homotopic to a closed curve in $\mathcal{M}_{m,k-1}$. □

Lemma 5.2 *The kernel of the map $\pi_1(\mathcal{M}_{m,k-1}) \to \pi_1(\mathcal{M}_{m,k})$ is generated by torsion elements.*

Proof. Let $\gamma : I \to \mathcal{M}_{m,k-1}$ be a closed curve and $\lambda : I \times I \to \mathcal{M}_{m,k}$ a null homotopy of γ.

Then there are finitely many discs $D_1^2, \ldots, D_j^2 \subset I \times I - \partial(I \times I)$ with piecewise C^1-boundary, such that:

1. $\lambda^{-1}(\mathcal{M}_{m,k} - \mathcal{M}_{m,k-1}) \subset \bigcup_i (D_i^2 - \partial D_i^2)$
2. $\lambda(D_i^2)$ is contained in an Ebin chart.

Without loss of generality we may assume, that there is a curve $\sigma : I \to I \times I$ from the base point to the boundary of D_k^2 such that $\sigma(t) \notin \bigcup D_i^2$ for $t \neq 1$.

Cutting $I \times I$ along σ leads to a homotopy from γ to $\lambda(\sigma * \partial D_k^2 * \sigma^{-1})$.

Now we can lift $\lambda(\partial D_k^2)$ to a curve in an Ebin slice and make it transversal to the minimal stratum. This leads to a homotopy $\lambda' : I \times I \to \mathcal{M}_{m,k}$ from γ to $\lambda(\sigma * \partial(D_k^2)' * \sigma^{-1})$ with

1. $\lambda'(\partial(I \times I)) \subset \mathcal{M}_{m,k-1}$

2. $\lambda'^{-1}(\mathcal{M}_{m,k} - \mathcal{M}_{m,k-1})$ is contained in $k-1$ discs as above.

By induction we see that γ is homotopic in $\mathcal{M}_{m,k-1}$ to

$$\sigma_1 * \partial D_1^2 * \sigma_1^{-1} * \sigma_2 * \partial D_2^2 \cdots * \sigma_k^{-1}.$$

Therefore it suffices to show that closed curves δ which are contained in an Ebin chart are torsion elements.

To do so, lift δ to an Ebin slice. Let δ' be the lift of δ. Let G be the compact Lie group which acts on this slice. Then there is a $g \in G$, such that $g\delta'(0) = \delta'(1)$. Let k be the order of the class of g in G/G^0.

Then

$$\delta' * g\delta' * \cdots * g^{k-1}\delta'$$

is a lift of δ^k, which ends and starts in the same component of $G\delta'(0)$. Therefore δ^k can be lifted to a closed curve in the Ebin slice. Since the complement of the minimal stratum in the Ebin slice is contractible, it follows that δ is torsion. $\square$

Corollary 5.3 *The kernel of $H_1(\mathcal{M}_{0,1}) \to H_1(\mathcal{M})$ is torsion.*

Proof. It follows from the two lemmas above that the kernels of the maps $H_1(\mathcal{M}_{m,k-1}) \to H_1(\mathcal{M}_{m,k})$ are torsion.

To see this note that $H_1(\mathcal{M}_{m,k}) = \pi_1(\mathcal{M}_{m,k})/C_{m,k}$, where $C_{m,k}$ denotes the commutator subgroup.

Let $\gamma \in \pi_1(\mathcal{M}_{m,k-1})$ such that $[\gamma] = 0 \in H_1(\mathcal{M}_{m,k})$. We have to show that γ is contained in a normal subgroup of $\pi_1(\mathcal{M}_{m,k-1})$ which is generated by torsion elements and $C_{m,k-1}$. Since $[\gamma] = 0 \in H_1(\mathcal{M}_{m,k})$, we have $\gamma \in C_{m,k} \subset \pi_1(\mathcal{M}_{m,k})$. Because $C_{m,k-1} \to C_{m,k}$ is surjective by the first lemma, it follows from the second lemma that up to torsion elements γ is contained in $C_{m,k-1}$. This proves the claim.

The next step is to show that the kernels of $H_1(\mathcal{M}_{0,1}) \to H_1(\mathcal{M}_{1,0})$ and $H_1(\mathcal{M}_{m-1,0}) \to H_1(\mathcal{M}_{m,0})$ are torsion. This follows from 5 and 7 in the previous section, by taking the limit.

Therefore the statement follows from 6 and 7 in the previous section, by taking the limit. $\square$

Corollary 5.4 *Let $n > 1$. Then in the image of the map*

$$\pi_1(\mathcal{R}^+(S^{4n-2})) \to \pi_1(\mathcal{M}_0^+(S^{4n-2}))$$

there are elements of infinite order.

