Multiscale Partition of Unity

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Abstract. We introduce a new Partition of Unity Method for the numerical homogenization of elliptic partial differential equations with arbitrarily rough coefficients. We do not restrict to a particular ansatz space or the existence of a finite element mesh. The method modifies a given partition of unity such that optimal convergence is achieved independent of oscillation or discontinuities of the diffusion coefficient. The modification is based on an orthogonal decomposition of the solution space while preserving the partition of unity property. This precomputation involves the solution of independent problems on local subdomains of selectable size. We deduce quantitative error estimates for the method that account for the chosen amount of localization. Numerical experiments illustrate the high approximation properties even for 'cheap' parameter choices.

Keywords. partition of unity method, multiscale method, LOD, upscaling, homogenization.

1 Introduction

In this paper, we present a novel Multiscale Partition of Unity Method for reliable numerical homogenization in the meshfree context.

The Partition of Unity Method (PUM) was introduced by Babuška and Melenk in [6, 29], with the motivation that known singularities of the solution of a given PDE can be embedded into the ansatz space. Examples of Partition of Unity Methods can be found in [11, 16, 17, 21, 25, 31, 36]. Specific realizations of methods that fit into the general PUM framework but which are formulated in the context of finite element methods are the Extended Finite Element Method (XFEM, cf. [7, 30]), the Generalized Finite Element Method (GFEM, cf. [9, 10, 12, 24, 34, 35]) and the Stable GFEM presented in [18]. More general surveys on XFEM and GFEM can be found in [3, 14, 33].

In contrast to local singularities (usually due to the shape of the domain), multiscale problems consider the issue of very rough coefficients all over the domain. In order to obtain a reliable numerical approximation to the solution of the multiscale problem, it is typically necessary to 'resolve the coefficient', whereas a simple local averaging of the coefficient leads to wrong approximations. This

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means that the discrete solution space in which we seek an adequate Galerkin approximation must be able to fully capture the fine structures of the coefficient. Practically, this often leads to very large spaces and therefore to tremendous computational efforts. One approach to overcome this difficulty is to construct a special low dimensional space that incorporates the relevant fine scale features in its basis functions and that exhibits high approximation properties. A locally supported basis of this space can be computed in parallel by solving fine scale problems in small patches. This approach has been studied extensively for Finite Elements in [19, 20, 28, 27].

Other numerical multiscale methods can be found in [1, 4, 13, 15, 22, 23, 26]. In the context of meshfree methods we refer to recent papers [5, 32] where elliptic problems with rough coefficients are treated by introducing special non-polynomial shape functions, i.e., local eigenfunctions in [5] and rough polyharmonic splines in [32].

This paper aims to generalize the mesh-based approach of [19, 20, 28, 27] to general ansatz spaces without the requirement of underlying finite element meshes.

Throughout the paper, our model problem consists of finding a stationary heat distribution in some heterogenous media. Let $A \in L^\infty(\Omega, \mathbb{R}^{d\times d}_{sym})$ be a symmetric coefficient with uniform spectral bounds $\beta \geq \alpha > 0$ in some bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ for $d = 1, 2, 3$, i.e.,

$$0 < \alpha := \operatorname*{ess\,inf}_{x\in\Omega} \inf_{v\in\mathbb{R}^d\setminus\{0\}} \frac{(A(x)v, v)}{(v, v)},$$
$$\infty > \beta := \operatorname*{ess\,sup}_{x\in\Omega} \sup_{v\in\mathbb{R}^d\setminus\{0\}} \frac{(A(x)v, v)}{(v, v)}.$$

This coefficient A may be strongly heterogenous and arbitrarily rough. We consider the prototypical second-order linear elliptic PDE

$$-\operatorname{div} A\nabla u = g \tag{1a}$$

with homogeneous Neumann boundary condition

$$A\nabla u \cdot \nu = 0 \quad \text{on } \partial\Omega, \tag{1b}$$

given the exterior normal vector ν on $\partial\Omega$ and compatible right-hand side $g \in L^2(\Omega)$ such that

$$\int_\Omega g \,\mathrm{d}x = 0.$$

We are looking for the unique (up to a constant) weak solution of problem (1a–b). This is, for $V := H^1(\Omega)$, find $u \in V/\mathbb{R} = \{v \in V \mid \int_\Omega v \,\mathrm{d}x = 0\}$ with

$$a(u, \phi) := \int_\Omega A\nabla u \cdot \nabla\phi \,\mathrm{d}x = \int_\Omega g\phi \,\mathrm{d}x \quad \text{for all } \phi \in V/\mathbb{R}. \tag{2}$$

2 Abstract Multiscale Partition of Unity

In this section, we propose a Multiscale Partition of Unity Method without restriction to a particular ansatz space or even the existence of a mesh. This method is built upon two abstract (and possibly equal) partitions of unity that will be introduced in Section 2.1. Another crucial tool for the design of the method and its error analysis is a quasi-interpolation operator presented in Section 2.2. In the third and last subsection, we finally define the novel multiscale partition of unity method based on a localized orthogonal decomposition of V.

2.1 Two Partitions of Unity

The subsequent derivation of the multiscale method is based upon two standard partitions of unity. One partition is regular and spans a coarse space V_c. The other partition may be discontinuous and is solely used for the localization of the corrector problems in Section 2.3.

Definition 1 (Partitions of Unity)**.**

(PU 1) Let $\mathcal{J}$ denote a finite index set and $\{\varphi_j \mid j \in \mathcal{J}\}$ a linearly independent Lipschitz partition of unity on Ω, i.e.

$$\sum_{j\in\mathcal{J}} \varphi_j = 1 \quad \text{with} \quad \forall\, j \in \mathcal{J}:\ \ 0 \le \varphi_j \in W^{1,\infty}(\Omega),$$

$$\text{s.t. for any } \lambda \in \mathbb{R}^{\mathcal{J}},\ \ \sum_{j\in\mathcal{J}} \lambda_j \varphi_j = 0 \quad \Leftrightarrow \quad \forall\, j \in \mathcal{J}:\ \ \lambda_j = 0.$$

We define $\omega_j := \operatorname{supp}(\varphi_j)$ and $H_j := \operatorname{diam}(\omega_j)$ for all $j \in \mathcal{J}$ and $H := \max_{j\in\mathcal{J}} H_j$. The partition of unity functions span a finite dimensional coarse space $V_c := \operatorname{span}\{\varphi_j \mid j \in \mathcal{J}\}$.

(PU 2) Let $\hat{\mathcal{J}}$ denote a finite index set and $\{\hat\varphi_{\hat\jmath} \mid \hat\jmath \in \hat{\mathcal{J}}\} \subseteq L^\infty(\Omega)$ a bounded and positive partition of unity on Ω, i.e.

$$\sum_{\hat\jmath\in\hat{\mathcal{J}}} \hat\varphi_{\hat\jmath} = 1 \quad \text{on}\ \ \Omega \quad \text{and} \quad \hat\varphi_{\hat\jmath} \ge 0.$$

We define $\hat\omega_{\hat\jmath} := \operatorname{supp}(\hat\varphi_{\hat\jmath})$ and $\hat H_{\hat\jmath} := \operatorname{diam}(\hat\omega_{\hat\jmath})$ for all $\hat\jmath \in \hat{\mathcal{J}}$. The maximum over all $\hat H_{\hat\jmath}$ is denoted by $\hat H$.

Example 1. The abstract definitions of (PU 1) and (PU 2) include the following special cases.

a) (PU 2) equals (PU 1).

b) Given some regular simplicial mesh $\mathcal{T}$ with vertices $\mathcal{N} = \mathcal{J}$, the partition (PU 1) is the continuous piecewise affine nodal basis functions φ_z, associated with vertices $z \in \mathcal{N}$. Recall that φ_z is defined by its values $\varphi_z(y) = \begin{cases} 1 & \text{if } y=z \\ 0 & \text{else} \end{cases}$ for vertices $y \in \mathcal{N}$. (PU 2) may be chosen as the characteristic (or 'indicator') functions of the triangles, i.e.

$$\hat{\mathcal{J}} = \mathcal{T} \quad \text{and} \quad \hat\varphi_T = \chi_T \text{ for all } T \in \mathcal{T}.$$

Definition 2 (extension patch)**.** For any patch ω_j in (PU 1) and $k \in \mathbb{N}$, we define the *k-th order extension patch* ω_j^k by

$$\omega_j^k := \bigcup_{x\in\omega_j} \overline{B_{k\cdot H}(x)} = \left\{x \in \overline{\Omega} \mid \operatorname{dist}(x, \omega_j) \le k \cdot H\right\}.$$

where $B_{k\cdot H}(x)$ denotes the ball with radius $k \cdot H$ around x and where "dist" denotes the set distance

$$\operatorname{dist}(x, B) := \inf_{b\in B} \|x - b\|\,.$$

For (PU 2), the extension patches $\hat\omega_{\hat\jmath}^k$, $k \in \mathbb{N}$ are defined analogously.

The subsequent definition serves only for the proofs. It has no practical relevance for the proposed method.

