Multiscale techniques for solving quadratic eigenvalue problems

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(joint work with Daniel Peterseim)

We consider numerical approximation of quadratic eigenvalue problems arising in structural mechanical systems with damping, using the Localized Orthogonal Decomposition (LOD) technique introduced in [5]. In this approach, a low dimensional generalized finite element space is constructed by solving localized (in space) independent linear stationary problems. The quadratic eigenvalue problem is then solved in the computed low dimensional space at a greatly reduced computational cost.

1. Problem formulation

The setting is as follows, let $V = \mathrm{H}^1_0(\Omega)$,

$$\Omega \subset \mathbb{R}^d \text{ be a polyhedral domain,}$$
$$0 < \kappa_0 \leq \kappa(\cdot) \leq \kappa_1 < \infty,$$
$$c : V \times V \to \mathbf{R} \text{ bounded.}$$

We seek $u \in V$ and $\lambda \in \mathbb{C}$ such that,

$$\langle \kappa \nabla u, \nabla w\rangle + \lambda\, c(u, w) + \lambda^2 \langle u, w\rangle = 0, \tag{1}$$

for all $w \in V$. A simple example of damping is the proportional damping where $c(u, w) = \langle \alpha\kappa\nabla u, \nabla w\rangle + \langle \beta u, w\rangle$ with $\alpha, \beta \in \mathbb{R}$. In this case the eigenmodes coincides with the eigenmodes of the linear generalized eigenvalue problem $\langle \kappa\nabla u, \nabla w\rangle = \lambda\langle u, w\rangle$ but with different eigenvalues. In a more realistic model α, β are allowed to vary in space. This corresponds to a system of proportionally damped components.

The analysis of quadratic eigenvalue problems is often done by linearization. This is achieved by introducing a new variable $v = \lambda u$ and rewriting the problem into a 2×2 system. The resulting linearized operator may or may not be compact and symmetric (depending of the damping), see [7]. Finite element convergence for these problems have been carefully studied e.g. in [1, 2] (compact operator) and [3, 4] (non-compact operator).

2. Generalized finite element approximation

We now present the LOD technique introduced in [5] and later applied to the linear eigenvalue problem in [6].

Let $\mathcal{T}_H$ denote a regular finite element mesh of Ω into closed simplices with mesh-size functions $0 < H \in L^\infty(\Omega)$. The first-order fine and coarse conforming finite element spaces are

$$V_{\mathrm{H}}^{\mathrm{FE}} := \{v \in V \mid \forall T \in \mathcal{T}_H, v|_T \text{ is polynomial of degree} \leq 1\}. \tag{2}$$

By $\mathcal{N}_H$ we denote the set of interior vertices of the mesh. For every vertex z, let ϕ_z denote the corresponding nodal basis function.

We apply a Clément-type interpolation operator $\mathcal{I}_H : V \to V_H$ defined in the following way. Given $v \in V$, $\mathcal{I}_H v := \sum_{z\in\mathcal{N}_H} (\mathcal{I}_H v)(z)\phi_z$ with $(\mathcal{I}_H v)(z) := \frac{(v,\phi_z)_{L^2(\Omega)}}{(1,\phi_z)_{L^2(\Omega)}}$ for $z \in \mathcal{N}_h$. This interpolation operator fulfills the usual interpolation and stability bounds.

We want to decompose V into a low dimensional part, with good approximation properties, and a remainder part. To this end we introduce the remainder space,

$$V_{\mathrm{f}} := \mathrm{kernel}\big(\mathcal{I}_H\big) \subset V,$$

representing the fine scales in the decomposition. The orthogonalization of the decomposition with respect to the scalar product $\langle\kappa\nabla\cdot,\nabla\cdot\rangle$ yields the definition of a modified coarse space $V_{\mathrm{H}}^{\mathrm{LOD}}$, that is the $\langle\kappa\nabla\cdot,\nabla\cdot\rangle$-orthogonal complement of V_{f} in V.