Proof. It follows from transversality considerations that $\pi_1(\mathcal{R}^+_{0,1}) \to \pi_1(\mathcal{R}^+)$ is an isomorphism. Using Theorem 2.1 one sees as in the proof of Theorem 2.3 that there are elements of infinite order in the image of the map $\pi_1(\mathcal{R}^+_{0,1}(S^{4n-2})) \to \pi_1(\mathcal{M}^+_{0,1}(S^{4n-2}))$, whose image in H_1 has infinite order. Therefore the statement follows from the above corollary. $\square$

6. Remarks on orientation-reversing isometries

In this section, we show that the index difference $A_k \otimes \mathbb{Q}$ cannot be used to detect non-trivial elements in $\pi_k(\mathcal{M}^+(M)) \otimes \mathbb{Q}$ if there is an orientation-reversing diffeomorphism φ of finite order in $\mathrm{Diff}(M)$ which leaves some metric g_0 of positive scalar curvature invariant. Here $\mathcal{M}^+(M)$ denotes the moduli space of positive scalar curvature metrics on the spin manifold M with respect to the full diffeomorphism group of M.

To be more precise we have the following theorem.

Theorem 6.1 *Let M, φ and g_0 as above. Assume $H^1(M;\mathbb{Z}_2) = 0$. Then any element in the image of the map $\pi_k(\mathcal{R}^+, g_0) \otimes \mathbb{Q} \to \pi_k(\mathcal{M}^+, g_0) \otimes \mathbb{Q}$ can be represented by an element $g' \in \pi_k(\mathcal{R}^+, g_0) \otimes \mathbb{Q}$ with $A_k(g') = 0$, $k \geq 2$.*

Proof. Let m be the order of φ and ζ a primitive m-th root of unity. We first prove the theorem with $\mathbb{Q}$ replaced by $\mathbb{Q}[\zeta]$ and will then indicate how the theorem follows.

Since g_0 is fixed by the action of φ on $\mathcal{R}^+$, there is a natural induced action of φ on $\pi_k(\mathcal{R}^+, g_0) \otimes \mathbb{Q}[\zeta]$. This $\mathbb{Q}[\zeta]$-vector space decomposes as a direct sum of the eigenspaces of φ_*.

To see this note that, for $v \in \pi_k(\mathcal{R}^+, g_0) \otimes \mathbb{Q}[\zeta]$,

$$W_v = \langle \varphi_*^l(v);\ l \in \mathbb{Z}\rangle$$

is a finite dimensional φ_*-invariant subvector space of $\pi_k(\mathcal{R}^+, g_0) \otimes \mathbb{Q}[\zeta]$ because φ has finite order. Since the minimal polynomial of the restriction of φ_* to W_v (a divisor of $X^m - 1$), splits over $\mathbb{Q}[\zeta]$ as a product of linear factors each appearing with multiplicity one, $\varphi_*|_{W_v}$ is diagonizable. Because $v \in W_v$, it follows that v is the sum of eigenvectors of φ_*.

The kernel of the natural map $\pi_k(\mathcal{R}^+, g_0) \otimes \mathbb{Q}[\zeta] \to \pi_k(\mathcal{M}^+, g_0) \otimes \mathbb{Q}[\zeta]$ contains all eigenspaces of φ_* except the one corresponding to the eigenvalue 1. This is because, for $g' \in \pi_k(\mathcal{R}^+) \otimes \mathbb{Q}[\zeta]$, $g' - \varphi_* g'$ is clearly contained in this kernel.

Therefore it suffices to show that $A_k \otimes \mathbb{Q}[\zeta]$ vanishes on this eigenspace. To see this we note that for $g' \in \pi_k(\mathcal{R}^+, g_0)$ we have

$$A_k(g') = -A_k(\varphi_* g') \tag{6.1}$$

by the Gromov-Lawson definition of the index difference and the fact that φ is an orientation-reversing isometry of g_0. Note here that since $H^1(M;\mathbb{Z}_2) = 0$ there is a unique spin structure on M. Therefore $\varphi : M \to -M$ is spin preserving.

To get the statement of the theorem with coefficients in $\mathbb{Q}$ and not in $\mathbb{Q}[\zeta]$, note that the Galois group H of the field extension $\mathbb{Q}[\zeta]|\mathbb{Q}$ is finite and acts with fixed point set $\pi_k(\mathcal{R}^+, g_0)\otimes\mathbb{Q}$ on $\pi_k(\mathcal{R}^+, g_0)\otimes\mathbb{Q}[\zeta]$. Therefore, if $g' \in \pi_k(\mathcal{R}^+, g_0)\otimes\mathbb{Q}[\zeta]$ with $A_k(g') = 0$ and $[g'] \in \pi_k(\mathcal{M}^+, g_0) \otimes \mathbb{Q}$, we have

$$A_k(\frac{1}{|H|}\sum_{h\in H} h(g')) = \frac{1}{|H|}\sum_{h\in H} h(A_k(g')) = 0,$$

$$\frac{1}{|H|}\sum_{h\in H} h(g') \in \pi_k(\mathcal{R}^+, g_0) \otimes \mathbb{Q}$$

and

$$[\frac{1}{|H|}\sum_{h\in H} h(g')] = \frac{1}{|H|}\sum_{h\in H} h([g']) = [g'] \in \pi_k(\mathcal{M}^+, g_0) \otimes \mathbb{Q}.$$

Hence, the theorem is proved. □

The examples of Puppe mentioned in the introduction have orientation-reversing involutions. Therefore with our methods one can not do better than in our Theorem 1.1. But we also want to mention that many positively curved manifolds such as spheres and complex projective spaces of odd complex dimensions admit orientation-reversing isometric involutions.

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