Definition 3 (quasi-inclusion)**.** Given two sets $B, C \subseteq \overline{\Omega}$, the set B is *n-quasi-included* in C (shorthand notation: $B \overset{n}{\subsetsim} C$) if

$$\forall\, j_1, \ldots, j_m \in \mathcal{J},\;\; k_1, \ldots, k_m \in \mathbb{N}: \;\; C \subseteq \bigcup_{i=1}^{m} \omega_{j_i}^{k_i} \;\;\Rightarrow\;\; B \subseteq \bigcup_{i=1}^{m} \omega_{j_i}^{k_i+n}.$$

Note that the shorthand notation allows for quantified transitivity

$$B \overset{n_1}{\subsetsim} C \overset{n_2}{\subsetsim} D \quad \Rightarrow \quad B \overset{n_1+n_2}{\subsetsim} D.$$

2.2 Abstract Quasi-Interpolation

Definition 4 (quasi-interpolation operator)**.** Throughout this paper, let $I : V \to V_{\mathrm{c}}$ denote an abstract quasi-interpolation operator which fulfills the following properties.

(I1) I is linear and continuous.

(I2) $I|_{V_{\mathrm{c}}} : V_{\mathrm{c}} \to V_{\mathrm{c}}$ is an isomorphism with H^1-stable inverse.

(I3) There exists a constant C_1 only depending on Ω and the shape of the patches ω_j such that for all $u \in H^1(\Omega)$ and all $j \in \mathcal{J}$

$$\|u - I(u)\|_{L^2(\omega_j)} \le C_1 H_j \|\nabla u\|_{L^2(\omega_j^1)},$$

and a constant C_2 that further depends on $\max_{j\in\mathcal{J}}(H_j \left\|\varphi_j\right\|_{W^{1,\infty}(\Omega)})$ such that

$$\|\nabla I(u)\|_{L^2(\omega_j)} \le C_2 \|\nabla u\|_{L^2(\omega_j^1)}.$$

(I4) There exists a constant C_3 with same dependencies as C_2 and some $\kappa \in \mathbb{N}$ depending on the overlapping of the supports $\{\omega_j\}_{j\in\mathcal{J}}$ such that for all $v_{\mathrm{c}} \in V_{\mathrm{c}}$ there exists $v \in V$ such that

$$I(v) = v_{\mathrm{c}}, \quad \|\nabla v\|_{L^2(\Omega)} \le C_3 \|\nabla v_{\mathrm{c}}\|_{L^2(\Omega)}, \;\; \text{and} \;\; \mathrm{supp}(v) \overset{\kappa}{\subsetsim} \mathrm{supp}(v_{\mathrm{c}}),$$

with the quasi-inclusion $\overset{\kappa}{\subsetsim}$ defined above.

A particular quasi-interpolation operator I is given in the subsequent definition.

Example 2 (Clement-type quasi-interpolation [8]). Define a weighted Clément-type quasi-interpolation operator

$$I : V \to V_{\mathrm{c}}, \quad v \mapsto I(v) := \sum_{j\in\mathcal{J}} v_j \varphi_j \quad \text{with } v_j := \frac{(v, \varphi_j)_{L^2(\Omega)}}{(1, \varphi_j)_{L^2(\Omega)}}.$$

This operator obviously satisfies **(I1)** and **(I2)**. The properties **(I3)** have been shown in [8] in the abstract setting of (PU 1). We verify that **(I4)** is satisfied for a particular choice of basis functions. The following result is similar to [28, Lemma 2.1].

Lemma 1. *For a given regular triangulation $\mathcal{T}$ with vertices $\mathcal{N} = \mathcal{J}$ and nodal basis functions $\{\varphi_z\}_{z\in\mathcal{N}}$ as in Example 1b), the quasi-interpolation operator from Example 2 satisfies **(I4)** with $\kappa = 1$.*

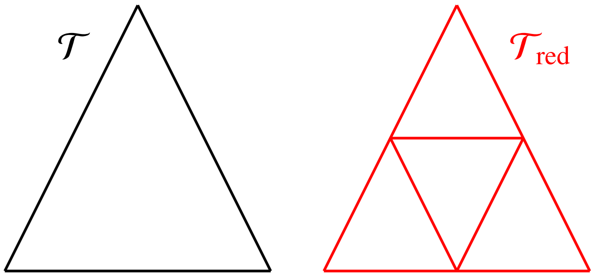

Figure 1: Sketch of a so-called *red refinement* of a single triangle in $2d$. In general, the red refinement is based on the bisection of all edges and yields at least $d+1$ simplices of same shape and half diameter.

Proof. For any basis function φ_z, we want to find $b_z \in H^1(\Omega)$ with

$$I(b_z) = \varphi_z, \quad |\nabla b_z| \leq C|\nabla \varphi_z| \text{ a.e. in } \Omega \text{ and } \quad \operatorname{supp}(b_z) \subseteq \operatorname{supp}(\varphi_z).$$

Consider the red refinement $\mathcal{T}_{\mathrm{red}}$ of $\mathcal{T}$ (cf. Figure 1) with nodal basis functions φ_z^{r}. If $\mathrm{nb}(z)$ denotes the set of all neighboring nodes of z in $\mathcal{T}_{\mathrm{red}}$ it can be verified that

$$b_z = (2^{d+1} - 1)\varphi_z^{\mathrm{r}} - \tfrac{1}{2} \sum_{y \in \mathrm{nb}(z)} \varphi_y^{\mathrm{r}}$$

satisfies the desired conditions.

To conclude the proof, set

$$w := \sum_{z \in \mathcal{N}} (v_{\mathrm{c}}(z) - I(v_{\mathrm{c}})(z))\, b_z$$

and observe that $v := v_{\mathrm{c}} + w$ satisfies $v_{\mathrm{c}} = I(v)$, with $\operatorname{supp}(v) \overset{1}{\underset{\sim}{\subset}} \operatorname{supp}(v_{\mathrm{c}})$ and

$$\|\nabla v\| \leq (1 + C + C_2 C)\, \|\nabla v_{\mathrm{c}}\|\,.$$

□

2.3 Definition of the method

The goal is the construction of a space V_{c}^m that is of the same dimension as the discrete coarse space $V_{\mathrm{c}} = \operatorname{span}\{\varphi_j \mid j \in \mathcal{J}\}$ (cf. Definition 1) but which exhibits high H^1-approximations that are inherited from the L^2-approximation properties of V_{c}. Furthermore, we wish to explicitly construct a partition of unity basis for V_{c}^m.

Under the conditions **(I1)** and **(I2)** on the abstract quasi-interpolation operator, the space V can be written as the direct sum

$$V = V_{\mathrm{c}} \oplus V_{\mathrm{f}}, \quad \text{with } \; V_{\mathrm{f}} := \{v \in V \mid I(v) = 0\}. \tag{3}$$

The subspace V_{f} contains the fine scale features in V that cannot be captured by the coarse space V_{c}.

Definition 5 (corrector)**.** For $\hat{\jmath} \in \hat{\mathcal{J}}$ and $m \in \mathbb{N}$, define the *local corrector* $Q_{\hat{\jmath}}^m : V_\mathrm{c} \to V_\mathrm{f}(\hat{\omega}_{\hat{\jmath}}^m)$ as the mapping of a given $v_\mathrm{c} \in V_\mathrm{c}$ onto the solution $Q_{\hat{\jmath}}^m(v_\mathrm{c}) \in V_\mathrm{f}(\hat{\omega}_{\hat{\jmath}}^m) := \{v \in V_\mathrm{f} \mid v = 0 \text{ in } \Omega \setminus \hat{\omega}_{\hat{\jmath}}^m\}$ of

$$\int_{\hat{\omega}_{\hat{\jmath}}^m} A\nabla Q_{\hat{\jmath}}^m(v_\mathrm{c}) \cdot \nabla w \,\mathrm{d}x = -\int_{\hat{\omega}_{\hat{\jmath}}} \hat{\varphi}_{\hat{\jmath}} A\nabla v_\mathrm{c} \cdot \nabla w \,\mathrm{d}x \qquad \text{for all } w \in V_\mathrm{f}(\hat{\omega}_{\hat{\jmath}}^m). \tag{4}$$

The global corrector is given by

$$Q^m(v_\mathrm{c}) := \sum_{\hat{\jmath} \in \hat{\mathcal{J}}} Q_{\hat{\jmath}}^m(v_\mathrm{c}).$$

For sufficiently large m such that $\hat{\omega}_{\hat{\jmath}}^m = \overline{\Omega}$ for all $\hat{\jmath} \in \hat{\mathcal{J}}$, we call $Q^\Omega := Q^m$ the *ideal corrector*.

The parameter m in Definition 5 reflects the locality of the method. The computational cost grows polynomially with m, while the error decays exponentially towards the error of the ideal (not localized) method.

Observe that the corrector problem (4) always yields a unique solution. Existence is clear by the Lax-Milgram theorem, because the zero function is the only constant function in V_f. For any $m \in \mathbb{N}$, the operator Q^m is linear and we denote the corrected discrete space

$$V_\mathrm{c}^m := \{v_\mathrm{c} + Q^m(v_\mathrm{c}) \mid v_\mathrm{c} \in V_\mathrm{c}\}, \quad V_\mathrm{c}^\Omega := \{v_\mathrm{c} + Q^\Omega(v_\mathrm{c}) \mid v_\mathrm{c} \in V_\mathrm{c}\}. \tag{5}$$

Note that V_c^m (and also V_c^Ω) satisfies

$$V = V_\mathrm{c}^m \oplus V_\mathrm{f}$$

and that $\{\varphi_j + Q^m(\varphi_j) \mid j \in \mathcal{J}\}$ is a basis of V_c^m. Moreover, the ideal method comes with a-orthogonality of V_c onto V_f, i.e.

$$a(V_\mathrm{c}^\Omega, V_\mathrm{f}) = 0. \tag{6}$$

Remark 1. The partition of unity property is preserved under correction. To prove this, it suffices to show $\sum_{j\in\mathcal{J}} Q^m(\varphi_j) = 0$. We compute

$$\sum_{j\in\mathcal{J}} Q^m(\varphi_j) = \sum_{j\in\mathcal{J}} \sum_{\hat{\jmath}\in\hat{\mathcal{J}}} Q_{\hat{\jmath}}^m(\varphi_j) = \sum_{\hat{\jmath}\in\hat{\mathcal{J}}} Q_{\hat{\jmath}}^m\Big(\sum_{j\in\mathcal{J}} \varphi_j\Big) = \sum_{\hat{\jmath}\in\hat{\mathcal{J}}} Q_{\hat{\jmath}}^m(1) = 0.$$

Hence $\{\varphi_j + Q^m(\varphi_j) \mid j \in \mathcal{J}\}$ is a partition of unity. This also holds for the ideal corrector Q^Ω.

