On the stability of the Rayleigh–Ritz method for eigenvalues

D. Gallistl[1] · P. Huber[2] · D. Peterseim[3]

Abstract This paper studies global stability properties of the Rayleigh–Ritz approximation of eigenvalues of the Laplace operator. The focus lies on the ratios $\hat{\lambda}_k/\lambda_k$ of the kth numerical eigenvalue $\hat{\lambda}_k$ and the kth exact eigenvalue λ_k. In the context of classical finite elements, the maximal ratio blows up with the polynomial degree. For B-splines of maximum smoothness, the ratios are uniformly bounded with respect to the degree except for a few instable numerical eigenvalues which are related to the presence of essential boundary conditions. These phenomena are linked to the inverse inequalities in the respective approximation spaces.

Mathematics Subject Classification 65N12 · 65N15 · 65N30

D. Gallistl is supported by the Deutsche Forschungsgemeinschaft (DFG) through CRC 1173.
D. Peterseim and P. Huber are supported by DFG in the Priority Program 1748 "Reliable simulation techniques in solid mechanics. Development of non-standard discretization methods, mechanical and mathematical analysis" under the project "Adaptive isogeometric modeling of propagating strong discontinuities in heterogeneous materials" (PE2143/2-1). The authors acknowledge the support given by the Hausdorff Center for Mathematics Bonn.

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1 Introduction

The accuracy of the Rayleigh–Ritz method for symmetric eigenvalue problems naturally depends on the approximation properties of the underlying ansatz space. In the case of finite elements, explicit convergence rates are known since [1,2,12]. However, because of smallness conditions on the finite element mesh, these results are restricted to the lower part of the discrete spectrum (cf. Fig. 1) and numerical experiments have shown that the remaining discrete eigenvalues are inaccurate, especially for high polynomial degrees (cf. [8,15]). A possible way to reduce these errors is to replace the finite element functions with splines of higher regularity, which is referred to as the concept of *isogeometric analysis* (IGA) (cf. [4,7]).

Numerical experiments in [5] indicate that except for a small number of so-called outlier frequencies the overall accuracy of the resulting discrete spectra is much greater than in the case of finite element spaces; see also Fig. 2. In other words, the isogeometric approach provides an accurate approximation of more eigenvalues compared with classical finite elements when the comparison is based on the same number of degrees of freedom.

This paper aims to explain these phenomena by investigating *global* properties of the discrete spectrum resulting from the Rayleigh–Ritz method with either classical finite element functions or splines of maximum smoothness. The term "global" refers to characteristics of the eigenvalues that concern the whole discrete spectrum. We address two questions: First, we study the stability of the method, i.e., we derive bounds of the form

$$\lambda_k \leq \hat{\lambda}_k \leq C\,\lambda_k,$$

where λ_k denotes the kth eigenvalue of the original differential operator and $\hat{\lambda}_k$ its discrete counterpart. We show that the constant $C > 0$ can be chosen uniformly in the case of splines (except for the aforementioned outlier eigenvalues) whereas it depends on the polynomial degree in the finite element framework.

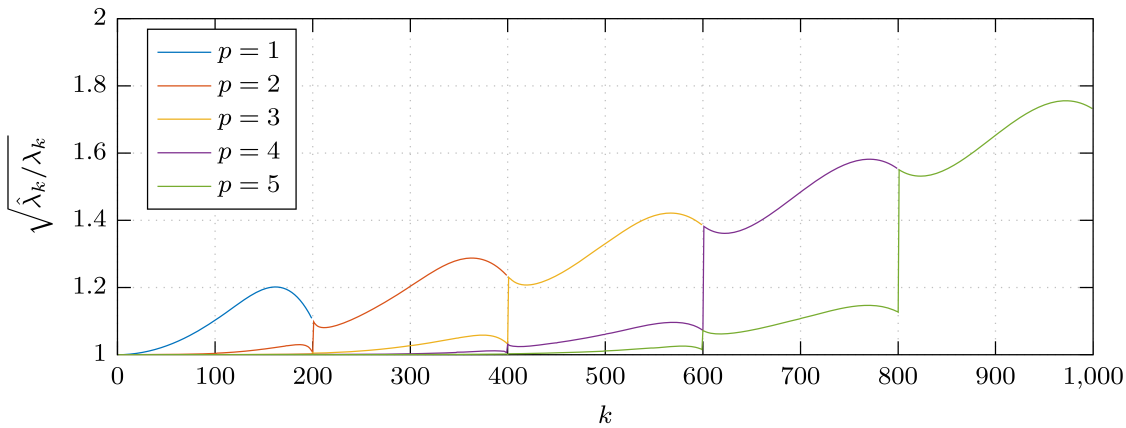

Fig. 1 Frequency ratios $\sqrt{\hat{\lambda}_k/\lambda_k}$ for the one-dimensional Laplace eigenvalue problem with Dirichlet boundary conditions computed with finite element functions of degree p on a one-dimensional grid consisting of 200 elements