Given $v \in V$, define the orthogonal fine scale projection operator $\mathcal{P}_{\mathrm{f}} v \in V_{\mathrm{f}}$ by

$$\langle\kappa\nabla\mathcal{P}_{\mathrm{f}} v,\nabla w\rangle = \langle\kappa\nabla v,\nabla w\rangle \quad \text{for all } w \in V_{\mathrm{f}}.$$

Lemma 1 (Orthogonal two-scale decomposition). *Any function $u \in V$ can be decomposed uniquely into $u = u_{\mathrm{c}} + u_{\mathrm{f}}$, where*

$$u_{\mathrm{c}} = \mathcal{P}_{\mathrm{c}} u := (1-\mathcal{P}_{\mathrm{f}})u \in (1-\mathcal{P}_{\mathrm{f}})V_H =: V_{\mathrm{H}}^{LOD}$$

and

$$u_{\mathrm{f}} := \mathcal{P}_{\mathrm{f}} u \in V_{\mathrm{f}} = \mathrm{kernel}\,\mathcal{I}_H.$$

The decomposition is orthogonal, $\langle\kappa\nabla u_{\mathrm{c}},\nabla u_{\mathrm{f}}\rangle = 0$.

Proof. See [MaPe12]. □

We are now ready to present the LOD approximation of the quadratic eigenvalue problem. We seek for an approximation of equation (1) in the space $V_{\mathrm{H}}^{\mathrm{LOD}}$. Find eigenfunctions $u_H \in V_{\mathrm{H}}^{\mathrm{LOD}}$ with associated eigenvalues $\lambda_H \in \mathbb{C}$ such that,

$$\langle\kappa\nabla u_H,\nabla w\rangle + \lambda_H\, c(u_H,w) + \lambda_H^2\,\langle u_H,w\rangle = 0 \quad \text{for all } w \in V_{\mathrm{H}}^{\mathrm{LOD}}. \tag{3}$$

If we assume, for some $0 \le s < 1$ that,

$$|c(u,w)| \le C\|u\|_{H^s(\Omega)}\|w\|_{H^s(\Omega)},$$

for all $u,w \in V$, the corresponding linearized operator becomes compact and the analysis in [1] is available. By applying their abstract result with the constructed space $V_{\mathrm{H}}^{\mathrm{LOD}}$ we get the following result.

Theorem 2. *Let λ be an isolated eigenvalue of algebraic multiplicity n and let λ_H be the harmonic mean of the n discrete eigenvalues approximating λ. Under the assumptions above it holds,*

$$|\lambda - \lambda_H| \le CH^{2-2s}, \tag{4}$$

for sufficiently small H.

3. Numerical Experiments

We present two numerical experiments. In both cases we let $\Omega = [0,1]^2$ and κ be piecewise constant on a uniform 64×64 grid taking (uniformly distributed random) values between 0.003 and 1.0. In the first case the damping is $c(u,v) = \langle(1 + sin(10x))u, v\rangle$ and in the second case $c(u,v) = \langle(2 - x - y)\nabla u, \nabla v\rangle$. We let $h = 2^{-6}$ be the reference mesh and vary H form 2^{-1} to 2^{-5}. We plot the relative error in eigenvalue (compared to the reference solution) for the ten lowest eigenvalues versus degrees of freedom N_H, see Figure 3. We detect H^4 convergence

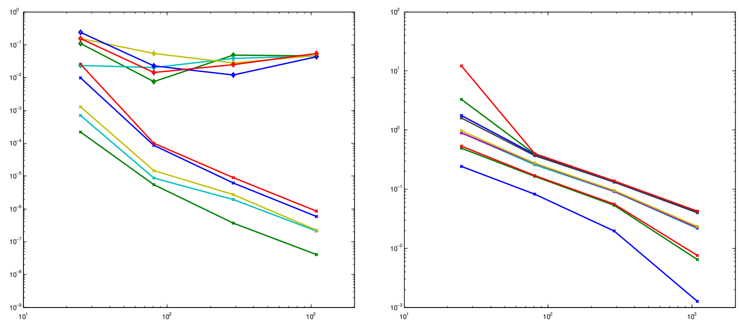

Figure 1. Relative error in the lowest eigenvalues for the two damping cases versus degrees of freedom N_H.

in the first case and H^2 in the second case. This is two orders better than our theoretical prediction. We believe this is because of the similarity between the first damping matrix and the mass matrix (for which the multiscale split is almost orthogonal) and the second damping matrix and the stiffness matrix for which the multiscale split is orthogonal.

References

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