The Galerkin discretization of (2) with respect to the corrected space V_c^m, $m \in \mathbb{N}$ reads as follows.

Definition 6 (Multiscale Partition of Unity Method)**.** Find $u_\mathrm{c}^m \in V_\mathrm{c}^m/\mathbb{R}$ such that

$$\int_\Omega A\nabla u_\mathrm{c}^m \cdot \nabla v_\mathrm{c} \,\mathrm{d}x = \int_\Omega g v_\mathrm{c} \,\mathrm{d}x \quad \text{for all } v_\mathrm{c} \in V_\mathrm{c}^m/\mathbb{R}. \tag{7}$$

The *ideal problem* seeks $u_\mathrm{c}^\Omega \in V_\mathrm{c}^\Omega/\mathbb{R}$ such that

$$\int_\Omega A\nabla u_\mathrm{c}^\Omega \cdot \nabla v_\mathrm{c} \,\mathrm{d}x = \int_\Omega g v_\mathrm{c} \,\mathrm{d}x \quad \text{for all } v_\mathrm{c} \in V_\mathrm{c}^\Omega/\mathbb{R}. \tag{8}$$

3 A priori error analysis

In this section, we prove error estimates for the discrete solution of (7). In the first subsection, we consider the ideal case with ansatz space V_{c}^{Ω} (cf. (5)). The second subsection yields an error estimate for the localized problem. We will use the notation "$a \lesssim b$" to state the existence of $C > 0$ such that $a \leq Cb$. The hidden constant C may depend on the Poincaré constant $C_{\mathrm{Poinc}}(\Omega)$, on the ratio $\hat{H}/H$, on the constants C_1, C_2, C_3 and κ from **(I1)**–**(I4)** in Definition 4, and on the operator norms of I and $(I|_{V_{\mathrm{c}}})^{-1}$ that result from **(I1)** and **(I2)**. The hidden constant does *not* depend on the data A and g, the spectral bounds α and β (in particular the contrast $\frac{\beta}{\alpha}$) or the patch sizes H and $\hat{H}$.

3.1 Error estimate for global basis functions

We consider the ideal (but expensive) case of no localization (i.e. $\hat{\omega}_{\hat{j}}^{m} = \overline{\Omega}$) and observe that the proposed method inherits the optimal approximation properties. This estimate is also important in the analysis of the localized method in Section 3.2.

Theorem 1 (A priori error estimate for the ideal case)**.** *Let u be the solution of* (2)*. Then the discrete solution u_{c}^{Ω} of* (8) *satisfies*

$$\alpha^{1/2}\left\|\nabla(u_{\mathrm{c}}^{\Omega} - u)\right\|_{L^2(\Omega)} \leq \left\|A^{1/2}\nabla(u_{\mathrm{c}}^{\Omega} - u)\right\|_{L^2(\Omega)} \lesssim \alpha^{-1/2}H\|g\|_{L^2(\Omega)}.$$

Proof. Observe that we can replace the test function space $V_{\mathrm{c}}^{\Omega}/\mathbb{R}$ by V_{c}^{Ω}, since we subsequently only consider gradients. Galerkin orthogonality, i.e.

$$a\left(u - u_{\mathrm{c}}^{\Omega}, v_{\mathrm{c}}\right) = 0 \quad \text{for all } v_{\mathrm{c}} \in V_{\mathrm{c}}^{\Omega} \tag{9}$$

and (6) imply that $e := u - u_{\mathrm{c}}^{\Omega} \in V_{\mathrm{f}}$ and therefore $I(e) = 0$. We get

$$\begin{aligned}
\|A^{1/2}\nabla e\|_{L^2(\Omega)}^2 &= a(e,e) \\
&\overset{(9)}{=} a(e,u) = a(u,e) \\
&= \int_{\Omega} g(e - I(e))\,\mathrm{d}x \\
&\overset{\textbf{(I3)}}{\lesssim} H\|g\|_{L^2(\Omega)}\|\nabla e\|_{L^2(\Omega)} \\
&\leq \alpha^{-1/2}H\|g\|_{L^2(\Omega)}\|A^{1/2}\nabla e\|_{L^2(\Omega)}.
\end{aligned}$$

□

3.2 Error estimate for local basis functions

In this final subsection, we give error estimates for the localized method. The main result is presented below.

Theorem 2 (A priori error estimates for the localized method)**.** *Assume that $u \in V$ solves* (2)*, then the discrete solution $u_{\mathrm{c}}^{m} \in V_{\mathrm{c}}^{m}$ of* (7) *satisfies*

$$\begin{aligned}
\|\nabla u - \nabla u_{\mathrm{c}}^{m}\|_{L^2(\Omega)} &\lesssim \alpha^{-1}(H + \tfrac{\beta}{\alpha}m^{d/2}\tilde{\theta}^{m})\|g\|_{L^2(\Omega)}, \\
\|u - u_{\mathrm{c}}^{m}\|_{L^2(\Omega)} &\lesssim \alpha^{-2}(H + \tfrac{\beta}{\alpha}m^{d/2}\tilde{\theta}^{m})^2\|g\|_{L^2(\Omega)},
\end{aligned}$$

with some generic constant $0 < \tilde{\theta} = \theta^{\lceil \hat{H}/H \rceil} < 1$ and θ depending on the contrast $\frac{\beta}{\alpha}$ (cf. Lemma 4 and 5 below).

Proof. Let u_c^Ω be the solution of the ideal problem with correction operator Q^Ω, and $u_c \in V_c/\mathbb{R}$ such that $u_c^\Omega = u_c + Q^\Omega u_c$. As a consequence of **(I3)**, all functions $v \in V_f$ satisfy $\int_\Omega v \,\mathrm{d}x = 0$. With $Q^m v_c \in V_f$, we get

$$
\begin{aligned}
\|\nabla u - \nabla u_c^m\|_{L^2(\Omega)} &\lesssim \min_{v_c^m \in V_c^m/\mathbb{R}} \left\|\nabla u - \nabla v_c^m\right\|_{L^2(\Omega)} \\
&\leq \left\|\nabla u - \nabla(u_c + Q^m u_c)\right\|_{L^2(\Omega)} \\
&\leq \left\|\nabla u - \nabla u_c^\Omega\right\|_{L^2(\Omega)} + \left\|\nabla Q^\Omega u_c - \nabla Q^m u_c\right\|_{L^2(\Omega)}.
\end{aligned}
$$

Lemma 5 will quantify the localization error

$$
\left\|\nabla Q^\Omega u_c - \nabla Q^m u_c\right\|_{L^2(\Omega)} \lesssim \tfrac{\beta}{\alpha} m^{d/2} \tilde{\theta}^m \Big(\sum_{\hat{\jmath} \in \hat{\mathcal{J}}} \|\nabla Q_{\hat{\jmath}}^\Omega u_c\|_{L^2(\Omega)}^2\Big)^{1/2}.
$$

This, Theorem 1 and the estimates

$$
\begin{aligned}
\sum_{\hat{\jmath} \in \hat{\mathcal{J}}} \|\nabla Q_{\hat{\jmath}}^\Omega u_c\|_{L^2(\Omega)}^2 &\overset{(4)}{\lesssim} \sum_{\hat{\jmath} \in \hat{\mathcal{J}}} \|\hat{\varphi}_{\hat{\jmath}} \nabla u_c\|_{L^2(\Omega)}^2 \leq \|\nabla u_c\|_{L^2(\Omega)}^2 \\
&= \left\|\nabla (I|_{V_c})^{-1} I(u_c^\Omega)\right\|_{L^2(\Omega)} \overset{\textbf{(I1)},\textbf{(I2)}}{\lesssim} \|\nabla u_c^\Omega\|_{L^2(\Omega)}^2 \leq \alpha^{-1} C_{\mathrm{Poinc}}(\Omega) \|g\|_{L^2(\Omega)}^2
\end{aligned}
$$

yield the H^1-error estimate. The L^2-error estimate is obtained by a standard Aubin-Nitsche argument. □

To prove Lemma 5, several tools are needed in addition to the preceding results. They will be discussed below.

