Numerical upscaling of eigenvalue problems

Axel Målqvist, Daniel Peterseim

Angaben zur Veröffentlichung / Publication details:

Målqvist, Axel, and Daniel Peterseim. 2013. "Numerical upscaling of eigenvalue problems." *Oberwolfach Reports* 10 (1): 402–4. https://doi.org/10.4171/owr/2013/07.

Numerical upscaling of eigenvalue problems

Axel Målqvist
(joint work with Daniel Peterseim)

1. Introduction

We present a numerical upscaling technique for computing eigenpairs of self-adjoint linear elliptic second order differential operators with arbitrary positive bounded coefficients. The precise setting is as follows. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with piecewise flat boundary and let $A \in L^\infty(\Omega, \mathbb{R}^{d\times d}_{\text{sym}})$ be a matrix-valued coefficient with uniform spectral bounds. Consider the self-adjoint eigenvalue problem: find eigenpairs (u, λ) such that

$$-\nabla \cdot (A\nabla u) = \lambda u. \tag{1}$$

A standard finite element approximation of these eigenvalues and eigenfunctions is constructed using a shape regular mesh $\mathcal{T}_h$ of Ω with a corresponding finite element space $V_h \subset V := H^1_0(\Omega)$: find eigenpairs $u_h^{(\ell)} \in V_h$ and $\lambda_h^{(\ell)} \in \mathbb{R}$ such that,

$$a(u_h^{(\ell)}, v) := (A\nabla u_h^{(\ell)}, \nabla v) = \lambda_h^{(\ell)}(u_h^{(\ell)}, v), \quad \forall v \in V_h. \tag{2}$$

We are mainly interested in the small eigenvalues. Popular approaches for the computation of these eigenvalues include e.g. Lanczos/Arnoldi-type iterations or the QR-algorithm applied directly to the N_h-dimensional finite element matrices, where $N_h = \dim(V_h)$.

In our approach we avoid the application of an eigenvalue solver to the large-scale problem (2) directly. Instead, inspired by [3], we compute a low-dimensional approximation space $V_{\text{cs}} \subset V_h$ first, with $N_H = \dim(V_{\text{cs}}) \ll N_h$. This preprocessing step is done by (approximately) inverting the operator for special right hand sides and subject to certain linear constraints. Having performed N_H of those computations, the solution of a low-dimensional $N_H \times N_H$ eigenvalue problem by standard solvers yields approximations of the first N_H eigenpairs. The linear problems needed to be solved on the fine scale are totally independent.

Our method is related to some coarse finite element mesh with maximal width H. The accuracy of the approximate eigenvalues is expressed in terms of H. Without any assumptions on the smoothness of eigenfunctions, we prove that the error scales like H^4. Note that a standard first-order conforming finite element computation yields H^2 under full $H^2(\Omega)$ regularity, see e.g. [2]. Under such strong assumption the two-grid method [5] allows certain postprocessing (solution of linear problems on the fine scale) of the coarse finite element eigenpairs to increase the accuracy to H^4. This is also possible to exploit for our proposed method to get even higher order convergence.

2. Galerkin approximation and main result

We let $\mathcal{T}_H$ denote an underlying coarse regular finite element mesh, with mesh function H defined by $H|_T = \text{diam}(T) := H_T$ for all $T \in \mathcal{T}_H$. We denote the

interior nodes of the mesh $\mathcal{N}$. We let $V_H = \text{span}(\{\phi_x\}_{x\in\mathcal{N}})$ be a finite element space such that $V_H \subset V_h$.

We recall the Clément type interpolant $\mathcal{I}_H : V \to V_H$ presented in [1]. Let $\mathcal{I}_H v = \sum_{x\in\mathcal{N}} (\mathcal{I}_H v)(x)\phi_x$, where,

$$(\mathcal{I}_H v)(x) = \frac{(v, \phi_x)}{(1, \phi_x)},$$

for all $x \in \mathcal{N}$. The following approximation and stability property holds,

$$H_T^{-1}\|v - \mathcal{I}_H v\|_{L^2(T)} + \|\nabla(v - \mathcal{I}_H v)\|_{L^2(T)} \leq C\|\nabla v\|_{L^2(\omega_T)}, \quad \forall v \in V,$$

where ω_T is the collection of elements in $\mathcal{T}$ overlapping T.

We are ready to present the decomposition of the space V_h into a coarse and a fine part. We let the fine scale space be defined by,

$$V_{\text{fs}} := \text{kernel } \mathcal{I}_H = \{v \in V_h : \mathcal{I}_H v = 0\},$$

and the coarse scale space by,

$$V_{\text{cs}} = \{v \in V_h : a(v, w) = 0, \text{ for all } w \in V_{\text{fs}}\}.$$