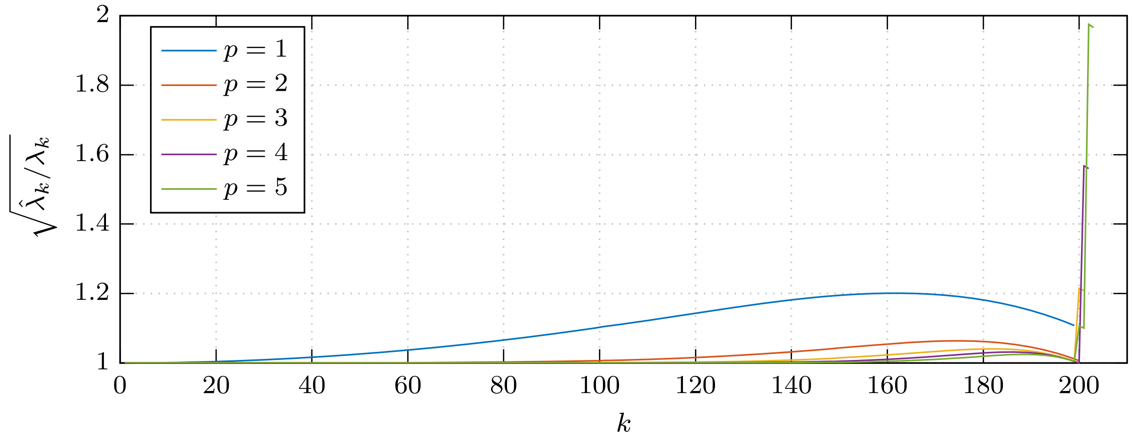

Fig. 2 Frequency ratios $\sqrt{\hat{\lambda}_k/\lambda_k}$ for the one-dimensional Laplace eigenvalue problem with Dirichlet boundary conditions computed with splines of maximum smoothness of degree p on a one-dimensional grid consisting of 200 elements

The second question is concerned with the behavior of the largest eigenvalues in the discrete spectrum. Using the sharpness of the inverse inequality, we will show that in the case of finite element spaces the ratio of the largest discrete eigenvalue and its corresponding exact eigenvalue $\hat{\lambda}_k/\lambda_k$ diverges with increasing polynomial degree. A similar statement can be derived in the isogeometric framework. Notably, these results show that in both frameworks the largest discrete eigenvalue diverges at a similar rate if the comparison is based on equal numbers of degrees of freedom. The analysis is restricted to the simple case of the Laplace eigenvalue problem on the unit cube and uses only uniform, rectangular meshes for the definition of the discrete spaces. For the case of *hp* finite elements, the arguments can be transferred to more general settings in a straightforward way. For tensor-product splines the situation appears to be more restrictive because the domain needs a tensor-product structure. Hence, in the case of splines, the results essentially hold for configurations with the unit cube as parameter domain.

A uniformly accurate approximation of the spectrum is desirable in several applications, e.g., in computational wave propagation. The work [9] established a relationship between the discrete spectrum and the wavenumber in Helmholtz problems, see also the dispersion analysis of [6]. The close connection of the discrete spectrum with the inverse inequality also shows that the CFL condition in explicit time-stepping methods is prescribed by the largest numerical eigenvalue. A uniformly stable numerical spectrum would therefore imply a relaxation of the CFL condition. This fact is exploited, e.g., in [10] where special operator-dependent spline-type basis functions replace classical finite elements to achieve feasible CFL numbers on adaptive spatial meshes. Despite their improved spectral properties, the standard IGA approximations are not yet sufficient for a CFL relaxation because of the outlier frequencies arising from the Dirichlet boundary condition. Based on numerical experience, the works [5,9] suggest a nonlinear parametrization of the control points in order to reduce the outlier modes.

The paper is structured as follows. Section 2 states the eigenvalue problem and an abstract stability result for the Rayleigh–Ritz method. This estimate is applied to *hp* finite elements and splines of maximum smoothness in the subsequent Sects. 3 and 4. The presentation is concluded with a numerical illustration for the two-dimensional model situation in Sect. 5.

2 The Rayleigh–Ritz method and its stability

Standard notation on Lebesgue and Sobolev spaces applies throughout this paper. Let $\Omega \subseteq \mathbb{R}^d$ for, $d \geq 1$, be a bounded Lipschitz domain and define $V := H_0^1(\Omega)$ along with the bilinear forms $a : V \times V \to \mathbb{R}$ and $b : V \times V \to \mathbb{R}$ given by

$$a(v, w) := \int_\Omega \nabla v \cdot \nabla w \, dx \quad \text{and} \quad b(v, w) := \int_\Omega vw \, dx \quad \text{for all } v, w \in V.$$