Lemma 2 (quasi-inclusion of intersecting patches)**.** *Let $i, j \in \mathcal{J}$ and $\ell, k, m \in \mathbb{N}$ with $k \geq \ell \geq 2$. Then*

$$
\textit{if} \quad \omega_i^m \cap \left(\omega_j^k \setminus \omega_j^\ell\right) \neq \emptyset \quad \textit{then} \quad \omega_i \subseteq \omega_j^{k+m+1} \setminus \omega_j^{\ell-m-1}.
$$

Proof. Consider $x \in \omega_i^m \cap \left(\omega_j^k \setminus \omega_j^\ell\right)$ and observe

$$
\omega_i \;\subseteq\; \overline{B_{(m+1)H}(x)} \;\subseteq\; \omega_j^{k+m+1} \setminus \omega_j^{\ell-m-1}.
$$

□

Definition 7 (cut-off functions)**.** For all $j \in \mathcal{J}$ and $\ell, k \in \mathbb{N}$ with $k > \ell$, we define the *cut-off function*

$$
\eta_j^{k,\ell}(x) = \frac{\operatorname{dist}(x, \omega_j^{k-\ell})}{\operatorname{dist}(x, \omega_j^{k-\ell}) + \operatorname{dist}(x, \Omega \setminus \omega_j^k)}.
$$

For $\Omega \setminus \omega_j^k = \emptyset$, we set $\eta_j^{k,\ell} \equiv 0$. Note that $\eta_j^{k,\ell} = 0$ in $\omega_j^{k-\ell}$ and $\eta_j^{k,\ell} = 1$ in $\Omega \setminus \omega_j^k$. Moreover, $\eta_j^{k,\ell}$ is bounded between 0 and 1 and Lipschitz continuous with

$$
\left\|\nabla \eta_j^{k,\ell}\right\|_{L^\infty(\Omega)} \leq \frac{1}{\ell H}. \tag{10}
$$

See [2, Theorem 8.5] for existence and boundedness of the weak derivative of Lipschitz-continuous functions.

Remark 2. The Lipschitz bound is shown as follows. For $x \in \mathbb{R}^d$ we have the triangle inequality

$$\operatorname{dist}(x, \omega_j^{k-\ell}) + \operatorname{dist}(x, \Omega \setminus \omega_j^k) \;\; \geq \;\; \operatorname{dist}(\omega_j^{k-\ell}, \Omega \setminus \omega_j^k) \;\; = \;\; \ell\, H.$$

Moreover, any nonemtpy set B in a metric space satisfies Lipschitz continuity of the distance function $\operatorname{dist}(\,\cdot\,, B)$ in the sense

$$|\operatorname{dist}(x, B) - \operatorname{dist}(y, B)| \;\; \leq \;\; \operatorname{dist}(x, y) \quad \text{for } x, y \in \mathbb{R}^d.$$

Altogether,

$$\begin{aligned} \frac{\left|\eta_j^{k,\ell}(x) - \eta_j^{k,\ell}(y)\right|}{\operatorname{dist}(x, y)} &\leq \frac{1}{\operatorname{dist}(x, y)} \cdot \frac{\left|\operatorname{dist}(x, \omega_j^{k-\ell}) - \operatorname{dist}(y, \omega_j^{k-\ell})\right|}{\ell\, H} \\ &\leq \frac{1}{\ell\, H}. \end{aligned}$$

A technical issue in our error analysis is that V_{f} is not invariant under multiplication by such cut-off functions. However, the product $\eta_j^{k,\ell} w$ for $w \in V_{\mathrm{f}}$ is close to V_{f} in the following sense.

Lemma 3 (quasi-invariance of V_{f} under multiplication by cut-off functions)**.** *Recall κ from **(I4)**. For any given $w \in V_{\mathrm{f}}$ and cutoff function $\eta_j^{k,\ell}$ with $k > \ell > 0$, there exists $\tilde{w} \in V_{\mathrm{f}}(\Omega \setminus \omega_j^{k-\ell-\kappa-2}) \subseteq V_{\mathrm{f}}$ such that*

$$\|\nabla(\eta_j^{k,\ell} w - \tilde{w})\|_{L^2(\Omega)} \lesssim \ell^{-1} \|\nabla w\|_{L^2(\omega_j^{k+2} \setminus \omega_j^{k-\ell-2})}.$$

Proof. We fix the $j \in \mathcal{J}$ and $k \in \mathbb{N}$ and denote $\eta_\ell := \eta_j^{k,\ell}$ and $c_i^\ell := \frac{1}{|\omega_i^1|} \int_{\omega_i^1} \eta_\ell \,\mathrm{d}x$ for $i \in \mathcal{J}$. The property **(I4)**, applied to $I(\eta_\ell w) \in V_{\mathrm{f}}$, yields $v \in V$ with

$$\begin{aligned} & I(v) = I(\eta_\ell w), \;\; \|\nabla v\|_{L^2(\Omega)} \lesssim \|\nabla I \eta_\ell w\|_{L^2(\Omega)}, \qquad (11)\\ \text{and} \quad & \operatorname{supp}(v) \overset{\kappa}{\underset{\sim}{\subset}} \operatorname{supp}(I(\eta_\ell w)) \overset{1}{\underset{\sim}{\subset}} \operatorname{supp}(\eta_\ell w) \subseteq \Omega \setminus \omega_j^{k-\ell}, \\ \text{which yields} \quad & \operatorname{supp}(v) \overset{\kappa+1}{\underset{\sim}{\subset}} \Omega \setminus \omega_j^{k-\ell} \Rightarrow \operatorname{supp}(v) \subseteq \Omega \setminus \omega_j^{k-\ell-\kappa-2}. \qquad (12) \end{aligned}$$

Note that $\operatorname{supp}(I(\eta_\ell w)) \overset{1}{\underset{\sim}{\subset}} \operatorname{supp}(\eta_\ell w)$ is a consequence of **(I3)**, and that (11) implies $I(v - \eta_\ell w) = 0$.

We define $\tilde{w} := \eta_\ell w - v \in V_{\mathrm{f}}(\Omega \setminus \omega_j^{k-\ell-\kappa-2})$. Using $I(w) = 0$, we obtain for any $i \in \mathcal{J}$

$$\|\nabla I(\eta_\ell w)\|_{L^2(\omega_i)} \overset{\textbf{(I1)}}{=} \left\|\nabla I((\eta_\ell - c_i^\ell) w)\right\|_{L^2(\omega_i)} \overset{\textbf{(I3)}}{\lesssim} \left\|\nabla((\eta_\ell - c_i^\ell) w)\right\|_{L^2(\omega_i^1)}. \qquad (13)$$

This gives us

$$
\begin{aligned}
\|\nabla I(\eta_\ell w)\|^2_{L^2(\Omega)} &\le \sum_{i\in\mathcal{J}} \|\nabla I(\eta_\ell w)\|^2_{L^2(\omega_i)} \\
&\overset{(13)}{\lesssim} \sum_{i\in\mathcal{J}} \big\|\nabla((\eta_\ell - c_i^\ell)w)\big\|^2_{L^2(\omega_i^1)} \\
&= \sum_{\substack{i\in\mathcal{J}:\\ \omega_i^1\cap(\omega_j^k\setminus\omega_j^{k-\ell})\neq\emptyset}} \big\|\nabla((\eta_\ell - c_i^\ell)w)\big\|^2_{L^2(\omega_i^1)} \\
&\overset{(2)}{\le} \sum_{\substack{i\in\mathcal{J}:\\ \omega_i\subseteq\omega_j^{k+2}\setminus\omega_j^{k-\ell-2}}} \big\|\nabla((\eta_\ell - c_i^\ell)w)\big\|^2_{L^2(\omega_i^1)} \\
&\lesssim \sum_{\substack{i\in\mathcal{J}:\\ \omega_i\subseteq\omega_j^{k+2}\setminus\omega_j^{k-\ell-2}}} \big\|(\nabla\eta_\ell)(w - Iw)\big\|^2_{L^2(\omega_i^1)} + \big\|(\eta_\ell - c_i^\ell)\nabla w\big\|^2_{L^2(\omega_i^1)}.
\end{aligned}
$$

Since $\nabla\eta_\ell \neq 0$ only in $\omega_j^k \setminus \omega_j^{k-\ell}$ and $(\eta_\ell - c_i^\ell)\big|_{\omega_i^1} \neq 0$ only if ω_i^1 intersects with $\omega_j^k \setminus \omega_j^{k-\ell}$, we have

$$
\begin{aligned}
&\lesssim \sum_{\substack{i\in\mathcal{J}:\\ \omega_i\subseteq\omega_j^{k+1}\setminus\omega_j^{k-\ell-1}}} \big\|(\nabla\eta_\ell)(w - Iw)\big\|^2_{L^2(\omega_i)} + \sum_{\substack{i\in\mathcal{J}:\\ \omega_i\subseteq\omega_j^{k+1}\setminus\omega_j^{k-\ell-1}}} \big\|(\eta_\ell - c_i^\ell)\nabla w\big\|^2_{L^2(\omega_i^1)} \\
&\lesssim H^2 \|\nabla\eta_\ell\|^2_{L^\infty(\Omega)} \|\nabla w\|^2_{L^2(\omega_j^{k+1}\setminus\omega_j^{k-\ell-1})} \\
&\qquad\qquad + \sum_{\substack{i\in\mathcal{J}:\\ \omega_i\subseteq\omega_j^{k+1}\setminus\omega_j^{k-\ell-1}}} \big\|(\eta_\ell - c_i^\ell)\nabla w\big\|^2_{L^2(\omega_i^1)} \\
&\overset{(10)}{\le} \ell^{-2} \big\|\nabla w\big\|^2_{L^2(\omega_j^{k+2}\setminus\omega_j^{k-\ell-2})},
\end{aligned} \tag{14}
$$

where we used the Lipschitz bound $\|\eta_\ell - c_i^\ell\|_{L^\infty(\omega_i^1)} \lesssim H\,\|\nabla\eta_\ell\|_{L^\infty(\omega_i^1)}$. The combination of (11) and (14) readily yields the assertion,

$$
\begin{aligned}
\|\nabla(\eta_\ell w - \tilde{w})\|^2_{L^2(\Omega)} = \|\nabla v\|^2_{L^2(\Omega)} &\overset{(11)}{\le} \|\nabla I(\eta_\ell w)\|^2_{L^2(\Omega)} \\
&\overset{(14)}{\lesssim} \ell^{-2}\,\|\nabla w\|^2_{L^2(\omega_j^{k+2}\setminus\omega_j^{k-\ell-2})}\,.
\end{aligned}
$$

□

A key result is the following.