This yields an a-orthogonal split of the space $V_h = V_{\text{cs}} \oplus V_{\text{fs}}$. We note that $\dim(V_{\text{cs}}) = \dim(V_H) = |\mathcal{N}|$. In order to compute a basis for V_{cs} we solve $|\mathcal{N}|$ corrector problems: find $\psi_x \in V_{\text{fs}}$ such that,

$$a(\psi_x, v) = a(\phi_x, v), \quad \text{for all } v \in V_{\text{fs}},$$

and let $V_{\text{cs}} = \text{span}(\{\phi_x - \psi_x\}_{x\in\mathcal{N}})$. The Galekin approximation of equation (2) in the space V_{cs} reads: find $u_{\text{cs}}^{(\ell)} \in V_{\text{cs}}$ and $\lambda_{\text{cs}}^{(\ell)} \in \mathbb{R}$ such that,

$$a(u_{\text{cs}}^{(\ell)}, v) = \lambda_{\text{cs}}^{(\ell)}(u_{\text{cs}}^{(\ell)}, v), \quad v \in V_{\text{cs}}.$$

We now present an error bound for the approximate eigenvalues.

Theorem 1. *Let H be sufficiently small so that $H \leq C\ell^{-1/4}\sqrt{\frac{\alpha}{\lambda_h^{(\ell)}}}$. Then it holds*

$$\frac{\lambda_{cs}^{(\ell)} - \lambda_h^{(\ell)}}{\lambda_h^{(\ell)}} \leq C\sqrt{\ell}\left(H\sqrt{\frac{\lambda_h^{(\ell)}}{\alpha}}\right)^4 \quad \text{for all } \ell = 1, \ldots, |\mathcal{N}|, \tag{3}$$

for some constant C only depending on Ω and the shape regularity constant and with α being the lower spectral bound of A.

Remark 1. In [3] it is shown that $\phi_x - \psi_x$ decays exponentially (in the number of coarse elements) away from node x. This allows the use of truncated patches of size $H\log(H^{-1})$, with Dirichlet boundary conditions, rather than solving for ψ_x on the entire domain Ω. Theorem 1 holds also when using truncated domains.

3. Numerical example

Let $\Omega := (-1,1)^2 \setminus [0,1]^2$ be the L-shaped domain. Consider the constant scalar coefficient $A = 1$ and consider uniform coarse meshes with maximal mesh widths $\sqrt{2}H = 2^{-1}, \dots, 2^{-4}$ of Ω. The reference mesh $\mathcal{T}_h$ has maximal mesh width $h = 2^{-7}/\sqrt{2}$. We consider a $P1$ conforming finite element approximation of the eigenvalues on the reference mesh $\mathcal{T}_h$ and compare these discrete eigenvalues $\lambda_h^{(\ell)}$ with coarse scale approximations depending on the coarse mesh size H.

Table 5 shows results for the case without truncation, i.e., all linear problems have been solved on the whole of Ω. For fixed ℓ, the rate of convergence of the

ℓ	$\lambda_h^{(\ell)}$	$e^{(\ell)}(1/2\sqrt{2})$	$e^{(\ell)}(1/4\sqrt{2})$	$e^{(\ell)}(1/8\sqrt{2})$	$e^{(\ell)}(1/16\sqrt{2})$
1	9.6436869	0.003494567	0.000034466	0.000000546	0.000000010
2	15.1989274	0.009621397	0.000079887	0.000000845	0.000000010
3	19.7421815	0.023813222	0.000213097	0.000002073	0.000000023
4	29.5281571	0.096910157	0.000724615	0.000006574	0.000000076
5	31.9265496	0.094454625	0.000874659	0.000009627	0.000000138
6	41.4922250	-	0.002395227	0.000019934	0.000000254
7	44.9604884	-	0.002443271	0.000019683	0.000000223
8	49.3631826	-	0.003651870	0.000028869	0.000000308
9	49.3655623	-	0.004266472	0.000032835	0.000000355
10	56.7389993	-	0.006863742	0.000055219	0.000000618

Table 5. Errors $e^{(\ell)}(H) =: \frac{\lambda_H^{(\ell)} - \lambda_h^{(\ell)}}{\lambda_h^{(\ell)}}$ for $\ell = 1, \dots, 10$, constant coefficient $A = 1$, and various choices of the coarse mesh size H.

eigenvalue error $\lambda_H^{(\ell)} - \lambda_h^{(\ell)}$ in terms of H observed in Table 5 is between 6 and 7 which is even better than predicted in Theorem 1. For more elaborate numerical results we refer to [4].

References

[1] C. Carstensen, *Quasi-interpolation and a posteriori error analysis in finite element methods*, M2AN Math. Model. Numer. Anal., 33, (1999), 1187–1202.

[2] M. G. Larson, *A posteriori and a priori error analysis for finite element approximations of self-adjoint elliptic eigenvalue problems*, SIAM J. Numer. Anal., 38, (2000), 608–625.

[3] A. Målqvist and D. Peterseim, *Localization of elliptic multiscale problems*, preprint `http://arxiv.org/abs/1110.0692`.

[4] A. Målqvist and D. Peterseim, *Computation of eigenvalues by numerical upscaling*, preprint `http://arxiv.org/abs/1212.0090`.

[5] J. Xu and A. Zhou, *A two-grid discretization scheme for eigenvalue problems*, Math. Comp., 70, (2001), 17–25.