The Laplace eigenvalue problem seeks eigenpairs $(\lambda, u) \in \mathbb{R} \times V$ such that

$$a(u, v) = \lambda b(u, v) \quad \text{for all } v \in V.$$

Given some finite-dimensional subspace $\widehat{V} \subseteq V$, the Rayleigh–Ritz method seeks eigenpairs $(\hat{\lambda}, \hat{u}) \in \mathbb{R} \times \widehat{V}$ such that

$$a(\hat{u}, \hat{v}) = \hat{\lambda} b(\hat{u}, \hat{v}) \quad \text{for all } \hat{v} \in \widehat{V}.$$

It is well-known that the eigenvalues are non-negative and have no finite accumulation point. They can be sorted in ascending order

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \quad \text{and} \quad 0 < \hat{\lambda}_1 \leq \hat{\lambda}_2 \leq \cdots \leq \hat{\lambda}_{\dim \widehat{V}}.$$

The Rayleigh quotient is defined by $R(v) := a(v, v)/b(v, v)$ for any $v \in V \backslash \{0\}$ and allows the characterization

$$\lambda_k = \min_{\substack{V^{(k)} \subseteq V \\ \dim V^{(k)} = k}} \max_{v \in V^{(k)} \backslash \{0\}} R(v) \quad \text{for all } k \in \mathbb{N}, \tag{2.1a}$$

$$\hat{\lambda}_k = \min_{\substack{\widehat{V}^{(k)} \subseteq \widehat{V} \\ \dim \widehat{V}^{(k)} = k}} \max_{\hat{v} \in \widehat{V}^{(k)} \backslash \{0\}} R(\hat{v}) \quad \text{for all } k \in \{1, 2, \ldots, \dim \widehat{V}\}. \tag{2.1b}$$

This *minmax* principle implies the well-known inequality $\lambda_k \leq \hat{\lambda}_k$ for all $k \in \{1, 2, \ldots, \dim \widehat{V}\}$. Therefore, defining the ratio $C(\widehat{V}, k) := \hat{\lambda}_k / \lambda_k$, we obtain the elementary two-sided estimate

$$\lambda_k \leq \hat{\lambda}_k \leq C(\widehat{V}, k)\, \lambda_k. \tag{2.2}$$

This means that from the knowledge of upper bounds for $C(\widehat{V}, k)$ we can deduce stability of the Rayleigh–Ritz method. Let, for example, $\Omega = (0, 1)^d$ be the hypercube and n be a positive integer. Let $N = 2^n$ and let $\mathcal{T}$ be a uniform rectangular grid with

$(N+1)$ vertices in each coordinate direction. Define $\widehat{V} \subseteq V$ to be the finite element subspace over $\mathcal{T}_h$ consisting of continuous and piecewise polynomial functions of a fixed maximal degree. Let $m \leq n$ and $M = 2^m$ and $k = (M-1)^d$. We note that the finite element space defined over the coarser grid of mesh size $h = 1/M$ is a subspace of $\widehat{V}$. Then, this subspace is an admissible choice for the minimum in (2.1b). The inverse inequality [3] then states that there is a constant C_{inv}, which depends on the used polynomial degree and the space dimension, such that $\hat{\lambda}_k \leq C_{\text{inv}} h^{-2}$. On the other hand, the classical asymptotic behavior of the Laplacian eigenvalues due to Weyl [14] states that

$$\lim_{k\to\infty} \frac{\lambda_k}{k^{2/d}} = \frac{4\pi^2}{(\omega_d \operatorname{meas}(\Omega))^{2/d}} \tag{2.3}$$

where ω_d denotes the volume if the unit ball in $\mathbb{R}^d$. (For the unit cube there are even explicit formulas for the eigenvalues λ_k. However, property (2.3) is valid for more general domains.) This means that for some constant C_d there holds that $\lambda_k \geq C_d k^{2/d}$, and, thus, with $h^{-1} = M = (k^{1/d} + 1)$, we obtain

$$C(\widehat{V}, k) \leq \frac{C_{\text{inv}}}{C_d} k^{-2/d} (k^{1/d} + 1)^2 \leq C_{\text{stab}} \tag{2.4}$$

for some (h, k)-independent constant C_{stab}. The constant C_{inv}, however, deteriorates for large polynomial degrees. Hence, the stability estimate is sensitive to the choice of the polynomial degree. Still, in Sect. 3 we will prove that, at least for the largest discrete eigenvalue, the stability estimate (2.4) is sharp by sketching a proof of the well-known sharpness [11] of the inverse inequality with respect to the polynomial degree p.

3 Application to *hp* FEM

Let us derive upper bounds for $C(\widehat{V}, k)$ in the finite element framework. We restrict ourselves to the Laplace eigenvalue problem on the hypercube $\Omega = (0,1)^d$. Let N be a positive integer and set $h = N^{-1}$. We then assume that $\mathcal{T}_h$ is a uniform rectangular grid with $N+1$ vertices in each coordinate direction. By $V_{h,p} \subseteq V$ we denote the finite element space over $\mathcal{T}_h$ of continuous and piecewise polynomial functions of maximum degree $p \in \mathbb{N}$. Note that for this setting the dimension of $V_{h,p}$ is given by $\dim V_{h,p} = (pN-1)^d$.