Lemma 4 (Exponential decay in the fine scale space)**.** *Consider some fixed $j \in \mathcal{J}$ and let $F \in (V_\mathrm{f})'$ satisfy $F(w) = 0$ for all $w \in V_\mathrm{f}(\Omega \setminus \omega_j^\varrho)$ with $\varrho := \lceil \frac{\hat{H}}{H} \rceil$. Let $p \in V_\mathrm{f}$ be the solution of*

$$
a(p, w) = F(w) \qquad \text{for all } w \in V_\mathrm{f}. \tag{15}
$$

Then there exists $0 < \theta < 1$ depending on the contrast $\frac{\beta}{\alpha}$ such that for all positive $k \in \mathbb{N}$ it holds

$$
\|\nabla p\|_{L^2(\Omega\setminus\omega_j^k)} \lesssim \theta^k\,\|\nabla p\|_{L^2(\Omega)}\,.
$$

Proof. We use a cut-off function as in the previous proof and denote $\eta_\ell := \eta_j^{k,\ell}$ with $\ell \leq k - \varrho - \kappa - 2$.

Applying Lemma 3 yields the existence of $\tilde{p} \in V_{\mathrm{f}}(\Omega \backslash \omega_j^{k-\ell-\kappa-2})$ with the estimate $\|\nabla(\eta_\ell p - \tilde{p})\|_{L^2(\Omega)} \lesssim \ell^{-1} \|\nabla p\|_{L^2(\omega_j^{k+2} \backslash \omega_j^{k-\ell-2})}$. Due to the property $\tilde{p} \in V_{\mathrm{f}}(\Omega \setminus \omega_j^{k-\ell-\kappa-2})$ and the assumptions on F we also have

$$\int_{\Omega \backslash \omega_j^{k-\ell-\kappa-2}} A \nabla p \cdot \nabla \tilde{p} \, \mathrm{d}x = \int_\Omega A \nabla p \cdot \nabla \tilde{p} \, \mathrm{d}x = F(\tilde{p}) = 0. \tag{16}$$

This leads to

$$\begin{aligned} \alpha \|\nabla p\|^2_{L^2(\Omega \backslash \omega_j^k)} &\leq \int_{\Omega \backslash \omega_j^k} A \nabla p \cdot \nabla p \, \mathrm{d}x \leq \int_{\Omega \backslash \omega_j^{k-\ell-\kappa-2}} \eta_\ell A \nabla p \cdot \nabla p \, \mathrm{d}x \\ &= \int_{\Omega \backslash \omega_j^{k-\ell-\kappa-2}} A \nabla p \cdot (\nabla(\eta_\ell p) - p \nabla \eta_\ell) \, \mathrm{d}x. \end{aligned}$$

With (16) and since $p \in V_{\mathrm{f}}$, this is

$$\begin{aligned} &= \int_{\Omega \backslash \omega_j^{k-\ell-\kappa-2}} A \nabla p \cdot (\nabla(\eta_\ell p - \tilde{p}) - (p - I(p)) \nabla \eta_\ell) \, \mathrm{d}x \\ &\lesssim \ell^{-1} \beta \Big(\|\nabla p\|^2_{L^2(\Omega \backslash \omega_j^{k-\ell-\kappa-2})} \\ &\qquad + H^{-1} \|\nabla p\|_{L^2(\Omega \backslash \omega_j^{k-\ell-\kappa-2})} \|p - I(p)\|_{L^2(\Omega \backslash \omega_j^{k-\ell-\kappa-2})} \Big) \\ &\overset{\textbf{(I3)}}{\lesssim} \ell^{-1} \beta \|\nabla p\|^2_{L^2(\Omega \backslash \omega_j^{k-\ell-\kappa-2})}. \end{aligned}$$

Hence, there exists a constant C independent of mesh size, contrast, number of patch extension layers, such that

$$\|\nabla p\|^2_{L^2(\Omega \backslash \omega_j^k)} \leq C \ell^{-1} \frac{\beta}{\alpha} \|\nabla p\|^2_{L^2(\Omega \backslash \omega_j^{k-\ell-\kappa-2})}. \tag{17}$$

Choose $\ell := \lceil eC\frac{\beta}{\alpha} \rceil$ and observe that successive use of (17) yields

$$\begin{aligned} \|\nabla p\|^2_{L^2(\Omega \backslash \omega_j^k)} &\leq e^{-1} \|\nabla p\|^2_{L^2(\Omega \backslash \omega_j^{k-\ell-\kappa-2})} \\ &\leq e^{-\lfloor \frac{k-\varrho}{\ell+\kappa+2} \rfloor} \|\nabla p\|^2_{L^2(\Omega \backslash \omega_j^\varrho)} \\ &\lesssim e^{-\frac{k}{\ell+\kappa+2}} \|\nabla p\|^2_{L^2(\Omega)}. \end{aligned}$$

The choice $\theta := e^{-(\lceil eC\beta/\alpha \rceil + \kappa + 2)^{-1}}$ concludes the proof. □

Lemma 5 (localization error)**.** *For $u_{\mathrm{c}} \in V_{\mathrm{c}}$, the correction operators Q^m and Q^Ω satisfy*

$$\left\|\nabla(Q^\Omega u_{\mathrm{c}} - Q^m u_{\mathrm{c}})\right\|_{L^2(\Omega)} \lesssim \tfrac{\beta}{\alpha} m^{d/2} \tilde{\theta}^m \left\|Q^\Omega u_{\mathrm{c}}^m\right\|_{L^2(\Omega)}$$

with $\tilde{\theta} := \theta^{\lceil \hat{H}/H \rceil} < 1$ and θ from Lemma 4.

Proof. Recall the definition $Q^m u_{\mathrm{c}} := \sum_{\hat{j} \in \hat{\mathcal{J}}} Q^m_{\hat{j}}(u_{\mathrm{c}})$ with

$$\int_{\hat{\omega}^m_{\hat{j}}} A \nabla Q^m_{\hat{j}}(u_{\mathrm{c}}) \cdot \nabla w \, \mathrm{d}x = \underbrace{-\int_\Omega \hat{\varphi}_{\hat{j}} A \nabla u_{\mathrm{c}} \cdot \nabla w \, \mathrm{d}x}_{F_{\hat{j}}(w)} \quad \text{for all } w \in V_{\mathrm{f}}(\hat{\omega}^m_{\hat{j}}), \quad \hat{j} \in \hat{\mathcal{J}}.$$

Note that the right-hand side $F_{\hat{j}}$ of the local problem is zero for $w \in V_{\mathrm{f}}(\Omega \setminus \hat{\omega}_{\hat{j}})$. Consider some fixed $\hat{j} \in \hat{\mathcal{J}}$ and choose $j \in \mathcal{J}$ such that $\omega_j \cap \hat{\omega}_{\hat{j}} \neq \emptyset$. Recall $\varrho = \lceil \frac{\hat{H}}{H} \rceil$, then we have $\hat{\omega}_{\hat{j}} \subseteq \omega_j^{\varrho}$ and thus $V_{\mathrm{f}}(\Omega \setminus \omega_j^{\varrho}) \subseteq V_{\mathrm{f}}(\Omega \setminus \hat{\omega}_{\hat{j}})$. Hence $F_{\hat{j}}$ satisfies the conditions from Lemma 4.

Moreover, we get $\omega_j^k \subseteq \hat{\omega}_{\hat{j}}^m$ for k satisfying

$$m = \lceil \tfrac{k \cdot H}{\hat{H}} \rceil \leq k \lceil \tfrac{H}{\hat{H}} \rceil. \tag{18}$$

Denote $v := Q^{\Omega} u_{\mathrm{c}} - Q^m u_{\mathrm{c}} \in V_{\mathrm{f}}$ and note that $I(v) = 0$. Using the cut-off functions $\eta_j^{k,1}$ from Definition 7, we obtain

$$\alpha \left\| \nabla v \right\|_{L^2(\Omega)}^2 \leq \sum_{\hat{j} \in \hat{\mathcal{J}}} \Big(\underbrace{(A \nabla (Q_{\hat{j}}^{\Omega} u_{\mathrm{c}} - Q_{\hat{j}}^m u_{\mathrm{c}}), \nabla (v(1 - \eta_j^{k,1})))_{L^2(\Omega)}}_{\mathrm{I}} + \underbrace{(A \nabla (Q_{\hat{j}}^{\Omega} u_{\mathrm{c}} - Q_{\hat{j}}^m u_{\mathrm{c}}), \nabla (v \eta_j^{k,1}))_{L^2(\Omega)}}_{\mathrm{II}} \Big).$$

We bound the term I by

$$\begin{aligned} \mathrm{I} &\leq \beta \left\| \nabla (Q_{\hat{j}}^{\Omega} u_{\mathrm{c}} - Q_{\hat{j}}^m u_{\mathrm{c}}) \right\|_{L^2(\Omega)} \left\| \nabla (v(1 - \eta_j^{k,1})) \right\|_{L^2(\omega_j^k)} \\ &\leq \beta \left\| \nabla (Q_{\hat{j}}^{\Omega} u_{\mathrm{c}} - Q_{\hat{j}}^m u_{\mathrm{c}}) \right\|_{L^2(\Omega)} (\|\nabla v\|_{L^2(\omega_j^k)} + \left\| v \nabla (1 - \eta_j^{k,1}) \right\|_{L^2(\omega_j^k \setminus \omega_j^{k-1})}) \\ &\lesssim \beta \left\| \nabla (Q_{\hat{j}}^{\Omega} u_{\mathrm{c}} - Q_{\hat{j}}^m u_{\mathrm{c}}) \right\|_{L^2(\Omega)} (\|\nabla v\|_{L^2(\omega_j^k)} + H^{-1} \|v - I(v)\|_{L^2(\omega_j^k \setminus \omega_j^{k-1})}) \\ &\lesssim \beta \left\| \nabla (Q_{\hat{j}}^{\Omega} u_{\mathrm{c}} - Q_{\hat{j}}^m u_{\mathrm{c}}) \right\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\omega_j^{k+1})}. \end{aligned}$$

Lemma 3 yields the existence of $\tilde{v} \in V_{\mathrm{f}}(\Omega \setminus \omega_j^{k-\kappa-3})$ with

$$\left\| \nabla (v \eta_j^{k,1} - \tilde{v}) \right\|_{L^2(\Omega)} \lesssim \|\nabla v\|_{L^2(\omega_j^{k+2})}.$$

We assume that m is large enough such that $k \geq \varrho + \kappa + 3$, then $\tilde{v} \in V_{\mathrm{f}}(\Omega \setminus \hat{\omega}_{\hat{j}})$ and hence

$$\int_{\Omega} A \nabla (Q_{\hat{j}}^{\Omega} u_{\mathrm{c}} - Q_{\hat{j}}^m u_{\mathrm{c}}) \cdot \nabla \tilde{v} \, \mathrm{d}x = 0.$$

It follows that

$$\begin{aligned} \mathrm{II} &= (A \nabla (Q_{\hat{j}}^{\Omega} u_{\mathrm{c}} - Q_{\hat{j}}^m u_{\mathrm{c}}), \nabla (v \eta_j^{k,1} - \tilde{v}))_{L^2(\Omega)} \\ &\lesssim \beta \left\| \nabla (Q_{\hat{j}}^{\Omega} u_{\mathrm{c}} - Q_{\hat{j}}^m u_{\mathrm{c}}) \right\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\omega_j^{k+2})}. \end{aligned}$$

Combining the estimates for I and II finally yields

$$\begin{aligned} \left\| \nabla v \right\|_{L^2(\Omega)}^2 &\lesssim \frac{\beta}{\alpha} \sum_{\hat{j} \in \hat{\mathcal{J}}} \left\| \nabla (Q_{\hat{j}}^{\Omega} u_{\mathrm{c}} - Q_{\hat{j}}^m u_{\mathrm{c}}) \right\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\omega_j^{k+2})} \\ &\lesssim \frac{\beta}{\alpha} k^{d/2} \Big(\sum_{\hat{j} \in \hat{\mathcal{J}}} \left\| \nabla (Q_{\hat{j}}^{\Omega} u_{\mathrm{c}} - Q_{\hat{j}}^m u_{\mathrm{c}}) \right\|_{L^2(\Omega)}^2 \Big)^{1/2} \|\nabla v\|_{L^2(\Omega)}, \end{aligned} \tag{19}$$

provided that $\left| \{ i \in \mathcal{J} \mid \omega_i \subseteq \omega_j^{k+2} \} \right| \lesssim k^{d/2}$.

In order to bound $\left\|\nabla(Q_{\hat{j}}^{\Omega}u_{\mathrm{c}} - Q_{\hat{j}}^{m}u_{\mathrm{c}})\right\|_{L^2(\Omega)}^2$, we use Galerkin orthogonality for the local problems, which is

$$\left\|\nabla(Q_{\hat{j}}^{\Omega}u_{\mathrm{c}} - Q_{\hat{j}}^{m}u_{\mathrm{c}})\right\|_{L^2(\Omega)}^2 \lesssim \inf_{q\in V_{\mathrm{f}}(\omega_j^k)}\left\|\nabla(Q_{\hat{j}}^{\Omega}u_{\mathrm{c}} - q)\right\|_{L^2(\Omega)}^2. \tag{20}$$

(I4) yields the existence of $\tilde{w} \in V_{\mathrm{f}}$ such that

$$I(\tilde{w}) = I((1-\eta_j^{k,1})Q_{\hat{j}}^{\Omega}u_{\mathrm{c}}), \quad \|\nabla\tilde{w}\|_{L^2(\Omega)} \lesssim \|\nabla I((1-\eta_j^{k,1})Q_{\hat{j}}^{\Omega}u_{\mathrm{c}})\|_{L^2(\Omega)},$$
$$\text{and} \quad \operatorname{supp}(\tilde{w}) \overset{\kappa}{\subseteq} \operatorname{supp}((1-\eta_j^{k,1})Q_{\hat{j}}^{\Omega}u_{\mathrm{c}}) \subseteq \omega_j^k.$$

We observe

$$\left\|\nabla I((1-\eta_j^{k,1})Q_{\hat{j}}^{\Omega}u_{\mathrm{c}})\right\|_{L^2(\omega_j^{k+\kappa})}^2 = \left\|\nabla I((1-\eta_j^{k,1})Q_{\hat{j}}^{\Omega}u_{\mathrm{c}})\right\|_{L^2(\omega_j^{k+1}\setminus\omega_j^{k-2})}^2. \tag{21}$$

With $p_{\hat{j}} := (1-\eta_j^{k,1})Q_{\hat{j}}^{\Omega}u_{\mathrm{c}} - \tilde{w} \in V_{\mathrm{f}}(\omega_j^{k+\kappa})$, we obtain

$$\begin{aligned}
\left\|\nabla(Q_{\hat{j}}^{\Omega}u_{\mathrm{c}} - Q_{\hat{j}}^{m}u_{\mathrm{c}})\right\|_{L^2(\Omega)}^2 &\overset{(20)}{\lesssim} \left\|\nabla(\eta_j^{k,1}Q_{\hat{j}}^{\Omega}u_{\mathrm{c}} + (1-\eta_j^{k,1})Q_{\hat{j}}^{\Omega}u_{\mathrm{c}} - p_{\hat{j}})\right\|_{L^2(\Omega)}^2 \\
&= \left\|\nabla(\eta_j^{k,1}Q_{\hat{j}}^{\Omega}u_{\mathrm{c}} - \tilde{w})\right\|_{L^2(\Omega)}^2 \\
&\lesssim \|\nabla Q_{\hat{j}}^{\Omega}u_{\mathrm{c}}\|_{L^2(\Omega\setminus\omega_j^{k-2})}^2 + \|\nabla\tilde{w}\|_{L^2(\omega_j^{k+\kappa})}^2 \\
&\lesssim \|\nabla Q_{\hat{j}}^{\Omega}u_{\mathrm{c}}\|_{L^2(\Omega\setminus\omega_j^{k-2})}^2 \\
&\qquad + \|\nabla I((1-\eta_j^{k,1})Q_{\hat{j}}^{\Omega}u_{\mathrm{c}})\|_{L^2(\omega_j^{k+\kappa})}^2 \\
&\overset{(21)}{\lesssim} \|\nabla Q_{\hat{j}}^{\Omega}u_{\mathrm{c}}\|_{L^2(\Omega\setminus\omega_j^{k-2})}^2 \\
&\qquad + \|\nabla I((1-\eta_j^{k,1})Q_{\hat{j}}^{\Omega}u_{\mathrm{c}})\|_{L^2(\omega_j^{k+1}\setminus\omega_j^{k-2})}^2 \\
&\overset{\mathbf{(I3)}}{\lesssim} \|\nabla Q_{\hat{j}}^{\Omega}u_{\mathrm{c}}\|_{L^2(\Omega\setminus\omega_j^{k-3})}^2 \\
&\overset{\text{Lemma 4}}{\lesssim} \theta^{2(k-3)}\left\|\nabla Q_{\hat{j}}^{\Omega}u_{\mathrm{c}}\right\|_{L^2(\Omega)}^2 \\
&\overset{(18)}{\lesssim} \tilde{\theta}^{2m}\left\|\nabla Q_{\hat{j}}^{\Omega}u_{\mathrm{c}}\right\|_{L^2(\Omega)}^2.
\end{aligned} \tag{22}$$

Combining (19) and (22) proves the lemma. □

4 Numerical Experiment

In this section, we present numerical results for a special realization of the Multiscale Partition of Unity Method. We consider a "coarse" regular triangulation $\mathcal{T}_H$ of Ω, where H denotes the maximum diameter of an element of $\mathcal{T}_H$. By $\mathcal{N}_H$ we denote the set of vertices of the triangulation. We choose the basis functions φ_z as in Example 1b), i.e., the continuous and piecewise affine nodal basis functions associated with vertices $z \in \mathcal{N} = \mathcal{J}$. The second partition of unity (PU 2) is given by the indicator functions of the elements of the triangulation, i.e. $\{\hat{\varphi}_{\hat{j}} \mid \hat{j} \in \hat{\mathcal{J}}\} := \{\chi_T \mid T \in \mathcal{T}_H\}$. The corrector problems given by (4) are solved with a P_1 Finite Element method on a fine grid with resolution $h = 2^{-8}$. The reference solution u_h is therefore the P_1 Finite Element approximation in a space with mesh size $h = 2^{-8}$.