Theorem 1 *For each* $k \in \{1, 2, \ldots, \dim V_{h,p}\}$ *the constant* $C(V_{h,p}, k)$ *in* (2.2) *can be bounded from above by*

$$C(V_{h,p}, k) \leq C_1\, q_k^2 \tag{3.1}$$

with some positive constant C_1 *and* $q_k \in \{1, 2, \ldots, p\}$ *being the smallest integer satisfying* $k \leq (q_k N - 1)^d$.

Proof We fix a pair (k, q_k) and note that there exists a unique

$$\kappa \in \{1, 2, \dots, \lfloor \log_2(N) \rfloor\}$$

such that

$$k_* := \left(q_k \left\lfloor \frac{N}{2^\kappa} \right\rfloor - 1\right)^d < k \le \left(q_k \left\lfloor \frac{N}{2^{\kappa-1}} \right\rfloor - 1\right)^d =: k^*. \tag{3.2}$$

We can choose a coarsening $\mathcal{T}_H$ of $\mathcal{T}_h$ such that the minimal mesh size in each coordinate direction is given by $H = 2^{\kappa-1}h$. Let $V_{H,q_k} \subseteq V_{h,p}$ be the finite element subspace of piecewise polynomial functions over $\mathcal{T}_H$ of maximal degree q_k. Note that by this choice the dimension of V_{H,q_k} is $\dim V_{H,q_k} = k^*$ and that consequently V_{H,q_k} is an admissible subspace for the minimum in the minmax characterization (2.1b) of $\hat{\lambda}_{k^*}$. By the inverse inequality [11] there exists a constant C_{inv} such that

$$\hat{\lambda}_k \le \hat{\lambda}_{k^*} \le C_{\text{inv}}\, d\, \frac{q_k^4}{H^2}.$$

By the choice of H, it is easy to see that we can bound q_k^2/H^2 by $C\, k_*^{2/d}$, where $C > 0$ is some universal constant. Finally, the application of Weyl's law (2.3) yields

$$\hat{\lambda}_k \le C\, C_{\text{inv}}\, d\, q_k^2\, k^{2/d} \le C_1\, q_k^2\, \lambda_k$$

with some constant $C_1 > 0$, which depends on d and Ω. □

Let $K_p = \dim V_{h,p}$. According to Theorem 1 the largest eigenvalue $\hat{\lambda}_{K_p}$ satisfies the estimate

$$\lambda_{K_p} \le \hat{\lambda}_K \le C_1\, p^2 \lambda_K.$$

As we have seen in the proof of Theorem 1, the crucial estimate for the upper bound of $\hat{\lambda}_{K_p}$ is the inverse inequality for finite element spaces. It is a well-known fact that this inequality is sharp with respect to p [11]. We show the sharpness by an explicit construction using Legendre polynomials.

The *Legendre polynomials* $((L_k)_{k\in\mathbb{N}_0})$ on the interval $\hat{\Omega} = [-1, 1]$ are given by the formula

$$L_k(x) = \frac{1}{k!2^k} \frac{d^k}{dx^k} \left[(x^2 - 1)^k\right], \quad x \in [-1, 1], \quad k = 0, 1, 2, \dots .$$

We recall that the Legendre polynomials are symmetric if k is even and antisymmetric if k is odd [11, (C.2.6)]. Moreover, the Legendre polynomials $((L_k)_{k\in\mathbb{N}_0})$ constitute a complete orthogonal system for $L^2(\hat{\Omega})$ with

$$(L_k, L_\ell)_{L^2(\hat{\Omega})} = \frac{2}{2k+1} \delta_{k\ell} \quad \text{for all } k, \ell \in \mathbb{N}_0. \tag{3.3}$$

Using the completeness of the Legendre basis and Eq. (3.3), it can be shown (following the lines of [11, p. 148]) that the derivatives of Legendre polynomials satisfy

$$\left(\frac{d}{dx}L_k, \frac{d}{dx}L_\ell\right)_{L^2(\hat{\Omega})} = \begin{cases} \ell(\ell+1), & \text{if } \ell \leq k \text{ and } (k+\ell) \in 2\mathbb{N}_0, \\ k(k+1), & \text{if } k < \ell \text{ and } (k+\ell) \in 2\mathbb{N}_0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

We use (3.3) and (3.4) to derive the sharpness of the inverse inequality.