In order to estimate the accuracy of u_h itself, we performed a second computation for the mesh size $h = 2^{-10}$. The relative L^2-error between the Finite Element approximation on a uniform mesh with resolution $h = 2^{-8}$ and the Finite Element approximation on a uniform mesh with resolution $h = 2^{-10}$ is 0.023. The relative H^1-error is 0.3204. However, we only compute the errors of u_c^m with respect to the reference solution (i.e. for $h = 2^{-8}$), since this is the relevant error for investigating the effect of the coarse grid resolution and the decay of the multiscale basis functions on u_c^m.

The extension patches $\hat{\omega}_{\hat{j}}^m$ can be defined by using the structure of the coarse grid by setting

$$\begin{aligned} \hat{\omega}_{\hat{j}}^0 &:= T_j \in \mathcal{T}_H, \\ \hat{\omega}_{\hat{j}}^m &:= \cup\{T \in \mathcal{T}_H \mid T \cap \hat{\omega}_{\hat{j}}^{m-1} \neq \emptyset\} \quad m = 1, 2, \dots . \end{aligned} \tag{23}$$

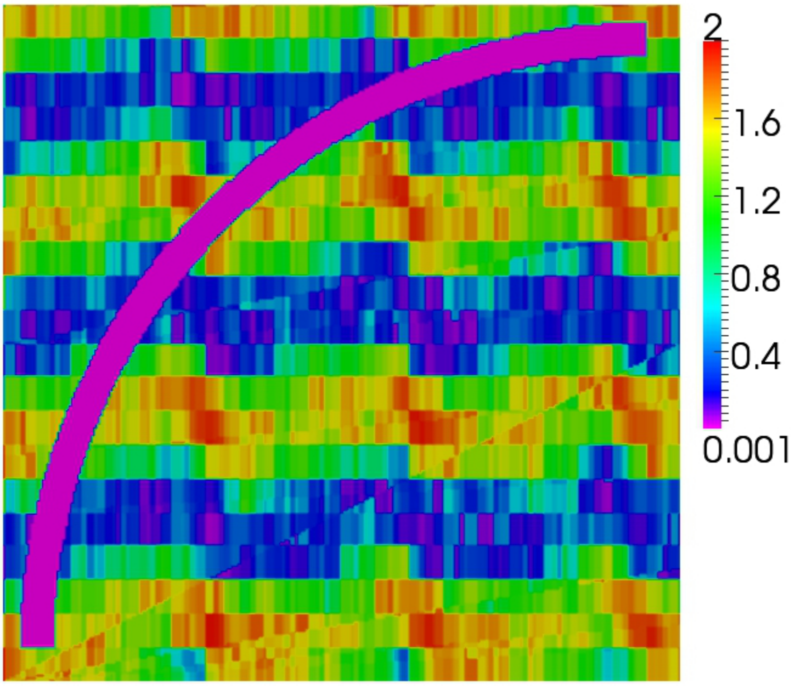

Figure 2: Plot of the rapidly varying and highly heterogeneous diffusion coefficient a_ε given by equation (25), which takes values between 0.01 and 2. The structure is disturbed by an isolating arc (purple) of thickness 0.05 and with conductivity 10^{-3}.

We consider the following model problem. Let $\Omega :=]0,1[^2$ and $\varepsilon := 0.05$. Find $u_\varepsilon \in V$ with

$$\begin{aligned} -\operatorname{div}(a_\varepsilon(x)\nabla u_\varepsilon(x)) &= x_1 - \tfrac{1}{2} && \text{in } \Omega \\ \nabla u_\varepsilon(x) \cdot \nu &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{24}$$

The scalar diffusion coefficient a_ε in equation (24) is depicted in Figure 2. It has a contrast of order 10^3 and is constructed from the highly heterogeneous distribution

$$c_\varepsilon(x_1, x_2) := 1 + \tfrac{1}{10} \sum_{j=0}^{4} \sum_{i=0}^{j} \left(\tfrac{2}{j+1} \cos\left(\lfloor i x_2 - \tfrac{x_1}{1+i} \rfloor + \left\lfloor \tfrac{i x_1}{\varepsilon} \right\rfloor + \left\lfloor \tfrac{x_2}{\varepsilon} \right\rfloor \right) \right)$$

and an isolating arc of radius $r := 0.9$, thickness $\frac{\varepsilon}{2}$ and center $c_0 := (1-\varepsilon, \varepsilon)$. The coefficient a_ε is then given by

$$a_\varepsilon(x) := \begin{cases} 10^{-3} & \text{if } \big||x-c_0| - r\big| < \frac{\varepsilon}{2}, \;\; x_2 > \varepsilon \text{ and } x_1 < 1-\varepsilon \\ (h \circ c_\varepsilon)(x) & \text{else,} \end{cases} \tag{25}$$

$$\text{with } h(t) := \begin{cases} t^4 & \text{for } \frac{1}{2} < t < 1 \\ t^{\frac{3}{2}} & \text{for } 1 < t < \frac{3}{2} \\ t & \text{else.} \end{cases}$$

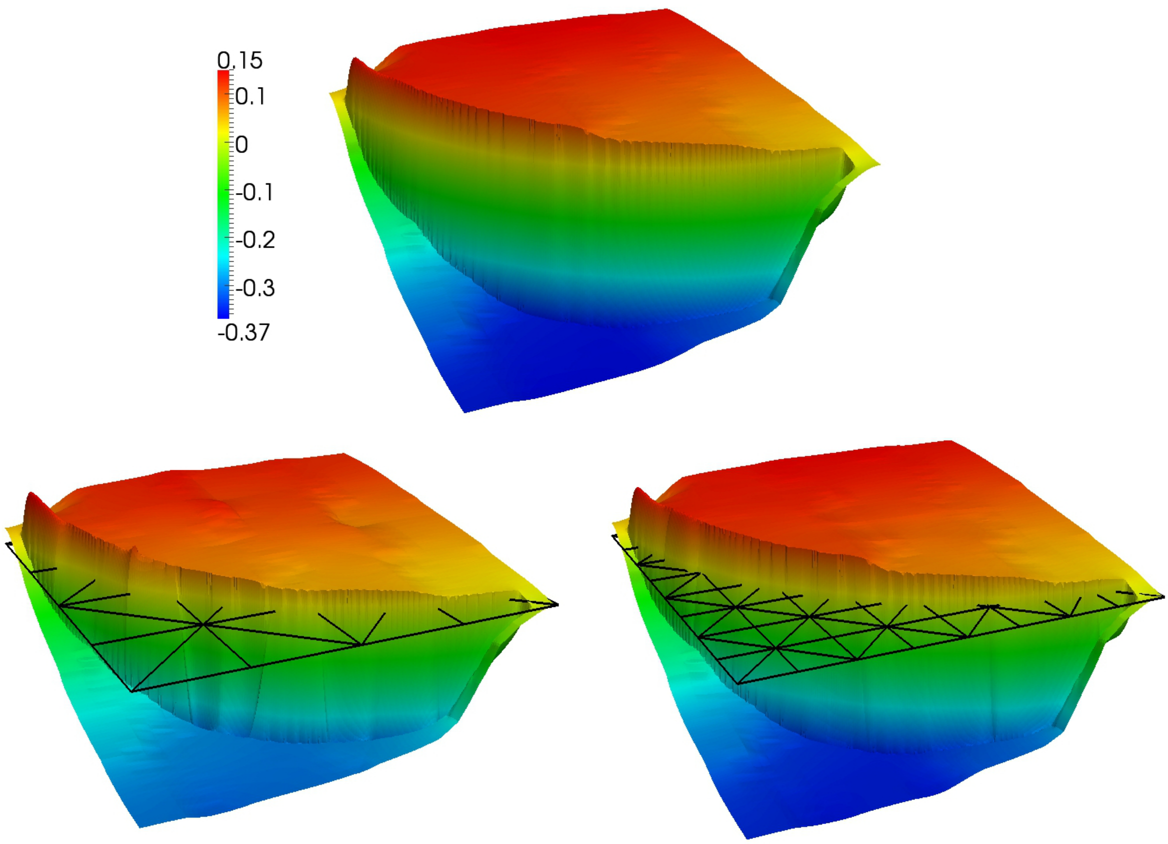

Figure 3: The top picture shows the P_1 finite element reference solution u_h for $h = 2^{-8}$. The left bottom picture shows the multiscale approximation u_c^m for $(H,m) = (2^{-2}, 1)$ together with the corresponding coarse grid. This solution already shows the essential features of u_h. The right bottom picture shows the multiscale approximation u_c^m for $(H,m) = (2^{-3}, 2)$ together with the corresponding coarse grid.