Lemma 1 *Let* $a, b \in \mathbb{R}$ *and assume that* $h = b - a > 0$. *There exists a constant* $c > 0$ *such that for all* $p \in \mathbb{N}$, *there exists a nonzero polynomial* v *of degree* p *on the interval* (a, b) *satisfying*

$$\frac{|v|^2_{H^1(a,b)}}{\|v\|^2_{L^2(a,b)}} \geq c\,\frac{p^4}{h^2} \quad \textit{and} \quad v(a) = 0. \tag{3.5}$$

Proof We first show Lemma 1 for the case $(a, b) = (0, 1)$ and deduce (3.5) by a scaling argument. Assume that $p = 2q + 1$ for $q \in \mathbb{N}$ and define the polynomial $\tilde{w}\colon (-1, 1) \to \mathbb{R}$ as a linear combination of Legendre polynomials:

$$\tilde{w}(x) := \sum_{k=0}^{q} a_k L_{2k+1}(x), \quad \text{with } a_k := \sqrt{4k+3}. \tag{3.6}$$

The orthogonality relation (3.3) immediately implies

$$\|\tilde{w}\|^2_{L^2(-1,1)} = 2(q+1) = p+1.$$

Using (3.4) and rearranging sums, we obtain

$$|\tilde{w}|^2_{H^1(-1,1)} \geq 2\sum_{k=0}^{q}\sum_{\ell=0}^{k-1} a_\ell^2(2\ell+1)(2\ell+2) + \sum_{k=0}^{q} a_k^2(2k+1)(2k+2).$$

For both terms the following relations can be shown by induction:

$$\sum_{k=0}^{q}\sum_{\ell=0}^{k-1}(4\ell+3)(2\ell+1)(2\ell+2) = \frac{1}{5}q(q+1)(q+2)(4q^2+3q-2),$$
$$\sum_{k=0}^{q}(4k+3)(2k+1)(2k+2) = (q+1)(q+2)(2q+1)(2q+3).$$

Summing up these terms and using $p = 2q + 1$ we obtain

$$\frac{|\tilde{w}|^2_{H^1(-1,1)}}{\|\tilde{w}\|^2_{L^2(-1,1)}} \geq \frac{1}{20}(p+1)^4.$$

Since $\tilde{w}$ is antisymmetric, its restriction to (0, 1) satisfies (3.5) for $(a, b) = (0, 1)$. The analogue result for even polynomial degree can be reduced to the case of odd degree by choosing $p+1$ instead of p. For general intervals (a, b) the statement follows from a scaling argument. □

An immediate consequence of Lemma 1 is the sharpness of the upper bound in Theorem 1 for the largest discrete eigenvalue.

Theorem 2 *There exists a constant* $C_2 > 0$ *such that*

$$\liminf_{p\to\infty} \frac{\hat{\lambda}_{K_p}}{\lambda_{K_p}} \frac{1}{p^2} \geq C_2. \tag{3.7}$$

Proof In the one-dimensional case $d = 1$, let x_k, x_{k+1} and x_{k+2} be neighboring vertices of the grid $\mathcal{T}_h$ for an arbitrary $k \in \{1, 2, \ldots, N-1\}$. Let $v\colon (0, h) \to \mathbb{R}$ be a polynomial function of degree p satisfying (3.5), with mesh size h. Then, the piecewise polynomial function defined by

$$\hat{w}_p(x) := \begin{cases} v(x - x_k), & \text{for } x \in (x_k, x_{k+1}), \\ v(x_{k+2} - x), & \text{for } x \in (x_{k+1}, x_{k+2}), \\ 0, & \text{otherwise}, \end{cases} \tag{3.8}$$

is contained in the finite element space $V_{h,p}$. For higher dimensions, we define w_h to be a suitable tensor product of univariate functions given by (3.8). The characterization (2.1b) and Lemma 1 yield

$$\hat{\lambda}_{K_p} = \max_{\hat{v}_p \in V_{h,p}} R(\hat{v}_p) \geq R(\hat{w}_p) \geq c\, d\, \frac{p^4}{h^2}.$$

As $K_p = (p/h - 1)$ we obtain with Weyl's law

$$\lim_{p\to\infty} \frac{K_p^{2/d}}{\lambda_{K_p}} = \frac{(\omega_d)^{2/d}}{4\pi^2}.$$

Finally, combining the last two expressions shows (3.7) for $d = 1$. In the case of higher dimension the proof follows the same line of arguments, where $\hat{w}_p$ is defined as a tensor product function of the univariate counterpart. □

4 Application to splines

With a similar reasoning as for finite element spaces, we can apply the elementary stability estimate (2.2) to spaces of splines of maximum smoothness. We employ the same notation as in Sect. 3 and denote by $S_{h,p} \subseteq V$ the space of all spline functions over $\mathcal{T}_h$ of degree $p \in \mathbb{N}$ that are $p-1$ times continuously differentiable in the hypercube Ω. We recall that for this setting the dimension of $S_{h,p}$ is given by $(N + p - 2)^d$ [13].

The following stability result relies on an enhanced inverse inequality for splines of maximum smoothness, stated in [13]:

$$\left|\tilde{v}_p\right|^2_{H^1(\Omega)} \leq \frac{12d}{h^2} \left\|\tilde{v}_p\right\|^2_{L^2(\Omega)}. \tag{4.1}$$

It is important to note that the inequality in (4.1) does only hold for spline functions $\tilde{v}_p$ contained in a certain subspace $\tilde{S}_{h,p} \subseteq S_{h,p}$ with dimension

$$\tilde{K} := \dim \tilde{S}_{h,p} = \begin{cases} (N-2)^d, & \text{if } p \text{ is even,} \\ (N-1)^d, & \text{if } p \text{ is odd.} \end{cases} \tag{4.2}$$

Theorem 3 *Let $\tilde{K}$ be given by (4.2). Then, for each $k \in \{1, 2, \ldots, \tilde{K}\}$ the constant $C(S_{h,p}, k)$ in (2.2) can be bounded from above uniformly by a positive constant C_3:*

$$C(S_{h,p}, k) \leq C_3. \tag{4.3}$$

Proof Without loss of generality we may assume that p is odd. The case that p is even can be treated in an analogous way. For fixed index $k \in \{1, 2, \ldots, \tilde{K}\}$ there exists a unique $\kappa \in \{1, 2, \ldots, \lfloor \log_2(N) \rfloor\}$ such that

$$k_* := \left(\left\lfloor \frac{n}{2^\kappa} \right\rfloor - 1 \right)^d < k \leq \left(\left\lfloor \frac{n}{2^{\kappa-1}} \right\rfloor - 1 \right)^d. \tag{4.4}$$

Similarly to the proof of Theorem 1, we choose a coarsening $\mathcal{T}_H$ of $\mathcal{T}_h$ such that the minimal mesh size is given by $H = 2^{\kappa-1}h$. Let $S_{H,p} \subseteq S_{h,p}$ be the corresponding subspace of splines of degree p over $\mathcal{T}_H$ and let $\tilde{S}_{H,p}$ denote the subspace of $S_{H,p}$ for which the inverse inequality [13, Theorem 8.2] holds. Since by (4.2) this space has dimension k^*, the minmax characterization of $\hat{\lambda}_{k^*}$ yields the existence of a constant C_{inv} such that

$$\hat{\lambda}_k \leq \hat{\lambda}_{k^*} \leq C_{\text{inv}}\, d\, H^{-2}.$$

Using standard estimates we can bound H^{-2} from above by a multiple of $k^{2/d}$ and derive (4.3) by an application of Weyl's law (2.3). □

Since the inverse inequality [13] applies only to a subspace $\tilde{S}_{h,p} \subseteq S_{h,p}$, the upper bound in Theorem 3 does not hold for all discrete eigenvalues if $p > 1$. We show, that there exists splines in $S_{h,p} \backslash \tilde{S}_{h,p}$ for which the Rayleigh quotient behaves like the square of the polynomial degree p.

Lemma 2 *For every $p \in \mathbb{N}$, there exists a spline function $\hat{w}_p \in S_{h,p}$ such that*

$$\lim_{p \to \infty} \frac{1}{p^2} R(\hat{w}_p) = \frac{d}{2h^2}. \tag{4.5}$$

Proof It suffices to consider the case $d = 1$. Let $\hat{w}_p \colon (0,1) \to \mathbb{R}$ be the spline function given by

$$\hat{w}_p(x) = \begin{cases} 2(1-\frac{x}{2h})^p - 2(1-\frac{x}{h})^p, & \text{in } [0,h), \\ 2(1-\frac{x}{2h})^p, & \text{in } [h,2h), \\ 0, & \text{otherwise.} \end{cases}$$

In fact, $\hat{w}_p$ is the unique B-spline basis function of degree p having support in the first two elements. The idea behind this choice is to exploit the steep slope of $\hat{w}_p$ near zero. An iterative application of the integration by parts formula yields explicit expressions for the norms of $\hat{w}_p$:

$$\|\hat{w}_p\|^2_{L^2(\Omega)} = \frac{12h}{2p+1} - \frac{8h}{p+1}A_p, \quad \text{and} \quad |\hat{w}_p|^2_{H^1(\Omega)} = \frac{6p^2}{(2p-1)h} - \frac{4p}{h}A_{p-1}.$$

Here, A_p is a perturbed partial sum of the geometric series

$$A_p = \sum_{j=0}^{p} \alpha_{p,j}\,(-2)^{-j} \quad \text{with} \quad \alpha_{p,j} := \frac{p!(p+1)!}{(p-j)!(p+j+1)!}.$$

Comparing the limit $\lim_{p\to\infty} A_p$ with the alternating geometric series $\sum_{j=0}^{\infty}(-2)^{-j}$ it can be shown that $\lim_{p\to\infty} A_p = 2/3$. The statement of Lemma 2 now follows from the expressions of the norms $\|\hat{w}_p\|_{L^2(\Omega)}$ and $|\hat{w}_p|_{H^1(\Omega)}$. □

Theorem 4 *Let $K_p := \dim S_{h,p}$. There exists a constant $C_4 > 0$ such that*

$$\liminf_{p\to\infty} \frac{\hat{\lambda}_{K_p}}{\lambda_{K_p}} \geq \frac{C_4}{h^2}. \tag{4.6}$$