In our computation, we picked the truncation parameter m (according to (23)) to be in the span between 0 and 2 and the coarse mesh size H to be in the span between 2^{-1} (i.e. $h = H^8$) and 2^{-4} (i.e. $h = H^2$). The results are depicted in Table 1. We observe that error stagnates if we decrease only H, without increasing m at the same time. However, already the modification $(H,m) = (2^{-m-1}, m) \mapsto (2^{-m-2}, m+1)$ leads to a dramatic error reduction. Despite the high contrast of order 10^3, we already

Table 1: Results for the relative error between the Multiscale Partition of Unity approximation u_c^m and a reference solution u_h on a fine grid of mesh size $h = 2^{-8} \approx 0.0039 \ll \varepsilon$ which fully resolves the micro structure of the coefficient a_ε. We use the notation $\|u_c^m - u_h\|^{\text{rel}}_{L^2(\Omega)} := \|u_c^m - u_h\|_{L^2(\Omega)}/\|u_h\|_{L^2(\Omega)}$ and analogously the same for $\|u_c^m - u_h\|^{\text{rel}}_{H^1(\Omega)}$. The truncation parameter m determines the patch size and is given by (23).

H	m	$\|u_c^m - u_h\|^{\text{rel}}_{L^2(\Omega)}$	$\|u_c^m - u_h\|^{\text{rel}}_{H^1(\Omega)}$
2^{-1}	0	0.867827	0.93475
2^{-2}	0	0.865630	0.96525
2^{-2}	1	0.167501	0.37387
2^{-3}	1	0.257826	0.61681
2^{-3}	2	0.037841	0.16525
2^{-4}	2	0.063645	0.25613

obtain a highly accurate approximation for $(H, m) = (2^{-3}, 2)$. In this case, the multiscale approximation looks almost identical to the FEM reference solution for $h = 2^{-8}$ (see Figure 3). Further numerical experiments can be found in [19, 20, 28].

References

[1] Assyr Abdulle, Weinan E, Björn Engquist, and Eric Vanden-Eijnden, *The heterogeneous multiscale method*, Acta Numer. **21** (2012), 1–87. MR 2916381

[2] Hans Wilhelm Alt, *Lineare Funktionalanalysis*, Springer-Verlag Berlin Heidelberg, 2006.

[3] Ivo Babuška, Uday Banerjee, and John E. Osborn, *Meshless and generalized finite element methods: a survey of some major results*, Meshfree methods for partial differential equations (Bonn, 2001), Lect. Notes Comput. Sci. Eng., vol. 26, Springer, Berlin, 2003, pp. 1–20. MR 2003426 (2004h:65116)

[4] Ivo Babuška, Gabriel Caloz, and John E. Osborn, *Special finite element methods for a class of second order elliptic problems with rough coefficients*, SIAM J. Numer. Anal. **31** (1994), no. 4, 945–981. MR 1286212 (95g:65146)

[5] Ivo Babuška and Robert Lipton, *Optimal local approximation spaces for generalized finite element methods with application to multiscale problems*, Multiscale Model. Simul. **9** (2011), no. 1, 373–406. MR 2801210 (2012e:65259)

[6] Ivo Babuška and Jens M. Melenk, *The partition of unity method*, International Journal of Numerical Methods in Engineering **40** (1996), 727–758.

[7] Ted Belytschko, Nicolas Moës, Shuji Usui, and Chandu Parimi, *Arbitrary discontinuities in finite elements*, Internat. J. Numer. Methods Engrg. **50** (2001), no. 4, 993–1013.

[8] Carsten Carstensen, *Quasi-interpolation and a posteriori error analysis in finite element methods*, M2AN Math. Model. Numer. Anal. **33** (1999), no. 6, 1187–1202.

[9] C. Armando Duarte, Ivo Babuška, and J. Tinsley Oden, *Generalized finite element methods for three-dimensional structural mechanics problems*, Comput. & Structures **77** (2000), no. 2, 215–232. MR 1768540 (2001b:74053)

[10] C. Armando Duarte and Dae-Jin Kim, *Analysis and applications of a generalized finite element method with globallocal enrichment functions*, Computer Methods in Applied Mechanics and Engineering **197** (2008), no. 68, 487–504.

[11] C. Armando Duarte and J. Tinsley Oden, *An h-p adaptive method using clouds*, Computer Methods in Applied Mechanics and Engineering **139** (1996), no. 14, 237–262.

[12] C. Armando Duarte, Luziana G. Reno, and Angelo Simone, *A high-order generalized FEM for through-the-thickness branched cracks*, Internat. J. Numer. Methods Engrg. **72** (2007), no. 3, 325–351. MR 2355178

[13] Weinan E and Bjorn Engquist, *The heterogeneous multiscale methods*, Commun. Math. Sci. **1** (2003), no. 1, 87–132. MR 1979846 (2004b:35019)

[14] Thomas-Peter Fries and Hermann-Georg Matthies, *Classification and overview of meshfree methods*, Tech. Report 2003-3, Technische Universitt Braunschweig, 2004.

[15] Antoine Gloria, *Reduction of the resonance error—Part 1: Approximation of homogenized coefficients*, Math. Models Methods Appl. Sci. **21** (2011), no. 8, 1601–1630. MR 2826466

[16] Michael Griebel and Marc Alexander Schweitzer, *A particle-partition of unity method for the solution of elliptic, parabolic, and hyperbolic PDEs*, SIAM J. Sci. Comput. **22** (2000), no. 3, 853–890 (electronic). MR 1785338 (2001i:65105)

[17] ———, *A particle-partition of unity method. II. Efficient cover construction and reliable integration*, SIAM J. Sci. Comput. **23** (2002), no. 5, 1655–1682 (electronic). MR 1885078 (2003b:65118)

[18] Varun Gupta, C. Armando Duarte, Babuška I., and Uday Banerjee, *A stable and optimally convergent generalized FEM (SGFEM) for linear elastic fracture mechanics*, Computer Methods in Applied Mechanics and Engineering **266** (2013), no. 0, 23–39.

[19] Patrick Henning, Axel Målqvist, and Daniel Peterseim, *A localized orthogonal decomposition method for semi-linear elliptic problems.*, ESAIM: Mathematical Modelling and Numerical Analysis **eFirst** (2013).

[20] Patrick Henning and Daniel Peterseim, *Oversampling for the Multiscale Finite Element Method*, Multiscale Model. Simul. **11** (2013), no. 4, 1149–1175. MR 3123820

[21] Michael Holst, *Application of domain decomposition and partition of unity methods in physics and geometry*, Domain decomposition methods in science and engineering, Natl. Auton. Univ. Mex., México, 2003, pp. 63–78 (electronic). MR 2093735

[22] Thomas Y. Hou and Xiao-Hui Wu, *A multiscale finite element method for elliptic problems in composite materials and porous media*, J. Comput. Phys. **134** (1997), no. 1, 169–189. MR MR1455261 (98e:73132)

[23] Thomas J. R. Hughes, Gonzalo R. Feijóo, Luca Mazzei, and Jean-Baptiste Quincy, *The variational multiscale method—a paradigm for computational mechanics*, Comput. Methods Appl. Mech. Engrg. **166** (1998), no. 1-2, 3–24. MR 1660141 (99m:65239)

[24] Dae-Jin Kim, C. Armando Duarte, and S. Pedro Proença, *A generalized finite element method with global-local enrichment functions for confined plasticity problems*, Computational Mechanics **50** (2012), no. 5, 563–578 (English).

[25] T. Jadwiga Liszka, C. Armando Duarte, and Woitek Tworzydlo, *hp-meshless cloud method*, Computer Methods in Applied Mechanics and Engineering **139** (1996), no. 14, 263–288.

[26] Axel Målqvist, *Multiscale methods for elliptic problems*, Multiscale Model. Simul. **9** (2011), no. 3, 1064–1086. MR 2831590 (2012j:65419)

[27] Axel Målqvist and Daniel Peterseim, *Computation of eigenvalues by numerical upscaling*, Numerische Mathematik (2014), 1–25 (English).

[28] ______, *Localization of elliptic multiscale problems*, Math. Comp. **83** (2014), no. 290, 2583–2603. MR 3246801

[29] Jens M. Melenk and Ivo Babuška, *The partition of unity finite element method: basic theory and applications*, Comput. Methods Appl. Mech. Engrg. **139(1-4)** (1996), 289–314.

[30] Nicolas Moës, John E. Dolbow, and Ted Belytschko, *A finite element method for crack growth without remeshing*, Internat. J. Numer. Methods Engrg. **46** (1999), no. 1, 131–150.

[31] J. Tinsley Oden, C. Armando Duarte, and Olek C. Zienkiewicz, *A new cloud-based hp finite element method*, Computer Methods in Applied Mechanics and Engineering **153** (1998), no. 12, 117–126.

[32] Houman Owhadi, Lei Zhang, and Leonid Berlyand, *Polyharmonic homogenization, rough polyharmonic splines and sparse super-localization*, ESAIM: Mathematical Modelling and Numerical Analysis **eFirst** (2013).

[33] Marc Alexander Schweitzer, *Generalizations of the finite element method*, Cent. Eur. J. Math. **10** (2012), no. 1, 3–24. MR 2863778 (2012k:65150)

[34] Theofanis Strouboulis, Ivo Babuška, and Kevin Copps, *The design and analysis of the generalized finite element method*, Comput. Methods Appl. Mech. Engrg. **181** (2000), no. 1-3, 43–69. MR 1734667 (2000h:74077)

[35] Theofanis Strouboulis, Kevin Copps, and Ivo Babuška, *The generalized finite element method*, Comput. Methods Appl. Mech. Engrg. **190** (2001), no. 32-33, 4081–4193. MR 1832655 (2002h:65195)

[36] Cheng Wang, Zi-ping Huang, and Li-kang Li, *Two-grid partition of unity method for second order elliptic problems*, Appl. Math. Mech. (English Ed.) **29** (2008), no. 4, 527–533. MR 2405141 (2009b:65329)