Proof This is merely a consequence of Lemma 2. The minmax characterization of $\hat{\lambda}_{K_p}$ yields

$$\frac{\hat{\lambda}_{K_p}}{\lambda_{K_p}} \geq \frac{1}{\lambda_{K_p}} R(\hat{w}_p) = \frac{K_p^{2/d}}{\lambda_{K_p}} \frac{R(\hat{w}_p)}{p^2} \frac{p^2}{K_p^{2/d}} \tag{4.7}$$

where $\hat{w}_p$ is a spline function in $S_{h,p}$ satisfying (4.5). According to Weyl's law (2.3), in the limit as $p \to \infty$, the first term converges to some constant $C_4 > 0$, while the limit of second fraction is given by Lemma 2. Finally, the last term of (4.7) converges to 1 as $K_p^{2/d} = (N+p-2)^2$. □

Remark 1 Numerical experiments indicate that the right-hand side in (4.6) may not be optimal (cf. Fig. 2) and that the eigenvalue ratio $\hat{\lambda}_{K_p}/\lambda_{K_p}$ diverges with rate p^2 as p tends to infinity.

Remark 2 The statement of Theorem 4 is formulated with respect to the mesh size h of the mesh $\mathcal{T}_h$. Note that in this case the dimension of the spline space $S_{h,p}$ is significantly smaller than the dimension of the corresponding finite element space $V_{h,p}$, in particular for large p. If the grid for $S_{h,p}$ is refined in such a way that the spline space has approximately the same dimension as $V_{h,p}$ (still defined with respect to the original mesh), then it is easy to show that $\hat{\lambda}_{K_p}/\lambda_{K_p}$ diverges with the same rate as for the finite element case, i.e., with rate p^2.

5 Numerical illustration

This section illustrates the previous analytical results in a two-dimensional model situation. We consider the unit square $\Omega = (0, 1)^2$ equipped with a fixed uniform rectangular grid $\mathcal{T}_h$ consisting of $N = h^{-1}$ elements in each coordinate direction (see Fig. 3).

The discrete eigenvalue problem is solved numerically using both the finite element spaces $V_{h,p}$ and the spline spaces $S_{h,p}$. In either case we first fix a grid width h and compute the discrete eigenvalue spectra for polynomial degrees $p = 1, 2, \ldots, 5$.

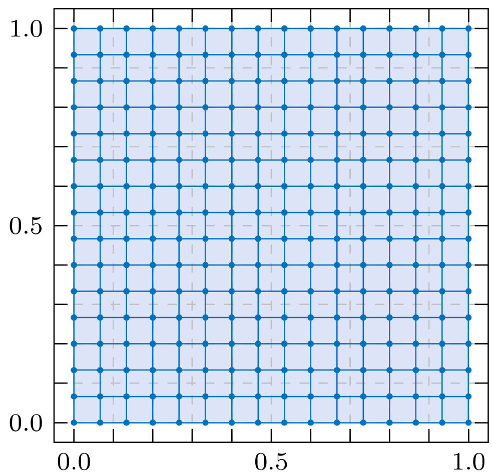

Fig. 3 Uniform rectangular grid $\mathcal{T}_h$ on $\Omega = (0, 1)^2$ with $N = h^{-1} = 15$

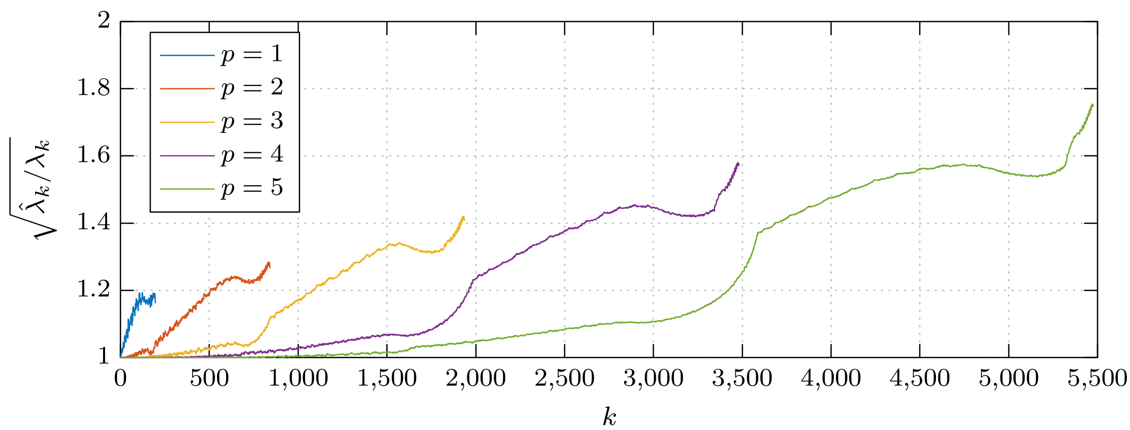

Fig. 4 Frequency ratios $\hat{\lambda}_k/\lambda_k$ for the Laplace eigenvalue problem on the unit square with Dirichlet boundary conditions computed with finite element functions of degree p on a uniform rectangular grid consisting of 15×15 elements

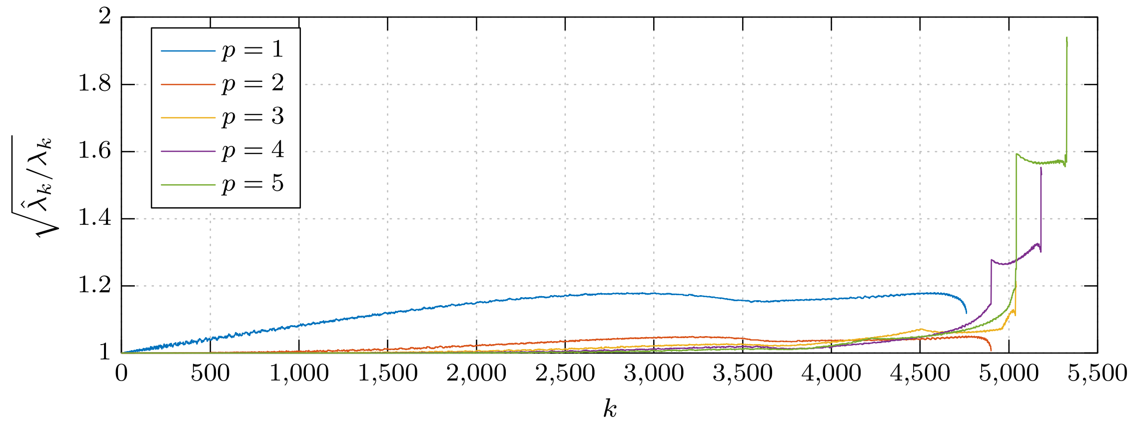

Fig. 5 Frequency ratios $\hat{\lambda}_k/\lambda_k$ for the Laplace eigenvalue problem on the unit square with Dirichlet boundary conditions computed with splines of maximum smoothness of degree p on a uniform rectangular grid consisting of 70 × 70 elements

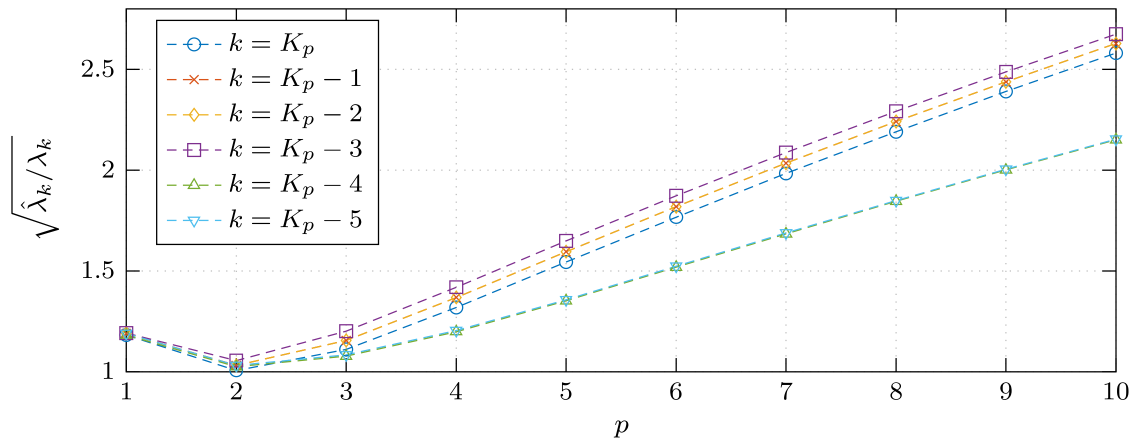

Fig. 6 Evolution of the frequency ratios corresponding to the six largest discrete eigenvalues evaluated with splines of maximum smoothness for increasing polynomial degrees p. All eigenvalues are computed on a uniform rectangular grid on the unit square consisting of 10 × 10 elements

Figure 4 depicts the resulting square roots of the eigenvalue ratios $\hat{\lambda}_k/\lambda_k$ for the computation with finite element functions on a grid consisting of $N^2 = 15^2$ elements. The numerical results illustrate the convergence of the lower part of the discrete spectrum for increasing polynomial degrees and confirm the divergence of the eigenvalue ratio associated to the largest discrete eigenvalue $\hat{\lambda}_{K_p}$ for growing polynomial degree as stated in Theorem 2.

The outcome of the analogous experiment involving the spline spaces $S_{h,p}$ are displayed in Fig. 5. In this case we used a grid of $N^2 = 70$ elements in order to obtain a comparable number of degrees of freedom. In accordance with Theorem 3 the bulk of the eigenvalue ratios is bounded while the upper part of the discrete spectrum exhibits a limited number of outlier frequencies. Note that like in the finite element setting the square root of the uppermost eigenvalue ratio $\hat{\lambda}_{K_p}/\lambda_{K_p}$ seems to increase linearly with the polynomial degree (cf. Fig. 6) which indicates that the right-hand side in (4.6) is not optimal and should exhibit some dependence on p (cf. Remark 1).

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